Varying slopes

Statistical Modeling

Robin Elahi

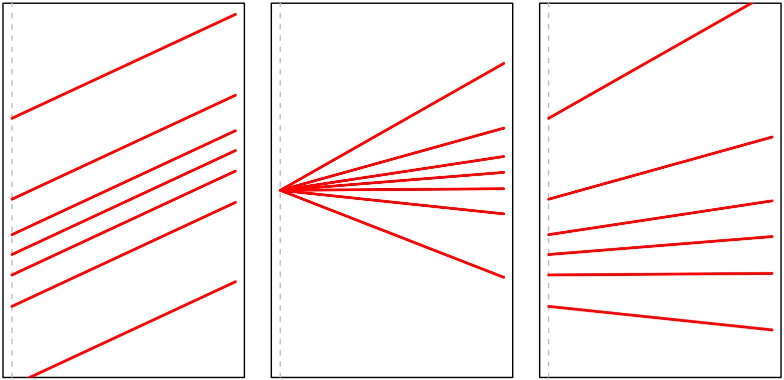
Winter 2025

Hopkins Marine Station, Stanford University

Learning objectives

- Covarying intercepts and slopes
- How to code with `ulam`

Varying intercepts, varying slopes, and *covarying* intercepts and slopes



Pooling *across* parameters

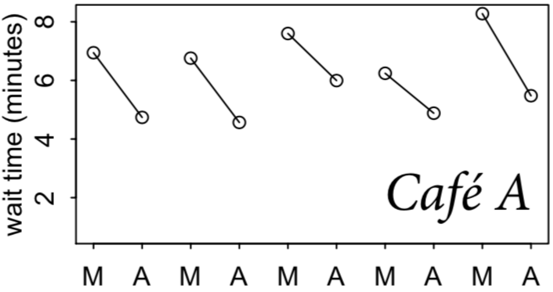
- If slopes and intercepts **covary**, then learning about one of these features gives us info about the other feature
- e.g., higher intercepts are associated with positive slopes, and lower intercepts are associated with negative slopes (previous slide)

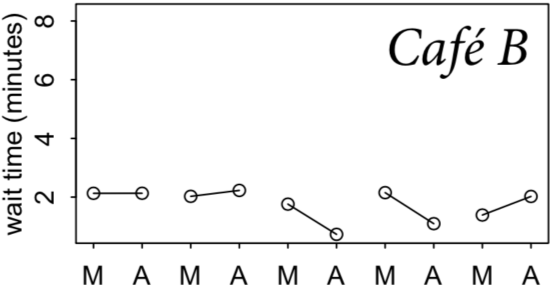
A robot walks into a cafe ...

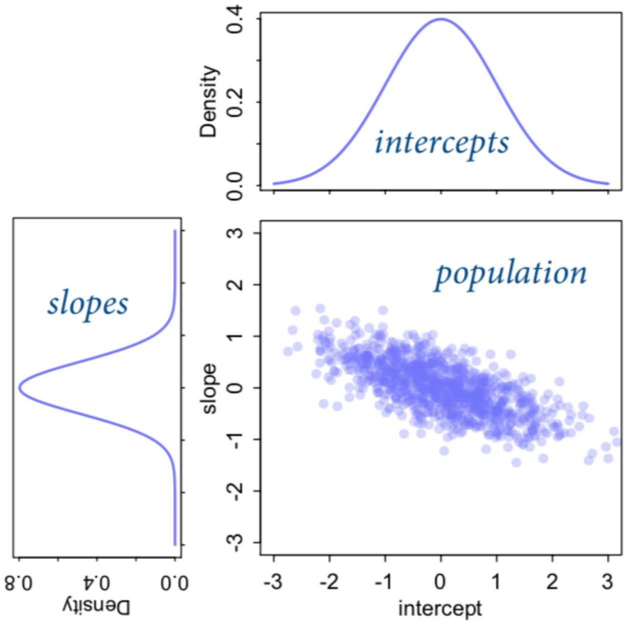
- orders coffee, records wait time (W_i)
- visits in morning (A = 0) and afternoon (A = 1)

$$\mu_i = \alpha_{cafe[i]} + \beta_{cafe[i]} A_i$$

- μ: avg wait time for a given cafe
- α: avg morning wait
- β: avg difference for afternoon wait
- intercepts and slopes are related!


wait time (minutes)
2
4
6
8
Café A
M A M A M A M A M A


wait time (minutes)
2
4
6
8
Café B
M A M A M A M A M A


intercepts
Density
0.0
0.2
0.4
slopes
Density
0.0
0.4
0.8
population
slope
intercept
-3
-2
-1
0
1
2
3

The model, in math

$$
\begin{aligned}
W_i &\sim \text{Normal}(\mu_i, \sigma) && \text{[likelihood]} \\
\mu_i &= \alpha_{cafe[i]} + \beta_{cafe[i]} A_i && \text{[linear model]}
\end{aligned}
$$

The model, in math

$$W_i \sim \text{Normal}(\mu_i, \sigma) \quad \text{[likelihood]}$$

$$\mu_i = \alpha_{cafe[i]} + \beta_{cafe[i]} A_i \quad \text{[linear model]}$$

$$\begin{bmatrix} \alpha_{\text{cafe}} \\ \beta_{\text{cafe}} \end{bmatrix} \sim \text{MVNormal}\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \mathbf{S} \right) \quad \text{[population of varying effects]}$$

