## 1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.
$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function $f$ has totally 3 zeros.

**Algorithm:** Bisection Method in the R file.

**Result:** Zeros: -4.837944, -2.259727, and 1.097664.

## 2: Poisson Regression - Newton's Method (25%)

**(1)** Since $y_i \sim Poisson(\lambda_i)$ and $log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihhod function:

$$L(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^{n} \frac{e^{(\alpha+\beta x_i+\gamma x_i^2)y_i} e^{-e^{\alpha+\beta x_i+\gamma x_i^2}}}{y_i!}$$

**(2)** The log-Likelihood function is

$$l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha+\beta x_i+\gamma x_i^2} - \log y_i!$$

Then we have:

$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \alpha}\right|_{\hat{\alpha}} = \left.\sum_{i=1}^{n} [y_i - e^{\alpha+\beta x_i+\gamma x_i^2}]\right|_{\hat{\alpha}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \beta}\right|_{\hat{\beta}} = \left.\sum_{i=1}^{n} [x_i y_i - x_i e^{\alpha+\beta x_i+\gamma x_i^2}]\right|_{\hat{\beta}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \gamma}\right|_{\hat{\gamma}} = \left.\sum_{i=1}^{n} [x_i^2 y_i - x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}]\right|_{\hat{\gamma}} = 0$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma^2} \end{pmatrix}$, in which

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha^2} = \sum_{i=1}^{n} -e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\beta} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\alpha} = \sum_{i=1}^{n} -x_i e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\alpha} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta^2} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\beta} = \sum_{i=1}^{n} -x_i^3 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma^2} = \sum_{i=1}^{n} -x_i^4 e^{\alpha+\beta x_i+\gamma x_i^2}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix} = \begin{pmatrix}\alpha^{(n-1)}\\ \beta^{(n-1)}\\ \gamma^{(n-1)}\end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

**(3)**

**Algorithm:** Newton's Method code in the R file.

**Result:** $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

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**3: Logistic Regression - Newton's Method (20%)**

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**(1)** Since $y_i \sim Bernoulli(p_i)$ and $logit(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i}(1-p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$. Then we can get the following Likelihood function:

$$\begin{aligned} L(\alpha,\beta|\mathbf{x},\mathbf{y}) &= \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i} \\ &= \prod_{i=1}^{n} (\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{y_i}(1-\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{1-y_i} \end{aligned}$$

**(2)** The log-Likelihood function is

$$\begin{aligned} l(\alpha, \beta|\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} y_i(\alpha + \beta x_i - \log(1 + e^{\alpha+\beta x_i})) + (1 - y_i)\log(\frac{1}{1 + e^{\alpha+\beta x_i}}) \\ &= \sum_{i=1}^{n} \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha+\beta x_i}) \end{aligned}$$

Then we have:

$$\begin{aligned} \frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha}\bigg|_{\hat{\alpha}} &= \sum_{i=1}^{n} [y_i - \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\alpha}} = 0 \\ \frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta}\bigg|_{\hat{\beta}} &= \sum_{i=1}^{n} [x_i y_i - \frac{x_i e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\beta}} = 0 \end{aligned}$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$\begin{aligned} \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^{n} -\frac{e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} -\frac{x_i e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^{n} -\frac{x_i^2 e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \end{aligned}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

**(3)**

**Algorithm:** Newton's Method code in the R file.

**Result:** $\alpha = 1.564284$ and $\beta = 1.771093$.

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## 4: EM Algorithm (30%)

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**(1)** Observed data: $Y_i$ for $i = 1, 2, \ldots, 8000$; Missing data: $Z_i$ for $i = 1, 2, \ldots, 8000.$, where $Z_i = 1$, 2, or 3 for low, middle, and high income respectively.

