

MATH3360 Homework 2

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Q1

(a)

Haar function:

$$H_m(t) = H_{2^p+n}(t) = \begin{cases} 2^{\frac{p}{2}} & \text{if } \frac{n}{2^p} \leq t \leq \frac{n+0.5}{2^p} \\ -2^{\frac{p}{2}} & \text{if } \frac{n+0.5}{2^p} \leq t \leq \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

Then we have:

$$\begin{aligned} \int_{\mathbb{R}} [H_m(t)]^2 dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^p dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} 2^p dt \\ &= \int_{\frac{n}{2^p}}^{\frac{n+1}{2^p}} 2^p dt \\ &= 2^p \left[\frac{n+1}{2^p} - \frac{n}{2^p} \right] \\ &= 1. \end{aligned}$$

Proof done.

(b)

i.

Note that $m = 2^p + n$ for any $m \in \mathbb{N} \setminus \{0\}$, then we have

$$\begin{aligned}
 \langle H_0, H_m \rangle &= \int_{\mathbb{R}} H_0 H_m \\
 &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}} dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} -2^{\frac{p}{2}} dt \\
 &= 2^{\frac{p}{2}} \left[\frac{n+0.5}{2^p} - \frac{n}{2^p} \right] + (-2^{\frac{p}{2}}) \left[\frac{n+1}{2^p} - \frac{n+0.5}{2^p} \right] \\
 &= 0.
 \end{aligned}$$

Proof done.

ii.(A)

Note that $m_1 = 2^{p_1} + n_1$ and $m_2 = 2^{p_2} + n_2$, where $0 \neq m_1 < m_2$, $p_1, p_2 \in \mathbb{N} \cup \{0\}$, $n_1 \in \mathbb{Z} \cap [0, 2^{p_1} - 1]$ and $n_2 \in \mathbb{Z} \cap [0, 2^{p_2} - 1]$.

Suppose $p_1 = p_2 = p$, then we have $n_1 < n_2$, furthermore $n_1 + 1 \leq n_2$. Then for piecewise Haar functions H_{m_1} and H_{m_2} , the boundary points are ordered as

$$\frac{n_1}{2^p} < \frac{n_1 + 0.5}{2^p} < \frac{n_1 + 1}{2^p} \leq \frac{n_2}{2^p} < \frac{n_2 + 0.5}{2^p} < \frac{n_2 + 1}{2^p}.$$

Then for the inner product of H_{m_1} and H_{m_2} :

$$\begin{aligned}
\langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{m_1} H_{m_2} \\
&= \int_{\frac{n_1}{2^p}}^{\frac{n_1+0.5}{2^p}} H_{m_1} H_{m_2} dt + \int_{\frac{n_1+0.5}{2^p}}^{\frac{n_1+1}{2^p}} H_{m_1} H_{m_2} dt \\
&\quad + \int_{\frac{n_1+1}{2^p}}^{\frac{n_2}{2^p}} H_{m_1} H_{m_2} dt + \int_{\frac{n_2}{2^p}}^{\frac{n_2+0.5}{2^p}} H_{m_1} H_{m_2} dt \\
&\quad + \int_{\frac{n_2+0.5}{2^p}}^{\frac{n_2+1}{2^p}} H_{m_1} H_{m_2} dt \\
&= 0 + 0 + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

Proof done.

ii.(B)

For any integers $a < b$, we must have $a + 1 \leq b$.

Then for $p_1 < p_2$, we have three cases of intersection of non-zero valued interval for H_{m_1} and H_{m_2} .

- $2^{p_2-p_1}n_1 \leq n_2 < 2^{p_2-p_1}(n_1 + 0.5)$
Then we have $n_2+1 \leq 2^{p_2-p_1}(n_1+0.5)$ and then $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}) \subseteq [\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}})$.
- $2^{p_2-p_1}(n_1 + 0.5) \leq n_2 < 2^{p_2-p_1}(n_1 + 1)$
Then we have $n_2+1 \leq 2^{p_2-p_1}(n_1+1)$ and then $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}) \subseteq [\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}})$.
- $n_2 < 2^{p_2-p_1}n_1$ or $n_2 \geq 2^{p_2-p_1}(n_1 + 1)$
 $n_2 < 2^{p_2-p_1}n_1$, we have $n_2 + 1 \leq 2^{p_2-p_1}n_1$.
 $n_2 \geq 2^{p_2-p_1}(n_1 + 1)$, we have $n_2 + 1 \geq 2^{p_2-p_1}(n_1 + 1)$.
Therefore, $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}) \cap [\frac{n_1}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}) = \emptyset$

