1: Hybrid Gibbs Sampler to estimate Poisson Distribution λ (30%)

Derivation:

For these Poisson distributed random variables (r.v.s) ($n = 500$) with mean parameter λ, unobserved variables are $\lambda, y_1, y_2, \ldots, y_{78}$, in which $y's$ denote the r.v.s which are larger than or equal to five.

$$\begin{aligned}
&\text{Prior} : \pi(\lambda) \propto \frac{1}{\lambda};\\
&P(X, Y|\lambda) = \prod_{i=1}^{422} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \prod_{j=1}^{78} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} I(y_i \geq 5);\\
&P(\lambda|X, Y) \propto P(X, Y|\lambda)P(\lambda) \propto e^{-n\lambda}\lambda^{\sum_i x_i + \sum_j y_j - 1}\\
&P(y_j|X, \lambda) = \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} I(y_i \geq 5) \propto \frac{\lambda^{y_i}}{y_i!} I(y_i \geq 5)
\end{aligned}$$

Then we can know that $\lambda|X, Y \propto Gamma(\sum_i x_i + \sum_j y_j, n)$.

The MH-Step to sample 78 unobserved $y's$ is

$$y_j^* = \begin{cases} y_j^{(t)} - 1, & \text{with probability } \frac{1}{3};\\ y_j^{(t)}, & \text{with probability } \frac{1}{3};\\ y_j^{(t)} + 1, & \text{with probability } \frac{1}{3}. \end{cases}$$

$$r = \min\left\{ \frac{[\lambda^{(t+1)}]^{y_j^*}/y_j^*}{[\lambda^{(t+1)}]^{y_j^{(t)}}/y_j^{(t)}} I(y_j^* \geq 5), 1 \right\}$$

where r is the accept-reject ratio.

Result: $\hat{\lambda} = 1.674$.

2: Gibbs Sampler for Clustering (30%)

Derivation:

We use $\{X_{ij}\}_{i=1,2,3,\ldots,1000}^{j=1,2,3}$ to denote the datum of i-th sample in j-th dimension. Then using the same notation in the question:

$$\begin{aligned}
f(X, Z|\Pi, \Theta) &= \prod_{i=1}^{1000} \prod_{k=1}^{3} P(Z_i = k|\Pi, \Theta)P(X_{ij}, j = 1, 2, 3|Z_j, \Pi, \Theta)\\
&= \prod_{i=1}^{1000} \prod_{k=1}^{3} \Big[P(Z_i = k|\Pi) \prod_{j=1}^{3} P(X_{ij}|Z_j, \Theta)\Big]^{I(Z_j = k)}\\
&= \prod_{i=1}^{1000} \prod_{k=1}^{3} \Big[\pi_k \prod_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}}\Big]^{I(Z_j = k)}
\end{aligned}$$

Note that given $Z_i = k$, for each sample i, $X_{ij} \sim Bino(10j, \theta_{jk})$; $P(\theta_{jk}) \propto 1$; and $(\pi_1, \pi_2, \pi_3) \sim Dirichlet(\alpha_1, \alpha_2, \alpha_3)$, we can derive by the following:

$$\begin{aligned}
P(\Pi, \Theta, Z|X) &\propto P(\Pi)P(\Theta)P(X, Z|\Pi, \Theta) \\
&\propto \sum_{k=1}^{3} \pi_k^{\alpha_k - 1} \sum_{i=1}^{1000} \sum_{k=1}^{3} \Big[\pi_k \sum_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]^{I(Z_j = k)} \\
&\propto \sum_{k=1}^{3} \pi_k^{\alpha_k - 1} \sum_{i=1}^{1000} \sum_{k=1}^{3} \pi_k^{I(Z_i=k)} \sum_{i=1}^{1000} \sum_{k=1}^{3} \sum_{j=1}^{3} \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij} I(Z_i=k)} [(1-\theta_{jk})^{10j - X_{ij}}]^{I(Z_i=k)}
\end{aligned}$$

Then for each parameter:

For π:

$$\begin{aligned}
f(\Pi|\Theta, Z) &\propto \sum_{k=1}^{3} \pi_k^{\alpha_k - 1} \sum_{i=1}^{1000} \sum_{k=1}^{3} \pi_k^{I(Z_i=k)} \\
&\propto \sum_{k=1}^{3} \pi_k^{\alpha_k - 1} \sum_{k=1}^{3} [\pi_k]^{\sum_{i=1}^{1000} I(Z_i=k)} \\
&\propto \sum_{k=1}^{3} \pi_k^{\alpha_k + \sum_{i=1}^{1000} I(Z_i=k) - 1}
\end{aligned}$$

Then it follows that:

$$\Pi|\Theta, Z \sim Dirichlet(\alpha_1 + \sum_{i=1}^{1000} I(Z_i = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i = 3))$$

For θ:

$$\begin{aligned}
\theta_{jk}|- &\propto \prod_{i=1}^{1000} \prod_{k=1}^{3} \Big[\prod_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]^{I(Z_j=k)} \\
&\propto \prod_{i=1}^{1000} \Big[\binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]
\end{aligned}$$

Then it follows that:

$$\theta_{jk}|- \sim Bino(10j, \theta_{jk})$$

For Z:

$$\begin{aligned}
Z_i|- &\propto \sum_{i=1}^{1000} \sum_{k=1}^{3} \pi_k^{I(Z_i=k)} \sum_{i=1}^{1000} \sum_{k=1}^{3} \sum_{j=1}^{3} \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij} I(Z_i=k)} [(1-\theta_{jk})^{10j - X_{ij}}]^{I(Z_i=k)} \\
&\propto \sum_{k=1}^{3} \pi_k^{I(Z_i=k)} \sum_{k=1}^{3} \sum_{j=1}^{3} \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij} I(Z_i=k)} [(1-\theta_{jk})^{10j - X_{ij}}]^{I(Z_i=k)} \\
&\propto \Bigg[\sum_{k=1}^{3} \pi_k \Big[\sum_{j=1}^{3} \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1-\theta_{jk})^{10j - X_{ij}}\Big]\Bigg]^{I(Z_i=k)}
\end{aligned}$$

Then for Gibbs Sampler algorithm:

Given $\Pi^{(t)}, \Theta^{(t)}, Z^{(t)}$, we update the parameters by the following:

$$\pi_1^{(t+1)}, \pi_2^{(t+1)}, \pi_3^{(t+1)}|- \sim Dirichlet(\alpha_1 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 3))$$

$$\theta_{jk}^{(t+1)} \sim Bino(10j, \theta_{jk}^{(t)}) \text{ for } j = 1,2,3 \text{ and } k = 1,2,3$$

$$P(Z_i^{(t+1)} = k|-) = \frac{\pi_k^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jk}^{(t+1)})^{X_{ij}} (1-\theta_{jk}^{(t+1)})^{10j - X_{ij}}}{\sum_{l=1}^3 \pi_l^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jl}^{(t+1)})^{X_{ij}} (1-\theta_{jl}^{(t+1)})^{10j - X_{ij}}}$$

Result:

3: Hybrid Gibbs Sampler (40%)
