

1: Hybrid Gibbs Sampler to estimate Poisson Distribution λ (30%)

Derivation:

For these Poisson distributed random variables (r.v.s) ($n = 500$) with mean parameter λ , unobserved variables are $\lambda, y_1, y_2, \dots, y_{78}$, in which y_i 's denote the r.v.s which are larger than or equal to five.

$$\begin{aligned} \text{Prior : } \pi(\lambda) &\propto \frac{1}{\lambda}; \\ P(X, Y|\lambda) &= \prod_{i=1}^{422} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \prod_{j=1}^{78} \frac{e^{-\lambda} \lambda^{y_j}}{y_j!} I(y_j \geq 5); \\ P(\lambda|X, Y) &\propto P(X, Y|\lambda) P(\lambda) \propto e^{-n\lambda} \lambda^{\sum_i x_i + \sum_j y_j - 1} \\ P(y_j|X, \lambda) &= \frac{e^{-\lambda} \lambda^{y_j}}{y_j!} I(y_j \geq 5) \propto \frac{\lambda^{y_j}}{y_j!} I(y_j \geq 5) \end{aligned}$$

Then we can know that $\lambda|X, Y \propto \text{Gamma}(\sum_i x_i + \sum_j y_j, n)$.

The MH-Step to sample 78 unobserved y_j 's is

$$\begin{aligned} y_j^* &= \begin{cases} y_j^{(t)} - 1, & \text{with probability } \frac{1}{3}; \\ y_j^{(t)}, & \text{with probability } \frac{1}{3}; \\ y_j^{(t)} + 1, & \text{with probability } \frac{1}{3}. \end{cases} \\ r &= \min \left\{ \frac{[\lambda^{(t+1)}]^{y_j^*} / y_j^{*!}}{[\lambda^{(t+1)}]^{y_j^{(t)}} / y_j^{(t)!}} I(y_j^* \geq 5), 1 \right\} \end{aligned}$$

where r is the accept-reject ratio.

Result: $\hat{\lambda} = 1.674$.

2: Gibbs Sampler for Clustering (30%)

Derivation:

We use $\{X_{ij}\}_{i=1,2,3,\dots,1000}^{j=1,2,3}$ to denote the datum of i -th sample in j -th dimension. Then using the same notation in the question, the complete-data likelihood function is:

$$\begin{aligned} f(X, Z|\Pi, \Theta) &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[P(Z_i = k|\Pi, \Theta) P(X_{ij}, j = 1, 2, 3|Z_j, \Pi, \Theta) \right]^{I(Z_j=k)} \\ &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[P(Z_i = k|\Pi) \prod_{j=1}^3 P(X_{ij}|Z_j, \Theta) \right]^{I(Z_j=k)} \\ &= \prod_{i=1}^{1000} \prod_{k=1}^3 \left[\pi_k \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right]^{I(Z_j=k)} \end{aligned}$$

Note that given $Z_i = k$, for each sample i , $X_{ij} \sim \text{Bino}(10j, \theta_{jk})$; $P(\theta_{jk}) \propto \text{Beta}(a, b)$ (Prior: $P(\theta_{jk}) \propto \text{Beta}(1, 1) \propto 1$); and $(\pi_1, \pi_2, \pi_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3)$, we can derive by the following:

$$P(\Pi, \Theta, Z|X) \propto P(\Pi)P(\Theta)P(X, Z|\Pi, \Theta)$$

$$\propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{j=1}^3 \prod_{k=1}^3 \theta_{jk}^{a-1} (1 - \theta_{jk})^{b-1} \prod_{i=1}^{1000} \prod_{k=1}^3 \left[\pi_k \prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right]^{I(Z_i=k)}$$

For π :

$$\begin{aligned} f(\Pi|\Theta, Z) &\propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{i=1}^{1000} \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \\ &\propto \prod_{k=1}^3 \pi_k^{\alpha_k-1} \prod_{k=1}^3 [\pi_k]^{\sum_{i=1}^{1000} I(Z_i=k)} \\ &\propto \prod_{k=1}^3 \pi_k^{\alpha_k + \sum_{i=1}^{1000} I(Z_i=k) - 1} \end{aligned}$$

Then it follows that:

$$\Pi|\Theta, Z \sim \text{Dirichlet}(\alpha_1 + \sum_{i=1}^{1000} I(Z_i = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i = 3))$$

