

1: The Bisection Method (25%)

For function $f(x) = x^3 + 6x^2 + \pi x - 12$, the derivative is $f'(x) = 3x^2 + 12x + \pi$. The we can calculate that zeros of the derivative are $\frac{-12-\sqrt{12(12-\pi)}}{6}$ and $\frac{-12+\sqrt{12(12-\pi)}}{6}$.

$f(\frac{-12-\sqrt{12(12-\pi)}}{6}) = 7.864841$ and $f(\frac{-12+\sqrt{12(12-\pi)}}{6}) = -12.43121$ Hence, the function f has totally 3 zeros.

Algorithm: Bisection Method in the R file.

Result: zeros -4.837944, -2.259727, and 1.097664.

2: Poisson Regression - Newton's Method (25%)

(1) Since $y_i \sim \text{Poisson}(\lambda_i)$ and $\log(\lambda_i) = \alpha + \beta x_i + \gamma x_i^2$, we can get the Likelihood function:

$$L(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \prod_{i=1}^n \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} = \prod_{i=1}^n \frac{e^{(\alpha + \beta x_i + \gamma x_i^2)y_i} e^{-e^{\alpha + \beta x_i + \gamma x_i^2}}}{y_i!}$$

(2) The log-Likelihood function is

$$l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (\alpha + \beta x_i + \gamma x_i^2) y_i - e^{\alpha + \beta x_i + \gamma x_i^2} - \log y_i!$$

Then we have:

$$\begin{aligned} \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \right|_{\hat{\alpha}} &= \sum_{i=1}^n [y_i - e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\alpha}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \right|_{\hat{\beta}} &= \sum_{i=1}^n [x_i y_i - x_i e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\beta}} = 0 \\ \left. \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \right|_{\hat{\gamma}} &= \sum_{i=1}^n [x_i^2 y_i - x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2}] \Big|_{\hat{\gamma}} = 0 \end{aligned}$$

$$\text{Let } \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta} \\ \frac{\partial l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma} \end{pmatrix}, \text{ then } \mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} & \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} \end{pmatrix}, \text{ in}$$

which

$$\begin{aligned}\frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha^2} &= \sum_{i=1}^n -e^{\alpha + \beta x_i + \gamma x_i^2} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \beta} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \alpha} = \sum_{i=1}^n -x_i e^{\alpha + \beta x_i + \gamma x_i^2} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \alpha \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \alpha} = \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta^2} &= \sum_{i=1}^n -x_i^2 e^{\alpha + \beta x_i + \gamma x_i^2} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \beta \partial \gamma} &= \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma \partial \beta} = \sum_{i=1}^n -x_i^3 e^{\alpha + \beta x_i + \gamma x_i^2} \\ \frac{\partial^2 l(\alpha, \beta, \gamma | \mathbf{x}, \mathbf{y})}{\partial \gamma^2} &= \sum_{i=1}^n -x_i^4 e^{\alpha + \beta x_i + \gamma x_i^2}\end{aligned}$$

Therefore, by Newton's Method, given initial guess $\alpha^{(0)}$, $\beta^{(0)}$, and $\gamma^{(0)}$, for each iteration:

$$\begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha^{(n-1)} \\ \beta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} - \mathbf{F}'[(\mathbf{x})]^{-1} \mathbf{F}(\mathbf{x}).$$

Hence, for the algorithm: $\mathbf{x}^{(n)} = \begin{pmatrix} \alpha^{(n)} \\ \beta^{(n)} \\ \gamma^{(n)} \end{pmatrix}$,

STEP 1: Solve $\mathbf{F}'(\mathbf{x}^{(n)}) \Delta \mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$;

STEP 2: Update by $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta \mathbf{x}^{(n)}$.

(3) Algorithm: Newton's Method code in the R file.

Result: $\alpha = 1.503533$, $\beta = 1.052351$, and $\gamma = 1.957396$.

3: Logistic Regression - Newton's Method (20%)

(1)

(2)

(3)

4: EM Algorithm (30%)

(1)

(2)

(3)

(4)