information is indicated to failure. Degraded plies are so indicated, but failures modes are only indicated to the extent that they are fiber or matrix in nature. Whether the matrix failures are associated with limiting transverse normal or shear conditions is not explicitly defined. It should be emphasized that whereas the maximum stress and strain theories involve several separate equations and no interaction of the stresses or failure modes, the Tsai-Hill quadratic criterion does provide for interaction. However, the interaction is fixed. Tsai and Wu subsequently demonstrated shortcomings of the Tsai-Hill type expressions due to Equation (7.91) not being invariant.

7.11 Summary

only interested in 3 of these

Six different failure criteria have been examined in this Chapter (1) Maximum Stress (2) Maximum Strain (3) Tsai-Wu Quadratic (4) Hashin Polynomial in 3-D and 2-D (5) Hill, and (6) Tsai-Hill. Also the concept of stress ratios (R) has been introduced in order to get an idea of the "safety factor" before first ply failure. There are numerous other failure criteria associated with composites which only attests to the complexity of the failure process. With experience, the analyst can better decide which failure criteria performs best for a given situation (class of material, type of loading, geometry, etc.).

1) Max stress criterion: { Strength Ratio (R) / mode of failure

Fiber mode: { fiber in tension: $R = X_T/\sigma_x$; " " comp: $R = X_c/|\sigma_x|$

Shear mode: $R = S/|\sigma_s|$

matrix mode: { $\sigma_y > 0$: $R = Y_T/\sigma_y$; $\sigma_y < 0$: $R = Y_c/|\sigma_y|$

Quadratic failure: (Tsai-Wu): only strength ratio (Eq. 7.8)

$AR^2 + BR - 1 = 0$ → get A, B from Eq. 7.8 we want $R > 0$ value

Hashin criterion (2D): { R / modal failure — $R > 1$, we are safe; $R = 1$, onset of failure; $R < 1$, already failed

$R < 0$ is the ⊕ R if $\sigma = -\sigma$.

- no shear (Pg. 158) Eq. 7.85

Fiber mode: { $\sigma_x > 0$: $R = \text{sqrt}\left(\left[\left(\frac{\sigma_x}{X_T}\right)^2 + \left(\frac{\sigma_s}{S}\right)^2\right]^{-1}\right)$; $\sigma_x < 0$; $R = X_c/|\sigma_x|$

Matrix mode: { $\sigma_y > 0$: $R = \text{sqrt}\left(\left[\left(\frac{\sigma_y}{Y_T}\right)^2 + \left(\frac{\sigma_s}{S}\right)^2\right]^{-1}\right)$; $\sigma_y < 0$: Eq. 7.88 where sub → $R\sigma \rightarrow \sigma$

Only report ⊕.

8.2 FPF: Symmetric Laminate - Membrane Load (N_1, N_2, N_6)

Let us now look at a specific example to show the FPF procedures. We will consider a symmetric laminate and a membrane load condition in the first instance. As the laminate is symmetric, all the coupling terms, $B_{ij} = 0$ and so we will not need to use the full laminate constitutive equation, eqn. (8.2). Furthermore, since we are looking at a membrane load only, we will only need the membrane (extensional) stiffnesses and compliances, A_{ij} and a_{ij}. The bending stiffnesses and compliances will not be required as there are no bending loads in this case.

* **Example 8.1** : FPF analysis

4 Plies total, balanced.

Let us consider a cross-ply laminate $[0/90]_s$ made from high strength carbon/epoxy unidirectional plies, and subjected to a tensile membrane longitudinal force intensity $N_1 = 100$ N/mm. The ply is 0.125 mm thick and the elastic properties of this material are $N_2 = N_6 = 0$

$$E_x = 140, \qquad E_y = 10, \qquad E_s = 5 \text{ kN/mn}^2; \qquad \nu_x = 0.3$$

The ply strengths for this material are

$$X_t = 1500; \qquad X_c = 1200; \qquad Y_t = 50; \qquad Y_c = 250; \qquad S = 70 \text{ N/mn}^2$$

The ply reduced stiffnesses, Q_{ij}, the membrane stiffnesses, A_{ij}, and the compliances, a_{ij}, for this cross-ply laminate are

$$Q_{ON} = \begin{bmatrix} 140.9 & 3.0 & 0 \\ 3.0 & 10.1 & 0 \\ 0 & 0 & 5.0 \end{bmatrix} = \begin{bmatrix} mE_x & m\nu_y E_x & 0 \\ & & 0 \\ & & E_s \end{bmatrix} \text{ kN/mm}^2 \text{ ("on-axis" material property)}$$

See Chapter 2

$$A = \begin{bmatrix} 37.8 & 1.5 & 0 \\ 1.5 & 37.8 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \text{ kN/mm}$$

$$a = \begin{bmatrix} 0.0265 & -0.0011 & 0 \\ -0.0011 & 0.0265 & 0 \\ 0 & 0 & 0.4000 \end{bmatrix} 1/(\text{kN/mm})$$

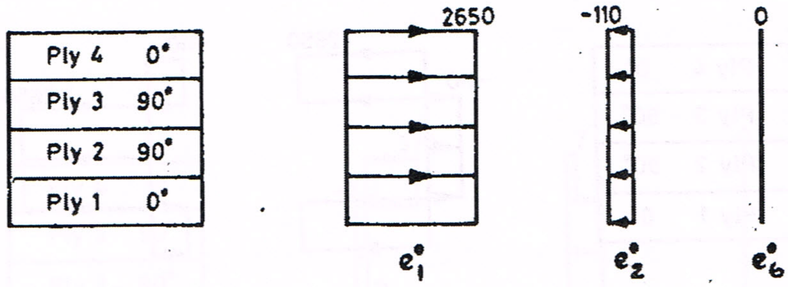


Figure 8.2: Midplane reference axes strains ($\times 10^{-6}$) through laminate thickness.

an applied M would NOT allow you to do this

Substituting the appropriate trigonometric values and the reference axes strain values into eqn. (8.9), we get

*do not have to calculate for entire laminate since loading is in-plane & symmetric laminate.

