MAT1856/APM466 Assignment 1

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Fundamental Questions - 25 points

1.

   (a) Printing money by Central Bank increases the money supply which leads to inflation and lower exchange rates. Issuing government bonds controls the liquidity of money (the money supply flows into the market) and creates upward pressure for interest rates. It also relieves federal deficits and assists in raising capital for projects spending and daily operations.

   (b) The long-term part of a yield curve might flatten if the market expects a decrease in the money demand in the future and a downward movement of economic/business activities. When there is a sign of recession, the long-term yield will fall and there will be rate hikes in the near term.

   (c) Quantitative easing is an unconventional monetary policy whereby the central bank buys long-term securities and other financial assets with the goal of injecting money into the market to expand economic activity and encourage lending. During COVID-19, the US Fed purchased hundreds of billions of Treasury securities and MBS which leads to a lowering of interest rate and costs of investment.

2. (CAN 0.50 Mar 1, 2022); (CAN 2.75 Jun 1, 2022); (CAN 1.75 Mar 1, 2023); (CAN 1.50 Jun 1, 2023); (CAN 2.25 Mar 1, 2024); (CAN 1.50 Sept 1, 2024); (CAN 1.25 Mar 1, 2025); (CAN 0.50 Sept 1, 2025); (CAN 0.25 Mar 1, 2026); (CAN 1.00 Sept 1, 2026); (CAN 1.25 Mar 1, 2027)

   Firstly, I abandon all bonds with a maturity date farther than Dec 1, 2027. Given that the Canadian Government Bonds are paid semi-annually, I try to find 10 bonds to make them as *evenly spread in maturity* as possible to cover the 5-year span so the interpolation will be more accurate. I select 8 bonds with coupon payment in March/September as it suffices the semi-annual condition. Since I don't have any bonds that mature on 9/1/2022 and 9/1/2023, I use 2 bonds with maturity on 6/1/2022 and 6/1/2023 as substitutes. I include one additional bond which matures in 62 months for interpolation purposes. Notice that the first bond could only have one cash flow, which happens at maturity. Otherwise, we have no proper discount rate to discount the value of the coupon payment that happens before maturity as there is no data to interpolate.

   Among the 36 bonds, there are multiple bonds with the same maturity date but different coupon rates, I pick the one that has *a similar coupon rate as others* as coupon rate may affect the bond price. One last thing to mention is that *on-the-run bonds are more preferable* than off-the-run bonds as they are more liquid and represent the yield more accurately. Thus, the bonds with high coupon rates issued in the 90s will not be used. In general, all my selected bonds have small coupon rates.

3. Principal Component Analysis evaluates the dispersion and dimensionality of the dataset, its direction and spread along variables is measured by the covariance matrix. Eigenvectors illustrate how directions of the variance of the data are dispersed, and the corresponding eigenvalues are coefficients applied to the eigenvectors which aim to measure the magnitude of the directions of the variance. The eigenvector with the largest eigenvalue is the first principal component in PCA, it implies the direction of the maximum variance. If each process among a series of stochastic processes represents a unique point along the curve, then the principal component (eigenvector) shows the direction of the largest variability of the points. The eigenvalue is the sum of the squared distance between each point and the eigenvector, showing how much variation has been captured by that principal component.

Empirical Questions - 75 points

4.

(a) Step 1: Calculate DP. Since the closed price determined by the market through the bid-ask process gives the clean price(CP), we should adjust it with the accrued interest (AI) to derive the dirty price (DP): $DP = AI + CP$ where $AI = \frac{n}{365} \times AnnualCouponRate$.
Step 2: Calculate YTM using the Newton-Rhapson Method for non-linear equations.
Since Canadian bonds are semi-annual bonds, the yield to maturity is based on the following discounted cash flow equation: $f(ytm) = \sum_{t=t_0}^{T} \frac{coupon}{(1+\frac{ytm}{2})^{2t}} + \frac{par}{(1+\frac{ytm}{2})^{2T}} - DP = 0$
Where t_0 is the time between the next payment and the data record date. $T = t_0 + (n-1)/2$, here n is the number of payments remaining.

