MATH3360 Homework 2

ZHANG Xinfang (Elaine) 1155141566

Saturday 9$^{\text{th}}$ October, 2021

Q1

(a)

Haar function:

$$H_m(t) = H_{2^p+n}(t) = \begin{cases} 2^{\frac{p}{2}} & \text{if } \frac{n}{2^p} \le t \le \frac{n+0.5}{2^p} \\ -2^{\frac{p}{2}} & \text{if } \frac{n+0.5}{2^p} \le t \le \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

Then we have:

$$\begin{aligned} \int_{\mathbb{R}} [H_m(t)]^2 \, dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^p \, dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} 2^p \, dt \\ &= \int_{\frac{n}{2^p}}^{\frac{n+1}{2^p}} 2^p \, dt \\ &= 2^p[\frac{n+1}{2^p} - \frac{n}{2^p}] \\ &= 1. \end{aligned}$$

Proof done.

(b)

i.

Note that $m = 2^p + n$ for any $m \in \mathbb{N}\backslash\{0\}$, then we have

$$\begin{aligned}
\langle H_0, H_m \rangle &= \int_{\mathbb{R}} H_0 H_m \\
&= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}}\, dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} -2^{\frac{p}{2}}\, dt \\
&= 2^{\frac{p}{2}}[\frac{n+0.5}{2^p} - \frac{n}{2^p}] + (-2^{\frac{p}{2}})[\frac{n+1}{2^p} - \frac{n+0.5}{2^p}] \\
&= 0.
\end{aligned}$$

Proof done.

ii.(A)

Note that $m_1 = 2^{p_1} + n_1$ and $m_2 = 2^{p_2} + n_2$, where $0 \neq m_1 < m_2$, $p_1, p_2 \in \mathbb{N} \cup \{0\}$, $n_1 \in \mathbb{Z} \cap [0, 2^{p_1} - 1]$ and $n_2 \in \mathbb{Z} \cap [0, 2^{p_2} - 1]$.
Suppose $p_1 = p_2 = p$, then we have $n_1 < n_2$, furthermore $n_1 + 1 \leq n_2$. Then for piecewise Haar functions H_{m_1} and H_{m_2}, the boundary points are ordered as

$$\frac{n_1}{2^p} < \frac{n_1 + 0.5}{2^p} < \frac{n_1 + 1}{2^p} \leq \frac{n_2}{2^p} < \frac{n_2 + 0.5}{2^p} < \frac{n_2 + 1}{2^p}.$$

Then for the inner product of H_{m_1} and H_{m_2}:

$$\begin{aligned}
\langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{m_1} H_{m_2} \\
&= \int_{\frac{n_1}{2^p}}^{\frac{n_1+0.5}{2^p}} H_{m_1} H_{m_2}\, dt + \int_{\frac{n_1+0.5}{2^p}}^{\frac{n_1+1}{2^p}} H_{m_1} H_{m_2}\, dt \\
&\quad + \int_{\frac{n_1+1}{2^p}}^{\frac{n_2}{2^p}} H_{m_1} H_{m_2}\, dt + \int_{\frac{n_2}{2^p}}^{\frac{n_2+0.5}{2^p}} H_{m_1} H_{m_2}\, dt \\
&\quad + \int_{\frac{n_2+0.5}{2^p}}^{\frac{n_2+1}{2^p}} H_{m_1} H_{m_2}\, dt \\
&= 0 + 0 + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

Proof done.

ii.(B)

For any integers $a < b$, we must have $a + 1 \leq b$.
Then for $p_1 < p_2$, we have three cases of intersection of non-zero valued interval for H_{m_1} and H_{m_2}.