For θ :

$$\begin{aligned} \theta_{jk}| - &\propto \prod_{j=1}^3 \prod_{k=1}^3 \theta_{jk}^{a-1} (1 - \theta_{jk})^{b-1} \prod_{i=1}^{1000} \prod_{k=1}^3 \left[\prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right]^{I(Z_i=k)} \\ &\propto \theta_{jk}^{a-1} (1 - \theta_{jk})^{b-1} \prod_{i=1}^{1000} \left[\theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right]^{I(Z_i=k)} \\ &\propto \theta_{jk}^{\sum_{i=1}^{1000} X_{ij} I(Z_i=k) + a - 1} (1 - \theta_{jk})^{\sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i=k) + b - 1} \end{aligned}$$

Then it follows that:

$$\theta_{jk}| - \sim \text{Beta}(a + \sum_{i=1}^{1000} X_{ij} I(Z_i = k), b + \sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i = k))$$

For Z :

$$\begin{aligned} Z_i| - &\propto \prod_{i=1}^{1000} \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \prod_{i=1}^{1000} \prod_{k=1}^3 \prod_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij} I(Z_i=k)} [(1 - \theta_{jk})^{10j - X_{ij}}]^{I(Z_i=k)} \\ &\propto \prod_{k=1}^3 \pi_k^{I(Z_i=k)} \prod_{k=1}^3 \prod_{j=1}^3 \binom{10j}{X_{ij}}^{I(Z_i=k)} \theta_{jk}^{X_{ij} I(Z_i=k)} [(1 - \theta_{jk})^{10j - X_{ij}}]^{I(Z_i=k)} \\ &\propto \prod_{k=1}^3 \left[\pi_k \left[\prod_{j=1}^3 \binom{10j}{X_{ij}} \theta_{jk}^{X_{ij}} (1 - \theta_{jk})^{10j - X_{ij}} \right] \right]^{I(Z_i=k)} \end{aligned}$$

Then for Gibbs Sampler algorithm:

Given $\Pi^{(t)}, \Theta^{(t)}, Z^{(t)}$, we update the parameters by the following:

$$\begin{aligned} \pi_1^{(t+1)}, \pi_2^{(t+1)}, \pi_3^{(t+1)} | - &\sim \text{Dirichlet}(\alpha_1 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 1), \alpha_2 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 2), \alpha_3 + \sum_{i=1}^{1000} I(Z_i^{(t)} = 3)) \\ \theta_{jk}^{(t+1)} | - &\sim \text{Beta}(a + \sum_{i=1}^{1000} X_{ij} I(Z_i^{(t)} = k), b + \sum_{i=1}^{1000} (10j - X_{ij}) I(Z_i^{(t)} = k)) \\ P(Z_i^{(t+1)} = k | -) &= \frac{\pi_k^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jk}^{(t+1)})^{X_{ij}} (1 - \theta_{jk}^{(t+1)})^{10j - X_{ij}}}{\sum_{l=1}^3 \pi_l^{(t+1)} \prod_{j=1}^3 \binom{10j}{X_{ij}} (\theta_{jl}^{(t+1)})^{X_{ij}} (1 - \theta_{jl}^{(t+1)})^{10j - X_{ij}}} \end{aligned}$$

Result:

Estimation of Π :

$\hat{\pi}_1 = 0.499$, $\hat{\pi}_2 = 0.298$, and $\hat{\pi}_3 = 0.203$.

Estimation of Θ :

$\theta_{11} = 0.807$, $\theta_{12} = 0.490$, $\theta_{13} = 0.196$, $\theta_{21} = 0.206$, $\theta_{22} = 0.804$, $\theta_{23} = 0.515$, $\theta_{31} = 0.502$, $\theta_{32} = 0.196$, and $\theta_{33} = 0.797$.