	2650	-110	0
e_x	0	1	0
e_y	1	0	0
e_s	0	0	-1

$\times 10^{-6}$

giving

$$e_x = -110 \times 10^{-6}, \quad e_y = 2650 \times 10^{-6}, \quad e_s = 0$$

In the case when the ply angle is 90°, the material axes strains can also be obtained by inspection; the orthogonal strains change directions when transposing from the reference axes to the material axes, and the shear strain changes sign.

There is no need to perform the calculations on Plies 3 and 4 because the laminate is symmetric and subjected to a membrane load only. Once the calculations are performed for a ply configuration, the rest of the plies with the same configuration will have identical values because, due to the membrane load, the strains in the reference axes are constant through the thickness. Thus, the strains in the material axes for Plies 1 and 4 with a ply angle of 0° will be the same, and the strains in the material axes for Plies 2 and 3 with a ply angle of 90° will be the same.

The distribution of strains in the material axes for all the plies are given in Fig. 8.3, where it is seen that the variation of the material axes strains through the laminate thickness is now in stepped form rather than being of constant form. This stepped kind of variation is to be expected as the material axes directions change from ply to ply, whereas in the case of the reference axes strain variation (see Fig. 8.2) the axes system 1 – 2 is common to all the plies. Next, we make use of eqn. (8.10) to determine the ply stresses in the material axes,

Exam Q: off-axis strain always linear and continuous.
on-axis strain always/could be jagged.

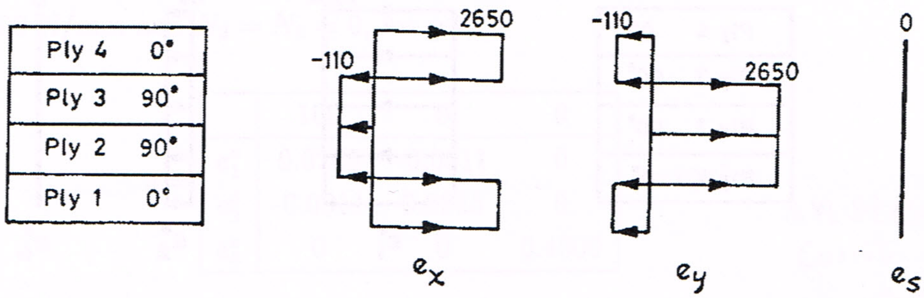


Figure 8.3: Midplane material axes strains ($\times 10^{-6}$) through laminate thickness.

having got the ply strains in the material axes. Substituting the values of the ply reduced stiffnesses given in the beginning of this example and converting the units to N and mm, and the ply strains in the material axes, for each ply into eqn. (8.10), we get:

Ply 1 at 0°

	2650	-110	0
σ_x	140 .9	3-0	0
σ_y	3 .0	10.1	0
σ_s	0	0	5 .0

$\times 10^{-6} \times 10^3$ N/mm^2

giving

$$\left.\begin{aligned} \sigma_x &= 373 \text{ N/mm}^2 \\ \sigma_y &= 7 \text{ N/mm}^2 \end{aligned}\right\} \text{tension}$$
$$\sigma_s = 0$$

Let us use the maximum stress failure criterion (Eqn. 7.8) for simplicity of calculations and determine the F.I. for this ply in the fibre, matrix and shear directions. We use the ply strengths given at the beginning of this example. Recall that, in the maximum stress theory, we use the tensile or compressive strength corresponding to the tensile or compressive stress condition.

(F.I. = 1/R)

$$\begin{aligned} \text{F.I. } x &= 373/\underset{X_T}{1500} = 0.25 \\ \text{F.I. } y &= 7/\underset{Y_T}{50} = 0.14 \\ \text{F.I. } s &= 0/\underset{S}{70} = 0 \end{aligned}$$

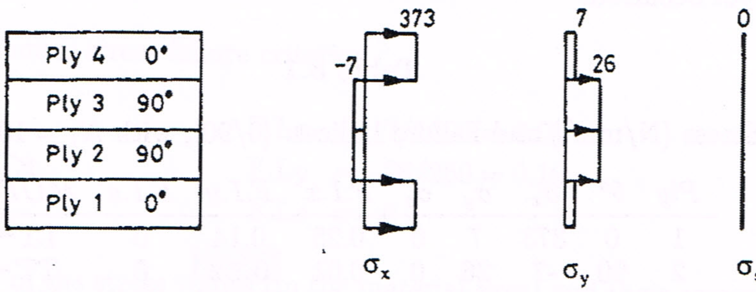


Figure 8.4: Material axes stress (N/mm^2) through laminate thickness.

Ply 2 at 90°

	-110	2650	0
σ_x	140.0	3. 0	0
σ_y	3.0	10.1	0
σ_s	0	0	5.0

$\times 10^{-6} \times 10^3$ N/mm^2

giving

$$\begin{aligned} \sigma_x &= -7 \text{ N/mm}^2 \\ \sigma_y &= 26 \text{ N/mm}^2 \\ \sigma_s &= 0 \end{aligned}$$

→ compression
→ tension

Using the maximum stress failure criterion (see Chapter 7):

$$\begin{aligned} \text{F.I. } x &= 7/\overset{X_C}{1200} = 0.01 \\ \text{F.I. } y &= 26/\overset{}{50} = 0.52 \quad Y_T \\ \text{F.I. } s &= 0/\underset{S}{70} = 0 \end{aligned}$$

The stresses in the rest of the plies in the top half of the laminate will be the same as their corresponding values for the plies in the lower half of the laminate (again, this is true only because of the laminate being symmetric and the load being a membrane one).

The stress distributions in the ply material axes through the laminate thickness, which are shown in Fig. 8.4, naturally show a stepped form variation because of the different material axes in the plies of the laminate.

