1: Hybrid Gibbs Sampler to estimate Poisson Distribution λ (30%)

Derivation:

For these Poisson distributed random variables (r.v.s) ($n = 500$) with mean parameter λ, unobserved variables are $\lambda, y_1, y_2, \ldots, y_{78}$, in which $y's$ denote the r.v.s which are larger than or equal to five.

$$
\begin{aligned}
&\text{Prior} : \pi(\lambda) \propto \frac{1}{\lambda};\\
&P(X,Y|\lambda) = \prod_{i=1}^{422} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \prod_{j=1}^{78} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} I(y_i \geq 5);\\
&P(\lambda|X,Y) \propto P(X,Y|\lambda)P(\lambda) \propto e^{-n\lambda}\lambda^{\sum_i x_i + \sum_j y_j - 1}\\
&P(y_j|X,\lambda) = \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} I(y_i \geq 5) \propto \frac{\lambda^{y_i}}{y_i!} I(y_i \geq 5)
\end{aligned}
$$

Then we can know that $\lambda|X,Y \propto Gamma(\sum_i x_i + \sum_j y_j, n)$.

The MH-Step to sample 78 unobserved $y's$ is

$$
y_j^* = \begin{cases} y_j^{(t)} - 1, & \text{with probability } \frac{1}{3};\\ y_j^{(t)}, & \text{with probability } \frac{1}{3};\\ y_j^{(t)} + 1, & \text{with probability } \frac{1}{3}. \end{cases}
$$

$$
r = \min\left\{ \frac{[\lambda^{(t+1)}]^{y_j^*}/y_j^*!}{[\lambda^{(t+1)}]^{y_j^{(t)}}/y_j^{(t)}!} I(y_j^* \geq 5), 1 \right\}
$$

where r is the accept-reject ratio.

Result: $\hat{\lambda} = 1.674$.

2: Gibbs Sampler for Clustering (30%)

Derivation:

We use $\{X_{ij}\}_{i=1,2,3,\ldots,1000}^{j=1,2,3}$ to denote the datum of i-th sample in j-th dimension. Then using the same notation in the question, the complete-data likelihood function is:

$$
\begin{aligned}
f(X,Z|\Pi,\Theta) &= \prod_{i=1}^{1000}\prod_{k=1}^{3}\Big[P(Z_i = k|\Pi,\Theta)P(X_{ij}, j=1,2,3|Z_j,\Pi,\Theta)\Big]^{I(Z_j=k)}\\
&= \prod_{i=1}^{1000}\prod_{k=1}^{3}\Big[P(Z_i = k|\Pi)\prod_{j=1}^{3}P(X_{ij}|Z_j,\Theta)\Big]^{I(Z_j=k)}\\
&= \prod_{i=1}^{1000}\prod_{k=1}^{3}\Big[\pi_k\prod_{j=1}^{3}\binom{10j}{X_{ij}}\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10j-X_{ij}}\Big]^{I(Z_j=k)}
\end{aligned}
$$

Note that given $Z_i = k$, for each sample i, $X_{ij} \sim Bino(10j, \theta_{jk})$; $P(\theta_{jk}) \propto 1$; and $(\pi_1, \pi_2, \pi_3) \sim Dirichlet(\alpha_1, \alpha_2, \alpha_3)$, we can derive by the following:

$$
\begin{aligned}
P(\Pi, \Theta, Z|X) &\propto P(\Pi)P(\Theta)P(X, Z|\Pi, \Theta) \\
&\propto \prod_{k=1}^{3} \pi_k^{\alpha_k - 1} \prod_{i=1}^{1000} \prod_{k=1}^{3} \Big[\pi_k \prod_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]^{I(Z_j = k)} \\
&\propto \prod_{k=1}^{3} \pi_k^{\alpha_k - 1} \prod_{i=1}^{1000} \prod_{k=1}^{3} \pi_k^{I(Z_i = k)} \prod_{i=1}^{1000} \prod_{k=1}^{3} \prod_{j=1}^{3} \binom{10j}{X_{ij}}^{I(Z_i = k)} \theta_{jk}^{X_{ij} I(Z_i = k)} [(1-\theta_{jk})^{10j - X_{ij}}]^{I(Z_i = k)}
\end{aligned}
$$

For π:

$$
\begin{aligned}
f(\Pi|\Theta, Z) &\propto \prod_{k=1}^{3} \pi_k^{\alpha_k - 1} \prod_{i=1}^{1000} \prod_{k=1}^{3} \pi_k^{I(Z_i = k)} \\
&\propto \prod_{k=1}^{3} \pi_k^{\alpha_k - 1} \prod_{k=1}^{3} [\pi_k]^{\sum_{i=1}^{1000} I(Z_i = k)} \\
&\propto \prod_{k=1}^{3} \pi_k^{\alpha_k + \sum_{i=1}^{1000} I(Z_i = k) - 1}
\end{aligned}
$$

Then it follows that:

$$
\Pi|\Theta, Z \sim Dirichlet(\alpha_1 + \sum_{i=1}^{1000} I(Z_i = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i = 3))
$$

For θ:

$$
\begin{aligned}
\theta_{jk}|- &\propto \prod_{i=1}^{1000} \prod_{k=1}^{3} \Big[\prod_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]^{I(Z_j = k)} \\
&\propto \prod_{i=1}^{1000} \Big[\theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]^{I(Z_j = k)} \\
&\propto \theta_{jk}^{\sum_{i=1}^{1000} X_{ij} I(Z_i = k)} (1-\theta_{jk})^{\sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i = k)}
\end{aligned}
$$

