1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.
$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function f has totally 3 zeros.

Algorithm: Bisection Method in the R file.

Result: Zeros: -4.837944, -2.259727, and 1.097664.

2: Poisson Regression - Newton's Method (25%)

(1) Since $y_i \sim Poisson(\lambda_i)$ and $log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihhod function:

$$L(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^{n} \frac{e^{(\alpha+\beta x_i+\gamma x_i^2)y_i} e^{-e^{\alpha+\beta x_i+\gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha+\beta x_i+\gamma x_i^2} - \log y_i!$$

Then we have:

$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \alpha}\right|_{\hat{\alpha}} = \sum_{i=1}^{n} [y_i - e^{\alpha+\beta x_i+\gamma x_i^2}]\Big|_{\hat{\alpha}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \beta}\right|_{\hat{\beta}} = \sum_{i=1}^{n} [x_i y_i - x_i e^{\alpha+\beta x_i+\gamma x_i^2}]\Big|_{\hat{\beta}} = 0$$
$$\left.\frac{\partial l(\alpha, \beta, \gamma|\mathbf{x}, \mathbf{y})}{\partial \gamma}\right|_{\hat{\gamma}} = \sum_{i=1}^{n} [x_i^2 y_i - x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}]\Big|_{\hat{\gamma}} = 0$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma^2} \end{pmatrix}$, in which

$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha^2} = \sum_{i=1}^{n} -e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\beta} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\alpha} = \sum_{i=1}^{n} -x_i e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\alpha\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\alpha} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta^2} = \sum_{i=1}^{n} -x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\beta\partial\gamma} = \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma\partial\beta} = \sum_{i=1}^{n} -x_i^3 e^{\alpha+\beta x_i+\gamma x_i^2}$$
$$\frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial\gamma^2} = \sum_{i=1}^{n} -x_i^4 e^{\alpha+\beta x_i+\gamma x_i^2}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix} = \begin{pmatrix}\alpha^{(n-1)}\\ \beta^{(n-1)}\\ \gamma^{(n-1)}\end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix}\alpha^{(n)}\\ \beta^{(n)}\\ \gamma^{(n)}\end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

3: Logistic Regression - Newton's Method (20%)

---

(1) Since $y_i \sim Bernoulli(p_i)$ and $logit(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i}(1-p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$. Then we can get the following Likelihood function:

$$\begin{aligned} L(\alpha,\beta|\mathbf{x},\mathbf{y}) &= \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i} \\ &= \prod_{i=1}^{n} (\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{y_i}(1-\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}})^{1-y_i} \end{aligned}$$

(2) The log-Likelihood function is

$$\begin{aligned} l(\alpha, \beta|\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} y_i(\alpha + \beta x_i - \log(1 + e^{\alpha+\beta x_i})) + (1 - y_i)\log(\frac{1}{1 + e^{\alpha+\beta x_i}}) \\ &= \sum_{i=1}^{n} \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha+\beta x_i}) \end{aligned}$$

Then we have:

$$\begin{aligned} \frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha}\bigg|_{\hat{\alpha}} &= \sum_{i=1}^{n}[y_i - \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\alpha}} = 0 \\ \frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta}\bigg|_{\hat{\beta}} &= \sum_{i=1}^{n}[x_i y_i - \frac{x_i e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\beta}} = 0 \end{aligned}$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$\begin{aligned} \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= -\frac{e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} &= \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = -\frac{x_i e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \\ \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta^2} &= -\frac{x_i^2 e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2} \end{aligned}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle \mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle \mathbf{x^{(n)}}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.564284$ and $\beta = 1.771093$.

4: EM Algorithm (30%)

(1) Observed data: Y_i for $i = 1, 2, \ldots, 8000$; Missing data: Z_i for $i = 1, 2, \ldots, 8000.$, where $Z_i = 1, 2,$ or 3 for low, middle, and high income respectively.