A summary of the stress values (in the material axes) and their corresponding F.I. for all the four plies in this cross-ply laminate configuration $[0/90]_s$ with $N_1 = 100$ N/mm is given in Table 8.1.

failure indice means reverse of R: high f.i. is bad.

From the summary of results in Table 8.1, it is seen that the maximum failure index occurs in the 90° plies in the transverse direction, F.I. $y = 0.52$, in tension mode as the stress in this direction σ_y is tensile. So, with the applied tensile load $N_1 = 100$ N/mm, no ply failure has yet occurred.

Table 8.1

Ply Stress (N/mm^2) and Failure Indices: $[0/90]_s$ with $N_1 = 100$ N/mm

mode of failure.

Ply	θ^o	σ_x	σ_y	σ_s	*F.I.x*	*F.I.y*	*F.I.s*	*MOF*
1	0	373	7	0	0.25	0.14	0	LT → longitudinal tension
2	90	-7	26	0	0.01	**0.52**	0	TT → transverse tension
3	90	-7	26	0	0.01	**0.52**	0	TT
4	0	373	7	0	0.25	0.14	0	LT

Therefore, the load can be increased by a R factor of $1/0.52 = 1.92$ before FPF is predicted by the maximum stress theory. Hence, the FPF load is $100 \times 1.92 = 192$ N/mm and this will occur in the 90° plies in the TT mode of failure (MOF). Note that the analysis predicts that both the 90° plies will fail simultaneously.

FPF: first instance/set of failure.

Example 8.2

Consider the same cross-ply laminate of Example 8.1 now subjected to a compressive membrane longitudinal force intensity $N_1 = -100$ N/mm.

There is no need to repeat the calculations to determine the ply stresses, as these can be obtained by inspection of the results from Example 8.1. All the ply stresses will now have the opposite signs to those in Example 8.1, as the load is now reversed in signs with no other change being effected. However, the ply strength analyses will alter as the ply tensile and compressive strengths are different in the fibre (longitudinal) or matrix (transverse) direction:

Ply 1 at 0°

$$\left.\begin{aligned} \sigma_x &= -373 \text{ N/mm}^2 \\ \sigma_y &= -7 \text{ N/mm}^2 \\ \sigma_s &= 0 \end{aligned}\right\}$$

Q: although stress sign is just -ve what we had before, failure indice is completely diff. since we use X_c, Y_c

Using the maximum stress failure criterion

$$\begin{aligned} \text{F.I.}x &= 373/1200 = 0.31 \\ \text{F.I.}y &= 7/250 = 0.03 \\ \text{F.I.}s &= 0 \end{aligned}$$

Ply 2 at 90°

$$\sigma_x = 7\ \text{N/mm}^2$$
$$\sigma_y = -26\ \text{N/mm}^2$$
$$\sigma_s = 0$$

Using the maximum stress failure criterion

$$\text{F.I.}x = 7/1500 = 0.01$$
$$\text{F.I.}y = 26/250 = 0.10$$
$$\text{F.I.}s = 0$$

A summary of the stress values (in the material axes) and their corresponding F.I. for all the four plies in this cross-ply laminate configuration $[0/90]_s$ for the compressive longitudinal force $N_1 = -100$ N/mm is given in Table 8.2.

It is seen from Table 8.2 that the maximum failure index now occurs in the 0^o plies in the longitudinal (fibre) direction, F.I. $x = 0.31$, in the compression mode. The FPF has now shifted from the 90^o ply, with a TT mode of failure for the tensile load case, to the 0^o ply, with a LC mode of failure for the compressive load case.

Table 8.2

Ply Stress (N/mm^2) and Failure Indices: $[0/90]_s$ with $N_1 = -100$ N/mm

Ply	θ^o	σ_x	σ_y	σ_s	*F.I.x*	*F.I.y*	*F.I.s*	*MOF*
1	0	-373	-7	0	**0.31**	0.03	0	LC
2	90	7	-26	0	0.01	0.10	0	TC
3	90	7	-26	0	0.01	0.10	0	TC
4	0	-373	-7	0	**0.31**	0.03	0	LC

N or M?

assume symmetric

if M, σ top -σ bot.

plies with same θ and only N applied, fail at same time.

The load can be increased by a factor of $1/0.31 = 3.23$ before FPF is predicted by the maximum stress theory. Hence, the FPF load is $-100 \times 3.23 = -323$ N/mm and this will occur in the 0^o plies in the longitudinal compression (LC) mode. So we see that the FPF and its mode of failure in a laminate is very much dependent on the signs as well as the magnitudes of the applied loads.

8.3 FPF: Symmetric Laminate – Bending Load

We now extend the analysis to a bending load, rather than a membrane load, but still keep the laminate symmetric. In the case of symmetric laminates with a membrane load, we only had to consider one half of the laminate configuration for ply-to-ply strain, strain and

Q: imagine you have table 8.2 but without θ values.

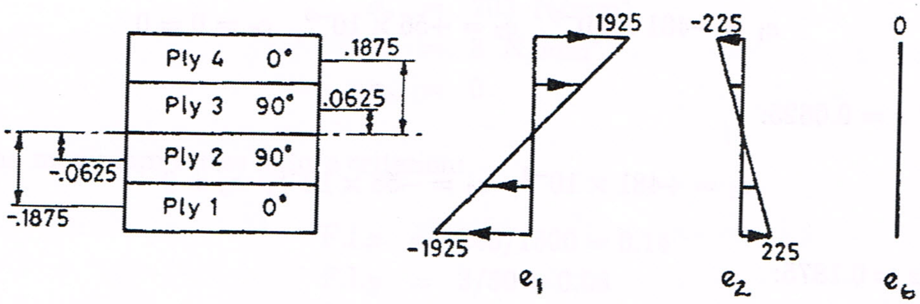


Figure 8.5: Bending reference axes strains ($\times 10^{-6}$) through laminate thickness.