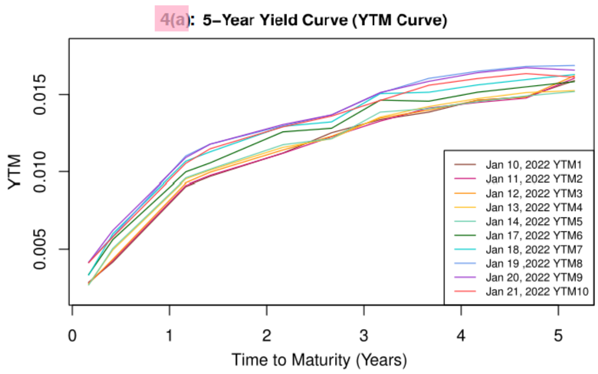


Figure 4a: 0-5 Year YTM Curve Using Data Collected from Jan10 to Jan 21, 2022

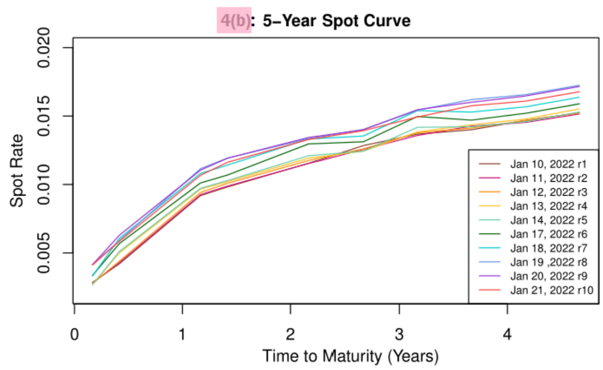


Figure 4b: Spot Curve with Terms Ranging from 1-5 Years

(b) Step 1: Since bond 1,2 mature less than six months (3/1/2022 & 6/1/2022) and have no intermediate coupon, we can derive the sport rate for bond 1,2 using the definition of yield of a zero coupon bond. That is, $r_{1,2} = -\frac{log(DP/FV)}{Time\,to\,Maturity}$ where $FV = Par + \frac{1}{2}Coupon$
Step 2: Estimate spot rate r_2^* for $t = 9/1/2022$ through linear extrapolation. We have:
$r_z = \frac{t_z - t_x}{t_y - t_x} r_y + \frac{t_y - t_z}{t_y - t_x} r_x$ for $t_x < t_y < t_z$. So $r_2^* = 2 \cdot r_2 - r_1$.
Step 3: Calculate r_3 for $t = 3/1/2023$ using the bootstrapping technique based on r_1 and r_2^*.
Step 4: Estimate r_4 with $t = 6/1/2023$ via linear interpolation. That is, estimate the yield for bonds with maturity in June/December by taking the mid-point of $\{r_1, r_2^*, r_3\}$ and then use the bootstrapping. Then apply linear extrapolation (illustated in step 2) again to derive r_4^* for $t = 9/1/2023$.
Step 5: Bonds 5-11 are all in March-September fashion, and could be calculated directly by bootstrapping method. That is, given $\{r_1, r_2^*, r_3, r_4^*, ..., r_{i-1}\}$, for $i \in [5, 11]$, r_i can be calculated using
$DP_i = \frac{1}{2}C_i \cdot e^{-r_1 \cdot t_1} + \sum_{w=2}^{i-1} \frac{1}{2}C_i \cdot e^{-r_w \cdot (\frac{w-1}{2} + t_1)} + FV_i \cdot e^{-r_i \cdot (\frac{i-1}{2} + t_1)}$

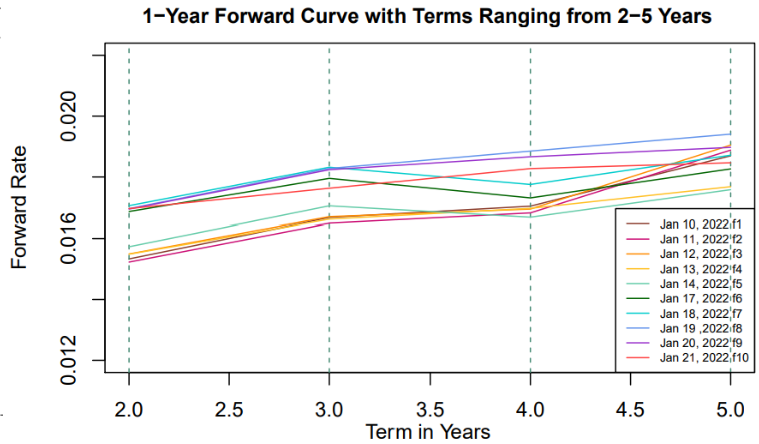
Algorithm 1 pseudo-code for calculating spot rate

t=1
while $t \leq 10$ **do**
$r_1 = -log(DP_{t1}/(0.5 * Coupon + Par))/t_1$
$r_2 = -log(DP_{t2}/(0.5 * Coupon + Par))/Time\,to\,t_1$
$r = c(r_1, r_2^*)$ # Here $r_2^* = 2 * r_2 - r_1$ by extrapolation
$pmt3$ =total discounted intermediate coupon of bond3
$r_3 = -log((DP_{t3} - pmt3)/(0.5 * Coupon + Par))/(t_1 + 1)$
$mid_r = middle\,point\,of\,\{r_1, r_2^*\}\,and\,\{r_2^*, r_3\}$
$pmt4$=total discounted intermediate coupon based on mid_r
$r_4 = -log((DP_{t4} - pmt_4)/(0.5 * Coupon + Par))/(t_2 + 1)$
$r = append(r, r_4^*)$ # Here $r_4^* = 2 * r_4 - r_3$ by extrapolation
the remaining bonds 5-11 in Mar/Sept fashion
solve r_i using formula stated in Step 5
end while

(c) To derive the 1-year forward curve, consider the formula: $e^{f\cdot(T_i-T_1)} = \frac{e^{r_i\cdot(T_i-T_0)}}{e^{r_1\cdot(T_1-T_0)}}$. Where $T_1 = 1, r_1$ is the estimated 1-year spot rate. Then we have: $f(t=0, T_1=1, T_2 \in [2,5]) = \frac{r_i\cdot(T_i-t_0)-r_1\cdot(1-t_0)}{T_i-1}$.

