

## 1: Hybrid Gibbs Sampler to estimate Poisson Distribution $\lambda$ (30%)

### Derivation:

For these Poisson distributed random variables (r.v.s) ( $n = 500$ ) with mean parameter  $\lambda$ , unobserved variables are  $\lambda, y_1, y_2, \dots, y_{78}$ , in which  $y_i$ 's denote the r.v.s which are larger than or equal to five.

$$\begin{aligned} \text{Prior : } \pi(\lambda) &\propto \frac{1}{\lambda}; \\ P(X, Y|\lambda) &= \prod_{i=1}^{422} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \prod_{j=1}^{78} \frac{e^{-\lambda} \lambda^{y_j}}{y_j!} I(y_j \geq 5); \\ P(\lambda|X, Y) &\propto P(X, Y|\lambda) P(\lambda) \propto e^{-n\lambda} \lambda^{\sum_i x_i + \sum_j y_j - 1} \\ P(y_j|X, \lambda) &= \frac{e^{-\lambda} \lambda^{y_j}}{y_j!} I(y_j \geq 5) \propto \frac{\lambda^{y_j}}{y_j!} I(y_j \geq 5) \end{aligned}$$

Then we can know that  $\lambda|X, Y \propto \text{Gamma}(\sum_i x_i + \sum_j y_j, n)$ .

The MH-Step to sample 78 unobserved  $y_i$ 's is

$$\begin{aligned} y_j^* &= \begin{cases} y_j^{(t)} - 1, & \text{with probability } \frac{1}{3}; \\ y_j^{(t)}, & \text{with probability } \frac{1}{3}; \\ y_j^{(t)} + 1, & \text{with probability } \frac{1}{3}. \end{cases} \\ r &= \min \left\{ \frac{[\lambda^{(t+1)}]^{y_j^*} / y_j^*}{[\lambda^{(t+1)}]^{y_j^{(t)}} / y_j^{(t)}} I(y_j^* \geq 5), 1 \right\} \end{aligned}$$

where  $r$  is the accept-reject ratio.

**Result:**  $\hat{\lambda} = 1.674$ .

## 2: Gibbs Sampler for Clustering (30%)

### Derivation:

We use  $\{X_{ij}\}_{i=1,2,3,\dots,1000}^{j=1,2,3}$  to denote the datum of  $i$ -th sample in  $j$ -th dimension. Then using the same notation in the question:

$$\begin{aligned} f(X, Z|\Pi, \Theta) &= \prod_{i=1}^{1000} \prod_{k=1}^3 P(Z_i = k|\Pi, \Theta) P(X_{ij}, j = 1, 2, 3|Z_j, \Pi, \Theta) \\ &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[ P(Z_i = k|\Pi) \prod_{j=1}^3 P(X_{ij}|Z_j, \Theta) \right]^{I(Z_j=k)} \\ &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[ \pi_k \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right]^{I(Z_j=k)} \end{aligned}$$

Note that given  $Z_i = k$ , for each sample  $i$ ,  $X_{ij} \sim \text{Bino}(10j, \theta_{jk})$ ;  $P(\theta_{jk}) \propto 1$ ; and  $(\pi_1, \pi_2, \pi_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3)$ , we can derive by the following:

$$\begin{aligned}
 P(\Pi, \Theta, Z|X) &\propto P(\Pi)P(\Theta)P(X, Z|\Pi, \Theta) \\
 &\propto \sum_{k=1}^3 \pi_k^{\alpha_k-1} \sum_{i=1}^{1000} \sum_{k=1}^3 \left[ \pi_k \sum_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right]^{I(Z_i=k)} \\
 &\propto \sum_{k=1}^3 \pi_k^{\alpha_k-1} \sum_{i=1}^{1000} \sum_{k=1}^3 \pi_k^{I(Z_i=k)} \sum_{i=1}^{1000} \sum_{k=1}^3 \sum_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij}I(Z_i=k)} [(1 - \theta_{jk})^{10j-X_{ij}}]^{I(Z_i=k)}
 \end{aligned}$$

Then for each parameter:

For  $\pi$ :

$$\begin{aligned}
 f(\Pi|\Theta, Z) &\propto \sum_{k=1}^3 \pi_k^{\alpha_k-1} \sum_{i=1}^{1000} \sum_{k=1}^3 \pi_k^{I(Z_i=k)} \\
 &\propto \sum_{k=1}^3 \pi_k^{\alpha_k-1} \sum_{k=1}^3 [\pi_k]^{\sum_{i=1}^{1000} I(Z_i=k)} \\
 &\propto \sum_{k=1}^3 \pi_k^{\alpha_k + \sum_{i=1}^{1000} I(Z_i=k) - 1}
 \end{aligned}$$

Then it follows that:

$$\Pi|\Theta, Z \sim \text{Dirichlet}(\alpha_1 + \sum_{i=1}^{1000} I(Z_i = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i = 3))$$

For  $\theta$ :

$$\begin{aligned}
 \theta_{jk}| - &\propto \prod_{i=1}^{1000} \prod_{k=1}^3 \left[ \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right]^{I(Z_i=k)} \\
 &\propto \prod_{i=1}^{1000} \left[ \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right]^{I(Z_i=k)}
 \end{aligned}$$

Then it follows that:

$$\theta_{jk}| - \sim \text{Bino}(10j \sum_{i=1}^{1000} I(Z_i = k), \theta_{jk})$$

For  $Z$ :

$$\begin{aligned}
 Z_i| - &\propto \sum_{i=1}^{1000} \sum_{k=1}^3 \pi_k^{I(Z_i=k)} \sum_{i=1}^{1000} \sum_{k=1}^3 \sum_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij}I(Z_i=k)} [(1 - \theta_{jk})^{10j-X_{ij}}]^{I(Z_i=k)} \\
 &\propto \sum_{k=1}^3 \pi_k^{I(Z_i=k)} \sum_{k=1}^3 \sum_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij}I(Z_i=k)} [(1 - \theta_{jk})^{10j-X_{ij}}]^{I(Z_i=k)} \\
 &\propto \left[ \sum_{k=1}^3 \pi_k \left[ \sum_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right] \right]^{I(Z_i=k)}
 \end{aligned}$$

Then for Gibbs Sampler algorithm:

Given  $\Pi^{(t)}, \Theta^{(t)}, Z^{(t)}$ , we update the parameters by the following:

$$\begin{aligned} \pi_1^{(t+1)}, \pi_2^{(t+1)}, \pi_3^{(t+1)} | - &\sim \text{Dirichlet}(\alpha_1 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 3)) \\ \theta_{jk}^{(t+1)} &\sim \text{Bino}(10j \sum_{i=1}^{1000} I(Z_i^{(t)} = k), \theta_{jk}) \text{ for } j = 1, 2, 3 \text{ and } k = 1, 2, 3 \\ P(Z_i^{(t+1)} = k | -) &= \frac{\pi_k^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}}}{\sum_{l=1}^3 \pi_l^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jl}^{X_{ij}} (1 - \theta_{jl})^{10j - X_{ij}}} \end{aligned}$$

**Result:**

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### 3: Hybrid Gibbs Sampler (40%)

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