Using the bending compliances, converted into units of N and mm, in eqn. (8.6), and with $M_1 = 10$ N mm/mm, $M_2 = M_6 = 0$, we get

	10	0	0
k_1	0.77	-0.09	0
k_2	-0.09	3.64	0
k_6	0	0	19.19

$\times 10^{-3}$

giving

$$
\begin{aligned}
k_1 &= 7700 \times 10^{-6} \text{ 1/mm} \\
k_2 &= -900 \times 10^{-6} \text{ 1/mm} \\
k_6 &= 0
\end{aligned}
$$

The strain induced by the curvatures is $+zk$, where z is the distance from the midplane. So, on the top surface where $z = +0.25$ mm top

$$
\begin{aligned}
e_1 &= (0.25) \times 7700 \times 10^{-6} = 1925 \times 10^{-6} \\
e_2 &= (0.25) \times -900 \times 10^{-6} = -225 \times 10^{-6} \\
e_6 &= (0.25) \times 0 = 0
\end{aligned}
$$

and on the bottom surface where $z = -0.25$, $e_1 = -1925 \times 10^{-6}$, $e_2 = 225 \times 10^{-6}$, $e_6 = 0$.

This is a linear strain distribution through the laminate thickness in all the four plies, as shown in Fig. 8.5. From Fig. 8.5, we get the ordinate distances for each ply centroid and hence the strains at the ply centroids:

center

Ply 1 at 0° $z = -0.1875$:

$$
\begin{aligned}
e_1 &= (-0.1875) \times 7700 \times 10^{-6} = -1444 \times 10^{6} \\
e_2 &= (-0.1875) \times -900 \times 10^{-6} = +169 \times 10^{-6} \\
e_6 &= (-0.1875) \times 0 = 0
\end{aligned}
$$

compare, these are radically different even though we've only gone 1/2 ply down.

Ply 1 at 0°

The ply stresses will be of opposing signs to those in the corresponding ply, Ply 4:

$$\begin{aligned} \sigma_x &= -203 \text{ N/mm}^2 \\ \sigma_y &= -3 \text{ N/mm}^2 \\ \sigma_s &= 0 \end{aligned}$$

Using the maximum stress failure criterion:

$$\begin{aligned} \text{F.I.}x &= 203/1200 = 0.17 \\ \text{F.I.}y &= 3/250 = 0.01 \\ \text{F.I.}s &= 0 \end{aligned}$$

Table 8.3

Ply Stresses (N/mm²) and Failure Indices: $[0/90]_s$ with $M_1 = 10$ N mm/mm

Ply	$\theta°$	σ_x	σ_y	σ_s	*F.I.x*	*F.I.y*	*F.I.s*	*MOF*
4	0	203	3	0	0.14	0.06	0	LT
3	90	-6	5	0	0.01	0.10	0	TT
2	90	6	-5	0	0.01	0.02	0	TC
1	0	-203	-3	0	**0.17**	0.01	0	LC

due to difference in tension, compression props

A summary of the stress values (in the material axes) and their corresponding F.I. for all the four plies in this cross-ply laminate configuration $[0/90]_s$ for a positive moment load $M_1 = 10$ N mm/mm is given in Table 8.3. From the summary of results in Table 8.3, it is seen that the maximum failure index occurs in the bottom ply, the 0° ply, Ply 1, in the longitudinal direction, F.I.$x = 0.17$, in the compression mode as the stress in this direction σ_x is compressive. The correspondingly placed top ply, Ply 4, has stress values of opposing signs to Ply 1, and the ply strength analyses give different F.I. because of the different tensile and compressive strengths. Since the compressive strength in the fibre direction is lower than its tensile strength, Ply 1 with a compressive fibre stress is the critical one, rather than Ply 4 with a tensile fibre stress.

So, with the applied positive moment $M_1 = 10$ N mm/mm, no ply failure has yet occurred. Therefore, the load can be increased by a factor of $1/0.17 = 5.88$ before FPF is predicted by the maximum stress theory. Hence, the FPF load is $10 \times 5.88 = 58.8$ N mm/mm and this will occur in the bottom ply, Ply 1: 0°, in the LC MOF.

8.4 Last-Ply-Failure (LPF) Procedure

The FPF analyses presented in the previous sections pose no problems as far as the analytical computations are involved. It is possible in practice to see the effect of the FPF load in a laminate made from MOPL (Multi-Oriented Plies Laminate). A kink in the load-strain graph in a laminate test setup is visually noticed. However, care must be taken in interpreting the load at which this kink occurs. In the analyses, we have considered so far for the FPF load, we have only considered mechanical (external) loads.

The MOPLs have residual stresses inbuilt into the plies before the external loads are applied. These residual stresses are present as a result of the curing processes of the prepreg plies, and will, therefore, influence the ply-to-ply stress and strength analysis. We will cover the subject of residual stresses in Chapter 10.

Assuming for the moment that the residual stresses in the plies can be neglected, then the FPF load can be predicted by the analytical method, demonstrated in the last three examples. The question we now have to ask is this: is the laminate able to carry any more loads after FPF has occurred? In other words, we are interested in the post-FPF mechanisms, in order to determine the load that will cause the last-ply-failure (LPF), and hence obtain the laminate strength.

There are three possible in-plane failure mechanisms, on the macroscopic scale, associated with a ply. These are the fibre, matrix and shear failures. After the FPF in a laminate, in which failure is caused by any of the three failure modes, the applied load gets redistributed in the remaining intact plies. There is a redistribution of the loads dues to the overall laminate stiffness reduction as a result of stiffness loss due to the FPF. As the laminate stiffness is reduced, the load gets redistributed according to the relative stiffnesses of the intact plies.

As the load gets increased, a second-ply-failure will occur associated with a failure mode; the laminate stiffness is further reduced, and the load is further redistributed according to the relative stiffnesses of the remaining intact plies. The load is further increased until a third-ply-failure and so on until a LPF of the laminate occurs. The problem is to model analytically this process of laminate stiffness reduction and the redistribution of the applied load in the intact plies due to a ply failure. Furthermore, having found a way to model this process, say, the question then is how true a model is this in terms of correlation with the real situation in practice?