- $2^{p_2-p_1} n_1 \leq n_2 < 2^{p_2-p_1}(n_1 + 0.5)$
  Then we have $n_2+1 \leq 2^{p_2-p_1}(n_1+0.5)$ and then $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}) \subseteq [\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}})$.
- $2^{p_2-p_1}(n_1 + 0.5) \leq n_2 < 2^{p_2-p_1}(n_1 + 1)$
  Then we have $n_2+1 \leq 2^{p_2-p_1}(n_1+1)$ and then $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}) \subseteq [\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}})$.
- $n_2 < 2^{p_2-p_1} n_1$ or $n_2 \geq 2^{p_2-p_1}(n_1 + 1)$

  $n_2 < 2^{p_2-p_1} n_1$, we have $n_2 + 1 \leq 2^{p_2-p_1} n_1$.

  $n_2 \geq 2^{p_2-p_1}(n_1 + 1)$, we have $n_2 + 1 \geq 2^{p_2-p_1}(n_1 + 1)$.

  Therefore, $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}) \cap [\frac{n_1}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}) = \emptyset$

Hence, in any case, $H_{m_1}(t) = c$ for $c \in \mathbb{R}$ and $t \in [\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$.

$$
\begin{aligned}
\langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{m_1} H_{m_2} \\
&= c \int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} H_{m_2}\, dt \\
&= c \int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}}\, dt + c \int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} -2^{\frac{p_2}{2}}\, dt \\
&= c[\frac{n_2+0.5}{2^{p_2}} - \frac{n_2}{2^{p_2}}]2^{\frac{p_2}{2}} - c[\frac{n_2+1}{2^{p_2}} - \frac{n_2+0.5}{2^{p_2}}]2^{\frac{p_2}{2}} \\
&= 0.
\end{aligned}
$$

Proof done.

Q2

(a)

The Walsh function is defined recursively as follows:

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q}\{W_j(2t) + (-1)^{j+q} W_j(2t-1)\},$$

where $q = 0$ or 1; $j = 0, 1, 2, \ldots$ and $W_0(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$

Firstly, we want to prove for any Walsh function, the non-zero valued interval is $[0, 1)$ by induction.

STEP 1: It's clear that W_0 satisfies the statement.

STEP 2: Suppose W_k satisfies the statement, which is to say $W_k(t) = 0$ for $t < 0$ or $t \ge 1$.

STEP 3: For W_{2k}:

if $t < 0$, we have $2t < 0$ and $2t - 1 < 0$, then $W_{2k}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor}\{W_k(2t) + (-1)^j W_k(2t-1)\} = (-1)^{\lfloor \frac{j}{2} \rfloor}\{0 + (-1)^j 0\} = 0$;

if $t \ge 1$, we have $2t \ge 2 > 1$ and $2t-1 \ge 1$, then $W_{2k}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor}\{W_k(2t) + (-1)^j W_k(2t-1)\} = (-1)^{\lfloor \frac{j}{2} \rfloor}\{0 + (-1)^j 0\} = 0$.

Hence, W_{2k} satisfies the statement, similarly, W_{2k+1} satisfies the statement. Therefore, by induction, we can conclude that for any Walsh function, the non-zero valued interval is $[0, 1)$.
Secondly, we want to prove $\int_{\mathbb{R}}[W_m(t)]^2\,dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$ by induction.
STEP 1: $\int_{\mathbb{R}}[W_0(t)]^2\,dt = \int_0^1 1\,dt = 1$, which satisfies the statement.
STEP 2: Suppose W_k satisfies the statement, which is to say $\int_{\mathbb{R}}[W_k(t)]^2\,dt = 1$.
STEP 3: For W_{2k}, $W_{2k}(t) = (-1)^{\lfloor \frac{k}{2} \rfloor}\{W_k(2t) + (-1)^k W_k(2t-1)\}$.
Since $W_k(2t)W_k(2t-1) = 0$ for any t, we have:

$$\begin{aligned}
\int_{\mathbb{R}}[W_{2k}(t)]^2\,dt &= \int_0^1 \{(-1)^{\lfloor \frac{k}{2} \rfloor}\{W_k(2t) + (-1)^k W_k(2t-1)\}\}^2\,dt \\
&= \int_0^1 \{W_k(2t) + (-1)^k W_k(2t-1)\}^2\,dt \\
&= \int_0^1 W_k^2(2t) + W_k^2(2t-1) + (-1)^{2k} W_k(2t)W_k(2t-1)\,dt \\
&= \int_0^1 W_k^2(2t)\,dt + \int_0^1 W_k^2(2t-1)\,dt + \int_0^1 (-1)^{2k} W_k(2t)W_k(2t-1)\,dt \\
&= \frac{1}{2}\int_0^1 W_k^2(2t)\,d(2t) + \frac{1}{2}\int_0^1 W_k^2(2t-1)\,d(2t-1) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

Hence, $\int_{\mathbb{R}}[W_{2k}(t)]^2\,dt = 1$, similarly, $\int_{\mathbb{R}}[W_{2k+1}(t)]^2\,dt = 1$.
Therefore, by induction, $\int_{\mathbb{R}}[W_m(t)]^2\,dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$.
Proof done.

(b)

i.

Since $j_1 = j_2 = j$ and $m_1 < m_2$, we can know that $q_1 = 0$ and $q_2 = 1$. Then $m_1 = 2j$ and $m_2 = 2j + 1$.

$$
\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_0^1 W_{m_1}(t) W_{m_2}(t)\, dt \\
&= \int_0^1 W_{2j}(t) W_{2j+1}(t)\, dt \\
&= \int_0^1 W_j^2(2t-1) - W_j^2(2t)\, dt \\
&= \frac{1}{2} \int_0^1 W_j^2(2t-1)\, d(2t-1) - \frac{1}{2} \int_0^1 W_j^2(2t)\, d(2t) \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0.
\end{aligned}
$$

Proof done.

ii.

Note that $W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} \{W_j(2t) + (-1)^{j+q} W_j(2t-1)\}$,
$W_0(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$ and $W_1(t) = \begin{cases} -1 & \text{if } 0 \le t < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$,
we prove this proposition by induction on n for $j_2 \le n$, $n \in \mathbb{N}$ below.
Before induction, we prove a lemma: if $m_1 < m_2$, $m_1 = 2j_1 + q1$, $m_2 = 2j_2 + q1$ and $j_1 < j_2$, then $\langle W_{m_1}, W_{m_2} \rangle = (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} [(-1)^{q_1+q_2} + (-1)^{j_1+j_2}] \langle W_{j_1}, W_{j_2} \rangle$.

Proof of Lemma:

$$\begin{aligned}
\langle W_{m_1}, W_{m_2}\rangle &= \langle W_{2j_1+q_1}, W_{2j_2+q_2}\rangle \\
&= \int_0^1 W_{2j_1+q_1}(t) W_{2j_2+q_2}(t)\,dt \\
&= \int_0^1 (-1)^{q_1+q_2+\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor}\{W_{j_1}(2t)W_{j_2}(2t) + (-1)^{j_2+q_2} W_{j_1}(2t)W_{j_2}(2t-1) \\
&\qquad + (-1)^{j_1+q_1} W_{j_2}(2t)W_{j_1}(2t-1) + (-1)^{j_1+j_2+q_1+q_2} W_{j_1}(2t-1)W_{j_2}(2t-1)\}\,dt \\
&= \int_0^1 (-1)^{\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor+q_1+q_2}\{W_{j_1}(2t)W_{j_2}(2t) \\
&\quad + (-1)^{j_1+j_2++q_1+q_2} W_{j_1}(2t-1)W_{j_2}(2t-1)\}\,dt \\
&= (-1)^{\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor+q_1+q_2}\int_0^{\frac{1}{2}} W_{j_1}(2t)W_{j_2}(2t)\,dt \\
&\quad + (-1)^{\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor+j_1+j_2}\int_{\frac{1}{2}}^1 W_{j_1}(2t-1)W_{j_2}(2t-1)\,dt \\
&= (-1)^{\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor+q_1+q_2}\int_0^1 W_{j_1}(t)W_{j_2}(t)\,dt \\
&\quad + (-1)^{\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor+j_1+j_2}\int_0^1 W_{j_1}(t)W_{j_2}(t)\,dt \\
&= (-1)^{\lfloor \frac{j_1}{2}\rfloor+\lfloor \frac{j_2}{2}\rfloor}[(-1)^{q_1+q_2} + (-1)^{j_1+j_2}]\langle W_{j_1}, W_{j_2}\rangle.
\end{aligned}$$

Lemma proof done.