Then it follows that:

$$
\theta_{jk}|- \sim Beta(1 + \sum_{i=1}^{1000} X_{ij} I(Z_i = k), 1 + \sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i = k))
$$

For Z:

$$
\begin{aligned}
Z_i|- &\propto \prod_{i=1}^{1000} \prod_{k=1}^{3} \pi_k^{I(Z_i = k)} \prod_{i=1}^{1000} \prod_{k=1}^{3} \prod_{j=1}^{3} \binom{10j}{X_{ij}}^{I(Z_i = k)} \theta_{jk}^{X_{ij} I(Z_i = k)} [(1-\theta_{jk})^{10j - X_{ij}}]^{I(Z_i = k)} \\
&\propto \prod_{k=1}^{3} \pi_k^{I(Z_i = k)} \prod_{k=1}^{3} \prod_{j=1}^{3} \binom{10j}{X_{ij}}^{I(Z_i = k)} \theta_{jk}^{X_{ij} I(Z_i = k)} [(1-\theta_{jk})^{10j - X_{ij}}]^{I(Z_i = k)} \\
&\propto \prod_{k=1}^{3} \Big[\pi_k \Big[\prod_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]\Big]^{I(Z_i = k)}
\end{aligned}
$$

Then for Gibbs Sampler algorithm:

Given $\Pi^{(t)}, \Theta^{(t)}, Z^{(t)}$, we update the parameters by the following:

$$\pi_1^{(t+1)}, \pi_2^{(t+1)}, \pi_3^{(t+1)}|- \sim Dirichlet(\alpha_1 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 3))$$

$$\theta_{jk}^{(t+1)}|- \sim Beta(1 + \sum_{i=1}^{1000} X_{ij} I(Z_i^{(t)} = k), 1 + \sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i^{(t)} = k))$$

$$P(Z_i^{(t+1)} = k|-) = \frac{\pi_k^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jk}^{(t+1)})^{X_{ij}} (1 - \theta_{jk}^{(t+1)})^{10j - X_{ij}}}{\sum_{l=1}^3 \pi_l^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jl}^{(t+1)})^{X_{ij}} (1 - \theta_{jl}^{(t+1)})^{10j - X_{ij}}}$$

Result:

3: Hybrid Gibbs Sampler (40%)

Note that for $i = 1, 2$, $\mathbf{Y}_i = (y_{i1}, y_{i2}, y_{i3}, y_{i4}) \sim multinomial(100, p_1, p_2, p_3, p_4)$, then:

$$\text{Prior} : \pi(\mathbf{p}) \propto Dirichlet(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \propto Dirichlet(2, 2, 2, 2) \propto p_1 p_2 p_3 p_4;$$

$$P(y_{i1}, y_{i2}, y_{i3}, y_{i4}|p_1, p_2, p_3, p_4) = \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1}} p_2^{y_{i2}} p_3^{y_{i3}} p_4^{y_{i4}}$$

$$\begin{aligned} P(p_1, p_2, p_3, p_4|y_{i1}, y_{i2}, y_{i3}, y_{i4}) &\propto P(y_{i1}, y_{i2}, y_{i3}, y_{i4}|p_1, p_2, p_3, p_4) f(p_1, p_2, p_3, p_4|\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &\propto \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1} + \alpha_1 - 1} p_2^{y_{i2} + \alpha_2 - 1} p_3^{y_{i3} + \alpha_3 - 1} p_4^{y_{i4} + \alpha_4 - 1} \end{aligned}$$

$$p_1, p_2, p_3, p_4|y_{i1}, y_{i2}, y_{i3}, y_{i4} \sim Dirichlet(y_{i1} + \alpha_1, y_{i2} + \alpha_2, y_{i3} + \alpha_3, y_{i4} + \alpha_4)$$

$$P(y_{i2}|\mathbf{Y}, \mathbf{P}) \propto \frac{p_2^{y_{i2}}}{y_{i2}!}$$

Then MH-Step to update y_{i2} (for $i = 1, 2$) is

$$y_{i2}^{(t+1)} = \begin{cases} y_{i2}^{(t)} + 1, & \text{with probability } \frac{1}{2} \\ y_{i2}^{(t)} - 1, & \text{with probability } \frac{1}{2} \end{cases}$$

$$y_{i2}^{(t+1)} = y_{i2}^{(t)} + 1, \qquad \text{if } y_{i2}^{(t)} = 15$$

$$y_{i2}^{(t+1)} = y_{i2}^{(t)} - 1, \qquad \text{if } y_{i2}^{(t)} = 32$$

$$r = \min\left\{ \frac{p_2^{y_{i2}^{(t+1)}} / y_{i2}^{(t+1)}!}{p_2^{y_{i2}^{(t)}} / y_{i2}^{(t)}!}, 1 \right\}$$

where r is the accept-reject ratio. Then for another two unobserved variables:

$$y_{11}^{(t+1)} = 100 - y_{13} - y_{14} - y_{12}^{(t+1)} = 100 - 22 - 31 - y_{12}^{(t+1)}$$

$$y_{24}^{(t+1)} = 100 - y_{21} - y_{23} - y_{22}^{(t+1)} = 100 - 28 - 26 - y_{22}^{(t+1)}$$

Result:

$p_1 = 0.277$, $p_2 = 0.197$, $p_3 = 0.219$, and $p_4 = 0.307$.