Estimation of Z :

Samples of cluster 1:

```
> which(estimated_z==1)
[1] 2 4 8 12 14 15 18 19 20 21 23 24 26 27 28 32 33 36 38 39 40 44 45 47 48 49 50
[28] 51 52 54 55 57 60 61 63 67 68 69 70 71 72 73 74 76 78 79 80 82 84 89 90 92 93 94
[55] 96 99 104 106 108 112 113 115 118 120 122 123 128 129 130 131 135 137 140 141 143 146 147 150 152 153 154
[82] 155 156 157 162 163 164 166 172 173 174 175 180 181 184 185 186 187 188 189 191 192 196 198 200 202 204 209
[109] 212 213 215 216 221 224 225 226 232 233 236 237 238 239 240 241 242 243 245 251 252 253 258 260 262 263 267
[136] 268 269 270 274 278 279 280 281 282 283 284 285 286 290 293 295 296 300 301 302 307 308 310 311 312 315 318
[163] 320 321 323 324 325 326 328 329 330 331 334 335 336 337 339 344 346 347 350 351 352 353 357 358 359 365 366
[190] 367 369 370 371 373 374 379 381 383 387 388 389 393 394 395 396 401 402 404 406 407 411 412 413 414 416 417
[217] 418 419 420 423 424 425 427 428 429 430 432 434 435 436 440 444 446 453 455 456 460 461 471 472 474 475 478
[244] 479 481 483 487 488 491 492 494 498 499 500 501 502 511 513 515 516 517 520 521 523 525 526 527 528 529 530
[271] 534 535 537 540 541 542 545 548 550 552 554 555 557 558 563 567 571 575 577 578 579 581 587 588 590 592 594
[298] 596 597 599 601 604 608 613 614 615 617 619 620 622 623 624 625 629 632 634 637 638 641 642 643 647 649 650
[325] 651 652 657 658 660 664 666 667 668 669 672 673 675 676 679 681 682 683 686 688 690 692 695 697 699 700 703
[352] 704 706 707 708 709 710 713 715 716 717 718 720 721 725 726 728 730 734 735 738 741 742 743 745 746 748 749
[379] 752 753 755 758 759 760 761 764 773 779 780 782 783 786 787 790 792 797 798 805 808 809 810 812 814 815 816
[406] 817 818 820 824 826 832 833 834 836 838 840 841 842 844 845 848 850 853 854 855 856 861 862 865 866 868 869
[433] 870 871 875 878 880 881 882 883 884 885 886 887 888 889 892 893 894 895 896 897 904 906 908 910 911 915 917