A method of simulating a failed ply in a laminate is to have the failed ply remain in the same position in the laminate configuration, but reduce the values of some of its elastic properties. This will reflect the practical case in which the failed ply is physically still there in the laminate configuration, but unable to carry any more load. By reducing the elastic

cannot set the elastic properties = 0 because then we have
CPF, PPF a non-invertible (ie: " ill-conditioned) matrix

constant values, we would have modelled the laminate reduced stiffness.

The next question which arises is by how much should these elastic values be reduced? It almost impossible to quantify this reduction for a general case of laminate configuration, in that the reduction will depend on the type of loading, material used and the orientation of the plies. A lot of research is being carried out in order to try and quantify this reduction in stiffness values. However, for initial design stressing, a conservative, simple and consistent approach is taken in which the elastic constants are reduced to having zero values. By setting the elastic constant values to zero, we are analytically simulating the assumed practical situation of the failed ply not carrying any stresses.

It should be noted that by numerically setting the failed ply elastic constant values to zero, it is the stresses, and not the strains, in the ply which will give zero values from laminate analysis computations.

For in the laminate analysis, the formulation is such that the strains in the reference axes are common to all plies, and thus nonzero values will be obtained from the laminate analysis for the plies with their elastic values set to zero. However, the stress values will be zero since the elastic constant values are zero. Hence, in post-FPF analysis, the assessment of ply failures must be performed with care. If strain values are to be considered, the modelled failed ply is assumed to have zero strains although nonzero values will be obtained from the laminate analysis. However, stress values can be taken as correctly calculated in that they will have zero values.

The elastic constants which are set to zero are chosen from the E_x, E_y, or E_s values, depending on the ply in-plane mode of failure and the assumptions we make in deciding the capability of the failed ply to carry any load in other than the failed direction.

Two approaches that we can take in terms of analytically modelling the post-FPF behaviour in a MOPL are called the complete ply failure approach and the partial ply failure approach. Let us look at the complete ply failure approach first. With this approach it is assumed that, after the FPF, the failed ply is no longer able to carry any more load regardless of the mode of failure in that ply. Thus, if a matrix or shear failure is predicted, it is assumed that the ply is not able to carry any load even in the fibre direction, even though no failure has occurred in the fibres. Also, if a fibre failure is predicted, then the ply is no longer able to carry out load in any mode; fibre, matrix or shear. Thus, upon FPF, for any mode of failure mechanism, the failed ply is assumed to have zero properties for the E_x, E_y and E_s constants. The thickness of the failed ply, and its position, is unaltered in the new laminate configuration, thereby modelling that the failed ply is still in the laminate, albeit with no load carrying capability.

With this new laminate configuration, the laminate stiffnesses are now recalculated. Note that since a ply has already failed and all its elastic properties are set to zero, the new laminate will now have reduced stiffness values. Moreover, in a test setup of a laminate under a tensile loading, say, a kink is observed in the load-strain graph, indicating the FPF load, and the curve continues with increasing load, but with a smaller slope, signifying a reduced stiffness in the direction of the load (see examples to follow).

The next step is to check that the load which caused the FPF in the original laminate is not causing a second-ply-failure in the now reduced stiffness laminate. If the second-ply-failure is predicted in this reduced stiffness laminate, then the laminate stiffness is recalculated with the elastic values of the failed second ply set to zero. This process is repeated until no ply failure, caused by the load level which initiated the previous ply failure, is predicted. If no second-ply-failure is predicted, then the load can be increased until the second-ply-failure is observed. The second failed ply is then set with zero E_x, E_y and E_s values, and the laminate stiffness, now reduced even further, is recalculated. A check is then performed to see whether the load which caused the second-ply-failure is still being carried by the laminate, which has a further reduced stiffness with two ply failures. If no further ply failure is predicted at this stage, the load is increased and the process is continued until the LPF load is reached.

The alternative analytical method in trying to model the post-FPF mechanism is the partial ply failure approach. With this approach, it is assumed that a matrix or shear failure will not affect the ply load carrying capability in its fibre direction. However, in the case of a fibre failure, then the ply is unable to carry any more load in any mechanism. Also, a matrix failure will incapacitate the shear capability at that point, and vice versa, i.e., a shear failure will also result in a loss of the matrix load carrying capacity.

Therefore, if a matrix or shear failure is predicted in the ply, then the elastic values E_y and E_s are set to zero, but the fibre dominated elastic value E_x is unaltered. If a fibre failure is predicted, than all the elastic values E_x, E_y and E_s are set to zero. The process of determining subsequent ply failures is then identical to the one discussed above when considering the complete ply failure approach.

The process of post-FPF to LPF analyses is shown schematically in the flowchart of Fig. 8.6.

In performing post-FPF analysis, the assumptions we make in terms of ply failures will depend on the loading condition. For symmetric laminates with a membrane load, the FPF theoretically occurs simultaneously in the two corresponding symmetric plies below and above the midplane - see Example 8.1 in the $[0/90]_s$ laminate configuration in which both the 90° plies are predicted to have failed at the same load level. Here, the terminology FPF is, therefore, taken to mean ply failures of the same configuration, although two 90° plies

