

1: Hybrid Gibbs Sampler to estimate Poisson Distribution λ (30%)**Derivation:**

For these Poisson distributed random variables (r.v.s) ($n = 500$) with mean parameter λ , unobserved variables are $\lambda, y_1, y_2, \dots, y_{78}$, in which y_i 's denote the r.v.s which are larger than or equal to five.

$$\begin{aligned} \text{Prior : } \pi(\lambda) &\propto \frac{1}{\lambda}; \\ P(X, Y|\lambda) &= \prod_{i=1}^{422} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \prod_{j=1}^{78} \frac{e^{-\lambda} \lambda^{y_j}}{y_j!} I(y_j \geq 5); \\ P(\lambda|X, Y) &\propto P(X, Y|\lambda) P(\lambda) \propto e^{-n\lambda} \lambda^{\sum_i x_i + \sum_j y_j - 1} \\ P(y_j|X, \lambda) &= \frac{e^{-\lambda} \lambda^{y_j}}{y_j!} I(y_j \geq 5) \propto \frac{\lambda^{y_j}}{y_j!} I(y_j \geq 5) \end{aligned}$$

Then we can know that $\lambda|X, Y \propto \text{Gamma}(\sum_i x_i + \sum_j y_j, n)$.

The MH-Step to sample 78 unobserved y_j 's is

$$\begin{aligned} y_j^* &= \begin{cases} y_j^{(t)} - 1, & \text{with probability } \frac{1}{3}; \\ y_j^{(t)}, & \text{with probability } \frac{1}{3}; \\ y_j^{(t)} + 1, & \text{with probability } \frac{1}{3}. \end{cases} \\ r &= \min \left\{ \frac{[\lambda^{(t+1)}]^{y_j^*} / y_j^{*!}}{[\lambda^{(t+1)}]^{y_j^{(t)}} / y_j^{(t)!}} I(y_j^* \geq 5), 1 \right\} \end{aligned}$$

where r is the accept-reject ratio.

Result: $\hat{\lambda} = 1.674$.

2: Gibbs Sampler for Clustering (30%)**Derivation:**

We use $\{X_{ij}\}_{i=1,2,3,\dots,1000}^{j=1,2,3}$ to denote the datum of i -th sample in j -th dimension. Then using the same notation in the question, the complete-data likelihood function is:

$$\begin{aligned} f(X, Z|\Pi, \Theta) &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[P(Z_i = k|\Pi, \Theta) P(X_{ij}, j = 1, 2, 3|Z_j, \Pi, \Theta) \right]^{I(Z_j=k)} \\ &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[P(Z_i = k|\Pi) \prod_{j=1}^3 P(X_{ij}|Z_j, \Theta) \right]^{I(Z_j=k)} \\ &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[\pi_k \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right]^{I(Z_j=k)} \end{aligned}$$

Note that given $Z_i = k$, for each sample i , $X_{ij} \sim \text{Bino}(10j, \theta_{jk})$; $P(\theta_{jk}) \propto 1$; and $(\pi_1, \pi_2, \pi_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3)$, we can derive by the following:

$$P(\Pi, \Theta, Z|X) \propto P(\Pi)P(\Theta)P(X, Z|\Pi, \Theta)$$

$$\begin{aligned} & \propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{i=1}^{1000} \prod_{k=1}^3 \left[\pi_k \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right]^{I(Z_i=k)} \\ & \propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{i=1}^{1000} \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \prod_{i=1}^{1000} \prod_{k=1}^3 \prod_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij}I(Z_i=k)} [(1 - \theta_{jk})^{10j-X_{ij}}]^{I(Z_i=k)} \end{aligned}$$

For π :

$$\begin{aligned} f(\Pi|\Theta, Z) & \propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{i=1}^{1000} \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \\ & \propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{k=1}^3 [\pi_k]^{\sum_{i=1}^{1000} I(Z_i=k)} \\ & \propto \prod_{k=1}^3 \pi_k^{\alpha_k + \sum_{i=1}^{1000} I(Z_i=k) - 1} \end{aligned}$$

Then it follows that:

$$\Pi|\Theta, Z \sim \text{Dirichlet}(\alpha_1 + \sum_{i=1}^{1000} I(Z_i = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i = 3))$$

For θ :

$$\begin{aligned} \theta_{jk}| - & \propto \prod_{i=1}^{1000} \prod_{k=1}^3 \left[\prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right]^{I(Z_i=k)} \\ & \propto \prod_{i=1}^{1000} \left[\theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right]^{I(Z_i=k)} \\ & \propto \theta_{jk}^{\sum_{i=1}^{1000} X_{ij}I(Z_i=k)} (1 - \theta_{jk})^{\sum_{i=1}^{1000} (10j-X_{ij})I(Z_i=k)} \end{aligned}$$

Then it follows that:

$$\theta_{jk}| - \sim \text{Beta}(1 + \sum_{i=1}^{1000} X_{ij}I(Z_i = k), 1 + \sum_{i=1}^{1000} (10j - X_{ij})I(Z_i = k))$$

