
1: Inverse method for Poisson Distribution (25%)

For discrete Poisson Distribution ($\lambda = 5$),

the p.m.f is $P(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!}$ and the c.d.f is $F(x|\lambda) = \sum_{t \leq x} e^{-\lambda} \frac{\lambda^t}{t!}$.

Algorithm: Inverse method for the Poisson Distribution:

To generate $X \sim F(x)$:

STEP 1: Generate $U \sim \text{unif}[0, 1]$;

STEP 2: Transform $X = F^{-1}(U)$: if $F(x|\lambda) < U \leq F(x+1|\lambda)$, let $X = x+1$.

Plot :

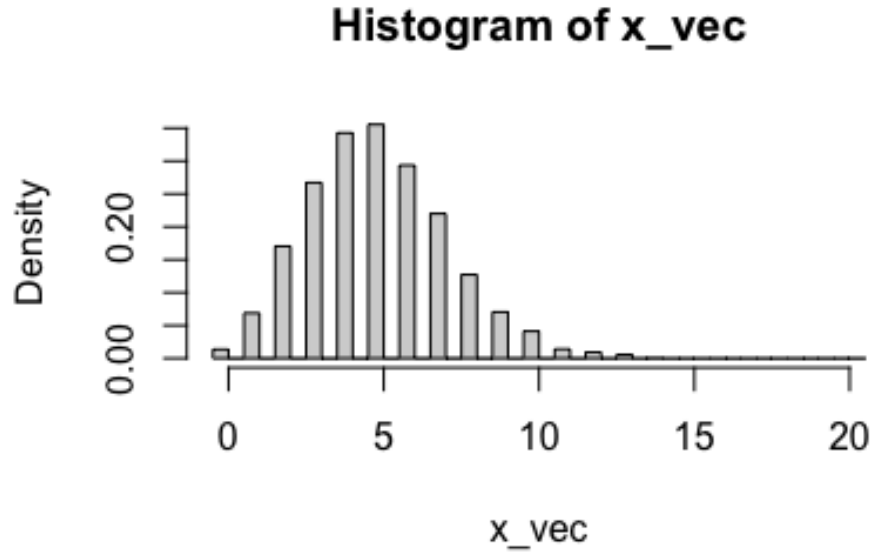


Figure 1: Histogram of 5000 samples

2: Accept-Reject method for truncated Gamma Distribution (25%)

For $X \sim \text{Gamma}(\frac{1}{2}, 1)I(x \geq 5)$, $f(x) = \frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}$.

We can define a shifted exponential distribution $g(x) = e^{-(x-5)} I(x \geq 5)$ and want to find a constant M such that $f(x) < Mg(x)$ for any x .

Then $M = \sup \frac{f(x)}{g(x)} = \sup \frac{\frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}}{e^{-(x-5)} I(x \geq 5)} = \frac{5^{-\frac{1}{2}} e^{-5}}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}$.

Algorithm: Accept-Reject method for truncated Gamma Distribution:

To generate $X \sim F(x) = \text{c.d.f of } f(x)$:

STEP 1: Generate $Y \sim g(y)$;

STEP 2: Generate $U \sim \text{unif}[0, 1]$;

STEP 3: Accept $X = Y$ if $U \leq \frac{f(Y)}{Mg(Y)}$.

Proof :

From the choice of constant M , we can know that $Mg(x) \geq f(x)$. The goal of this method is to generate $X \sim F(x) = \text{c.d.f of } f(x)$.

For the generating algorithm:

$$\begin{aligned}
 P(X \leq x) &= P(Y \leq x | Y \text{ is accepted}) \\
 &= P(Y \leq x | U \leq \frac{f(Y)}{Mg(Y)}) \\
 &= \frac{P(Y \leq x, U \leq \frac{f(Y)}{Mg(Y)})}{P(U \leq \frac{f(Y)}{Mg(Y)})} \\
 &= \frac{\int_{-\infty}^x g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1 dudy}{\int_{-\infty}^{+\infty} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1 dudy} \\
 &= \frac{\int_{-\infty}^x g(y) \frac{f(y)}{Mg(y)} dy}{\int_{-\infty}^{+\infty} g(y) \frac{f(y)}{Mg(y)} dy} \\
 &= \frac{\int_{-\infty}^x f(y) dy}{\int_{-\infty}^{+\infty} f(y) dy} \\
 &= \frac{\int_{-\infty}^x f(y) dy}{1} \\
 &= F(x)
 \end{aligned}$$

Therefore, this AR method works.

Comparison :

Theoretical acceptance probability:

$$\begin{aligned}
 P(U \leq \frac{f(Y)}{Mg(Y)}) &= \int_{-\infty}^{+\infty} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1 dudy \\
 &= \frac{1}{M} \int_{-\infty}^{+\infty} f(y) dy \\
 &= \frac{1}{M}
 \end{aligned}$$

After computation, this acceptance probability is 0.184157.

The actual acceptance rate is 0.1854, which is a little bit higher than the theoretical value.

3: Importance Sampling for Estimation (25%)

(1) Using 5000 samples from Q2 (l = length of samples obtained in Q2), the Monte Carlo estimate is

$$\begin{aligned} \int_5^{+\infty} \cos(x) x^{-\frac{1}{2}} e^{-x} dx &= \int_{-\infty}^{+\infty} \cos(x) \frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} dx \times \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \\ &= \frac{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}{l} \sum_{i=1}^l \cos(x_i) \\ &= 0.001708 \end{aligned}$$

(2) Using the same notations in Q2, we define $h(x) = \cos(x)$.

$$\begin{aligned} \int_5^{+\infty} \cos(x) x^{-\frac{1}{2}} e^{-x} dx &= \int_{-\infty}^{+\infty} \cos(x) \frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} dx \times \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \\ &= \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \int_{-\infty}^{+\infty} \frac{h(x) f(x)}{g(x)} dx \end{aligned}$$

Note that $\frac{f(x)}{g(x)} \leq M < \infty$, and $E_g h^2(x) = \int_{-\infty}^{+\infty} g(x) h^2(x) dx \leq \int_{-\infty}^{+\infty} g(x) dx = 1 < \infty$, we can use the importance sampling as follows:

Algorithm:

STEP 1: Generate $n = 5000$ samples from $g(x)$ by Accept-Reject Method;

STEP 2: Compute the Monte Carlo estimate:

$$m(x) = \frac{\sum_{i=1}^n \frac{h(x_i) f(x_i)}{g(x_i)}}{n} = \frac{e^{-5}}{n \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} \sum_{i=1}^n \cos(x_i) x_i^{-\frac{1}{2}}$$

4: Stratified Sampling (25%)

(1)

(2)

(3)