

## 1: The Bisection Method (25%)

For function  $f(x) = x^3 + 6x^2 + \pi x - 12$ , the derivative is  $f'(x) = 3x^2 + 12x + \pi$ . The we can calculate that zeros of the derivative are  $\frac{-12-\sqrt{12(12-\pi)}}{6}$  and  $\frac{-12+\sqrt{12(12-\pi)}}{6}$ .

$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$  and  $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$  Hence, the function  $f$  has totally 3 zeros.

**Algorithm:** Bisection Method in the R file.

**Result:** Zeros: -4.837944, -2.259727, and 1.097664.

## 2: Poisson Regression - Newton's Method (25%)

(1) Since  $y_i \sim \text{Poisson}(\lambda_i)$  and  $\log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$ , we can get the Likelihood function:

$$L(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \prod_{i=1}^n \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^n \frac{e^{(\alpha + \beta x_i + \gamma x_i^2)y_i} e^{-e^{\alpha + \beta x_i + \gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha + \beta x_i + \gamma x_i^2} - \log y_i!$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n [y_i - e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n [x_i y_i - x_i e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\beta}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \right|_{\hat{\gamma}} &= \sum_{i=1}^n [x_i^2 y_i - x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\gamma}} = 0 \end{aligned}$$

$$\text{Let } \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \end{pmatrix}, \text{ then } \mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} \end{pmatrix}, \text{ in}$$

which

$$\begin{aligned}
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^n -e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} = \sum_{i=1}^n -x_i e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} = \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} = \sum_{i=1}^n -x_i^3 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} &= \sum_{i=1}^n -x_i^4 e^{\alpha + \beta x_i + \gamma x_i^2}
\end{aligned}$$

Therefore, by Newton's Method, given initial guess  $\alpha^{(0)}$ ,  $\beta^{(0)}$ , and  $\gamma^{(0)}$ , for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm:  $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix}$ ,

STEP 1: Solve  $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$ ;

STEP 2: Update by  $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$ .

(3)

**Algorithm:** Newton's Method code in the R file.

**Result:**  $\hat{\alpha} = 1.503533$ ,  $\hat{\beta} = 1.052351$ , and  $\hat{\gamma} = 1.957396$ .

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### 3: Logistic Regression - Newton's Method (20%)

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(1) Since  $y_i \sim \text{Bernoulli}(p_i)$  and  $\text{logit}(p_i) = \alpha + \beta x_i$ , we can know that  $f(y_i, p_i) = p_i^{y_i} (1 - p_i)^{1-y_i}$ , and  $p_i = \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}$ . Then we can get the following Likelihood function:

$$\begin{aligned}
L(\alpha, \beta | \mathbf{x}, \mathbf{y}) &= \prod_{i=1}^n p_i^{y_i} (1 - p_i)^{1-y_i} \\
&= \prod_{i=1}^n \left( \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right)^{y_i} \left( 1 - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right)^{1-y_i}
\end{aligned}$$

(2) The log-Likelihood function is

$$\begin{aligned} l(\alpha, \beta | \mathbf{x}, \mathbf{y}) &= \sum_{i=1}^n y_i (\alpha + \beta x_i - \log(1 + e^{\alpha + \beta x_i})) + (1 - y_i) \log\left(\frac{1}{1 + e^{\alpha + \beta x_i}}\right) \\ &= \sum_{i=1}^n \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha + \beta x_i}) \end{aligned}$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n \left[ y_i - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n \left[ x_i y_i - \frac{x_i e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right] \Big|_{\hat{\beta}} = 0 \end{aligned}$$

Let  $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta} \end{pmatrix}$ , then  $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta^2} \end{pmatrix}$ , in which

$$\begin{aligned} \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^n -\frac{e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = \sum_{i=1}^n -\frac{x_i e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^n -\frac{x_i^2 e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \end{aligned}$$

Therefore, by Newton's Method, given initial guess  $\alpha^{(0)}$  and  $\beta^{(0)}$ , for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm:  $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$ ,

STEP 1: Solve  $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$ ;

STEP 2: Update by  $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$ .

(3)

**Algorithm:** Newton's Method code in the R file.

**Result:**  $\hat{\alpha} = 1.564284$  and  $\hat{\beta} = 1.771093$ .

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#### 4: EM Algorithm (30%)

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(1) Observed data:  $Y_i$  for  $i = 1, 2, \dots, 8000$ ; Missing data:  $Z_i$  for  $i = 1, 2, \dots, 8000$ , where  $Z_i = 1, 2$ , or  $3$  for low, middle, and high income respectively.

