1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.
$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function f has totally 3 zeros.

Algorithm: Bisection Method in the R file.

Result: Zeros: -4.837944, -2.259727, and 1.097664.

2: Poisson Regression - Newton's Method (25%)

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(1) Since $y_i \sim Poisson(\lambda_i)$ and $log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihhod function:

$$L(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^{n} \frac{e^{(\alpha+\beta x_i+\gamma x_i^2)y_i} e^{-e^{\alpha+\beta x_i+\gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha+\beta x_i+\gamma x_i^2} - \log y_i!$$

Then we have:

$$\left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} = \sum_{i=1}^{n} [y_i - e^{\alpha+\beta x_i+\gamma x_i^2}] \Big|_{\hat{\alpha}} = 0$$

$$\left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} = \sum_{i=1}^{n} [x_i y_i - x_i e^{\alpha+\beta x_i+\gamma x_i^2}] \Big|_{\hat{\beta}} = 0$$

$$\left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \right|_{\hat{\gamma}} = \sum_{i=1}^{n} [x_i^2 y_i - x_i^2 e^{\alpha+\beta x_i+\gamma x_i^2}] \Big|_{\hat{\gamma}} = 0$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha,\beta,\gamma|\mathbf{x},\mathbf{y})}{\partial \gamma^2} \end{pmatrix}$, in

which

$$\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} = \sum_{i=1}^{n} -e^{\alpha + \beta x_i + \gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} -x_i e^{\alpha + \beta x_i + \gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} = \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} = \sum_{i=1}^{n} -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} = \sum_{i=1}^{n} -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} = \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} = \sum_{i=1}^{n} -x_i^3 e^{\alpha + \beta x_i + \gamma x_i^2}$$

$$\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} = \sum_{i=1}^{n} -x_i^4 e^{\alpha + \beta x_i + \gamma x_i^2}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}}) \triangle \mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle \mathbf{x^{(n)}}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

3: Logistic Regression - Newton's Method (20%)

---

(1) Since $y_i \sim Bernoulli(p_i)$ and $logit(p_i) = \alpha + \beta x_i$, we can know that $f(y_i, p_i) = p_i^{y_i}(1 - p_i)^{1-y_i}$, and $p_i = \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}$. Then we can get the following Likelihood function:

$$L(\alpha, \beta | \mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} p_i^{y_i}(1 - p_i)^{1-y_i}$$

$$= \prod_{i=1}^{n} \left(\frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}\right)^{y_i} \left(1 - \frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}\right)^{1-y_i}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta|\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} y_i(\alpha + \beta x_i - \log(1 + e^{\alpha+\beta x_i})) + (1 - y_i)\log(\frac{1}{1 + e^{\alpha+\beta x_i}})$$

$$= \sum_{i=1}^{n} \alpha x_i + \beta x_i y_i - \log(1 + e^{\alpha+\beta x_i})$$

Then we have:

$$\frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha}\bigg|_{\hat{\alpha}} = \sum_{i=1}^{n}[y_i - \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\alpha}} = 0$$

$$\frac{\partial l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta}\bigg|_{\hat{\beta}} = \sum_{i=1}^{n}[x_i y_i - \frac{x_i e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}]\bigg|_{\hat{\beta}} = 0$$

Let $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta} \end{pmatrix}$, then $\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha,\beta|\mathbf{x},\mathbf{y})}{\partial \beta^2} \end{pmatrix}$, in which

$$\frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha^2} = \sum_{i=1}^{n} -\frac{e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2}$$

$$\frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} = \frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} -\frac{x_i e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2}$$

$$\frac{\partial^2 l(\alpha, \beta|\mathbf{x}, \mathbf{y})}{\partial \beta^2} = \sum_{i=1}^{n} -\frac{x_i^2 e^{\alpha+\beta x_i}}{(1 + e^{\alpha+\beta x_i})^2}$$

Therefore, by Newton's Methhod, given initial guess $\alpha^{(0)}$ and $\beta^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1}\mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x^{(n)}} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x^{(n)}})\triangle\mathbf{x^{(n)}} = -\mathbf{F}(\mathbf{x^{(n)}})$;

STEP 2: Update by $\mathbf{x^{(n+1)}} = \mathbf{x^{(n)}} + \triangle\mathbf{x^{(n)}}$.

