## 1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.
$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function $f$ has totally 3 zeros.

**Algorithm:** Bisection Method in the R file.

**Result:** Zeros: -4.837944, -2.259727, and 1.097664.

## 2: Poisson Regression - Newton's Method (25%)

**(1)** Since $y_i \sim Poisson(\lambda_i)$ and $log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihhod function:

$$L(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^{n} \frac{e^{(\alpha+\beta x_i+\gamma x_i^2)y_i} e^{-e^{\alpha+\beta x_i+\gamma x_i^2}}}{y_i!}$$

**(2)** The log-Likelihood function is

$$l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha+\beta x_i+\gamma x_i^2} - \log y_i!$$

Then we have:

$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \alpha}\right|_{\hat{\alpha}} = \left.\sum_{i=1}^{n} [y_i - e^{\alpha+\beta x_i+\gamma x_i^2}]\right|_{\hat{\alpha}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \beta}\right|_{\hat{\beta}} = \left.\sum_{i=1}^{n} [x_i y_i - x_i e^{\alpha+\beta x_i+\gamma x_i^2}]\right|_{\hat{\beta}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \gamma}\right|_{\hat{\gamma}} = \left.\sum_{i=1}^{n} [x_i^2 y_i - x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}]\right|_{\hat{\gamma}} = 0$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma^2} \end{pmatrix}$, in which

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha^2} = \sum_{i=1}^{n} -e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\beta} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\alpha} = \sum_{i=1}^{n} -x_i e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\alpha} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta^2} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\beta} = \sum_{i=1}^{n} -x_i^3 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma^2} = \sum_{i=1}^{n} -x_i^4 e^{\alpha+\beta x_i+\gamma x_i^2}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix} = \begin{pmatrix}\alpha^{(n-1)}\\ \beta^{(n-1)}\\ \gamma^{(n-1)}\end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

**(3)**

**Algorithm:** Newton's Method code in the R file.

**Result:** $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

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**3: Logistic Regression - Newton's Method (20%)**

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**(1)** Since $y_i \sim Bernoulli(p_i)$ and $logit(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i}(1-p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$. Then we can get the following Likelihood function:

$$\begin{aligned} L(\alpha,\beta|\mathbf{x},\mathbf{y}) &= \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i} \\ &= \prod_{i=1}^{n} (\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{y_i}(1-\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{1-y_i} \end{aligned}$$

**(2)** The log-Likelihood function is

$$\begin{aligned} l(\alpha, \beta|\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} y_i(\alpha + \beta x_i - \log(1 + e^{\alpha+\beta x_i})) + (1 - y_i) \log(\frac{1}{1 + e^{\alpha+\beta x_i}}) \\ &= \sum_{i=1}^{n} \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha+\beta x_i}) \end{aligned}$$

Then we have:

$$\begin{aligned} \frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha}\bigg|_{\hat{\alpha}} &= \sum_{i=1}^{n}[y_i - \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\alpha}} = 0 \\ \frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta}\bigg|_{\hat{\beta}} &= \sum_{i=1}^{n}[x_i y_i - \frac{x_i e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\beta}} = 0 \end{aligned}$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$\begin{aligned} \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= -\frac{e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = -\frac{x_i e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta^2} &= -\frac{x_i^2 e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \end{aligned}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle \mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle \mathbf{x^{(n)}}$.

**(3)**

**Algorithm:** Newton's Method code in the R file.

**Result:** $\alpha = 1.564284$ and $\beta = 1.771093$.

## 4: EM Algorithm (30%)

**(1)** Observed data: $Y_i$ for $i = 1, 2, \ldots, 8000$; Missing data: $Z_i$ for $i = 1, 2, \ldots, 8000.$, where $Z_i = 1, 2,$ or $3$ for low, middle, and high income respectively.

Since $Y_i\big|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$, with proportion $\pi_k$ ($\pi_3 = 1 - (\pi_1 + \pi_2)$), we can formulate the **complete − data Likelihhod function** as:

$$\begin{aligned}
&L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}, \mathbf{Z}) \\
=&\prod_{i=1}^{n}\left[\pi_1\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}}\right]^{I(Z_i=1)}\left[\pi_2\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}\right]^{I(Z_i=2)}\left[(1-\pi_1-\pi_2)\frac{1}{\sqrt{2\pi}\sigma_3}e^{-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}}\right]^{I(Z_i=3)}
\end{aligned}$$

The observed-data Likelihhod function is:

$$\begin{aligned}
&L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}) \\
=&\prod_{i=1}^{n}\left[\left[\pi_1\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}}\right]+\left[\pi_2\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}\right]+\left[(1-\pi_1-\pi_2)\frac{1}{\sqrt{2\pi}\sigma_3}e^{-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}}\right]\right]
\end{aligned}$$

**(2)** The complete-data log-Likelihhod function is:

$$\begin{aligned}
l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{n}\Bigg[&I(Z_i=1)\left[\log\pi_1 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_1^2 - \frac{(y_i-\mu_1)^2}{2\sigma_1^2}\right] \\
&+I(Z_i=2)\left[\log\pi_2 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_2^2 - \frac{(y_i-\mu_2)^2}{2\sigma_2^2}\right] \\
&+I(Z_i=3)\left[\log(1-\pi_1-\pi_2) - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_3^2 - \frac{(y_i-\mu_3)^2}{2\sigma_3^2}\right]\Bigg]
\end{aligned}$$