Since $Y_i\big|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$, with proportion $\pi_k$ $(\pi_3 = 1 - (\pi_1 + \pi_2))$, we can formulate the **complete − data Likelihhod function** as:

$$\begin{aligned}&L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}, \mathbf{Z})\\ =&\prod_{i=1}^{n}\left[\pi_1\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}}\right]^{I(Z_i=1)}\left[\pi_2\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}\right]^{I(Z_i=2)}\left[(1-\pi_1-\pi_2)\frac{1}{\sqrt{2\pi}\sigma_3}e^{-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}}\right]^{I(Z_i=3)}\end{aligned}$$

**(2)** The complete-data log-Likelihhod function is:

$$\begin{aligned}l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{n}\Bigg[&I(Z_i = 1)\left[\log\pi_1 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_1^2 - \frac{(y_i-\mu_1)^2}{2\sigma_1^2}\right]\\ &+I(Z_i = 2)\left[\log\pi_2 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_2^2 - \frac{(y_i-\mu_2)^2}{2\sigma_2^2}\right]\\ &+I(Z_i = 3)\left[\log(1-\pi_1-\pi_2) - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_3^2 - \frac{(y_i-\mu_3)^2}{2\sigma_3^2}\right]\Bigg]\end{aligned}$$

Then given initial guess $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, we define a $Q$ function by $Q(\Theta; \Theta^{(t)}) = E_{\mathbf{\Theta^{(t)}}}\big(l(\mathbf{\Theta}|\mathbf{Y}, \mathbf{Z})|\mathbf{Y}\big)$:

$$\begin{aligned}Q(\Theta; \Theta^{(t)}) &= E_{\pi_1^{(t)},\pi_2^{(t)},\mu_1^{(t)},\mu_2^{(t)},\mu_3^{(t)},\sigma_1^{2(t)},\sigma_2^{2(t)},\sigma_3^{2(t)}}\big(l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3|\mathbf{Y}, \mathbf{Z})|\mathbf{Y}\big)\\ &= \sum_{i=1}^{n}\Bigg[\widehat{Z_{i1}}^{(t)}\left[\log\pi_1 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_1^2 - \frac{(y_i-\mu_1)^2}{2\sigma_1^2}\right]\\ &+ \widehat{Z_{i2}}^{(t)}\left[\log\pi_2 - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_2^2 - \frac{(y_i-\mu_2)^2}{2\sigma_2^2}\right]\\ &+ \widehat{Z_{i3}}^{(t)}\left[\log(1-\pi_1-\pi_2) - \log\sqrt{2\pi} - \frac{1}{2}\log\sigma_3^2 - \frac{(y_i-\mu_3)^2}{2\sigma_3^2}\right]\Bigg]\end{aligned}$$

where $(\pi_3^{(t)} = 1 - \pi_1^{(t)} - \pi_2^{(t)})$

$$\begin{aligned}\widehat{Z_{ik}}^{(t)} &= E(Z_i = k|\Theta^{(t)}) = E(Z_i^{(t)}|\pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)})\\ &= \frac{k\pi_k^{(t)}\frac{1}{\sqrt{2\pi}\sigma_k{}^{(t)}}e^{-\frac{(y_i-\mu_k^{(t)})^2}{2\sigma_k^{2(t)}}}}{\pi_1^{(t)}\frac{1}{\sqrt{2\pi}\sigma_1{}^{(t)}}e^{-\frac{(y_i-\mu_1^{(t)})^2}{2\sigma_1^{2(t)}}} + \pi_2^{(t)}\frac{1}{\sqrt{2\pi}\sigma_2{}^{(t)}}e^{-\frac{(y_i-\mu_2^{(t)})^2}{2\sigma_2^{2(t)}}} + (1-\pi_1^{(t)}-\pi_2^{(t)})\frac{1}{\sqrt{2\pi}\sigma_3{}^{(t)}}e^{-\frac{(y_i-\mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}\end{aligned}$$