Induction below:

STEP 1: For $n = 1$: since $j_1 < j_2 \le 1$, we have $j_1 = 0$ and $j_2 = 1$.

Then

$$\begin{aligned}
\langle W_{m_1}, W_{m_2}\rangle &= \langle W_{q_1}, W_{2+q_2}\rangle \\
&= [(-1)^{q_1+q_2} - 1]\langle W_0, W_1\rangle \\
&= [(-1)^{q_1+q_2} - 1]\int_0^1 W_1(t)\,dt \\
&= 0.
\end{aligned}$$

STEP 2: Suppose the proposition is true for $j_1 < j_2 \le k$, $k \in \mathbb{N}\cup\{0\}$.

Then we have $\langle W_{2j_1+q_1}, W_{2j_2+q_2}\rangle = 0$ for any $j_1 < j_2 \le k$.

STEP 3: Therefore, for any $j_1 < j_2 \le k+1$: write $j_1 = 2a_1 + b1$ and $j_2 = 2a_2 + b2$, then we have $a_1, a_2 < k$.

$$
\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \langle W_{2j_1+q_1}, W_{2j_2+q_2} \rangle \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor}[(-1)^{q_1+q_2} + (-1)^{j_1+j_2}]\langle W_{j_1}, W_{j_2} \rangle \\
&= c_1 c_2 \langle W_{a_1}, W_{a_2} \rangle
\end{aligned}
$$

where $c_1 = (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor}[(-1)^{q_1+q_2} + (-1)^{j_1+j_2}]$,
$c_2 = (-1)^{\lfloor \frac{a_1}{2} \rfloor + \lfloor \frac{a_2}{2} \rfloor}[(-1)^{b_1+b_2} + (-1)^{a_1+a_2}]$.
From STEP 2,
for $a_1 \neq a_2$, $\langle W_{a_1}, W_{a_2} \rangle = 0$, then $\langle W_{m_1}, W_{m_2} \rangle = 0$;
for $a_1 = a_2$, since $j_1 < j_2$, then $b_1 = 0$, $b_2 = 1$, we have

$$
\begin{aligned}
c_2 &= (-1)^{\lfloor \frac{a_1}{2} \rfloor + \lfloor \frac{a_2}{2} \rfloor}[(-1)^{b_1+b_2} + (-1)^{a_1+a_2}] \\
&= (-1)^{2\lfloor \frac{a_1}{2} \rfloor}[(-1)^{1} + (-1)^{2a_1}] \\
&= -1 + 1 \\
&= 0. \\
\langle W_{m_1}, W_{m_2} \rangle &= c_1 c_2 \langle W_{a_1}, W_{a_2} \rangle \\
&= 0.
\end{aligned}
$$

Then $\langle W_{m_1}, W_{m_2} \rangle = 0$ for $j_1 < j_2 \le k+1$.
Therefore, by mathematical induction, $\langle W_{m_1}, W_{m_2} \rangle = 0$ for any $j_1 < j_2$.

Q3

(a)

For 4×4 matrix, define a matrix U by $U_{x\alpha} = \frac{1}{4} e^{-2\pi j \frac{x\alpha}{4}}$.

Then $U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$. Then DFT of an image g is UgU.

DFT of A is $M = UAU = \frac{1}{8}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -j & 1+j & -1 \\ 0 & 1+j & 2 & 1-j \\ 0 & -1 & 1-j & j \end{pmatrix}$.