Then given initial guess $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, we define a $Q$ function by $Q(\Theta; \Theta^{(t)}) = E_{\mathbf{\Theta^{(t)}}}\big(l(\mathbf{\Theta}|\mathbf{Y}, \mathbf{Z})|\mathbf{Y}\big)$:

$$\begin{aligned}
Q(\Theta;\Theta^{(t)}) &= E_{\pi_1^{(t)},\pi_2^{(t)},\mu_1^{(t)},\mu_2^{(t)},\mu_3^{(t)},\sigma_1^{2(t)},\sigma_2^{2(t)},\sigma_3^{2(t)}}\big(l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}, \mathbf{Z})|\mathbf{Y}\big) \\
&= \sum_{i=1}^{n}\Bigg[\widehat{Z_{i1}}^{(t)}\left[\log\pi_1 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_1^2 - \frac{(y_i-\mu_1)^2}{2\sigma_1^2}\right] \\
&+ \widehat{Z_{i2}}^{(t)}\left[\log\pi_2 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_2^2 - \frac{(y_i-\mu_2)^2}{2\sigma_2^2}\right] \\
&+ \widehat{Z_{i3}}^{(t)}\left[\log(1-\pi_1-\pi_2) - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_3^2 - \frac{(y_i-\mu_3)^2}{2\sigma_3^2}\right]\Bigg]
\end{aligned}$$

where ($\pi_3^{(t)} = 1 - \pi_1^{(t)} - \pi_2^{(t)}$)

$$\begin{aligned}
\widehat{Z_{ik}}^{(t)} &= E(Z_i = k|\Theta^{(t)}) = E(Z_i^{(t)}|\pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}) \\
&= \frac{k\pi_k^{(t)}\frac{1}{\sqrt{2\pi}\sigma_k{}^{(t)}}e^{-\frac{(y_i-\mu_k^{(t)})^2}{2\sigma_k^{2(t)}}}}{\pi_1^{(t)}\frac{1}{\sqrt{2\pi}\sigma_1{}^{(t)}}e^{-\frac{(y_i-\mu_1^{(t)})^2}{2\sigma_1^{2(t)}}}+\pi_2^{(t)}\frac{1}{\sqrt{2\pi}\sigma_2{}^{(t)}}e^{-\frac{(y_i-\mu_2^{(t)})^2}{2\sigma_2^{2(t)}}}+(1-\pi_1^{(t)}-\pi_2^{(t)})\frac{1}{\sqrt{2\pi}\sigma_3{}^{(t)}}e^{-\frac{(y_i-\mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}
\end{aligned}$$

Then we can calculate:

$$
\begin{aligned}
\frac{\partial Q}{\partial \pi_1}\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\frac{1}{\pi_1} - \widehat{Z_{i3}}^{(t)}\frac{1}{1-\pi_1-\pi_2}\right]\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \pi_2}\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\frac{1}{\pi_2} - \widehat{Z_{i3}}^{(t)}\frac{1}{1-\pi_1-\pi_2}\right]\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_1}\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\frac{y_i-\mu_1}{\sigma_1^2}\right]\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_2}\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\frac{y_i-\mu_2}{\sigma_2^2}\right]\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_3}\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i3}}^{(t)}\frac{y_i-\mu_3}{\sigma_3^2}\right]\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_1^2}\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_1^2}+\frac{(y_i-\mu_1)^2}{2(\sigma_1^2)^2}\right)\right]\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_2^2}\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_2^2}+\frac{(y_i-\mu_2)^2}{2(\sigma_2^2)^2}\right)\right]\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_3^2}\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} &= \sum_{i=1}^{n}\left[\widehat{Z_{i3}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_3^2}+\frac{(y_i-\mu_3)^2}{2(\sigma_3^2)^2}\right)\right]\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = 0
\end{aligned}
$$

Then for iteration in **EM Algorithm**:

Given initial guess: $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, for $t \geq 0$ and $t \in \mathbb{Z}$:

$\mathbf{E-step}$: Calculate $E(Z_i^{(t)}|\Theta^{(t)})$, where $\Theta^{(t)} = \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}$.

$\mathbf{M-step}$: Update $\Theta^{(t+1)}$ by equations (1) to (8) listed at the next page.

