1: Inverse method for Poissson Distribution (25%)

For discrete Poisson Distribution ($\lambda = 5$),
the p.m.f is $P(x|\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}$ and the c.d.f is $F(x|\lambda) = \sum_{t \le x} e^{-\lambda}\frac{\lambda^t}{t!}$.

Algorithm: Inverse method for the Poisson Distribution:

To generate $X \sim F(x)$:

STEP 1: Generate $U \sim unif[0, 1]$;

STEP 2: Transform $X = F^{-}(U)$: if $F(x|\lambda) < U \le F(x + 1|\lambda)$, let $X = x + 1$.

Plot :

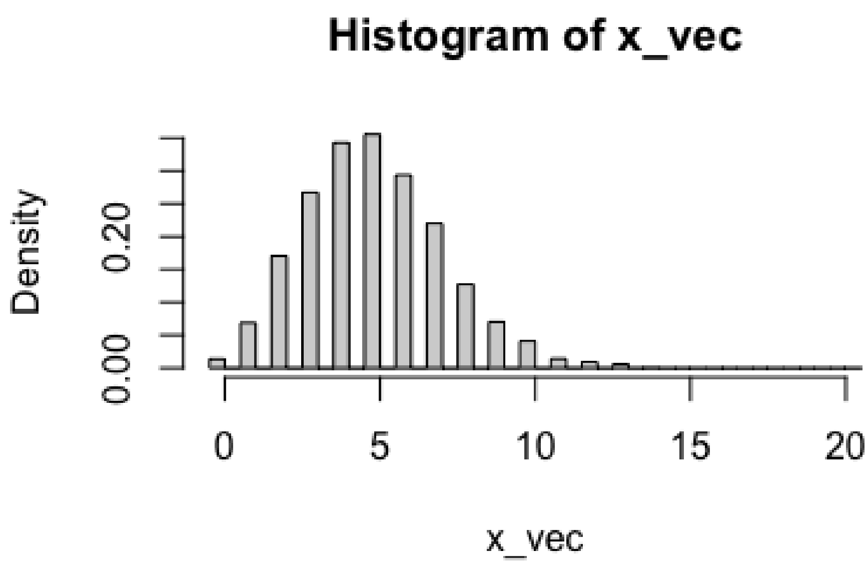

Figure 1: Histogram of 5000 samples

2: Accept-Reject method for truncated Gamma Distribution (25%)

For $X \sim Gamma(\frac{1}{2}, 1)I(x \ge 5)$, $f(x) = \frac{x^{-\frac{1}{2}}e^{-x}I(x\ge 5)}{\int_5^{+\infty} y^{-\frac{1}{2}}e^{-y}dy}$.

We can define a shifted exponential distribution $g(x) = e^{-(x-5)}I(x \ge 5)$ and want to find a constant M such that $f(x) < Mg(x)$ for any x.

Then $M = \sup\frac{f(x)}{g(x)} = \sup\frac{\frac{x^{-\frac{1}{2}}e^{-x}I(x\ge 5)}{\int_5^{+\infty} y^{-\frac{1}{2}}e^{-y}dy}}{e^{-(x-5)}I(x\ge 5)} = \frac{5^{-\frac{1}{2}}e^{-5}}{\int_5^{+\infty} y^{-\frac{1}{2}}e^{-y}dy}$.

Algorithm: Accept-Reject method for truncated Gamma Distribution:

To generate $X \sim F(x) =$ c.d.f of $f(x)$:

STEP 1: Generate $Y \sim g(y)$;

STEP 2: Generate $U \sim unif[0,1]$;

STEP 3: Accept $X = Y$ if $U \leq \frac{f(Y)}{Mg(Y)}$.

Proof :

From the choice of constant M, we can know that $Mg(x) \geq f(x)$. The goal of this method is to generate $X \sim F(x) =$ c.d.f of $f(x)$.

For the generating algorithm:

$$
\begin{aligned}
P(X \leq x) &= P(Y \leq x | Y \text{ is accepted}) \\
&= P(Y \leq x | U \leq \frac{f(Y)}{Mg(Y)}) \\
&= \frac{P(Y \leq x, U \leq \frac{f(Y)}{Mg(Y)})}{P(U \leq \frac{f(Y)}{Mg(Y)})} \\
&= \frac{\int_{-\infty}^{x} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1dudy}{\int_{-\infty}^{+\infty} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1dudy} \\
&= \frac{\int_{-\infty}^{x} g(y) \frac{f(y)}{Mg(y)} dy}{\int_{-\infty}^{+\infty} g(y) \frac{f(y)}{Mg(y)} dy} \\
&= \frac{\int_{-\infty}^{x} f(y)dy}{\int_{-\infty}^{+\infty} f(y)dy} \\
&= \frac{\int_{-\infty}^{x} f(y)dy}{1} \\
&= F(x)
\end{aligned}
$$

Therefore, this AR method works.

Comparison :

Theoretical acceptance probability:

$$
\begin{aligned}
P(U \leq \frac{f(Y)}{Mg(Y)}) &= \int_{-\infty}^{+\infty} g(y) \int_0^{\frac{f(y)}{Mg(y)}} 1dudy \\
&= \frac{1}{M} \int_{-\infty}^{+\infty} f(y)dy \\
&= \frac{1}{M}
\end{aligned}
$$

After computation, $M = 0.184157$.

The actual acceptance rate is 0.1854.

3: Importance Sampling for Estimation (25%)

(1)

(2)

Algorithm:

Result:

4: Stratified Sampling (25%)

(1)

(2)

(3)