Since $Y_i\big|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$, with proportion π_k ($\pi_3 = 1-(\pi_1+\pi_2)$), we can formulate the **complete − data Likelihhod function** as:

$$
\begin{aligned}
&L(\pi_1,\pi_2,\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3|\mathbf{Y},\mathbf{Z})\\
=&\prod_{i=1}^{n}\left[\pi_1\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}}\right]^{I(Z_i=1)}\left[\pi_2\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}\right]^{I(Z_i=2)}\left[(1-\pi_1-\pi_2)\frac{1}{\sqrt{2\pi}\sigma_3}e^{-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}}\right]^{I(Z_i=3)}
\end{aligned}
$$

The observed-data Likelihhod function is:

$$
\begin{aligned}
&L(\pi_1,\pi_2,\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3|\mathbf{Y})\\
=&\prod_{i=1}^{n}\left[\left[\pi_1\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}}\right]+\left[\pi_2\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}\right]+\left[(1-\pi_1-\pi_2)\frac{1}{\sqrt{2\pi}\sigma_3}e^{-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}}\right]\right]
\end{aligned}
$$

(2) The complete-data log-Likelihhod function is:

$$
\begin{aligned}
l(\pi_1,\pi_2,\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3|\mathbf{Y},\mathbf{Z}) = \sum_{i=1}^{n}\Bigg[&I(Z_i=1)\left[\log\pi_1-\log\sqrt{2\pi}-\frac{1}{2}\log\sigma_1^2-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}\right]\\
&+I(Z_i=2)\left[\log\pi_2-\log\sqrt{2\pi}-\frac{1}{2}\log\sigma_2^2-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}\right]\\
&+I(Z_i=3)\left[\log(1-\pi_1-\pi_2)-\log\sqrt{2\pi}-\frac{1}{2}\log\sigma_3^2-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}\right]\Bigg]
\end{aligned}
$$

Then given initial guess $\pi_1^{(0)},\pi_2^{(0)},\mu_1^{(0)},\mu_2^{(0)},\mu_3^{(0)},\sigma_1^{(0)},\sigma_2^{(0)},\sigma_3^{(0)}$, we define a Q function by $Q(\Theta;\Theta^{(t)}) = E_{\mathbf{\Theta^{(t)}}}\big(l(\mathbf{\Theta}|\mathbf{Y},\mathbf{Z})|\mathbf{Y}\big)$:

$$
\begin{aligned}
Q(\Theta;\Theta^{(t)}) &= E_{\pi_1^{(t)},\pi_2^{(t)},\mu_1^{(t)},\mu_2^{(t)},\mu_3^{(t)},\sigma_1^{2(t)},\sigma_2^{2(t)},\sigma_3^{2(t)}}\big(l(\pi_1,\pi_2,\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3|\mathbf{Y},\mathbf{Z})|\mathbf{Y}\big)\\
&=\sum_{i=1}^{n}\Bigg[I(Z_i^{(t)}=1)\left[\log\pi_1-\log\sqrt{2\pi}-\frac{1}{2}\log\sigma_1^2-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}\right]\\
&+I(Z_i^{(t)}=2)\left[\log\pi_2-\log\sqrt{2\pi}-\frac{1}{2}\log\sigma_2^2-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}\right]\\
&+I(Z_i^{(t)}=3)\left[\log(1-\pi_1-\pi_2)-\log\sqrt{2\pi}-\frac{1}{2}\log\sigma_3^2-\frac{(y_i-\mu_3)^2}{2\sigma_3^2}\right]\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
Z_i^{(t)} &= E(Z_i^{(t)}|\Theta^{(t)}) = E(Z_i^{(t)}|\pi_1^{(t)},\pi_2^{(t)},\mu_1^{(t)},\mu_2^{(t)},\mu_3^{(t)},\sigma_1^{2(t)},\sigma_2^{2(t)},\sigma_3^{2(t)})\\
&= p(Z_i^{(t)}=1|\Theta^{(t)})+2p(Z_i^{(t)}=2|\Theta^{(t)})+3p(Z_i^{(t)}=3|\Theta^{(t)})\\
&= 1+\frac{\pi_2^{(t)}\frac{1}{\sqrt{2\pi}\sigma_2{}^{(t)}}e^{-\frac{(y_i-\mu_2^{(t)})^2}{2\sigma_2^{2(t)}}}+2(1-\pi_1^{(t)}-\pi_2^{(t)})\frac{1}{\sqrt{2\pi}\sigma_3{}^{(t)}}e^{-\frac{(y_i-\mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}{\pi_1^{(t)}\frac{1}{\sqrt{2\pi}\sigma_1{}^{(t)}}e^{-\frac{(y_i-\mu_1^{(t)})^2}{2\sigma_1^{2(t)}}}+\pi_2^{(t)}\frac{1}{\sqrt{2\pi}\sigma_2{}^{(t)}}e^{-\frac{(y_i-\mu_2^{(t)})^2}{2\sigma_2^{2(t)}}}+(1-\pi_1^{(t)}-\pi_2^{(t)})\frac{1}{\sqrt{2\pi}\sigma_3{}^{(t)}}e^{-\frac{(y_i-\mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}
\end{aligned}
$$