Hence, in any case, $H_{m_1}(t) = c$ for $c \in \mathbb{R}$ and $t \in [\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$.

$$\begin{aligned}
\langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{m_1} H_{m_2} \\
&= c \int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} H_{m_2} dt \\
&= c \int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}} dt + c \int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} -2^{\frac{p_2}{2}} dt \\
&= c \left[\frac{n_2+0.5}{2^{p_2}} - \frac{n_2}{2^{p_2}} \right] 2^{\frac{p_2}{2}} - c \left[\frac{n_2+1}{2^{p_2}} - \frac{n_2+0.5}{2^{p_2}} \right] 2^{\frac{p_2}{2}} \\
&= 0.
\end{aligned}$$

Proof done.

Q2

(a)

The Walsh function is defined recursively as follows:

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} \{W_j(2t) + (-1)^{j+q} W_j(2t-1)\},$$

where $q = 0$ or 1 ; $j = 0, 1, 2, \dots$ and $W_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases}$

Firstly, we want to prove for any Walsh function, the non-zero valued interval is $[0, 1)$ by induction.

STEP 1: It's clear that W_0 satisfies the statement.

STEP 2: Suppose W_k satisfies the statement, which is to say $W_k(t) = 0$ for $t < 0$ or $t \geq 1$.

STEP 3: For W_{2k} :

if $t < 0$, we have $2t < 0$ and $2t-1 < 0$, then $W_{2k}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor} \{W_k(2t) + (-1)^j W_k(2t-1)\} = (-1)^{\lfloor \frac{j}{2} \rfloor} \{0 + (-1)^j 0\} = 0$;

if $t \geq 1$, we have $2t \geq 2 > 1$ and $2t-1 \geq 1$, then $W_{2k}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor} \{W_k(2t) + (-1)^j W_k(2t-1)\} = (-1)^{\lfloor \frac{j}{2} \rfloor} \{0 + (-1)^j 0\} = 0$.

Hence, W_{2k} satisfies the statement, similarly, W_{2k+1} satisfies the statement. Therefore, by induction, we can conclude that for any Walsh function, the non-zero valued interval is $[0, 1)$.

Secondly, we want to prove $\int_{\mathbb{R}} [W_m(t)]^2 dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$ by induction.

STEP 1: $\int_{\mathbb{R}} [W_0(t)]^2 dt = \int_0^1 1 dt = 1$, which satisfies the statement.

STEP 2: Suppose W_k satisfies the statement, which is to say $\int_{\mathbb{R}} [W_k(t)]^2 dt = 1$.

STEP 3: For W_{2k} , $W_{2k}(t) = (-1)^{\lfloor \frac{k}{2} \rfloor} \{W_k(2t) + (-1)^k W_k(2t-1)\}$.

Since $W_k(2t)W_k(2t-1) = 0$ for any t , we have:

$$\begin{aligned}
\int_{\mathbb{R}} [W_{2k}(t)]^2 dt &= \int_0^1 \{(-1)^{\lfloor \frac{k}{2} \rfloor} \{W_k(2t) + (-1)^k W_k(2t-1)\}\}^2 dt \\
&= \int_0^1 \{W_k(2t) + (-1)^k W_k(2t-1)\}^2 dt \\
&= \int_0^1 W_k^2(2t) + W_k^2(2t-1) + (-1)^{2k} W_k(2t)W_k(2t-1) dt \\
&= \int_0^1 W_k^2(2t) dt + \int_0^1 W_k^2(2t-1) dt + \int_0^1 (-1)^{2k} W_k(2t)W_k(2t-1) dt \\
&= \frac{1}{2} \int_0^1 W_k^2(2t) d(2t) + \frac{1}{2} \int_0^1 W_k^2(2t-1) d(2t-1) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

Hence, $\int_{\mathbb{R}} [W_{2k}(t)]^2 dt = 1$, similarly, $\int_{\mathbb{R}} [W_{2k+1}(t)]^2 dt = 1$.

Therefore, by induction, $\int_{\mathbb{R}} [W_m(t)]^2 dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$.

Proof done.

(b)

i.

Since $j_1 = j_2 = j$ and $m_1 < m_2$, we can know that $q_1 = 0$ and $q_2 = 1$. Then $m_1 = 2j$ and $m_2 = 2j + 1$.

