

1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.

$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function f has totally 3 zeros.

Algorithm: Bisection Method in the R file.

Result: Zeros: -4.837944, -2.259727, and 1.097664.

2: Poisson Regression - Newton's Method (25%)

(1) Since $y_i \sim \text{Poisson}(\lambda_i)$ and $\log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihood function:

$$L(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \prod_{i=1}^n \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^n \frac{e^{(\alpha + \beta x_i + \gamma x_i^2)y_i} e^{-e^{\alpha + \beta x_i + \gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha + \beta x_i + \gamma x_i^2} - \log y_i!$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n [y_i - e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n [x_i y_i - x_i e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\beta}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \right|_{\hat{\gamma}} &= \sum_{i=1}^n [x_i^2 y_i - x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\gamma}} = 0 \end{aligned}$$

$$\text{Let } \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \end{pmatrix}, \text{ then } \mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} \end{pmatrix}, \text{ in}$$

which

$$\begin{aligned}
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^n -e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} = \sum_{i=1}^n -x_i e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} = \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} = \sum_{i=1}^n -x_i^3 e^{\alpha + \beta x_i + \gamma x_i^2} \\
\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} &= \sum_{i=1}^n -x_i^4 e^{\alpha + \beta x_i + \gamma x_i^2}
\end{aligned}$$

Therefore, by Newton's Method, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$;

STEP 2: Update by $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\hat{\alpha} = 1.503533$, $\hat{\beta} = 1.052351$, and $\hat{\gamma} = 1.957396$.

3: Logistic Regression - Newton's Method (20%)

(1) Since $y_i \sim \text{Bernoulli}(p_i)$ and $\text{logit}(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i} (1 - p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}$. Then we can get the following Likelihood function:

$$\begin{aligned}
L(\alpha, \beta | \mathbf{x}, \mathbf{y}) &= \prod_{i=1}^n p_i^{y_i} (1 - p_i)^{1-y_i} \\
&= \prod_{i=1}^n \left(\frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right)^{y_i} \left(1 - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right)^{1-y_i}
\end{aligned}$$

(2) The log-Likelihood function is

$$\begin{aligned} l(\alpha, \beta | \mathbf{x}, \mathbf{y}) &= \sum_{i=1}^n y_i (\alpha + \beta x_i - \log(1 + e^{\alpha + \beta x_i})) + (1 - y_i) \log\left(\frac{1}{1 + e^{\alpha + \beta x_i}}\right) \\ &= \sum_{i=1}^n \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha + \beta x_i}) \end{aligned}$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n \left[y_i - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n \left[x_i y_i - \frac{x_i e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}} \right] \Big|_{\hat{\beta}} = 0 \end{aligned}$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$\begin{aligned} \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^n -\frac{e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = \sum_{i=1}^n -\frac{x_i e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^n -\frac{x_i^2 e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2} \end{aligned}$$

Therefore, by Newton's Method, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$;

STEP 2: Update by $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\hat{\alpha} = 1.564284$ and $\hat{\beta} = 1.771093$.

4: EM Algorithm (30%)

(1) Observed data: Y_i for $i = 1, 2, \dots, 8000$; Missing data: Z_i for $i = 1, 2, \dots, 8000$, where $Z_i = 1, 2$, or 3 for low, middle, and high income respectively.

Since $Y_i|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$, with proportion π_k ($\pi_3 = 1 - (\pi_1 + \pi_2)$), we can formulate the **complete – data Likelihood function** as:

$$L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) \\ = \prod_{i=1}^n \left[\pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right]^{I(Z_i=1)} \left[\pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right]^{I(Z_i=2)} \left[(1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right]^{I(Z_i=3)}$$

(2) The complete-data log-Likelihood function is:

$$l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^n \left[I(Z_i = 1) \left[\log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right. \\ \left. + I(Z_i = 2) \left[\log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \right. \\ \left. + I(Z_i = 3) \left[\log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