[460] 919 920 922 924 925 926 930 931 934 936 940 941 946 952 953 954 955 957 960 961 962 963 964 965 968 972 973
[487] 976 977 981 982 985 988 992 994 995 996 997 998 999
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Figure 1: Samples in cluster 1

Samples of cluster 2:

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> which(estimated_z==2)
[1] 3 5 6 7 10 11 13 16 17 25 29 31 34 37 42 43 46 56 58 65 66
[22] 75 81 88 91 97 98 100 105 107 109 114 116 119 121 124 126 127 133 134 136 139
[43] 142 144 145 148 151 159 160 161 165 169 170 171 176 182 183 194 195 197 199 203 205
[64] 206 210 217 219 220 223 227 229 230 231 234 235 247 249 254 264 271 272 273 275 276
[85] 277 289 291 292 294 298 304 305 309 313 316 317 319 322 332 342 345 348 349 354 360
[106] 361 362 363 368 372 375 376 378 382 391 397 405 409 415 422 431 433 437 438 439 441
[127] 443 445 447 448 449 450 451 454 457 462 463 473 485 486 489 490 495 496 497 504 506
[148] 507 512 531 532 536 539 543 546 549 553 560 561 562 564 568 573 580 584 585 586 591
[169] 593 595 598 600 602 605 606 609 610 612 618 626 633 639 640 646 656 659 661 663 665
[190] 670 674 677 680 684 687 689 693 696 701 705 711 712 714 719 724 727 733 737 739 744
[211] 750 754 757 762 763 766 770 771 774 775 777 784 785 788 791 793 795 796 799 800 802
[232] 803 806 811 821 822 823 825 827 828 829 831 835 837 839 843 846 849 852 857 863 864
[253] 867 872 873 874 876 879 890 898 902 903 905 907 912 913 914 918 921 928 929 932 933
[274] 935 937 939 942 943 944 945 947 948 949 951 966 967 969 970 974 975 978 979 984 987
[295] 989 990 993 1000
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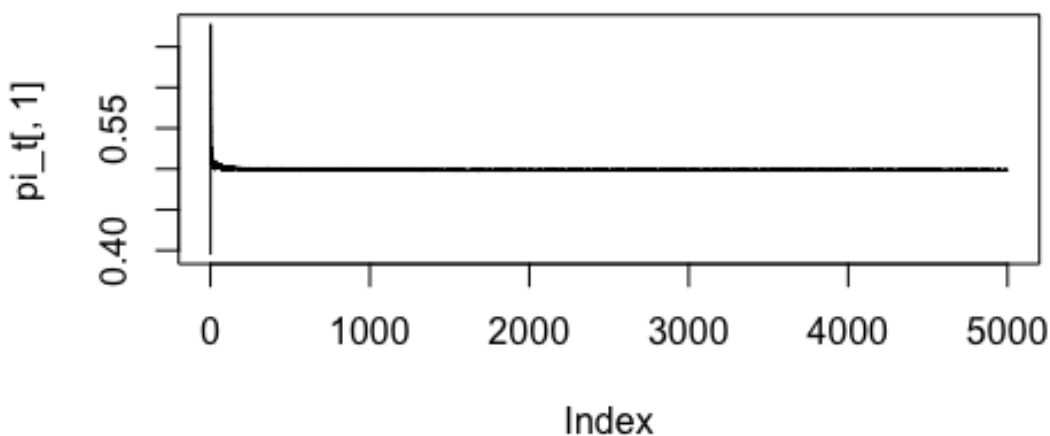
Figure 2: Samples in cluster 2

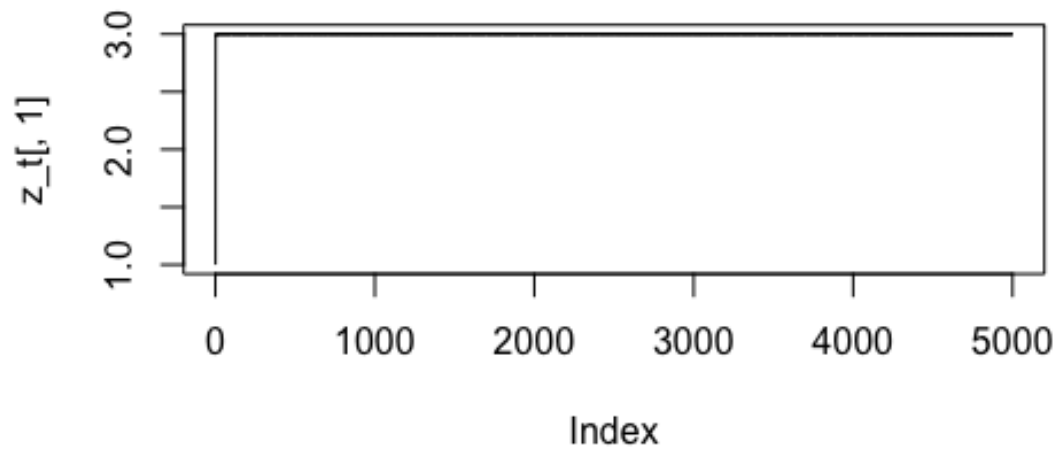
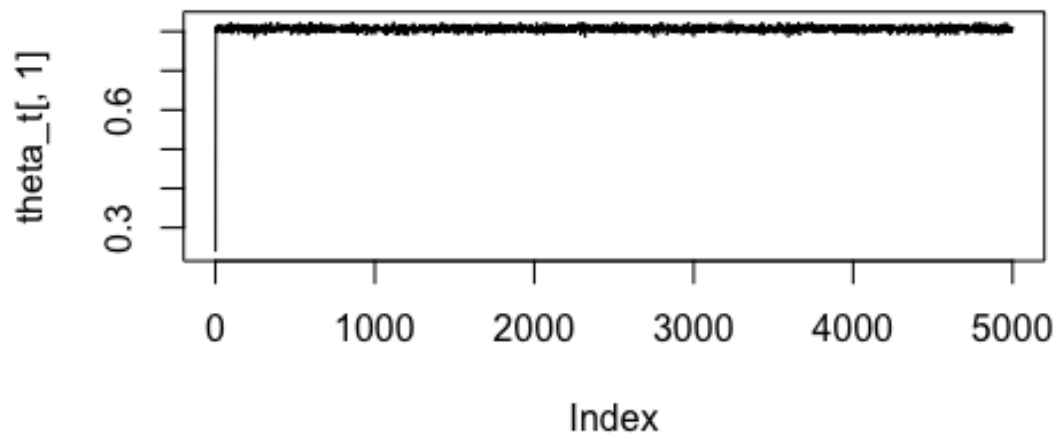
Samples of cluster 3:

```
> which(estimated_z==3)
[1] 1 9 22 30 35 41 53 59 62 64 77 83 85 86 87 95 101 102 103 110 111 117 125 132 138 149 158
[28] 167 168 177 178 179 190 193 201 207 208 211 214 218 222 228 244 246 248 250 255 256 257 259 261 265 266 287
[55] 288 297 299 303 306 314 327 333 338 340 341 343 355 356 364 377 380 384 385 386 390 392 398 399 400 403 408
[82] 410 421 426 442 452 458 459 464 465 466 467 468 469 470 476 477 480 482 484 493 503 505 508 509 510 514 518
[109] 519 522 524 533 538 544 547 551 556 559 565 566 569 570 572 574 576 582 583 589 603 607 611 616 621 627 628
[136] 630 631 635 636 644 645 648 653 654 655 662 671 678 685 691 694 698 702 722 723 729 731 732 736 740 747 751
[163] 756 765 767 768 769 772 776 778 781 789 794 801 804 807 813 819 830 847 851 858 859 860 877 891 899 900 901
[190] 909 916 923 927 938 950 956 958 959 971 980 983 986 991
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Figure 3: Samples in cluster 3

Traceplots:

Figure 4: Traceplot of π_1

Figure 5: Traceplot of Z_1 Figure 6: Traceplot of θ_{11}

3: Hybrid Gibbs Sampler (40%)

Note that for $i = 1, 2$, $\mathbf{Y}_i = (y_{i1}, y_{i2}, y_{i3}, y_{i4}) \sim \text{multinomial}(100, p_1, p_2, p_3, p_4)$, then:

$$\text{Prior : } \pi(\mathbf{p}) \propto \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \propto \text{Dirichlet}(2, 2, 2, 2) \propto p_1 p_2 p_3 p_4;$$

$$P(y_{i1}, y_{i2}, y_{i3}, y_{i4} | p_1, p_2, p_3, p_4) = \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1}} p_2^{y_{i2}} p_3^{y_{i3}} p_4^{y_{i4}}$$

$$\begin{aligned} P(p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4}) &\propto P(y_{i1}, y_{i2}, y_{i3}, y_{i4} | p_1, p_2, p_3, p_4) f(p_1, p_2, p_3, p_4 | \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &\propto \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1} + \alpha_1 - 1} p_2^{y_{i2} + \alpha_2 - 1} p_3^{y_{i3} + \alpha_3 - 1} p_4^{y_{i4} + \alpha_4 - 1} \end{aligned}$$

$$p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4} \sim \text{Dirichlet}(y_{i1} + \alpha_1, y_{i2} + \alpha_2, y_{i3} + \alpha_3, y_{i4} + \alpha_4)$$

$$P(y_{i2} | \mathbf{Y}, \mathbf{P}) \propto \frac{p_2^{y_{i2}}}{y_{i2}!}$$

Then MH-Step to update y_{i2} (for $i = 1, 2$) is

$$\begin{aligned} y_{i2}^{(t+1)} &= \begin{cases} y_{i2}^{(t)} + 1, & \text{with probability } \frac{1}{2} \\ y_{i2}^{(t)} - 1, & \text{with probability } \frac{1}{2} \end{cases} \\ y_{i2}^{(t+1)} &= y_{i2}^{(t)} + 1, & \text{if } y_{i2}^{(t)} = 15 \\ y_{i2}^{(t+1)} &= y_{i2}^{(t)} - 1, & \text{if } y_{i2}^{(t)} = 32 \\ r &= \min \left\{ \frac{p_2^{y_{i2}^{(t+1)}} / y_{i2}^{(t+1)}!}{p_2^{y_{i2}^{(t)}} / y_{i2}^{(t)}!}, 1 \right\} \end{aligned}$$

where r is the accept-reject ratio. Then for another two unobserved variables:

$$\begin{aligned} y_{11}^{(t+1)} &= 100 - y_{13} - y_{14} - y_{12}^{(t+1)} = 100 - 22 - 31 - y_{12}^{(t+1)} \\ y_{24}^{(t+1)} &= 100 - y_{21} - y_{23} - y_{22}^{(t+1)} = 100 - 28 - 26 - y_{22}^{(t+1)} \end{aligned}$$

Result:

$p_1 = 0.301$, $p_2 = 0.149$, $p_3 = 0.239$, and $p_4 = 0.311$.