Then we can calculate:

$$\frac{\partial Q}{\partial \pi_1}\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\frac{1}{\pi_1} - \widehat{Z_{i3}}^{(t)}\frac{1}{1-\pi_1-\pi_2}\right]\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \pi_2}\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\frac{1}{\pi_2} - \widehat{Z_{i3}}^{(t)}\frac{1}{1-\pi_1-\pi_2}\right]\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \mu_1}\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\frac{y_i-\mu_1}{\sigma_1^2}\right]\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \mu_2}\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\frac{y_i-\mu_2}{\sigma_2^2}\right]\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \mu_3}\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i3}}^{(t)}\frac{y_i-\mu_3}{\sigma_3^2}\right]\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \sigma_1^2}\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_1^2}+\frac{(y_i-\mu_1)^2}{2(\sigma_1^2)^2}\right)\right]\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \sigma_2^2}\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_2^2}+\frac{(y_i-\mu_2)^2}{2(\sigma_2^2)^2}\right)\right]\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = 0$$
$$\frac{\partial Q}{\partial \sigma_3^2}\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i3}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_3^2}+\frac{(y_i-\mu_3)^2}{2(\sigma_3^2)^2}\right)\right]\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = 0$$

The observed-data Likelihhod function is:

$$L(\pi_1,\pi_2,\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3|\mathbf{Y})$$
$$=\prod_{i=1}^{n}\left[\left[\pi_1\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}}\right]+\left[\pi_2\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}\right]+\left[(1-\pi_1-\pi_2)\frac{1}{\sqrt{2\pi}\sigma_3}e^{-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}}\right]\right]$$

Then for iteration in **EM Algorithm**:

Given initial guess: $\pi_1^{(0)},\pi_2^{(0)},\mu_1^{(0)},\mu_2^{(0)},\mu_3^{(0)},\sigma_1^{(0)},\sigma_2^{(0)},\sigma_3^{(0)}$, for $t \geq 0$ and $t \in \mathbb{Z}$:

$\mathbf{E-step}$: Calculate $E(Z_i^{(t)}|\Theta^{(t)})$, where $\Theta^{(t)} = \pi_1^{(t)},\pi_2^{(t)},\mu_1^{(t)},\mu_2^{(t)},\mu_3^{(t)},\sigma_1^{2(t)},\sigma_2^{2(t)},\sigma_3^{2(t)}$.

$\mathbf{M-step}$: Update $\Theta^{(t+1)}$ by equations (1) to (8) listed at the next page.

**Stopping criterion**: $|L(\Theta^{(t)}|\mathbf{Y}) - L(\Theta^{(T+1)}|\mathbf{Y}))| < tolerance$

Iterative scheme:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)}}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i2}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{1}$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)}}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i2}}^{(t)} + \sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{2}$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)}} \tag{3}$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)}} \tag{4}$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{5}$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)} (y_i - \mu_1^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z_{i1}}^{(t)}} \tag{6}$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)} (y_i - \mu_2^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z_{i2}}^{(t)}} \tag{7}$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)} (y_i - \mu_3^{(t+1)})^2}{\sum_{i=1}^n \widehat{Z_{i3}}^{(t)}} \tag{8}$$

**(3)**

**Algorithm:** EM Algorithm code in the R file.

**Result:** $\pi_1 = 0.6464652, \pi_2 = 0.2498821, \mu_1 = 2999.048, \mu_2 = 8004.365, \mu_3 = 29848.78, \sigma_1 = 303.3709, \sigma_2 = 1515.854$ and $\sigma_3 = 7963.118$