Then given initial guess $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, we define a Q function by $Q(\Theta; \Theta^{(t)}) = E_{\Theta^{(t)}}(l(\Theta | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y})$:

$$Q(\Theta; \Theta^{(t)}) = E_{\pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}}(l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y}) \\ = \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \left[\log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right. \\ \left. + \widehat{Z}_{i2}^{(t)} \left[\log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \right. \\ \left. + \widehat{Z}_{i3}^{(t)} \left[\log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

where $(\pi_3^{(t)} = 1 - \pi_1^{(t)} - \pi_2^{(t)})$

$$\widehat{Z}_{ik}^{(t)} = E(Z_i = k | \Theta^{(t)}) = E(Z_i^{(t)} | \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}) \\ = \frac{\pi_k^{(t)} \frac{1}{\sqrt{2\pi}\sigma_k^{(t)}} e^{-\frac{(y_i - \mu_k^{(t)})^2}{2\sigma_k^{2(t)}}}}{\pi_1^{(t)} \frac{1}{\sqrt{2\pi}\sigma_1^{(t)}} e^{-\frac{(y_i - \mu_1^{(t)})^2}{2\sigma_1^{2(t)}}} + \pi_2^{(t)} \frac{1}{\sqrt{2\pi}\sigma_2^{(t)}} e^{-\frac{(y_i - \mu_2^{(t)})^2}{2\sigma_2^{2(t)}}} + (1 - \pi_1^{(t)} - \pi_2^{(t)}) \frac{1}{\sqrt{2\pi}\sigma_3^{(t)}} e^{-\frac{(y_i - \mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}$$

Then we can calculate:

$$\begin{aligned}
\frac{\partial Q}{\partial \pi_1} \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \frac{1}{\pi_1} - \widehat{Z}_{i3}^{(t)} \frac{1}{1 - \pi_1 - \pi_2} \right] \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \pi_2} \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i2}^{(t)} \frac{1}{\pi_2} - \widehat{Z}_{i3}^{(t)} \frac{1}{1 - \pi_1 - \pi_2} \right] \Big|_{\pi_1^{(t+1)}, \pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_1} \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \frac{y_i - \mu_1}{\sigma_1^2} \right] \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_2} \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i2}^{(t)} \frac{y_i - \mu_2}{\sigma_2^2} \right] \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_3} \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i3}^{(t)} \frac{y_i - \mu_3}{\sigma_3^2} \right] \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_1^2} \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i1}^{(t)} \left(-\frac{1}{2} \frac{1}{\sigma_1^2} + \frac{(y_i - \mu_1)^2}{2(\sigma_1^2)^2} \right) \right] \Big|_{\mu_1^{(t+1)}, \sigma_1^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_2^2} \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i2}^{(t)} \left(-\frac{1}{2} \frac{1}{\sigma_2^2} + \frac{(y_i - \mu_2)^2}{2(\sigma_2^2)^2} \right) \right] \Big|_{\mu_2^{(t+1)}, \sigma_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_3^2} \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} &= \sum_{i=1}^n \left[\widehat{Z}_{i3}^{(t)} \left(-\frac{1}{2} \frac{1}{\sigma_3^2} + \frac{(y_i - \mu_3)^2}{2(\sigma_3^2)^2} \right) \right] \Big|_{\mu_3^{(t+1)}, \sigma_3^{(t+1)}} = 0
\end{aligned}$$

The observed-data Likelihood function is:

$$\begin{aligned}
&L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}) \\
&= \prod_{i=1}^n \left[\left[\pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right] + \left[\pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right] + \left[(1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right] \right]
\end{aligned}$$

Then for iteration in **EM Algorithm**:

Given initial guess: $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, for $t \geq 0$ and $t \in \mathbb{Z}$:

E – step: Calculate $E(Z_i^{(t)} | \Theta^{(t)})$, where $\Theta^{(t)} = \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}$.

M – step: Update $\Theta^{(t+1)}$ by equations (1) to (8) listed at the next page.

Stopping criterion: $|L(\Theta^{(t)} | \mathbf{Y}) - L(\Theta^{(T+1)} | \mathbf{Y})| < \text{tolerance}$.

Iterative scheme:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (1)$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (2)$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}} \quad (3)$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}} \quad (4)$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)} y_i}{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (5)$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)} (y_i - \mu_1^{(t)})^2}{\sum_{i=1}^n \widehat{Z}_{i1}^{(t)}} \quad (6)$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)} (y_i - \mu_2^{(t)})^2}{\sum_{i=1}^n \widehat{Z}_{i2}^{(t)}} \quad (7)$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)} (y_i - \mu_3^{(t)})^2}{\sum_{i=1}^n \widehat{Z}_{i3}^{(t)}} \quad (8)$$

where $\widehat{Z}_{i1}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i2}^{(t)} + \sum_{i=1}^n \widehat{Z}_{i3}^{(t)} = n$ (the sample size).