Since  $Y_i|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$ , with proportion  $\pi_k$  ( $\pi_3 = 1 - (\pi_1 + \pi_2)$ ), we can formulate the **complete – data Likelihood function** as:

$$L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) \\ = \prod_{i=1}^n \left[ \pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right]^{I(Z_i=1)} \left[ \pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right]^{I(Z_i=2)} \left[ (1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right]^{I(Z_i=3)}$$

(2) The complete-data log-Likelihood function is:

$$l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^n \left[ I(Z_i = 1) \left[ \log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right. \\ \left. + I(Z_i = 2) \left[ \log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \right. \\ \left. + I(Z_i = 3) \left[ \log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

Then given initial guess  $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$ , we define a  $Q$  function by  $Q(\Theta; \Theta^{(t)}) = E_{\Theta^{(t)}}(l(\Theta | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y})$ :

$$Q(\Theta; \Theta^{(t)}) = E_{\pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}}(l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y}) \\ = \sum_{i=1}^n \left[ \widehat{Z}_{i1}^{(t)} \left[ \log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right. \\ \left. + \widehat{Z}_{i2}^{(t)} \left[ \log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \right. \\ \left. + \widehat{Z}_{i3}^{(t)} \left[ \log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

where  $(\pi_3^{(t)} = 1 - \pi_1^{(t)} - \pi_2^{(t)})$

$$\widehat{Z}_{ik}^{(t)} = E(Z_i = k | \Theta^{(t)}) = E(Z_i^{(t)} | \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}) \\ = \frac{\pi_k^{(t)} \frac{1}{\sqrt{2\pi}\sigma_k^{(t)}} e^{-\frac{(y_i - \mu_k^{(t)})^2}{2\sigma_k^{2(t)}}}}{\pi_1^{(t)} \frac{1}{\sqrt{2\pi}\sigma_1^{(t)}} e^{-\frac{(y_i - \mu_1^{(t)})^2}{2\sigma_1^{2(t)}}} + \pi_2^{(t)} \frac{1}{\sqrt{2\pi}\sigma_2^{(t)}} e^{-\frac{(y_i - \mu_2^{(t)})^2}{2\sigma_2^{2(t)}}} + (1 - \pi_1^{(t)} - \pi_2^{(t)}) \frac{1}{\sqrt{2\pi}\sigma_3^{(t)}} e^{-\frac{(y_i - \mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}$$

Then we can calculate:

$$\begin{aligned}
\frac{\partial Q}{\partial \pi_1} \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i1}^{(t)} \frac{1}{\pi_1} - \widehat{Z}_{i3}^{(t)} \frac{1}{1 - \pi_1 - \pi_2} \right] \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \pi_2} \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i2}^{(t)} \frac{1}{\pi_2} - \widehat{Z}_{i3}^{(t)} \frac{1}{1 - \pi_1 - \pi_2} \right] \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_1} \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i1}^{(t)} \frac{y_i - \mu_1}{\sigma_1^2} \right] \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_2} \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i2}^{(t)} \frac{y_i - \mu_2}{\sigma_2^2} \right] \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_3} \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i3}^{(t)} \frac{y_i - \mu_3}{\sigma_3^2} \right] \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_1^2} \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i1}^{(t)} \left( -\frac{1}{2} \frac{1}{\sigma_1^2} + \frac{(y_i - \mu_1)^2}{2(\sigma_1^2)^2} \right) \right] \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_2^2} \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i2}^{(t)} \left( -\frac{1}{2} \frac{1}{\sigma_2^2} + \frac{(y_i - \mu_2)^2}{2(\sigma_2^2)^2} \right) \right] \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_3^2} \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} &= \sum_{i=1}^n \left[ \widehat{Z}_{i3}^{(t)} \left( -\frac{1}{2} \frac{1}{\sigma_3^2} + \frac{(y_i - \mu_3)^2}{2(\sigma_3^2)^2} \right) \right] \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} = 0
\end{aligned}$$

The observed-data Likelihood function is:

$$\begin{aligned}
&L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}) \\
&= \prod_{i=1}^n \left[ \left[ \pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right] + \left[ \pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right] + \left[ (1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right] \right]
\end{aligned}$$

Then for iteration in **EM Algorithm**:

Given initial guess:  $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$ , for  $t \geq 0$  and  $t \in \mathbb{Z}$ :

**E – step:** Calculate  $E(Z_i^{(t)} | \Theta^{(t)})$ , where  $\Theta^{(t)} = \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}$ .

