

1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.

$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function f has totally 3 zeros.

Algorithm: Bisection Method in the R file.

Result: Zeros: -4.837944, -2.259727, and 1.097664.

2: Poisson Regression - Newton's Method (25%)

(1) Since $y_i \sim \text{Poisson}(\lambda_i)$ and $\log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihood function:

$$L(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \prod_{i=1}^n \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^n \frac{e^{(\alpha + \beta x_i + \gamma x_i^2)y_i} e^{-e^{\alpha + \beta x_i + \gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha + \beta x_i + \gamma x_i^2} - \log y_i!$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n [y_i - e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n [x_i y_i - x_i e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\beta}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \right|_{\hat{\gamma}} &= \sum_{i=1}^n [x_i^2 y_i - x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\gamma}} = 0 \end{aligned}$$

$$\text{Let } \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \end{pmatrix}, \text{ then } \mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} \end{pmatrix}, \text{ in}$$

which

$$\begin{aligned}
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^n -e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} = \sum_{i=1}^n -x_i e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} = \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} = \sum_{i=1}^n -x_i^3 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} &= \sum_{i=1}^n -x_i^4 e^{\alpha + \beta x_i + \gamma x_i^2}
\end{aligned}$$

Therefore, by Newton's Method, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$;

STEP 2: Update by $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

3: Logistic Regression - Newton's Method (20%)

(1) Since $y_i \sim \text{Bernoulli}(p_i)$ and $\text{logit}(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i} (1 - p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}$. Then we can get the following Likelihood function:

$$\begin{aligned}
L(\alpha, \beta | \mathbf{x}, \mathbf{y}) &= \prod_{i=1}^n p_i^{y_i} (1 - p_i)^{1-y_i} \\
&= \prod_{i=1}^n \left(\frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right)^{y_i} \left(1 - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right)^{1-y_i}
\end{aligned}$$

(2) The log-Likelihood function is

$$\begin{aligned} l(\alpha, \beta | \mathbf{x}, \mathbf{y}) &= \sum_{i=1}^n y_i(\alpha + \beta x_i - \log(1 + e^{\alpha + \beta x_i})) + (1 - y_i) \log\left(\frac{1}{1 + e^{\alpha + \beta x_i}}\right) \\ &= \sum_{i=1}^n \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha + \beta x_i}) \end{aligned}$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n \left[y_i - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n \left[x_i y_i - \frac{x_i e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right] \Big|_{\hat{\beta}} = 0 \end{aligned}$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$\begin{aligned} \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= -\frac{e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = -\frac{x_i e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= -\frac{x_i^2 e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \end{aligned}$$

Therefore, by Newton's Method, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'(\mathbf{x})^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$;

STEP 2: Update by $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.564284$ and $\beta = 1.771093$.

4: EM Algorithm (30%)

(1) Observed data: Y_i for $i = 1, 2, \dots, 8000$; Missing data: Z_i for $i = 1, 2, \dots, 8000$, where $Z_i = 1, 2$, or 3 for low, middle, and high income respectively.

Since $Y_i|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$, with proportion π_k ($\pi_3 = 1 - (\pi_1 + \pi_2)$), we can formulate the **complete – data Likelihood function** as:

$$L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) \\ = \prod_{i=1}^n \left[\pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right]^{I(Z_i=1)} \left[\pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right]^{I(Z_i=2)} \left[(1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right]^{I(Z_i=3)}$$

The observed-data Likelihood function is:

$$L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}) \\ = \prod_{i=1}^n \left[\left[\pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right] + \left[\pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right] + \left[(1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right] \right]$$

(2) The complete-data log-Likelihood function is:

$$l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^n \left[I(Z_i = 1) \left[\log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right. \\ \left. + I(Z_i = 2) \left[\log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \right. \\ \left. + I(Z_i = 3) \left[\log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

Then given initial guess $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, we define a Q function by $Q(\Theta; \Theta^{(t)}) = E_{\Theta^{(t)}}(l(\Theta | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y})$:

$$Q(\Theta; \Theta^{(t)}) = E_{\pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}}(l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y}) \\ = \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \left[\log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right. \\ \left. + \widehat{Z}_{i2}^{(t)} \left[\log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \right. \\ \left. + \widehat{Z}_{i3}^{(t)} \left[\log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

where $(\pi_3^{(t)} = 1 - \pi_1^{(t)} - \pi_2^{(t)})$

$$\widehat{Z}_{ik}^{(t)} = E(Z_i = k | \Theta^{(t)}) = E(Z_i^{(t)} | \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}) \\ = \frac{\pi_k^{(t)} \frac{1}{\sqrt{2\pi}\sigma_k^{(t)}} e^{-\frac{(y_i - \mu_k^{(t)})^2}{2\sigma_k^{2(t)}}}}{\pi_1^{(t)} \frac{1}{\sqrt{2\pi}\sigma_1^{(t)}} e^{-\frac{(y_i - \mu_1^{(t)})^2}{2\sigma_1^{2(t)}}} + \pi_2^{(t)} \frac{1}{\sqrt{2\pi}\sigma_2^{(t)}} e^{-\frac{(y_i - \mu_2^{(t)})^2}{2\sigma_2^{2(t)}}} + (1 - \pi_1^{(t)} - \pi_2^{(t)}) \frac{1}{\sqrt{2\pi}\sigma_3^{(t)}} e^{-\frac{(y_i - \mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}$$