Algorithm 2 Pseudo-code for 1-year forward curve

$t = 1$
while $t \leq 10$ **do**
  estimate annual r for $yr1 = (r_{t2} + r_{t4})/2$
  $i = 1$
  while $i \leq 11$ **do**
    create empty vector forward
    $f_{01i} = \frac{r_i\cdot(T_i-t_0)-r_1\cdot(1-t_0)}{T_i-1}$
    forward=append(forward,f_{01i})
    i=i+1
  end while
  t=t+1
end while
We calculate all forwards over the 0-5 year span,
but we only need T2={2,3,4,5}
so select bonds with maturity in march 2024-2027.



5. The covariance matrix for the time series of daily log-returns of **yield** is

$$\begin{bmatrix} 0.011326872 & 0.0010629733 & 0.0011841299 & 0.0010780640 & 0.0022814963 \\ 0.001062973 & 0.0008579410 & 0.0004315453 & 0.0005414344 & 0.0006173138 \\ 0.001184130 & 0.0004315453 & 0.0005342130 & 0.0005366878 & 0.0003448639 \\ 0.001078064 & 0.0005414344 & 0.0005366878 & 0.0006242361 & 0.0003866526 \\ 0.002281496 & 0.0006173138 & 0.0003448639 & 0.0003866526 & 0.0007234129 \end{bmatrix}$$

The covariance matrix for the time series of daily log-returns of **forward rates** is

$$\begin{bmatrix} 0.0004046832 & 0.0004360880 & 0.0003337612 & 0.0004485543 \\ 0.0004360880 & 0.0005228843 & 0.0003493111 & 0.0004738488 \\ 0.0003337612 & 0.0003493111 & 0.0007197948 & 0.0006319142 \\ 0.0004485543 & 0.0004738488 & 0.0006319142 & 0.0007325842 \end{bmatrix}$$

6. **YTM**: The eigenvalues and their associated eigenvectors for the time series of daily log-returns of yields are:

$\lambda_1 = 1.219243e-02, \quad \vec{v_1} = [0.9602722, 0.1105478, 0.1125727, 0.1067035, 0.2039565]^T$

$\lambda_2 = 1.435480e-03, \quad \vec{v_2} = [0.2401391, -0.6511283, -0.4125453, -0.5206825, -0.2775993]^T$

$\lambda_3 = 3.846263e-04, \quad \vec{v_3} = [-0.04440926, 0.49310976, -0.55543191, -0.45692235, 0.48742993]^T$

$\lambda_4 = 3.292288e-05$, $\vec{v_4} = [0.05264139, 0.03437068, -0.71252424, 0.66323132, -0.22018393]^T$

$\lambda_5 = 2.121836e-05 \quad \vec{v_5} = [0.12436723, 0.56521909, 0.03063289, -0.26239338, -0.77153841]^T$

Interpretation: The first/largest eigenvalue $\lambda_1 = 1.219243e-02$ accounts for $\frac{0.01219243}{0.01219243+0.00143548+0.0003846263+0.00003292288+0.00002121836} = 0.8667597 \approx 86.68\%$ in magnitude (sum of total eigenvalues), meaning that the corresponding first eigenvector represents the direction of the largest variance after orthogonal decomposition and it can explain 86.68% of the total variation of the log-return of yield.

Forward Rate: The eigenvalues and the associated eigenvectors for the time series of daily log-returns of forward rates are:

$\lambda_1 = 1.968373e-03, \quad \vec{v_1} = [-0.4079717, -0.4459639, -0.5333365, -0.5918002]^T$

$\lambda_2 = 3.330345e-04, \quad \vec{v_2} = [0.4393583, 0.5925921, -0.6564089, -0.1578804]^T$

$\lambda_3 = 5.859162e-05, \quad \vec{v_3} = [-0.005141119, -0.376611025, -0.526723779, 0.762036591]^T$

$\lambda_4 = 1.994755e-05, \quad \vec{v_4} = [0.80031055, -0.55508094, 0.08509762, -0.21011087]^T$

Interpretation: The first/largest eigenvalue $\lambda_1 = 1.968373e-03$ accounts for $\frac{0.001968373}{0.002379946} = 0.827066 \approx 82.71\%$ in magnitude (sum of total eigenvalues), meaning that the corresponding eigenvector represents the direction of the largest variance after orthogonal decomposition and it can explain 82.71% of the total variation of the log-return of forward rates.

References and GitHub Link to Code

1. **References**:

   The Investopedia Team. (October 15, 2021). *Quantitative Easing (QE)*. Retrieved from: https://www.investopedia.com/terms/q/quantitative-easing.asp

2. **GitHub Link to Code**

   https://github.com/elaine0325/APM466-A1.git