(3)

Algorithm: EM Algorithm code in the R file.

Result: $\hat{\pi}_1 = 0.6464046$, $\hat{\pi}_2 = 0.2508577$, $\hat{\mu}_1 = 2998.948$, $\hat{\mu}_2 = 8008.473$, $\hat{\mu}_3 = 30030.97$, $\hat{\sigma}_1 = 303.3189$, $\hat{\sigma}_2 = 1525.052$ and $\hat{\sigma}_3 = 7755.734$

(4) **Result:** in the following page:

	low_income_1	middle_income_2	high_income_3	class
x1	9.991958e-01	7.473953e-04	5.678040e-05	1
x2	9.997040e-01	2.475613e-04	4.846901e-05	1
x3	0.000000e+00	2.883362e-47	1.000000e+00	3
x4	9.997088e-01	2.451883e-04	4.603783e-05	1
x5	1.091611e-39	9.975983e-01	2.401733e-03	2
x6	0.000000e+00	7.379262e-54	1.000000e+00	3
x7	3.508001e-137	9.730584e-01	2.694163e-02	2
x8	9.991552e-01	7.861646e-04	5.867285e-05	1
x9	9.988827e-01	1.046195e-03	7.112928e-05	1
x10	9.996717e-01	2.919623e-04	3.636031e-05	1
x11	9.995750e-01	3.857297e-04	3.928783e-05	1
x12	8.309440e-101	9.927352e-01	7.264784e-03	2
x13	1.840195e-44	9.976051e-01	2.394850e-03	2
x14	9.994094e-01	5.438131e-04	4.675592e-05	1
x15	1.062045e-153	9.474148e-01	5.258523e-02	2
x16	9.882324e-01	1.131743e-02	4.502208e-04	1
x17	0.000000e+00	7.196126e-19	1.000000e+00	3
x18	9.994248e-01	5.292086e-04	4.603881e-05	1
x19	0.000000e+00	2.324951e-107	1.000000e+00	3
x20	3.084091e-20	9.965954e-01	3.404619e-03	2
x21	9.989492e-01	9.826852e-04	6.812581e-05	1
x22	9.997055e-01	2.570307e-04	3.751380e-05	1
x23	1.246812e-16	9.960167e-01	3.983289e-03	2
x24	0.000000e+00	7.884756e-82	1.000000e+00	3
x25	9.996514e-01	2.804111e-04	6.815332e-05	1
x26	9.990114e-01	6.894410e-04	2.991983e-04	1
x27	9.993308e-01	6.187803e-04	5.045323e-05	1
x28	9.997141e-01	2.448797e-04	4.098493e-05	1
x29	9.994920e-01	4.650407e-04	4.292448e-05	1
x30	9.997139e-01	2.458929e-04	4.024660e-05	1
x31	1.114517e-117	9.871259e-01	1.287412e-02	2
x32	0.000000e+00	9.164527e-69	1.000000e+00	3
x33	9.997104e-01	2.445317e-04	4.508595e-05	1
x34	9.997140e-01	2.456565e-04	4.039182e-05	1
x35	9.996539e-01	3.094729e-04	3.661519e-05	1
x36	9.997087e-01	2.452019e-04	4.605523e-05	1
x37	9.974392e-01	2.429040e-03	1.317922e-04	1
x38	9.954162e-01	4.375381e-03	2.084620e-04	1
x39	9.986961e-01	1.224447e-03	7.943071e-05	1
x40	9.996930e-01	2.538918e-04	5.313980e-05	1
x41	0.000000e+00	3.431441e-75	1.000000e+00	3
x42	9.996570e-01	3.064784e-04	3.655429e-05	1
x43	4.687180e-44	9.976068e-01	2.393242e-03	2
x44	9.996299e-01	2.945703e-04	7.550999e-05	1
x45	9.996646e-01	2.989785e-04	3.643057e-05	1
x46	0.000000e+00	7.681117e-24	1.000000e+00	3
x47	9.993030e-01	6.452097e-04	5.175772e-05	1
x48	5.729718e-74	9.964873e-01	3.512736e-03	2
x49	0.000000e+00	1.139030e-40	1.000000e+00	3
x50	9.995960e-01	3.655035e-04	3.844810e-05	1

Figure 1: Classification of first 50 individuals