**(4) Result:** in the following page:

| | low_income_1 | middle_income_2 | high_income_3 | class |
|---|---|---|---|---|
| x1 | 9.991958e-01 | 7.473953e-04 | 5.678040e-05 | 1 |
| x2 | 9.997040e-01 | 2.475613e-04 | 4.846901e-05 | 1 |
| x3 | 0.000000e+00 | 2.883362e-47 | 1.000000e+00 | 3 |
| x4 | 9.997088e-01 | 2.451883e-04 | 4.603783e-05 | 1 |
| x5 | 1.091611e-39 | 9.975983e-01 | 2.401733e-03 | 2 |
| x6 | 0.000000e+00 | 7.379262e-54 | 1.000000e+00 | 3 |
| x7 | 3.508001e-137 | 9.730584e-01 | 2.694163e-02 | 2 |
| x8 | 9.991552e-01 | 7.861646e-04 | 5.867285e-05 | 1 |
| x9 | 9.988827e-01 | 1.046195e-03 | 7.112928e-05 | 1 |
| x10 | 9.996717e-01 | 2.919623e-04 | 3.636031e-05 | 1 |
| x11 | 9.995750e-01 | 3.857297e-04 | 3.928783e-05 | 1 |
| x12 | 8.309440e-101 | 9.927352e-01 | 7.264784e-03 | 2 |
| x13 | 1.840195e-44 | 9.976051e-01 | 2.394850e-03 | 2 |
| x14 | 9.994094e-01 | 5.438131e-04 | 4.675592e-05 | 1 |
| x15 | 1.062045e-153 | 9.474148e-01 | 5.258523e-02 | 2 |
| x16 | 9.882324e-01 | 1.131743e-02 | 4.502208e-04 | 1 |
| x17 | 0.000000e+00 | 7.196126e-19 | 1.000000e+00 | 3 |
| x18 | 9.994248e-01 | 5.292086e-04 | 4.603881e-05 | 1 |
| x19 | 0.000000e+00 | 2.324951e-107 | 1.000000e+00 | 3 |
| x20 | 3.084091e-20 | 9.965954e-01 | 3.404619e-03 | 2 |
| x21 | 9.989492e-01 | 9.826852e-04 | 6.812581e-05 | 1 |
| x22 | 9.997055e-01 | 2.570307e-04 | 3.751380e-05 | 1 |
| x23 | 1.246812e-16 | 9.960167e-01 | 3.983289e-03 | 2 |
| x24 | 0.000000e+00 | 7.884756e-82 | 1.000000e+00 | 3 |
| x25 | 9.996514e-01 | 2.804111e-04 | 6.815332e-05 | 1 |
| x26 | 9.990114e-01 | 6.894410e-04 | 2.991983e-04 | 1 |
| x27 | 9.993308e-01 | 6.187803e-04 | 5.045323e-05 | 1 |
| x28 | 9.997141e-01 | 2.448797e-04 | 4.098493e-05 | 1 |
| x29 | 9.994920e-01 | 4.650407e-04 | 4.292448e-05 | 1 |
| x30 | 9.997139e-01 | 2.458929e-04 | 4.024660e-05 | 1 |
| x31 | 1.114517e-117 | 9.871259e-01 | 1.287412e-02 | 2 |
| x32 | 0.000000e+00 | 9.164527e-69 | 1.000000e+00 | 3 |
| x33 | 9.997104e-01 | 2.445317e-04 | 4.508595e-05 | 1 |
| x34 | 9.997140e-01 | 2.456565e-04 | 4.039182e-05 | 1 |
| x35 | 9.996539e-01 | 3.094729e-04 | 3.661519e-05 | 1 |
| x36 | 9.997087e-01 | 2.452019e-04 | 4.605523e-05 | 1 |
| x37 | 9.974392e-01 | 2.429040e-03 | 1.317922e-04 | 1 |
| x38 | 9.954162e-01 | 4.375381e-03 | 2.084620e-04 | 1 |
| x39 | 9.986961e-01 | 1.224447e-03 | 7.943071e-05 | 1 |
| x40 | 9.996930e-01 | 2.538918e-04 | 5.313980e-05 | 1 |
| x41 | 0.000000e+00 | 3.431441e-75 | 1.000000e+00 | 3 |
| x42 | 9.996570e-01 | 3.064784e-04 | 3.655429e-05 | 1 |
| x43 | 4.687180e-44 | 9.976068e-01 | 2.393242e-03 | 2 |
| x44 | 9.996299e-01 | 2.945703e-04 | 7.550999e-05 | 1 |
| x45 | 9.996646e-01 | 2.989785e-04 | 3.643057e-05 | 1 |
| x46 | 0.000000e+00 | 7.681117e-24 | 1.000000e+00 | 3 |
| x47 | 9.993030e-01 | 6.452097e-04 | 5.175772e-05 | 1 |
| x48 | 5.729718e-74 | 9.964873e-01 | 3.512736e-03 | 2 |
| x49 | 0.000000e+00 | 1.139030e-40 | 1.000000e+00 | 3 |
| x50 | 9.995960e-01 | 3.655035e-04 | 3.844810e-05 | 1 |

Figure 1: Classification of first 50 individuals