DFT of B is $N = UBU = \frac{1}{8}\begin{pmatrix} 2 & -1-j & 0 & -1+j \\ -1-j & j & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1+j & 1 & 0 & -j \end{pmatrix}$.

(b)

After discarding 4 smallest non-zero coefficients in DFT of A, we have the new DFT of A is $C = \frac{1}{8}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1+j & 0 \\ 0 & 1+j & 2 & 1-j \\ 0 & 0 & 1-j & 0 \end{pmatrix}$.

Denote the reconstructed A as D, then

$D = (4U^*)C(4U^*) = 16U^*CU^* = \frac{1}{4}\begin{pmatrix} 5 & -1 & 3 & 1 \\ -1 & 5 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$.

(c)

$A * B = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}$. $\widehat{A * B} = U(A * B)U = \frac{1}{4}\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$.

The point-wise product of $\hat{A}$ and $\hat{B}$ is $\frac{1}{64}\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$.

Therefore, $\widehat{A * B}(p, q) = 16\hat{A}(p, q)\hat{B}(p, q)$ for all p, q.

Q4

(a)

For the DFT formula, we multiply both sides by $\frac{1}{N}e^{-2\pi j(\frac{pm+qn}{N})}$ and sum m over 0 to $N-1$, and n over 0 to $N-1$, then

$$\begin{aligned}
&\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\hat{f}(m,n)e^{-2\pi j(\frac{pm+qn}{N})} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(k,l)e^{2\pi j(\frac{(km+ln)-(pm+qn)}{N})} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\sum_{m=0}^{N-1} e^{2\pi j(\frac{(k-p)m}{N})}\sum_{n=0}^{N-1} e^{2\pi j(\frac{(l-q)n}{N})}. \qquad (*)
\end{aligned}$$

For $s \in \mathbb{Z}\backslash\{0\}$ and $t \in \mathbb{Z}$,

$$\sum_{m=0}^{s-1} e^{2\pi j\frac{tm}{s}} = s\mathbf{1}_{s\mathbb{Z}}(t) = \begin{cases} s & \text{if } t \in s\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, R.H.S of (*):

$$\begin{aligned}
&\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)M\mathbf{1}_{N\mathbb{Z}}(k-p)M\mathbf{1}_{N\mathbb{Z}}(l-q) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\delta(k-p)\delta(l-q) = f(p,q)
\end{aligned}$$

Hence, the inverse DFT is defined by

$$f(p,q) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\hat{f}(m,n)e^{-2\pi j(\frac{pm+qn}{N})}.$$

(b)

Define $U_{x\alpha} = \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}}$, where $0 \leq x,\ \alpha \leq N-1$ and $U = (U_{x\alpha})_{0 \leq x, \alpha \leq N-1}$.
Next, we want to prove U is unitary.
Note that $U_{mn} = \frac{1}{\sqrt{N}} e^{2\pi j \frac{mn}{N}}$, and $U^*_{nm} = \frac{1}{\sqrt{N}} e^{-2\pi j \frac{mn}{N}}$.
For $0 \leq a,\ b \leq N-1$, $a, b \in \mathbb{Z}$, if $a - b = t\mathbb{Z}$ for $t \in \mathbb{Z}$, we can know that $a - b = 0$.

$$
\begin{aligned}
UU^*(a,b) &= \frac{1}{N} \sum_{d=0}^{N-1} e^{2\pi j \frac{ad}{N}} e^{-\pi j \frac{bd}{N}} \\
&= \frac{1}{N} \sum_{d=0}^{N-1} e^{2\pi j \frac{(a-b)d}{N}} \\
&= \frac{1}{N} N \mathbf{1}_{N\mathbb{Z}}(a-b) \\
&= \mathbf{1}_{N\mathbb{Z}}(a-b) \\
&= \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Hence, $UU^* = I$ and $U^*U = I$, which is to say U is unitary.