PPF: ① if laminate fails in matrix mode, set $E_y, E_S = 0$

② if laminate fails in shear mode,

SUMMARY: approach

before failure: $Q = \begin{bmatrix} mE_x & m\nu_y E_x & 0 \\ m\nu_x E_y & mE_y & 0 \\ 0 & 0 & E_s \end{bmatrix}$

can't use quad poly here

doesn't tell us anything about modal failure

After failure:

a) complete ply failure (conservative approach)
$Q = [0]$

b) partial ply failure

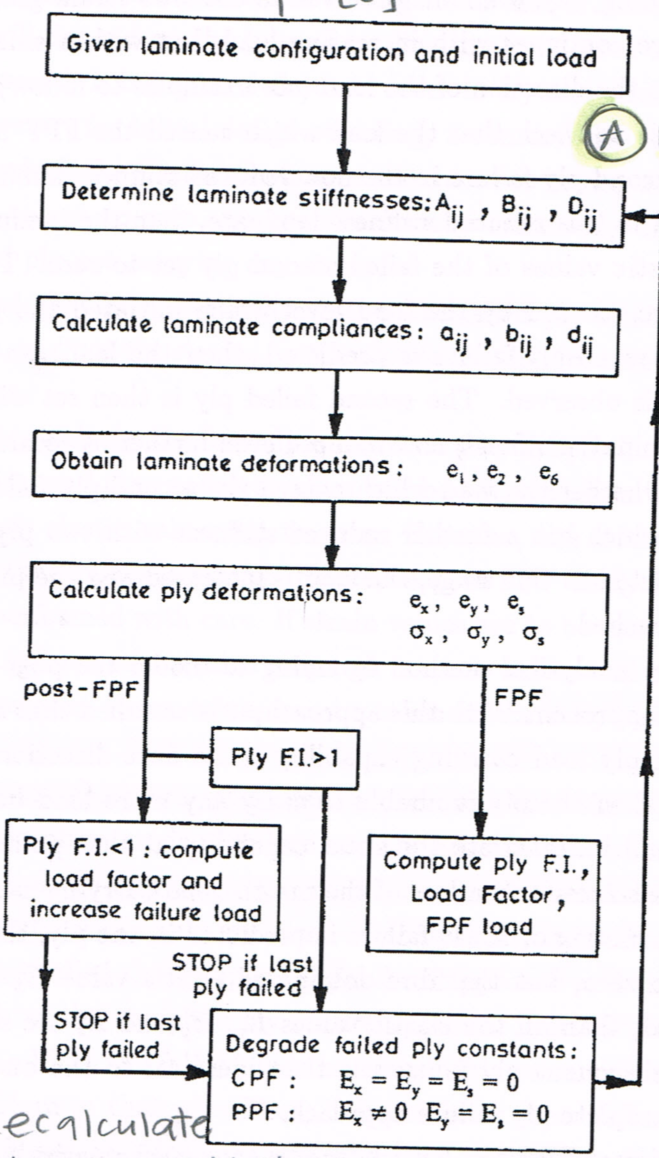
(i) if fiber failure: $Q = [0]$

(ii) matrix or shear failure: $E_y = E_s = 0$, $E_x \neq 0$

or alternatively

(A) matrix failure $E_x, E_s \neq 0$, $E_y = 0$

(B) shear failure $E_s = E_y = 0$, $E_x \neq 0$



R, RN, RM

then recalculate elastic properties and redo analysis

Figure 8.6: LPF procedure

continuos distribution of off-axis strain.
But generally failure criterion are ON-axis.

have failed. The subsequent ply failure is, therefore, termed the second-ply-failure (which will be of a different ply configuration), but numerically it is the third ply in the laminate which will fail. By then setting these two plies as having failed, subsequent ply failure will again be predicted in pairs, for the corresponding plies below and above the midplane. Thus, symmetry of the laminate configuration is maintained in the post-FPF analyses.

In the case of a symmetric laminate with a bending load, FPF will occur in a single ply without the accompanying failure of the corresponding ply on the other side of the midplane – see Example 8.3 in the $[0/90]_s$ laminate configuration under a bending load in which the bottom ply, Ply 1 at 0^o is going to fail first, whereas the corresponding top ply, Ply 4 would still be intact. Now, by setting some or all of the elastic values of this failed ply to zero, the laminate symmetry is no longer maintained, and the couplings B_{ij} will come into ply as they will now have nonzero values. Thus, a laminate configuration which started out as being symmetric, is no longer so after the FPF. An assumption can be made, in order to simplify calculations, that the ply corresponding to the failed ply, on the other side of the midplane, has also failed, thereby making the laminate symmetric in which case then the coupling B_{ij} stiffnesses will be zero.

So, by using the complete ply failure or partial ply failure approach, we are in a position to determine analytically the load to cause the LPF in the laminate, and hence the laminate strength.

(N_1, N_2, N_6).

Nov 4

8.5 LPF: Complete Ply Failure – Membrane Load

We will now consider a specific example to show the process of determining the LPF load, and hence the laminate strength. In the first instance, we will consider a membrane loading condition on a symmetric laminate and assume the complete ply failure approach. As the laminate is symmetric all the coupling terms $B_{ij} = 0$ and so we will not need to use the full laminate constitutive equation (eqn. 8.2). Furthermore, since we are looking at a membrane load only, we will only need the membrane (extensional) stiffnesses and compliances, A_{ij} and a_{ij}. The bending stiffnesses and compliances will not be required as there are no bending loads in this case.

Example 8.4

$\frac{\pi}{4}$

Determine the longitudinal tensile load N_1 that will cause the LPF in the symmetric quasi-isotropic laminate $[0/45/-45/90]_s$. Use the maximum stress failure theory and the complete ply failure approach. The laminate is made from high strength carbon/epoxy