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_0^1 W_{m_1}(t) W_{m_2}(t) dt \\
&= \int_0^1 W_{2j}(t) W_{2j+1}(t) dt \\
&= \int_0^1 W_j^2(2t-1) - W_j^2(2t) dt \\
&= \frac{1}{2} \int_0^1 W_j^2(2t-1) d(2t-1) - \frac{1}{2} \int_0^1 W_j^2(2t) d(2t) \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0.
\end{aligned}$$

Proof done.

ii.

Note that $W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} \{W_j(2t) + (-1)^{j+q} W_j(2t-1)\}$,

$$W_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases} \text{ and } W_1(t) = \begin{cases} -1 & \text{if } 0 \leq t < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

we prove this proposition by induction on n for $j_2 \leq n$, $n \in \mathbb{N}$ below.

Before induction, we prove a lemma: if $m_1 < m_2$, $m_1 = 2j_1 + q_1$, $m_2 = 2j_2 + q_2$ and $j_1 < j_2$, then $\langle W_{m_1}, W_{m_2} \rangle = (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} [(-1)^{q_1+q_2} + (-1)^{j_1+j_2}] \langle W_{j_1}, W_{j_2} \rangle$.

Proof of Lemma:

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \langle W_{2j_1+q_1}, W_{2j_2+q_2} \rangle \\
&= \int_0^1 W_{2j_1+q_1}(t) W_{2j_2+q_2}(t) dt \\
&= \int_0^1 (-1)^{q_1+q_2+\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \{ W_{j_1}(2t) W_{j_2}(2t) + (-1)^{j_2+q_2} W_{j_1}(2t) W_{j_2}(2t-1) \\
&\quad + (-1)^{j_1+q_1} W_{j_2}(2t) W_{j_1}(2t-1) + (-1)^{j_1+j_2+q_1+q_2} W_{j_1}(2t-1) W_{j_2}(2t-1) \} dt \\
&= \int_0^1 (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1+q_2} \{ W_{j_1}(2t) W_{j_2}(2t) \\
&\quad + (-1)^{j_1+j_2+q_1+q_2} W_{j_1}(2t-1) W_{j_2}(2t-1) \} dt \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1+q_2} \int_0^{\frac{1}{2}} W_{j_1}(2t) W_{j_2}(2t) dt \\
&\quad + (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + j_1+j_2} \int_{\frac{1}{2}}^1 W_{j_1}(2t-1) W_{j_2}(2t-1) dt \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1+q_2} \int_0^1 W_{j_1}(t) W_{j_2}(t) dt \\
&\quad + (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + j_1+j_2} \int_0^1 W_{j_1}(t) W_{j_2}(t) dt \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} [(-1)^{q_1+q_2} + (-1)^{j_1+j_2}] \langle W_{j_1}, W_{j_2} \rangle.
\end{aligned}$$

Lemma proof done.

Induction below:

STEP 1: For $n = 1$: since $j_1 < j_2 \leq 1$, we have $j_1 = 0$ and $j_2 = 1$.

Then

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \langle W_{q_1}, W_{2+q_2} \rangle \\
&= [(-1)^{q_1+q_2} - 1] \langle W_0, W_1 \rangle \\
&= [(-1)^{q_1+q_2} - 1] \int_0^1 W_1(t) dt \\
&= 0.
\end{aligned}$$

STEP 2: Suppose the proposition is true for $j_1 < j_2 \leq k$, $k \in \mathbb{N} \cup \{0\}$.

Then we have $\langle W_{2j_1+q_1}, W_{2j_2+q_2} \rangle = 0$ for any $j_1 < j_2 \leq k$.

STEP 3: Therefore, for any $j_1 < j_2 \leq k + 1$: write $j_1 = 2a_1 + b_1$ and $j_2 = 2a_2 + b_2$, then we have $a_1, a_2 < k$.

$$\begin{aligned}\langle W_{m_1}, W_{m_2} \rangle &= \langle W_{2j_1+q_1}, W_{2j_2+q_2} \rangle \\ &= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} [(-1)^{q_1+q_2} + (-1)^{j_1+j_2}] \langle W_{j_1}, W_{j_2} \rangle \\ &= c_1 c_2 \langle W_{a_1}, W_{a_2} \rangle\end{aligned}$$

$$\begin{aligned}\text{where } c_1 &= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} [(-1)^{q_1+q_2} + (-1)^{j_1+j_2}], \\ c_2 &= (-1)^{\lfloor \frac{a_1}{2} \rfloor + \lfloor \frac{a_2}{2} \rfloor} [(-1)^{b_1+b_2} + (-1)^{a_1+a_2}].\end{aligned}$$

