1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.
$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function f has totally 3 zeros.

Algorithm: Bisection Method in the R file.

Result: zeros -4.837944, -2.259727, and 1.097664.

2: Poisson Regression - Newton's Method (25%)

(1) Since $y_i \sim Poisson(\lambda_i)$ and $log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihhod function:

$$L(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^{n} \frac{e^{(\alpha+\beta x_i+\gamma x_i^2)y_i} e^{-e^{\alpha+\beta x_i+\gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha+\beta x_i+\gamma x_i^2} - \log y_i!$$

Then we have:

$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \alpha}\right|_{\hat{\alpha}} = \sum_{i=1}^{n} [y_i - e^{\alpha+\beta x_i+\gamma x_i^2}]\Big|_{\hat{\alpha}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \beta}\right|_{\hat{\beta}} = \sum_{i=1}^{n} [x_i y_i - x_i e^{\alpha+\beta x_i+\gamma x_i^2}]\Big|_{\hat{\beta}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \gamma}\right|_{\hat{\gamma}} = \sum_{i=1}^{n} [x_i^2 y_i - x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}]\Big|_{\hat{\gamma}} = 0$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma^2} \end{pmatrix}$, in which

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha^2} = \sum_{i=1}^{n} -e^{\alpha+\beta x_i+\gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\beta} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\alpha} = \sum_{i=1}^{n} -x_i e^{\alpha+\beta x_i+\gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\alpha} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta^2} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\beta} = \sum_{i=1}^{n} -x_i^3 e^{\alpha+\beta x_i+\gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma^2} = \sum_{i=1}^{n} -x_i^4 e^{\alpha+\beta x_i+\gamma x_i^2}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix} = \begin{pmatrix}\alpha^{(n-1)}\\ \beta^{(n-1)}\\ \gamma^{(n-1)}\end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

(3) Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

3: Logistic Regression - Newton's Method (20%)

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(1) Since $y_i \sim Bernoulli(p_i)$ and $logit(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i}(1-p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$. Then we can get the following Likelihood function:

$$\begin{aligned} L(\alpha,\beta|\mathbf{x},\mathbf{y}) &= \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i} \\ &= \prod_{i=1}^{n} (\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{y_i}(1-\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{1-y_i} \end{aligned}$$

(2) The log-Likelihood function is

$$
\begin{aligned}
l(\alpha,\beta|\mathbf{x},\mathbf{y}) &= \sum_{i=1}^{n} y_i(\alpha+\beta x_i - \log(1+e^{\alpha+\beta x_i})) + (1-y_i)\log(\frac{1}{1+e^{\alpha+\beta x_i}}) \\
&= \sum_{i=1}^{n} \alpha x_i + \beta x_i y_i - \log(1+e^{\alpha+\beta x_i})
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
\frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha}\bigg|_{\hat{\alpha}} &= \sum_{i=1}^{n}[y_i - \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}]\bigg|_{\hat{\alpha}} = 0 \\
\frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta}\bigg|_{\hat{\beta}} &= \sum_{i=1}^{n}[x_i y_i - \frac{x_i e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}]\bigg|_{\hat{\beta}} = 0
\end{aligned}
$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$
\begin{aligned}
\frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha^2} &= -\frac{e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2} \\
\frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} = -\frac{x_i e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2} \\
\frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta^2} &= -\frac{x_i^2 e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2}
\end{aligned}
$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$
\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).
$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

(3)Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.564284$ and $beta = 1.771093$.

4: EM Algorithm (30%)

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(1)

(2)

(3)

(4)