1) X, Y are two jointly distributed random variables.

$$\underbrace{f_{X,Y}(-1,1) = f_{X,Y}(1,1) = f_{X,Y}(0,-2)}_{\text{these 3 pairs of coordinates are equally likely}} = \frac{1}{3} \quad ; \; 0 \text{ everywhere else}$$

What is $\mathrm{cov}(X,Y)$?

use eq. 5.15:

$$\mathrm{Cov}(X,Y) = \underbrace{E(XY)}_{\text{this is the average of } X \text{ times } Y} - \underbrace{E(X)}\underbrace{E(Y)}$$

to get these, we need to calculate the average X across all Y's, and the average Y across all X's

outcome	prob	X	Y	XY
$(-1,1)$	$\frac{1}{3}$	-1	1	$(-1)(1) = -1$
$(1,1)$	$\frac{1}{3}$	1	1	$(1)(1) = 1$
$(0,-2)$	$\frac{1}{3}$	0	-2	$(0)(-2) = 0$

$$E(X) = \frac{(-1+1+0)}{3} = 0 \qquad E(Y) = \frac{(1+1-2)}{3} = 0 \qquad E(XY) = \frac{(-1+1+0)}{3} = 0$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$\therefore \mathrm{cov}(X,Y) = 0$

(1b) Are X and Y independent?

If yes, then $\underbrace{P(A \cap B)}_{\text{we know that this is } \frac{1}{3} \text{ for each possible outcome}} = P(A)\,P(B)$

outcome	X	$P(x)$	Y	$P(y)$	$P(x)\,P(y)$
$(-1, 1)$	-1	$\frac{1}{3}$	1	$\frac{2}{3}$	$= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
$(1, 1)$	1	$\frac{1}{3}$	1	$\frac{2}{3}$	$= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
$(0, -2)$	0	$\frac{1}{3}$	-2	$\frac{1}{3}$	$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

None of these

$= P(XY)$

$= \frac{1}{3}$

2) $\mathrm{Cov}(X,Y) = \boxed{\gamma}$

$\mathrm{Cor}(X,Y) = \rho$

If $Z = \underbrace{a + bX}_{\text{a and b are constants}}$, what is $\mathrm{Cov}(Z,Y)$ and $\mathrm{Cor}(Z,Y)$?

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= E([a+bX][Y]) - E(a+bX)E(Y)$$

$$= E(aY + bXY) - [(E(a) + E(bX))E(Y)]$$

$$= E(aY) + E(bXY) - [(a + bE(X))E(Y)]$$

$$= aE(Y) + bE(XY) - [aE(Y) + bE(X)E(Y)]$$

$$= \cancel{aE(Y)} + bE(XY) - \cancel{aE(Y)} - bE(X)E(Y)$$

$$= bE(XY) - bE(X)E(Y)$$

$$= b(E(XY) - E(X)E(Y))$$

$$= b\,\boxed{\mathrm{Cov}(X,Y)}$$

$$= b\boxed{\gamma}$$

☆ If we add a constant a to X, it does not change cov(X,Y)

☆ if we multiply X by a constant b, Cov(X,Y) is now multiplied by b.

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$\mathrm{Cor}(Z,Y) = \frac{\mathrm{Cov}(Z,Y)}{\sqrt{\mathrm{Var}(a+bX)\,\mathrm{Var}(Y)}}$$

$$= \frac{b\gamma}{\sqrt{b^2\,\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$= \frac{\cancel{b}\gamma}{\cancel{b}\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$= \frac{\boxed{\gamma}}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \qquad \mathrm{cov}(X,Y)$$

$$= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$= \mathrm{Cor}(X,Y)$$

* the correlation between X and Y is unchanged by adding a constant or multiply a constant (to X)

(3) Prove $E(X+Y) = E(X) + E(Y)$

$$X \in \{x_1 \dots x_n\}$$

$$Y \in \{y_1 \dots y_n\}$$

$$E(X+Y) = \sum_{i=1}^{k} \sum_{j=1}^{m} (x_i + y_j)\, f_{X,Y}(x_i, y_j)$$

$$= \sum_{i}^{k} \sum_{j}^{m} x_i f_{X,Y}(x_i, y_j) + y_i f_{X,Y}(x_i, y_j)$$

$$= \sum_{i}^{k} \sum_{j}^{m} x_i f_{X,Y}(x_i, y_j) + \sum_{i}^{k} \sum_{j}^{m} y_i f_{X,Y}(x_i, y_j)$$

$$= \sum_{i}^{k} x_i \sum_{j}^{m} f_{X,Y}(x_i, y_j) + \sum_{i}^{k} f_{X,Y}(x_i, y_j) \sum_{j}^{m} y_j$$

$$= \sum_{i}^{k} x_i f_X(x) + f_Y(y) \sum_{j}^{m} y_j$$

$$= E(X) + E(Y)$$