The model, in math

$$W_i \sim \text{Normal}(\mu_i, \sigma) \qquad \text{[likelihood]}$$

$$\mu_i = \alpha_{cafe[i]} + \beta_{cafe[i]} A_i \qquad \text{[linear model]}$$

$$\begin{bmatrix} \alpha_{\text{cafe}} \\ \beta_{\text{cafe}} \end{bmatrix} \sim \text{MVNormal}\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \mathbf{S} \right) \qquad \text{[population of varying effects]}$$

$$\mathbf{S} = \begin{pmatrix} \sigma_\alpha & \sigma_\alpha \sigma_\beta \rho \\ \sigma_\alpha \sigma_\beta \rho & \sigma_\beta \end{pmatrix} \qquad \text{[covariance matrix]}$$

Decomposing S

$$\mathbf{S} = \begin{pmatrix} \sigma_\alpha & \sigma_\alpha \sigma_\beta \rho \\ \sigma_\alpha \sigma_\beta \rho & \sigma_\beta \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\beta \end{pmatrix} \mathbf{R} \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\beta \end{pmatrix}$$

Decomposing S

$$\mathbf{S} = \begin{pmatrix} \sigma_\alpha & \sigma_\alpha \sigma_\beta \rho \\ \sigma_\alpha \sigma_\beta \rho & \sigma_\beta \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\beta \end{pmatrix} \mathbf{R} \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\beta \end{pmatrix}$$

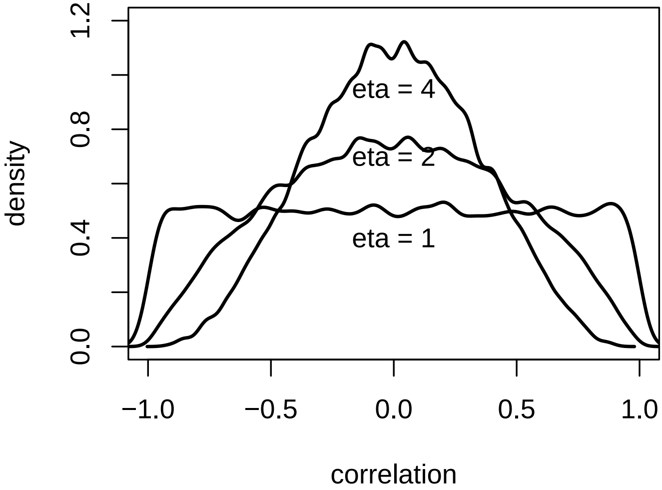
$$\mathbf{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

$\rho =$ correlation coefficient

Priors

$$
\begin{aligned}
\alpha &\sim \text{Normal}(5, 2) && \text{[prior for average intercept]} \\
\beta &\sim \text{Normal}(-1, 0.5) && \text{[prior for average slope]} \\
\sigma &\sim \text{Exponential}(1) && \text{[prior sd within cafes]} \\
\sigma_\alpha &\sim \text{Exponential}(1) && \text{[prior sd among intercepts]} \\
\sigma_\beta &\sim \text{Exponential}(1) && \text{[prior sd among slopes]} \\
\mathbf{R} &\sim \text{LKJcorr}(2) && \text{[prior for correlation matrix]}
\end{aligned}
$$

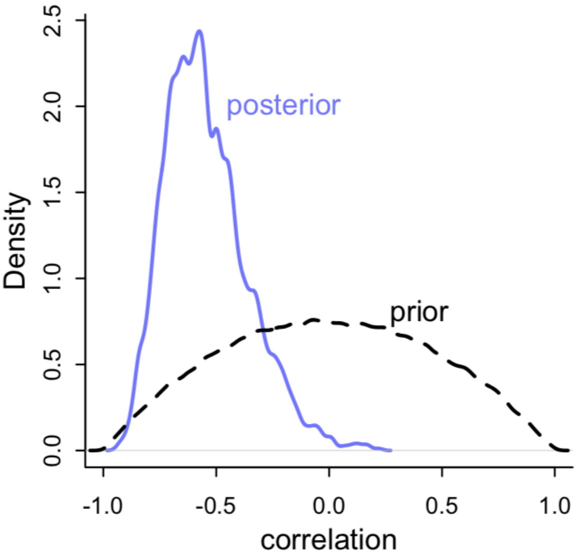
LKJ correlation prior



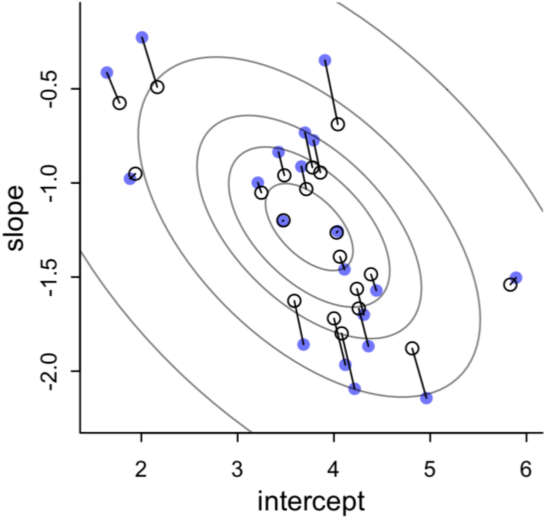
The model, in code

```
m14.1 <- ulam(
    alist(
        wait ~ normal(mu, sigma),
        mu <- a_cafe[cafe] + b_cafe[cafe]*afternoon,
        c(a_cafe,b_cafe)[cafe] ~
          multi_normal(c(a,b), Rho ,sigma_cafe),
        a ~ normal(5,2),
        b ~ normal(-1,0.5),
        sigma_cafe ~ exponential(1),
        sigma ~ exponential(1),
        Rho ~ lkj_corr(2)
    ), data = d, chains = 4, cores = 4)
```

Correlation, prior and posterior



Shrinkage in two dimensions



Exercise

1. Work through code in SR2 14.1
2. Start homework

References

McElreath, Richard. 2020. *Statistical Rethinking: A Bayesian Course with Examples in r and Stan*. CRC Press.