Table 8.4

$\bar{Q}_{ij}$ Values (kN/mm^2): $[0/45/-45/90]_s$; All Plies Intact

$Q_{11} > Q_{22}$: 0°
$Q_{11} = Q_{22} = 45°$
$Q_{22} > Q_{11}$: 90°

Ply	$\theta°$	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{33}$	$\bar{Q}_{12}$	$\bar{Q}_{1\cancel{3}6}$	$\bar{Q}_{2\cancel{3}6}$
1	0	140.9	10.1	5.0	3.0	0	0
2	45	44.3	44.3	36.3	34.3	32.7	32.7
3	-45	44.3	44.3	36.3	34.3	-32.7	-32.7
4	90	10.1	140.9	5.0	3.0	0	0
5	90	10.1	140.9	5.0	3.0	0	0
6	-45	44.3	44.3	36.3	34.3	-32.7	-32.7
7	45	44.3	44.3	36.3	34.3	32.7	32.7
8	0	140.9	10.1	5.0	3.0	0	0

fibers aligned with 1,2

The process of determining the membrane (extensional) stiffness terms have been gone through in detail in Chapter 4, and hence only the essentials of the computations will be shown from now on without any further explanations. With a ply thickness of 0.125 mm, and the transformed reduced stiffness values from Table 8.1, we get

symmetric – h_0 Q_{11} Q_{11}

$$A_{11} = 2\{(0.125 \times 140.9)_{\text{Ply1}} + (0.125 \times 44.3)_{\text{Ply2}} + (0.125 \times 44.3)_{\text{Ply3}} + (0.125 \times 10.1)_{\text{Ply4}}\} = 59.9 \text{ kN/mm}$$

$$A_{22} = 2\{(0.125 \times 10.1)_{\text{Ply1}} + (0.125 \times 44.3)_{\text{Ply2}} + (0.125 \times 44.3)_{\text{Ply3}} + (0.125 \times 140.9)_{\text{Ply4}}\} = 59.9 \text{ kN/mm}$$

$$A_{\cancel{33}66} = 2\{(0.125 \times 5.0)_{\text{Ply1}} + (0.125 \times 36.3)_{\text{Ply2}} + (0.125 \times 36.3)_{\text{Ply3}} + (0.125 \times 5.0)_{\text{Ply4}}\} = 20.7 \text{ kN/mm}$$

$$A_{12} = 2\{(0.125 \times 3.0)_{\text{Ply1}} + (0.125 \times 34.3)_{\text{Ply2}} + (0.125 \times 34.3)_{\text{Ply3}} + (0.125 \times 3.0)_{\text{Ply4}}\} = 18.7 \text{ kN/mm}$$

$$A_{1\cancel{3}6} = 2\{(0.125 \times 0)_{\text{Ply1}} + (0.125 \times 32.7)_{\text{Ply2}} + (0.125 \times -32.7)_{\text{Ply3}} + (0.125 \times 0)_{\text{Ply4}}\} = 0$$

$$A_{2\cancel{3}6} = 2\{(0.125 \times 0)_{\text{Ply1}} + (0.125 \times 32.7)_{\text{Ply2}} + (0.125 \times -32.7)_{\text{Ply3}} + (0.125 \times 0)_{\text{Ply4}}\} = 0$$

The extensional stiffness terms A_{ij} may be written in a boxed matrix notation form:

$$A = \boxed{\begin{matrix} 59.9 & 18.7 & 0 \\ 18.7 & 59.9 & 0 \\ 0 & 0 & 20.7 \end{matrix}} \text{ kN/mm}$$

Plies 2 and 7 at 45°

$$\begin{aligned} m = \cos 45° = 1/\sqrt{2} \quad &\text{and} \quad n = \sin 45° = 1/\sqrt{2} \\ m^2 = 0.5 \quad &\quad n^2 = 0.5 \\ mn = 0.5 \quad &\quad 2mn = 1 \\ m^2 - n^2 = 0 \quad & \end{aligned}$$

and from eqn. (8.9):

	1850	-580	0
e_x	0.5	0.5	0.5
e_y	0.5	0.5	-0.5
e_s	-1	1	0

$\times 10^{-6}$

using m, n or invariants with θ = θ

giving

$$e_x = 635 \times 10^{-6}, \quad e_y = 635 \times 10^{-6}, \quad e_s = -2430 \times 10^{-6}$$

these are = because there is no shear and θ = 45°

Plies 3 and 6 at −45°

$$\begin{aligned} m = \cos -45° = 1/\sqrt{2} \quad &\text{and} \quad n = \sin -45° = -1/\sqrt{2} \\ m^2 = 0.5 \quad &\quad n^2 = 0.5 \\ mn = -0.5 \quad &\quad 2mn = -1 \\ m^2 - n^2 = 0 \quad & \end{aligned}$$

and from eqn. (8.9):

	1850	-580	0
e_x	0.5	0.5	-0.5
e_y	0.5	0.5	0.5
e_s	1	-1	0

$\times 10^{-6}$

giving

$$e_x = 635 \times 10^{-6}, \quad e_y = 635 \times 10^{-6}, \quad e_s = 2430 \times 10^{-6}$$

Plies 4 and 5 at 90°

The strains in the material axes can be obtained by inspection for a ply angle of 90° (see Example 8.1):

$$e_x = -580 \times 10^{-6}, \quad e_y = 1850 \times 10^{-6}, \quad e_s = 0$$

Next, we make use of eqn. (8.10) to determine the ply stresses in the material axes, having got the ply strains in the material axes. Substituting the values of the ply reduced stiffnesses given in the beginning of this example and converting the units to N and mm, and the ply strains in the material axes, for each ply into eqn. (8.10), we get:

Table 8.5

Ply Stresses (N/mm^2) and Failure Indices: $[0/45/-45/90]_s$, with $N_1 = 100$ N/mm; All Plies Intact

(Low R) or (high F.I.) is BAD!

Ply	θ^o	σ_x	σ_y	σ_s	*F.I.x*	*F.I.y*	*F.I.s*	*MOF*
1	0	259	-0.3	0	0.17	0.01	0	LT
2	45	91	8	-12	0.06	0.16	0.17	S
3	-45	91	8	12	0.06	0.16	0.17	S
4	90	-76	17	0	0.06	**0.34**	0	TT
5	90	-76	17	0	0.06	**0.34**	0	TT
6	-45	91	8	12	0.06	0.16	0.17	S
7	45	91	8	-12	0.06	0.16	0.17	S
8	0	259	-0.3	0	0.17	0.01	0	LT

Note that since the laminate is symmetric and only a membrane load is present, then the failure of the first ply is predicted simultaneously for the two 90° plies, Plies 4 and 5.