For Z :

$$\begin{aligned} Z_i| - & \propto \prod_{i=1}^{1000} \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \prod_{i=1}^{1000} \prod_{k=1}^3 \prod_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij}I(Z_i=k)} [(1 - \theta_{jk})^{10j-X_{ij}}]^{I(Z_i=k)} \\ & \propto \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \prod_{k=1}^3 \prod_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij}I(Z_i=k)} [(1 - \theta_{jk})^{10j-X_{ij}}]^{I(Z_i=k)} \\ & \propto \prod_{k=1}^3 \left[\pi_k \left[\prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j-X_{ij}} \right] \right]^{I(Z_i=k)} \end{aligned}$$

Then for Gibbs Sampler algorithm:

Given $\Pi^{(t)}, \Theta^{(t)}, Z^{(t)}$, we update the parameters by the following:

$$\begin{aligned} \pi_1^{(t+1)}, \pi_2^{(t+1)}, \pi_3^{(t+1)} | - &\sim \text{Dirichlet}(\alpha_1 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 3)) \\ \theta_{jk}^{(t+1)} | - &\sim \text{Beta}(1 + \sum_{i=1}^{1000} X_{ij} I(Z_i^{(t)} = k), 1 + \sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i^{(t)} = k)) \\ P(Z_i^{(t+1)} = k | -) &= \frac{\pi_k^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jk}^{(t+1)})^{X_{ij}} (1 - \theta_{jk}^{(t+1)})^{10j - X_{ij}}}{\sum_{l=1}^3 \pi_l^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jl}^{(t+1)})^{X_{ij}} (1 - \theta_{jl}^{(t+1)})^{10j - X_{ij}}} \end{aligned}$$

Result:

3: Hybrid Gibbs Sampler (40%)

Note that for $i = 1, 2$, $\mathbf{Y}_i = (y_{i1}, y_{i2}, y_{i3}, y_{i4}) \sim \text{multinomial}(100, p_1, p_2, p_3, p_4)$, then:

Prior : $\pi(\mathbf{p}) \propto \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \propto \text{Dirichlet}(2, 2, 2, 2) \propto p_1 p_2 p_3 p_4$;

$$P(y_{i1}, y_{i2}, y_{i3}, y_{i4} | p_1, p_2, p_3, p_4) = \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1}} p_2^{y_{i2}} p_3^{y_{i3}} p_4^{y_{i4}}$$

$$P(p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4}) \propto P(y_{i1}, y_{i2}, y_{i3}, y_{i4} | p_1, p_2, p_3, p_4) f(p_1, p_2, p_3, p_4 | \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$\propto \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1} + \alpha_1 - 1} p_2^{y_{i2} + \alpha_2 - 1} p_3^{y_{i3} + \alpha_3 - 1} p_4^{y_{i4} + \alpha_4 - 1}$$

$$p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4} \sim \text{Dirichlet}(y_{i1} + \alpha_1, y_{i2} + \alpha_2, y_{i3} + \alpha_3, y_{i4} + \alpha_4)$$

$$P(y_{i2} | \mathbf{Y}, \mathbf{P}) \propto \frac{p_2^{y_{i2}}}{y_{i2}!}$$

Then MH-Step to update y_{i2} (for $i = 1, 2$) is

$$\begin{aligned} y_{i2}^{(t+1)} &= \begin{cases} y_{i2}^{(t)} + 1, & \text{with probability } \frac{1}{2} \\ y_{i2}^{(t)} - 1, & \text{with probability } \frac{1}{2} \end{cases} \\ y_{i2}^{(t+1)} &= y_{i2}^{(t)} + 1, \quad \text{if } y_{i2}^{(t)} = 15 \\ y_{i2}^{(t+1)} &= y_{i2}^{(t)} - 1, \quad \text{if } y_{i2}^{(t)} = 32 \\ r &= \min \left\{ \frac{p_2^{y_{i2}^{(t+1)}} / y_{i2}^{(t+1)}!}{p_2^{y_{i2}^{(t)}} / y_{i2}^{(t)}!}, 1 \right\} \end{aligned}$$

where r is the accept-reject ratio. Then for another two unobserved variables:

$$\begin{aligned} y_{11}^{(t+1)} &= 100 - y_{13} - y_{14} - y_{12}^{(t+1)} = 100 - 22 - 31 - y_{12}^{(t+1)} \\ y_{24}^{(t+1)} &= 100 - y_{21} - y_{23} - y_{22}^{(t+1)} = 100 - 28 - 26 - y_{22}^{(t+1)} \end{aligned}$$

Result:

For Stage 1: $p_1 = 0.27$, $p_2 = 0.199$, $p_3 = 0.22$, and $p_4 = 0.311$.

For Stage 2: $p_1 = 0.281$, $p_2 = 0.2$, $p_3 = 0.259$, and $p_4 = 0.259$.