Then we can calculate:

$$
\begin{aligned}
\frac{\partial Q}{\partial \pi_1}\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=1)\frac{1}{\pi_1} - I(Z_i^{(t)}=3)\frac{1}{1-\pi_1-\pi_2}\right]\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \pi_2}\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=2)\frac{1}{\pi_2} - I(Z_i^{(t)}=3)\frac{1}{1-\pi_1-\pi_2}\right]\bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_1}\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=1)\frac{y_i-\mu_1}{\sigma_1^2}\right]\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_2}\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=2)\frac{y_i-\mu_2}{\sigma_2^2}\right]\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \mu_3}\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=3)\frac{y_i-\mu_3}{\sigma_3^2}\right]\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_1^2}\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=1)\left(-\frac{1}{2}\frac{1}{\sigma_1^2}+\frac{(y_i-\mu_1)^2}{2(\sigma_1^2)^2}\right)\right]\bigg|_{\mu_1^{(t+1)},\sigma_1{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_2^2}\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=2)\left(-\frac{1}{2}\frac{1}{\sigma_2^2}+\frac{(y_i-\mu_2)^2}{2(\sigma_2^2)^2}\right)\right]\bigg|_{\mu_2^{(t+1)},\sigma_2{}^{(t+1)}} = 0 \\
\frac{\partial Q}{\partial \sigma_3^2}\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} &= \sum_{i=1}^{n}\left[I(Z_i^{(t)}=3)\left(-\frac{1}{2}\frac{1}{\sigma_3^2}+\frac{(y_i-\mu_3)^2}{2(\sigma_3^2)^2}\right)\right]\bigg|_{\mu_3^{(t+1)},\sigma_3{}^{(t+1)}} = 0
\end{aligned}
$$

Hence, we can conclude that:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=1)}{\sum_{i=1}^n I(Z_i^{(t)}=1)+\sum_{i=1}^n I(Z_i^{(t)}=2)+\sum_{i=1}^n I(Z_i^{(t)}=3)} \tag{1}$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=2)}{\sum_{i=1}^n I(Z_i^{(t)}=1)+\sum_{i=1}^n I(Z_i^{(t)}=2)+\sum_{i=1}^n I(Z_i^{(t)}=3)} \tag{2}$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=1)y_i}{\sum_{i=1}^n I(Z_i^{(t)}=1)} \tag{3}$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=2)y_i}{\sum_{i=1}^n I(Z_i^{(t)}=2)} \tag{4}$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=3)y_i}{\sum_{i=1}^n I(Z_i^{(t)}=3)} \tag{5}$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=1)(y_i-\mu_1^{(t+1)})^2}{\sum_{i=1}^n I(Z_i^{(t)}=1)} \tag{6}$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=2)(y_i-\mu_2^{(t+1)})^2}{\sum_{i=1}^n I(Z_i^{(t)}=2)} \tag{7}$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^n I(Z_i^{(t)}=3)(y_i-\mu_3^{(t+1)})^2}{\sum_{i=1}^n I(Z_i^{(t)}=3)} \tag{8}$$

Then for iteration in **EM Algorithm**:

Given initial guess: $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, for $t \geq 0$ and $t \in \mathbb{Z}$:

$\mathbf{E-step}$: Calculate $E(Z_i^{(t)}|\Theta^{(t)})$, where $\Theta^{(t)} = \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}$.

$\mathbf{M-step}$: Update $\Theta^{(t+1)}$ by equations (1) to (8) that we have obtained before.

(3)

Algorithm: EM Algorithm code in the R file.

Result:

(4)