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**1: Inverse method for Poisson Distribution (25%)**


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For discrete Poisson Distribution ( $\lambda = 5$ ),

the p.m.f is  $P(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!}$  and the c.d.f is  $F(x|\lambda) = \sum_{t \leq x} e^{-\lambda} \frac{\lambda^t}{t!}$ .

**Algorithm:** Inverse method for the Poisson Distribution:

To generate  $X \sim F(x)$ :

STEP 1: Generate  $U \sim \text{unif}[0, 1]$ ;

STEP 2: Transform  $X = F^{-1}(U)$ : if  $F(x|\lambda) < U \leq F(x+1|\lambda)$ , let  $X = x+1$ .

**Plot :**

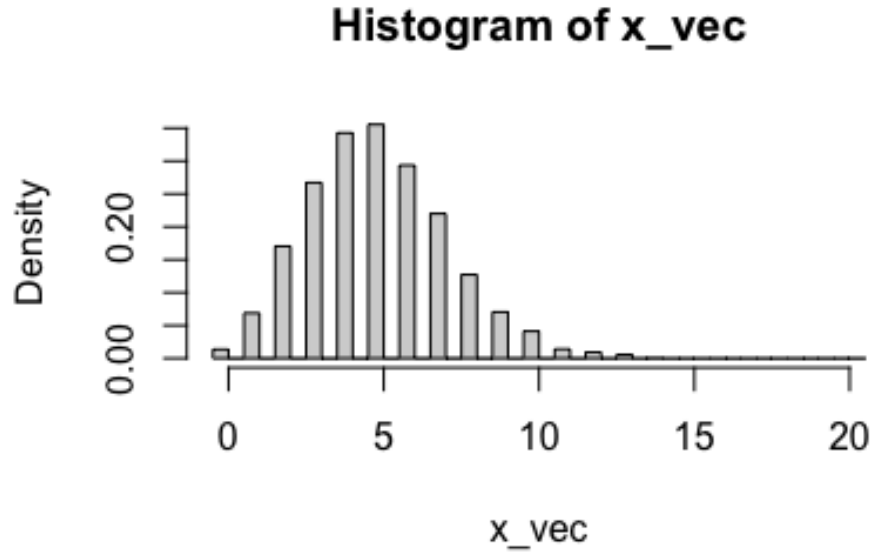


Figure 1: Histogram of 5000 samples

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**2: Accept-Reject method for truncated Gamma Distribution (25%)**


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For  $X \sim \text{Gamma}(\frac{1}{2}, 1)I(x \geq 5)$ ,  $f(x) = \frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}$ .

We can define a shifted exponential distribution  $g(x) = e^{-(x-5)} I(x \geq 5)$  and want to find a constant  $M$  such that  $f(x) < Mg(x)$  for any  $x$ .

Then  $M = \sup \frac{f(x)}{g(x)} = \sup \frac{\frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}}{e^{-(x-5)} I(x \geq 5)} = \frac{5^{-\frac{1}{2}} e^{-5}}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}$ .

**Algorithm:** Accept-Reject method for truncated Gamma Distribution:

To generate  $X \sim F(x) = \text{c.d.f of } f(x)$ :

STEP 1: Generate  $Y \sim g(y)$ ;

STEP 2: Generate  $U \sim \text{unif}[0, 1]$ ;

STEP 3: Accept  $X = Y$  if  $U \leq \frac{f(Y)}{Mg(Y)}$ .

**Proof :**

From the choice of constant  $M$ , we can know that  $Mg(x) \geq f(x)$ . The goal of this method is to generate  $X \sim F(x) = \text{c.d.f of } f(x)$ .