(3)

Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.564284$ and $\beta = 1.771093$.

4: EM Algorithm (30%)

---

(1) Observed data: Y_i for $i = 1, 2, \ldots, 8000$; Missing data: Z_i for $i = 1, 2, \ldots, 8000$., where $Z_i = 1, 2,$ or 3 for low, middle, and high income respectively.

Since $Y_i\big|(Z_i = k) \sim N(\mu_k, \sigma_k^2)$, with proportion π_k $(\pi_3 = 1 - (\pi_1 + \pi_2))$, we can formulate the **complete − data Likelihhod function** as:

$$L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z})$$

$$= \prod_{i=1}^{n} \left[\pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right]^{I(Z_i=1)} \left[\pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right]^{I(Z_i=2)} \left[(1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right]^{I(Z_i=3)}$$

The observed-data Likelihhod function is:

$$L(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y})$$

$$= \prod_{i=1}^{n} \left[\left[\pi_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(y_i - \mu_1)^2}{2\sigma_1^2}} \right] + \left[\pi_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y_i - \mu_2)^2}{2\sigma_2^2}} \right] + \left[(1 - \pi_1 - \pi_2) \frac{1}{\sqrt{2\pi}\sigma_3} e^{-\frac{(y_i - \mu_3)^2}{2\sigma_3^2}} \right] \right]$$

(2) The complete-data log-Likelihhod function is:

$$l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{n} \left[I(Z_i = 1) \left[\log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right.$$

$$+ I(Z_i = 2) \left[\log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right]$$

$$\left. + I(Z_i = 3) \left[\log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

Then given initial guess $\pi_1^{(0)}, \pi_2^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, \sigma_3^{(0)}$, we define a Q function by $Q(\Theta; \Theta^{(t)}) = E_{\Theta^{(t)}}\big(l(\Theta | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y}\big)$:

$$Q(\Theta; \Theta^{(t)}) = E_{\pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)}} \big(l(\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3 | \mathbf{Y}, \mathbf{Z}) | \mathbf{Y} \big)$$

$$= \sum_{i=1}^{n} \left[\widehat{Z_{i1}}^{(t)} \left[\log \pi_1 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_1^2 - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \right.$$

$$+ \widehat{Z_{i2}}^{(t)} \left[\log \pi_2 - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_2^2 - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right]$$

$$\left. + \widehat{Z_{i3}}^{(t)} \left[\log(1 - \pi_1 - \pi_2) - \log \sqrt{2\pi} - \frac{1}{2} \log \sigma_3^2 - \frac{(y_i - \mu_3)^2}{2\sigma_3^2} \right] \right]$$

where $(\pi_3^{(t)} = 1 - \pi_1^{(t)} - \pi_2^{(t)})$

$$\widehat{Z_{ik}}^{(t)} = E(Z_i = k | \Theta^{(t)}) = E(Z_i^{(t)} | \pi_1^{(t)}, \pi_2^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \mu_3^{(t)}, \sigma_1^{2(t)}, \sigma_2^{2(t)}, \sigma_3^{2(t)})$$

$$= \frac{k\pi_k^{(t)} \frac{1}{\sqrt{2\pi}\sigma_k^{(t)}} e^{-\frac{(y_i - \mu_k^{(t)})^2}{2\sigma_k^{2(t)}}}}{\pi_1^{(t)} \frac{1}{\sqrt{2\pi}\sigma_1^{(t)}} e^{-\frac{(y_i - \mu_1^{(t)})^2}{2\sigma_1^{2(t)}}} + \pi_2^{(t)} \frac{1}{\sqrt{2\pi}\sigma_2^{(t)}} e^{-\frac{(y_i - \mu_2^{(t)})^2}{2\sigma_2^{2(t)}}} + (1 - \pi_1^{(t)} - \pi_2^{(t)}) \frac{1}{\sqrt{2\pi}\sigma_3^{(t)}} e^{-\frac{(y_i - \mu_3^{(t)})^2}{2\sigma_3^{2(t)}}}}$$