**M – step:** Update  $\Theta^{(t+1)}$  by equations (1) to (8) listed at the next page.

**Stopping criterion:**  $|L(\Theta^{(t)} | \mathbf{Y}) - L(\Theta^{(t+1)} | \mathbf{Y})| < \text{tolerance}$ .

Iterative scheme:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (1)$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (2)$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}} \quad (3)$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}} \quad (4)$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (5)$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} (y_i - \mu_1^{(t)})^2}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}} \quad (6)$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)} (y_i - \mu_2^{(t)})^2}{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}} \quad (7)$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)} (y_i - \mu_3^{(t)})^2}{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (8)$$

where  $\widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)} = n$  (the sample size).

(3)

**Algorithm:** EM Algorithm code in the R file.

**Result:**  $\hat{\pi}_1 = 0.6464652$ ,  $\hat{\pi}_2 = 0.2498821$ ,  $\hat{\mu}_1 = 2999.048$ ,  $\hat{\mu}_2 = 8004.365$ ,  $\hat{\mu}_3 = 29848.78$ ,  $\hat{\sigma}_1 = 303.3709$ ,  $\hat{\sigma}_2 = 1515.854$  and  $\hat{\sigma}_3 = 7963.118$

(4) **Result:** in the following page:

	low_income_1	middle_income_2	high_income_3	class
x1	9.991958e-01	7.473953e-04	5.678040e-05	1
x2	9.997040e-01	2.475613e-04	4.846901e-05	1
x3	0.000000e+00	2.883362e-47	1.000000e+00	3
x4	9.997088e-01	2.451883e-04	4.603783e-05	1
x5	1.091611e-39	9.975983e-01	2.401733e-03	2
x6	0.000000e+00	7.379262e-54	1.000000e+00	3
x7	3.508001e-137	9.730584e-01	2.694163e-02	2
x8	9.991552e-01	7.861646e-04	5.867285e-05	1
x9	9.988827e-01	1.046195e-03	7.112928e-05	1
x10	9.996717e-01	2.919623e-04	3.636031e-05	1
x11	9.995750e-01	3.857297e-04	3.928783e-05	1
x12	8.309440e-101	9.927352e-01	7.264784e-03	2
x13	1.840195e-44	9.976051e-01	2.394850e-03	2
x14	9.994094e-01	5.438131e-04	4.675592e-05	1
x15	1.062045e-153	9.474148e-01	5.258523e-02	2
x16	9.882324e-01	1.131743e-02	4.502208e-04	1
x17	0.000000e+00	7.196126e-19	1.000000e+00	3
x18	9.994248e-01	5.292086e-04	4.603881e-05	1
x19	0.000000e+00	2.324951e-107	1.000000e+00	3
x20	3.084091e-20	9.965954e-01	3.404619e-03	2
x21	9.989492e-01	9.826852e-04	6.812581e-05	1
x22	9.997055e-01	2.570307e-04	3.751380e-05	1
x23	1.246812e-16	9.960167e-01	3.983289e-03	2
x24	0.000000e+00	7.884756e-82	1.000000e+00	3
x25	9.996514e-01	2.804111e-04	6.815332e-05	1
x26	9.990114e-01	6.894410e-04	2.991983e-04	1
x27	9.993308e-01	6.187803e-04	5.045323e-05	1
x28	9.997141e-01	2.448797e-04	4.098493e-05	1
x29	9.994920e-01	4.650407e-04	4.292448e-05	1
x30	9.997139e-01	2.458929e-04	4.024660e-05	1
x31	1.114517e-117	9.871259e-01	1.287412e-02	2
x32	0.000000e+00	9.164527e-69	1.000000e+00	3
x33	9.997104e-01	2.445317e-04	4.508595e-05	1
x34	9.997140e-01	2.456565e-04	4.039182e-05	1
x35	9.996539e-01	3.094729e-04	3.661519e-05	1
x36	9.997087e-01	2.452019e-04	4.605523e-05	1
x37	9.974392e-01	2.429040e-03	1.317922e-04	1
x38	9.954162e-01	4.375381e-03	2.084620e-04	1
x39	9.986961e-01	1.224447e-03	7.943071e-05	1
x40	9.996930e-01	2.538918e-04	5.313980e-05	1
x41	0.000000e+00	3.431441e-75	1.000000e+00	3
x42	9.996570e-01	3.064784e-04	3.655429e-05	1
x43	4.687180e-44	9.976068e-01	2.393242e-03	2
x44	9.996299e-01	2.945703e-04	7.550999e-05	1
x45	9.996646e-01	2.989785e-04	3.643057e-05	1
x46	0.000000e+00	7.681117e-24	1.000000e+00	3
x47	9.993030e-01	6.452097e-04	5.175772e-05	1
x48	5.729718e-74	9.964873e-01	3.512736e-03	2
x49	0.000000e+00	1.139030e-40	1.000000e+00	3
x50	9.995960e-01	3.655035e-04	3.844810e-05	1

Figure 1: Classification of first 50 individuals