For the generating algorithm:

$$\begin{aligned}
 P(X \leq x) &= P(Y \leq x | Y \text{ is accepted}) \\
 &= P(Y \leq x | U \leq \frac{f(Y)}{Mg(Y)}) \\
 &= \frac{P(Y \leq x, U \leq \frac{f(Y)}{Mg(Y)})}{P(U \leq \frac{f(Y)}{Mg(Y)})} \\
 &= \frac{\int_{-\infty}^x g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1 dudy}{\int_{-\infty}^{+\infty} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1 dudy} \\
 &= \frac{\int_{-\infty}^x g(y) \frac{f(y)}{Mg(y)} dy}{\int_{-\infty}^{+\infty} g(y) \frac{f(y)}{Mg(y)} dy} \\
 &= \frac{\int_{-\infty}^x f(y) dy}{\int_{-\infty}^{+\infty} f(y) dy} \\
 &= \frac{\int_{-\infty}^x f(y) dy}{1} \\
 &= F(x)
 \end{aligned}$$

Therefore, this AR method works.

**Comparison :**

Theoretical acceptance probability:

$$\begin{aligned}
 P(U \leq \frac{f(Y)}{Mg(Y)}) &= \int_{-\infty}^{+\infty} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1 dudy \\
 &= \frac{1}{M} \int_{-\infty}^{+\infty} f(y) dy \\
 &= \frac{1}{M}
 \end{aligned}$$

After computation, this acceptance probability is 0.184157.

The actual acceptance rate is 0.1854, which is a little bit higher than the theoretical value.

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**3: Importance Sampling for Estimation (25%)**


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(1) Using 5000 samples from Q2 ( $l$  = length of samples obtained in Q2), the Monte Carlo estimate is

$$\begin{aligned}\int_5^{+\infty} \cos(x) x^{-\frac{1}{2}} e^{-x} dx &= \int_{-\infty}^{+\infty} \cos(x) \frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} dx \times \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \\ &= \frac{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy}{l} \sum_{i=1}^l \cos(x_i) \\ &= 0.001708\end{aligned}$$

(2) Using the same notations in Q2, we define  $h(x) = \cos(x)$ .

$$\begin{aligned}\int_5^{+\infty} \cos(x) x^{-\frac{1}{2}} e^{-x} dx &= \int_{-\infty}^{+\infty} \cos(x) \frac{x^{-\frac{1}{2}} e^{-x} I(x \geq 5)}{\int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} dx \times \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \\ &= \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \int_{-\infty}^{+\infty} \frac{h(x)f(x)}{g(x)} dx\end{aligned}$$

Note that  $\frac{f(x)}{g(x)} \leq M < \infty$ , and  $E_g h^2(x) = \int_{-\infty}^{+\infty} g(x) h^2(x) dx \leq \int_{-\infty}^{+\infty} g(x) dx = 1 < \infty$ , we can use the importance sampling as follows:

**Algorithm:**

STEP 1: Generate  $n = 5000$  samples from  $g(x)$ ;

STEP 2: Compute the Monte Carlo estimate:

$$\int_{-\infty}^{+\infty} \frac{h(x)f(x)}{g(x)} dx = \frac{\sum_{i=1}^n \frac{h(x_i)f(x_i)}{g(x_i)}}{n} = \frac{e^{-5}}{n \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} \sum_{i=1}^n \cos(x_i) x_i^{-\frac{1}{2}}$$

Therefore,

$$\begin{aligned}\int_5^{+\infty} \cos(x) x^{-\frac{1}{2}} e^{-x} dx &= \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \int_{-\infty}^{+\infty} \frac{h(x)f(x)}{g(x)} dx \\ &= \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy \frac{e^{-5}}{n \int_5^{+\infty} y^{-\frac{1}{2}} e^{-y} dy} \sum_{i=1}^n \cos(x_i) x_i^{-\frac{1}{2}} \\ &= \frac{e^{-5}}{n} \sum_{i=1}^n \cos(x_i) x_i^{-\frac{1}{2}} \\ &= 0.00174\end{aligned}$$

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**4: Stratified Sampling (25%)**

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(1) Randomly draw 100 samples from the data set, for each subpopulation:

Standard deviation for age interval 1 is 98.9192658;

Standard deviation for age interval 2 is 121.0536453;

Standard deviation for age interval 3 is 200.7419485.

(2)

(3)