Iterative scheme:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)}}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i2}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{1}$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)}}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i2}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{2}$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)}} \tag{3}$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)}} \tag{4}$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{5}$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} (y_i - \mu_1^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)}} \tag{6}$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)} (y_i - \mu_2^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)}} \tag{7}$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)} (y_i - \mu_3^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{8}$$

**(3)**

**Algorithm:** EM Algorithm code in the R file.

**Result:** $\pi_1 = 0.4416, \pi_2 = 0.3450, \mu_1 = 2997.46, \mu_2 = 7969.146. \mu_3 = 29817.81, \sigma_1 = 301.2129, \sigma_2 = 1560.899$ and $\sigma_3 = 8002.694$

**(4) Result:** in the following page:

| | low_income_1 | middle_income_2 | high_income_3 | class |
|---|---|---|---|---|
| x1 | 9.976749e-01 | 4.441287e-03 | 0.0003134518 | 1 |
| x2 | 9.991160e-01 | 1.594289e-03 | 0.0002606041 | 1 |
| x3 | 0.000000e+00 | 7.620788e-44 | 3.0000000000 | 3 |
| x4 | 9.991313e-01 | 1.572229e-03 | 0.0002476155 | 1 |
| x5 | 1.376108e-40 | 1.995002e+00 | 0.0074976378 | 2 |
| x6 | 0.000000e+00 | 5.709167e-50 | 3.0000000000 | 3 |
| x7 | 1.605130e-139 | 1.941673e+00 | 0.0874902893 | 2 |
| x8 | 9.975588e-01 | 4.666248e-03 | 0.0003241864 | 1 |
| x9 | 9.967818e-01 | 6.173050e-03 | 0.0003949243 | 1 |
| x10 | 9.990338e-01 | 1.801087e-03 | 0.0001969761 | 1 |
| x11 | 9.987583e-01 | 2.340676e-03 | 0.0002141768 | 1 |
| x12 | 1.366895e-102 | 1.983752e+00 | 0.0243716695 | 2 |
| x13 | 1.996536e-45 | 1.994937e+00 | 0.0075938199 | 2 |
| x14 | 9.982850e-01 | 3.258826e-03 | 0.0002566222 | 1 |
| x15 | 2.693409e-156 | 1.888978e+00 | 0.1665332214 | 2 |
| x16 | 9.670800e-01 | 6.413560e-02 | 0.0025567223 | 1 |
| x17 | 0.000000e+00 | 2.068191e-17 | 3.0000000000 | 3 |
| x18 | 9.983288e-01 | 3.173947e-03 | 0.0002525564 | 1 |
| x19 | 0.000000e+00 | 1.278875e-99 | 3.0000000000 | 3 |
| x20 | 6.929102e-21 | 1.993596e+00 | 0.0096063794 | 2 |
| x21 | 9.969714e-01 | 5.805350e-03 | 0.0003778551 | 1 |
| x22 | 9.991284e-01 | 1.608230e-03 | 0.0002024855 | 1 |
| x23 | 3.089608e-17 | 1.992721e+00 | 0.0109182757 | 2 |
| x24 | 0.000000e+00 | 6.134380e-76 | 3.0000000000 | 3 |
| x25 | 9.989538e-01 | 1.848234e-03 | 0.0003661978 | 1 |
| x26 | 9.970267e-01 | 4.868615e-03 | 0.0016169156 | 1 |
| x27 | 9.980602e-01 | 3.694451e-03 | 0.0002775799 | 1 |
| x28 | 9.991503e-01 | 1.552303e-03 | 0.0002207225 | 1 |
| x29 | 9.985212e-01 | 2.801044e-03 | 0.0002348864 | 1 |
| x30 | 9.991501e-01 | 1.555343e-03 | 0.0002168142 | 1 |
| x31 | 1.016778e-119 | 1.971510e+00 | 0.0427355847 | 2 |
| x32 | 0.000000e+00 | 8.200779e-64 | 3.0000000000 | 3 |
| x33 | 9.991366e-01 | 1.565117e-03 | 0.0002425366 | 1 |
| x34 | 9.991502e-01 | 1.554535e-03 | 0.0002175823 | 1 |
| x35 | 9.989834e-01 | 1.900805e-03 | 0.0001986392 | 1 |
| x36 | 9.991312e-01 | 1.572369e-03 | 0.0002477084 | 1 |
| x37 | 9.926841e-01 | 1.413777e-02 | 0.0007410024 | 1 |
| x38 | 9.869832e-01 | 2.524698e-02 | 0.0011798788 | 1 |
| x39 | 9.962506e-01 | 7.204020e-03 | 0.0004421441 | 1 |
| x40 | 9.990815e-01 | 1.646530e-03 | 0.0002856051 | 1 |
| x41 | 0.000000e+00 | 8.881480e-70 | 3.0000000000 | 3 |
| x42 | 9.989921e-01 | 1.883689e-03 | 0.0001982629 | 1 |
| x43 | 5.151205e-45 | 1.994947e+00 | 0.0075794189 | 2 |
| x44 | 9.988882e-01 | 1.953147e-03 | 0.0004057667 | 1 |
| x45 | 9.990137e-01 | 1.840926e-03 | 0.0001974733 | 1 |
| x46 | 0.000000e+00 | 4.782532e-22 | 3.0000000000 | 3 |
| x47 | 9.979810e-01 | 3.847983e-03 | 0.0002849741 | 1 |
| x48 | 2.361706e-75 | 1.992179e+00 | 0.0117309531 | 2 |
| x49 | 0.000000e+00 | 1.034641e-37 | 3.0000000000 | 3 |
| x50 | 9.988184e-01 | 2.223548e-03 | 0.0002093611 | 1 |

Figure 1: Classification of first 50 individuals