4

Then we can calculate:

$$\left.\frac{\partial Q}{\partial \pi_1}\right|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\frac{1}{\pi_1} - \widehat{Z_{i3}}^{(t)}\frac{1}{1-\pi_1-\pi_2}\right]\Bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \pi_2}\right|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\frac{1}{\pi_2} - \widehat{Z_{i3}}^{(t)}\frac{1}{1-\pi_1-\pi_2}\right]\Bigg|_{\pi_1^{(t+1)},\pi_2^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \mu_1}\right|_{\mu_1^{(t+1)},\sigma_1^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\frac{y_i - \mu_1}{\sigma_1^2}\right]\Bigg|_{\mu_1^{(t+1)},\sigma_1^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \mu_2}\right|_{\mu_2^{(t+1)},\sigma_2^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\frac{y_i - \mu_2}{\sigma_2^2}\right]\Bigg|_{\mu_2^{(t+1)},\sigma_2^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \mu_3}\right|_{\mu_3^{(t+1)},\sigma_3^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i3}}^{(t)}\frac{y_i - \mu_3}{\sigma_3^2}\right]\Bigg|_{\mu_3^{(t+1)},\sigma_3^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \sigma_1^2}\right|_{\mu_1^{(t+1)},\sigma_1^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i1}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_1^2} + \frac{(y_i-\mu_1)^2}{2(\sigma_1^2)^2}\right)\right]\Bigg|_{\mu_1^{(t+1)},\sigma_1^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \sigma_2^2}\right|_{\mu_2^{(t+1)},\sigma_2^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i2}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_2^2} + \frac{(y_i-\mu_2)^2}{2(\sigma_2^2)^2}\right)\right]\Bigg|_{\mu_2^{(t+1)},\sigma_2^{(t+1)}} = 0$$

$$\left.\frac{\partial Q}{\partial \sigma_3^2}\right|_{\mu_3^{(t+1)},\sigma_3^{(t+1)}} = \sum_{i=1}^{n}\left[\widehat{Z_{i3}}^{(t)}\left(-\frac{1}{2}\frac{1}{\sigma_3^2} + \frac{(y_i-\mu_3)^2}{2(\sigma_3^2)^2}\right)\right]\Bigg|_{\mu_3^{(t+1)},\sigma_3^{(t+1)}} = 0$$

Then for iteration in **EM Algorithm**:

Given initial guess: $\pi_1^{(0)},\pi_2^{(0)},\mu_1^{(0)},\mu_2^{(0)},\mu_3^{(0)},\sigma_1^{(0)},\sigma_2^{(0)},\sigma_3^{(0)}$, for $t \geq 0$ and $t \in \mathbb{Z}$:

$\mathbf{E-step}$: Calculate $E(Z_i^{(t)}|\Theta^{(t)})$, where $\Theta^{(t)} = \pi_1^{(t)},\pi_2^{(t)},\mu_1^{(t)},\mu_2^{(t)},\mu_3^{(t)},\sigma_1^{2(t)},\sigma_2^{2(t)},\sigma_3^{2(t)}$.

$\mathbf{M-step}$: Update $\Theta^{(t+1)}$ by equations (1) to (8) listed at the next page.

Iterative scheme:

$$\pi_1^{(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)}}{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)} + \sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)} + \sum_{i=1}^{n} \widehat{Z_{i3}}^{(t)}} \tag{1}$$

$$\pi_2^{(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)}}{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)} + \sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)} + \sum_{i=1}^{n} \widehat{Z_{i3}}^{(t)}} \tag{2}$$

$$\mu_1^{(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)} y_i}{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)}} \tag{3}$$

$$\mu_2^{(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)} y_i}{\sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)}} \tag{4}$$

$$\mu_3^{(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i3}}^{(t)} y_i}{\sum_{i=1}^{n} \widehat{Z_{i3}}^{(t)}} \tag{5}$$

$$\sigma_1^{2(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)} (y_i - \mu_1^{(t+1)})^2}{\sum_{i=1}^{n} \widehat{Z_{i1}}^{(t)}} \tag{6}$$

$$\sigma_2^{2(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)} (y_i - \mu_2^{(t+1)})^2}{\sum_{i=1}^{n} \widehat{Z_{i2}}^{(t)}} \tag{7}$$

$$\sigma_3^{2(t+1)} = \frac{\sum_{i=1}^{n} \widehat{Z_{i3}}^{(t)} (y_i - \mu_3^{(t+1)})^2}{\sum_{i=1}^{n} \widehat{Z_{i3}}^{(t)}} \tag{8}$$

(3)

Algorithm: EM Algorithm code in the R file.