From STEP 2,

for $a_1 \neq a_2$, $\langle W_{a_1}, W_{a_2} \rangle = 0$, then $\langle W_{m_1}, W_{m_2} \rangle = 0$;

for $a_1 = a_2$, since $j_1 < j_2$, then $b_1 = 0$, $b_2 = 1$, we have

$$\begin{aligned}c_2 &= (-1)^{\lfloor \frac{a_1}{2} \rfloor + \lfloor \frac{a_2}{2} \rfloor} [(-1)^{b_1+b_2} + (-1)^{a_1+a_2}] \\ &= (-1)^{2\lfloor \frac{a_1}{2} \rfloor} [(-1)^1 + (-1)^{2a_1}] \\ &= -1 + 1 \\ &= 0. \\ \langle W_{m_1}, W_{m_2} \rangle &= c_1 c_2 \langle W_{a_1}, W_{a_2} \rangle \\ &= 0.\end{aligned}$$

Then $\langle W_{m_1}, W_{m_2} \rangle = 0$ for $j_1 < j_2 \leq k + 1$.

Therefore, by mathematical induction, $\langle W_{m_1}, W_{m_2} \rangle = 0$ for any $j_1 < j_2$.

Q3

(a)

For 4×4 matrix, define a matrix U by $U_{x\alpha} = \frac{1}{4}e^{-2\pi j \frac{x\alpha}{4}}$.

Then $U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$. Then DFT of an image g is UgU .

$$\begin{aligned} \text{DFT of } A \text{ is } M = UAU &= \frac{1}{8} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -j & 1+j & -1 \\ 0 & 1+j & 2 & 1-j \\ 0 & -1 & 1-j & j \end{pmatrix}. \\ \text{DFT of } B \text{ is } N = UBU &= \frac{1}{8} \begin{pmatrix} 2 & -1-j & 0 & -1+j \\ -1-j & j & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1+j & 1 & 0 & -j \end{pmatrix}. \end{aligned}$$

(b)

After discarding 4 smallest non-zero coefficients in DFT of A, we have the

$$\text{new DFT of A is } C = \frac{1}{8} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1+j & 0 \\ 0 & 1+j & 2 & 1-j \\ 0 & 0 & 1-j & 0 \end{pmatrix}.$$

Denote the reconstructed A as D , then

$$D = (4U^*)C(4U^*) = 16U^*CU^* = \frac{1}{4} \begin{pmatrix} 5 & -1 & 3 & 1 \\ -1 & 5 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(c)

$$A * B = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}. \widehat{A * B} = U(A * B)U = \frac{1}{4} \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

$$\text{The point-wise product of } \hat{A} \text{ and } \hat{B} \text{ is } \frac{1}{64} \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Therefore, $\widehat{A * B}(p, q) = 16\hat{A}(p, q)\hat{B}(p, q)$ for all p, q .

Q4

(a)

For the DFT formula, we multiply both sides by $\frac{1}{N}e^{-2\pi j(\frac{pm+qn}{N})}$ and sum m over 0 to $N-1$, and n over 0 to $N-1$, then

$$\begin{aligned}
& \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m, n) e^{-2\pi j(\frac{pm+qn}{N})} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(k, l) e^{2\pi j(\frac{(km+ln)-(pm+qn)}{N})} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) \sum_{m=0}^{N-1} e^{2\pi j(\frac{(k-p)m}{N})} \sum_{n=0}^{N-1} e^{2\pi j(\frac{(l-q)n}{N})}. \quad (*)
\end{aligned}$$

For $s \in \mathbb{Z} \setminus \{0\}$ and $t \in \mathbb{Z}$,

$$\sum_{m=0}^{s-1} e^{2\pi j \frac{tm}{s}} = s \mathbf{1}_{s\mathbb{Z}}(t) = \begin{cases} s & \text{if } t \in s\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, R.H.S of (*):

$$\begin{aligned}
& \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) M \mathbf{1}_{N\mathbb{Z}}(k-p) M \mathbf{1}_{N\mathbb{Z}}(l-q) \\
&= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) \delta(k-p) \delta(l-q) = f(p, q)
\end{aligned}$$

Hence, the inverse DFT is defined by

$$f(p, q) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m, n) e^{-2\pi j(\frac{pm+qn}{N})}.$$

(b)

Define $U_{x\alpha} = \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}}$, where $0 \leq x, \alpha \leq N-1$ and $U = (U_{x\alpha})_{0 \leq x, \alpha \leq N-1}$. Next, we want to prove U is unitary. Note that $U_{mn} = \frac{1}{\sqrt{N}} e^{2\pi j \frac{mn}{N}}$, and $U_{nm}^* = \frac{1}{\sqrt{N}} e^{-2\pi j \frac{mn}{N}}$. For $0 \leq a, b \leq N-1$, $a, b \in \mathbb{Z}$, if $a - b = tN$ for $t \in \mathbb{Z}$, we can know that $a - b = 0$.