Then we can calculate:

$$\begin{aligned}
\frac{\partial Q}{\partial \pi_1} \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \frac{1}{\pi_1} - \widehat{Z}_{i3}^{(t)} \frac{1}{1 - \pi_1 - \pi_2} \right] \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \pi_2} \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i2}^{(t)} \frac{1}{\pi_2} - \widehat{Z}_{i3}^{(t)} \frac{1}{1 - \pi_1 - \pi_2} \right] \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_1} \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \frac{y_i - \mu_1}{\sigma_1^2} \right] \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_2} \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i2}^{(t)} \frac{y_i - \mu_2}{\sigma_2^2} \right] \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_3} \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i3}^{(t)} \frac{y_i - \mu_3}{\sigma_3^2} \right] \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_1^2} \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \left(-\frac{1}{2} \frac{1}{\sigma_1^2} + \frac{(y_i - \mu_1)^2}{2(\sigma_1^2)^2} \right) \right] \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_2^2} \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i2}^{(t)} \left(-\frac{1}{2} \frac{1}{\sigma_2^2} + \frac{(y_i - \mu_2)^2}{2(\sigma_2^2)^2} \right) \right] \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_3^2} \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i3}^{(t)} \left(-\frac{1}{2} \frac{1}{\sigma_3^2} + \frac{(y_i - \mu_3)^2}{2(\sigma_3^2)^2} \right) \right] \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} = 0
\end{aligned}$$

Then for iteration in **EM Algorithm**:

Given initial guess: $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, for $t \geq 0$ and $t \in \mathbb{Z}$:

E – step: Calculate $E(Z_i^{(t)} | \Theta^{(t)})$, where $\Theta^{(t)} = \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}$.

M – step: Update $\Theta^{(t+1)}$ by equations (1) to (8) listed at the next page.

Iterative scheme:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (1)$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (2)$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}} \quad (3)$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}} \quad (4)$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (5)$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} (y_i - \mu_1^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}} \quad (6)$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)} (y_i - \mu_2^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}} \quad (7)$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)} (y_i - \mu_3^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (8)$$

(3)

Algorithm: EM Algorithm code in the R file.

Result: $\pi_1 = 0.4416, \pi_2 = 0.3450, \mu_1 = 2997.46, \mu_2 = 7969.146, \mu_3 = 29817.81, \sigma_1 = 301.2129, \sigma_2 = 1560.899$ and $\sigma_3 = 8002.694$

(4) **Result:** in the following page:

	low_income_1	middle_income_2	high_income_3	class
x1	9.976749e-01	4.441287e-03	0.0003134518	1
x2	9.991160e-01	1.594289e-03	0.0002606041	1
x3	0.000000e+00	7.620788e-44	3.0000000000	3
x4	9.991313e-01	1.572229e-03	0.0002476155	1
x5	1.376108e-40	1.995002e+00	0.0074976378	2
x6	0.000000e+00	5.709167e-50	3.0000000000	3
x7	1.605130e-139	1.941673e+00	0.0874902893	2
x8	9.975588e-01	4.666248e-03	0.0003241864	1
x9	9.967818e-01	6.173050e-03	0.0003949243	1
x10	9.990338e-01	1.801087e-03	0.0001969761	1
x11	9.987583e-01	2.340676e-03	0.0002141768	1
x12	1.366895e-102	1.983752e+00	0.0243716695	2
x13	1.996536e-45	1.994937e+00	0.0075938199	2
x14	9.982850e-01	3.258826e-03	0.0002566222	1
x15	2.693409e-156	1.888978e+00	0.1665332214	2
x16	9.670800e-01	6.413560e-02	0.0025567223	1
x17	0.000000e+00	2.068191e-17	3.0000000000	3
x18	9.983288e-01	3.173947e-03	0.0002525564	1
x19	0.000000e+00	1.278875e-99	3.0000000000	3
x20	6.929102e-21	1.993596e+00	0.0096063794	2
x21	9.969714e-01	5.805350e-03	0.0003778551	1
x22	9.991284e-01	1.608230e-03	0.0002024855	1
x23	3.089608e-17	1.992721e+00	0.0109182757	2
x24	0.000000e+00	6.134380e-76	3.0000000000	3
x25	9.989538e-01	1.848234e-03	0.0003661978	1
x26	9.970267e-01	4.868615e-03	0.0016169156	1
x27	9.980602e-01	3.694451e-03	0.0002775799	1
x28	9.991503e-01	1.552303e-03	0.0002207225	1
x29	9.985212e-01	2.801044e-03	0.0002348864	1
x30	9.991501e-01	1.555343e-03	0.0002168142	1
x31	1.016778e-119	1.971510e+00	0.0427355847	2
x32	0.000000e+00	8.200779e-64	3.0000000000	3
x33	9.991366e-01	1.565117e-03	0.0002425366	1
x34	9.991502e-01	1.554535e-03	0.0002175823	1
x35	9.989834e-01	1.900805e-03	0.0001986392	1
x36	9.991312e-01	1.572369e-03	0.0002477084	1
x37	9.926841e-01	1.413777e-02	0.0007410024	1
x38	9.869832e-01	2.524698e-02	0.0011798788	1
x39	9.962506e-01	7.204020e-03	0.0004421441	1
x40	9.990815e-01	1.646530e-03	0.0002856051	1
x41	0.000000e+00	8.881480e-70	3.0000000000	3
x42	9.989921e-01	1.883689e-03	0.0001982629	1
x43	5.151205e-45	1.994947e+00	0.0075794189	2
x44	9.988882e-01	1.953147e-03	0.0004057667	1
x45	9.990137e-01	1.840926e-03	0.0001974733	1
x46	0.000000e+00	4.782532e-22	3.0000000000	3
x47	9.979810e-01	3.847983e-03	0.0002849741	1
x48	2.361706e-75	1.992179e+00	0.0117309531	2
x49	0.000000e+00	1.034641e-37	3.0000000000	3
x50	9.988184e-01	2.223548e-03	0.0002093611	1

Figure 1: Classification of first 50 individuals