Result: $\pi_1 = 0.4416, \pi_2 = 0.3450, \mu_1 = 2997.46, \mu_2 = 7969.146. \mu_3 = 29817.81, \sigma_1 = 301.2129, \sigma_2 = 1560.899$ and $\sigma_3 = 8002.694$

(4) Result: in the following page:

```
      low_income_1 middle_income_2 high_income_3 class
x1    9.976749e-01    4.441287e-03   0.0003134518     1
x2    9.991160e-01    1.594289e-03   0.0002606041     1
x3    0.000000e+00    7.620788e-44   3.0000000000     3
x4    9.991313e-01    1.572229e-03   0.0002476155     1
x5    1.376108e-40    1.995002e+00   0.0074976378     2
x6    0.000000e+00    5.709167e-50   3.0000000000     3
x7   1.605130e-139    1.941673e+00   0.0874902893     2
x8    9.975588e-01    4.666248e-03   0.0003241864     1
x9    9.967818e-01    6.173050e-03   0.0003949243     1
x10   9.990338e-01    1.801087e-03   0.0001969761     1
x11   9.987583e-01    2.340676e-03   0.0002141768     1
x12  1.366895e-102    1.983752e+00   0.0243716695     2
x13   1.996536e-45    1.994937e+00   0.0075938199     2
x14   9.982850e-01    3.258826e-03   0.0002566222     1
x15  2.693409e-156    1.888978e+00   0.1665332214     2
x16   9.670800e-01    6.413560e-02   0.0025567223     1
x17   0.000000e+00    2.068191e-17   3.0000000000     3
x18   9.983288e-01    3.173947e-03   0.0002525564     1
x19   0.000000e+00    1.278875e-99   3.0000000000     3
x20   6.929102e-21    1.993596e+00   0.0096063794     2
x21   9.969714e-01    5.805350e-03   0.0003778551     1
x22   9.991284e-01    1.608230e-03   0.0002024855     1
x23   3.089608e-17    1.992721e+00   0.0109182757     2
x24   0.000000e+00    6.134380e-76   3.0000000000     3
x25   9.989538e-01    1.848234e-03   0.0003661978     1
x26   9.970267e-01    4.868615e-03   0.0016169156     1
x27   9.980602e-01    3.694451e-03   0.0002775799     1
x28   9.991503e-01    1.552303e-03   0.0002207225     1
x29   9.985212e-01    2.801044e-03   0.0002348864     1
x30   9.991501e-01    1.555343e-03   0.0002168142     1
x31  1.016778e-119    1.971510e+00   0.0427355847     2
x32   0.000000e+00    8.200779e-64   3.0000000000     3
x33   9.991366e-01    1.565117e-03   0.0002425366     1
x34   9.991502e-01    1.554535e-03   0.0002175823     1
x35   9.989834e-01    1.900805e-03   0.0001986392     1
x36   9.991312e-01    1.572369e-03   0.0002477084     1
x37   9.926841e-01    1.413777e-02   0.0007410024     1
x38   9.869832e-01    2.524698e-02   0.0011798788     1
x39   9.962506e-01    7.204020e-03   0.0004421441     1
x40   9.990815e-01    1.646530e-03   0.0002856051     1
x41   0.000000e+00    8.881480e-70   3.0000000000     3
x42   9.989921e-01    1.883689e-03   0.0001982629     1
x43   5.151205e-45    1.994947e+00   0.0075794189     2
x44   9.988882e-01    1.953147e-03   0.0004057667     1
x45   9.990137e-01    1.840926e-03   0.0001974733     1
x46   0.000000e+00    4.782532e-22   3.0000000000     3
x47   9.979810e-01    3.847983e-03   0.0002849741     1
x48   2.361706e-75    1.992179e+00   0.0117309531     2
x49   0.000000e+00    1.034641e-37   3.0000000000     3
x50   9.988184e-01    2.223548e-03   0.0002093611     1
```

Figure 1: Classification of first 50 individuals