$$\begin{aligned} UU^*(a, b) &= \frac{1}{N} \sum_{d=0}^{N-1} e^{2\pi j \frac{ad}{N}} e^{-\pi j \frac{bd}{N}} \\ &= \frac{1}{N} \sum_{d=0}^{N-1} e^{2\pi j \frac{(a-b)d}{N}} \\ &= \frac{1}{N} N \mathbf{1}_{N\mathbb{Z}}(a - b) \\ &= \mathbf{1}_{N\mathbb{Z}}(a - b) \\ &= \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence, $UU^* = I$ and $U^*U = I$, which is to say U is unitary.

(c)

For $N \times N$ images g and f , assuming that they are periodically extended, the convolution of them is

$$v(n, m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n - n', m - m') f(n', m')$$

Then the DFT of v :

$$\begin{aligned}
& \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} v(n, m) e^{2\pi j \frac{pn+qm}{N}} \\
&= \frac{1}{N} \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} g(n-n', m-m') f(n', m') e^{2\pi j \frac{pn+qm}{N}} \\
&= \frac{1}{N} \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} f(n', m') e^{2\pi j \frac{pn'+qm'}{N}} \sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{N-1-m'} g(n'', m'') e^{2\pi j \frac{pn''+qm''}{N}} \\
&= \hat{f}(p, q) \sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{N-1-m'} g(n'', m'') e^{2\pi j \frac{pn''+qm''}{N}}
\end{aligned}$$

Since g is periodically extended, $g(n-N, m) = g(n, m)$ and $g(n, m-N) = g(n, m)$. Then

$$\begin{aligned}
& \sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{N-1-m'} g(n'', m'') e^{2\pi j \frac{pn''+qm''}{N}} \\
&= \sum_{m''=-m'}^{N-1-m'} e^{2\pi j \frac{qm''}{N}} \sum_{n''=-n'}^{N-1-n'} g(n'', m'') e^{2\pi j \frac{pn''}{N}} \\
&= \sum_{m''=-m'}^{N-1-m'} e^{2\pi j \frac{qm''}{N}} \sum_{n''=-n'}^{-1} g(n'', m'') e^{2\pi j \frac{pn''}{N}} + \sum_{m''=-m'}^{N-1-m'} e^{2\pi j \frac{qm''}{N}} \sum_{n''=0}^{N-1-n'} g(n'', m'') e^{2\pi j \frac{pn''}{N}}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n''=-n'}^{-1} g(n'', m'') e^{2\pi j \frac{pn''}{N}} &= \sum_{n'''=N-n'}^{N-1} g(n'''-N, m'') e^{2\pi j \frac{pn'''}{N}} e^{-2\pi j p} \\
&= \sum_{n'''=N-n'}^{N-1} g(n''', m'') e^{2\pi j \frac{pn'''}{N}}
\end{aligned}$$

After doing the similar operation on m'' , we have

$$\begin{aligned}
& \sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{N-1-m'} g(n'', m'') e^{2\pi j \frac{pn''+qm''}{N}} \\
&= \sum_{n'''=N-n'}^{2N-1-n'} \sum_{m'''=N-m'}^{2N-1-m'} g(n''', m''') e^{2\pi j \frac{pn'''+qm'''}{N}} \\
&= N\hat{g}(p, q)
\end{aligned}$$

Therefore,

$$\hat{v}(p, q) = N\hat{g}(p, q)\hat{f}(p, q).$$

(d)

Let \tilde{g} be the shifted image. $\tilde{g}(k, l) = g(k - k_0, l - l_0)$, for $0 \leq k, l \leq N - 1$. Then

$$\begin{aligned}
\hat{\tilde{g}}(m, n) &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k - k_0, l - l_0) e^{2\pi j \frac{km+ln}{N}} \\
&= \frac{1}{N} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l'=-l_0}^{N-1-l_0} g(k', l') e^{2\pi j \frac{k'm+l'n}{N}} e^{2\pi j \frac{k_0m+l_0n}{N}} \\
&= \hat{g}(m, n) e^{2\pi j \frac{k_0m+l_0n}{N}}
\end{aligned}$$

Therefore, $\hat{\tilde{g}}(m, n) = \hat{g}(m, n) e^{2\pi j \frac{k_0m+l_0n}{N}}$.