(c)

For $N \times N$ images g and f, assuming that they are periodically extended, the convolution of them is

$$
v(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m') f(n', m')
$$

Then the DFT of v:

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} v(n,m)e^{2\pi j\frac{pn+qm}{N}} \\
&= \frac{1}{N}\sum_{n'=0}^{N-1}\sum_{m'=0}^{N-1}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} g(n-n',m-m')f(n',m')e^{2\pi j\frac{pn+qm}{N}} \\
&= \frac{1}{N}\sum_{n'=0}^{N-1}\sum_{m'=0}^{N-1} f(n',m')e^{2\pi j\frac{pn'+qm'}{N}} \sum_{n''=-n'}^{N-1-n'}\sum_{m''=-m'}^{N-1-m'} g(n'',m'')e^{2\pi j\frac{pn''+qm''}{N}} \\
&= \hat{f}(p,q)\sum_{n''=-n'}^{N-1-n'}\sum_{m''=-m'}^{N-1-m'} g(n'',m'')e^{2\pi j\frac{pn''+qm''}{N}}
\end{aligned}$$

Since g is periodically extended, $g(n-N,m)=g(n,m)$ and $g(n,m-N)=g(n,m)$. Then

$$\begin{aligned}
&\sum_{n''=-n'}^{N-1-n'}\sum_{m''=-m'}^{N-1-m'} g(n'',m'')e^{2\pi j\frac{pn''+qm''}{N}} \\
&= \sum_{m''=-m'}^{N-1-m'} e^{2\pi j\frac{qm''}{N}} \sum_{n''=-n'}^{N-1-n'} g(n'',m'')e^{2\pi j\frac{pn''}{N}} \\
&= \sum_{m''=-m'}^{N-1-m'} e^{2\pi j\frac{qm''}{N}} \sum_{n''=-n'}^{-1} g(n'',m'')e^{2\pi j\frac{pn''}{N}} + \sum_{m''=-m'}^{N-1-m'} e^{2\pi j\frac{qm''}{N}} \sum_{n''=0}^{N-1-n'} g(n'',m'')e^{2\pi j\frac{pn''}{N}}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n''=-n'}^{-1} g(n'',m'')e^{2\pi j\frac{pn''}{N}} &= \sum_{n'''=N-n'}^{N-1} g(n'''-N,m'')e^{2\pi j\frac{pn'''}{N}}e^{-2\pi jp} \\
&= \sum_{n'''=N-n'}^{N-1} g(n''',m'')e^{2\pi j\frac{pn'''}{N}}
\end{aligned}$$

After doing the similar operation on m'', we have

$$
\begin{aligned}
&\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{N-1-m'} g(n'', m'') e^{2\pi j \frac{pn''+qm''}{N}} \\
&= \sum_{n'''=N-n'}^{2N-1-n'} \sum_{m'''=N-m'}^{2N-1-m'} g(n''', m''') e^{2\pi j \frac{pn'''+qm'''}{N}} \\
&= N\hat{g}(p, q)
\end{aligned}
$$

Therefore,

$$\hat{v}(p, q) = N\hat{g}(p, q)\hat{f}(p, q).$$

(d)

Let $\tilde{g}$ be the shifted image. $\tilde{g}(k, l) = g(k - k_0, l - l_0)$, for $0 \leq k,\ l \leq N - 1$. Then

$$
\begin{aligned}
\hat{\tilde{g}}(m, n) &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k - k_0, l - l_0) e^{2\pi j \frac{km+ln}{N}} \\
&= \frac{1}{N} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l'=-l_0}^{N-1-l_0} g(k', l') e^{2\pi j \frac{k'm+l'n}{N}} e^{2\pi j \frac{k_0m+l_0n}{N}} \\
&= \hat{g}(m, n) e^{2\pi j \frac{k_0m+l_0n}{N}}
\end{aligned}
$$

Therefore, $\hat{\tilde{g}}(m, n) = \hat{g}(m, n) e^{2\pi j \frac{k_0m+l_0n}{N}}$.