Second-ply-failure

$\left.\frac{100\ \text{N/mm}}{\text{F.I}}\right\}$ inverse of using R!

The FPF occurs at the load level of $N_1 = 294$ N/mm with failure being in the transverse directions of the 90° plies, plies 4 and 5. Although two plies have failed simultaneously, they are of the same ply configurations and hence the subsequent ply failure after the FPF is termed the second-ply-failure; this is merely an agreed terminology used in this book as mentioned earlier in Section 8.4.

Using the complete ply failure approach, we now assume that all the elastic values E_x, E_y and E_s for the two failed 90° plies are set to zero; the reduced and the transformed reduced stiffnesses for these failed plies will therefore be zero. The new assumed transformed reduced stiffness terms for all the plies in the laminate are given in Table 8.6.

The failed plies are, therefore, maintained at their positions, but with altered elastic values. This altered laminate configuration is still symmetric, and hence we only need the membrane stiffnesses and compliances. Using the transformed stiffness values from Table 8.6 and with a ply thickness of 0.125 mm, we get

$$
\begin{aligned}
A_{11} &= 2\{(0.125 \times 140.9)_{\text{Ply1}} + (0.125 \times 44.3)_{\text{Ply2}} + (0.125 \times 44.3)_{\text{Ply3}} \\
&\quad + (0.125 \times 0)_{\text{Ply4}}\} = 57.4 \text{ kN/mm} \\
A_{22} &= 2\{(0.125 \times 10.1)_{\text{Ply1}} + (0.125 \times 44.3)_{\text{Ply2}} + (0.125 \times 44.3)_{\text{Ply3}} \\
&\quad + (0.125 \times 0)_{\text{Ply4}}\} = 24.7 \text{ kN/mm} \\
A_{33} &= 2\{(0.125 \times 5.0)_{\text{Ply1}} + (0.125 \times 36.3)_{\text{Ply2}} + (0.125 \times 36.3)_{\text{Ply3}}
\end{aligned}
$$

	6615	-4792	0	
e_x	0.5	0.5	0.5	$\times 10^{-6}$
e_y	0.5	0.5	-0.5	
e_s	-1	0	0	

giving

$$e_x = 912 \times 10^{-6}, \quad e_y = 912 \times 10^{-6}, \quad e_s = -11\,407 \times 10^{-6}$$

Plies 3 and 6 at −45°

Using the appropriate trigonometric values for $\theta = -45^o$ obtained previously in this example when performing the intact plies analysis, we get

Failure criteria $\begin{cases} \text{on-axis stress} \\ \text{on-axis strain} \end{cases}$

	6615	-4792	0	
e_x	0.5	0.5	-0.5	$\times 10^{-6}$
e_y	0.5	0.5	0.5	
e_s	1	-1	0	

* off axis strain will still be = for all plies but we need to manually set the stress of the failed plies to zero.

giving

$$e_x = 912 \times 10^{-6}, \quad e_y = 912 \times 10^{-6}, \quad e_s = 11\,407 \times 10^{-6}$$

Plies 4 and 5 at 90°

Note that these are the failed plies. The strains in the material axes can be obtained by inspection for a ply angle of 90° (see Example 8.1):

$$e_x = -4792 \times 10^{-6}, \quad e_y = 6615 \times 10^{-6}, \quad e_s = 0$$

Next, we make use of eqn. (8.10) to determine the ply stresses in the material axes, having got the ply strains in the material axes. Substituting the values of the ply reduced stiffnesses for the intact plies giving in the beginning of this example (and converting the units to N and mm), and the ply strains in the material axes, for each ply, into eqn. (8.10), we get:

Plies 1 and 8 at 0°

	6615	-4792	0	
σ_x	140.9	3.0	0	$\times 10^{-6} \times 10^3$ N/mm^2
σ_y	3.0	10.1	0	
σ_s	0	0	5.0	

Using the maximum stress failure criterion:

$$
\begin{aligned}
\text{F.I.}x &= 131/1500 = 0.09 \\
\text{F.I.}y &= 12/50 = 0.24 \\
\text{F.I.}s &= 57/70 = 0.81
\end{aligned}
$$

Plies 4 and 5 at 90°

These are the failed plies, and assuming the complete ply failure method, the reduced stiffness and hence the transformed reduced stiffnesses are set to zero.

	-4792	6615	0
σ_x	0	0	0
σ_y	0	0	0
σ_s	0	0	0

$\times 10^{-6} \times 10^3$ N/mm^2

given

$$\sigma_x = \sigma_y = \sigma_s = 0$$

and hence the analytical method to model the failed plies whereby we get zero ply stress values implying the inability of the ply to carry any loads. A summary of the stress values (in the material axes), and their corresponding F.I. for all the eight plies in the laminate configuration $[0/45/-45/90]_s$ with the two failed 90° plies and $N_1 = 294$ N/mm is given in Table 8.7.

Table 8.7

Ply Stresses (N/mm^2) and Failure Indices: $[0/45/-45/90]_s$, with $N_1 = 294$ N/mm; Plies Completely Failed: 90°

Ply	θ_o	σ_x	σ_y	σ_s	*F.I.x*	*F.I.y*	*F.I.s*	*MOF*
1	0	918	-29	0	0.61	0.12	0	LT
2	45	131	12	-57	0.09	0.24	0.81	S
3	-45	131	12	57	0.09	0.24	0.81	S
4	90	0	0	0	0	0	0	
5	90	0	0	0	0	0	0	
6	-45	131	12	57	0.09	0.24	0.81	S
7	45	131	12	-57	0.09	0.24	0.81	S
8	0	918	-29	0	0.61	0.12	0	LT

note that Regar
Less of how ma
plies actually fa
we still call the ste
"first, second, thi

From the summary of results in Table 8.7, it is seen that the maximum failure index occurs in the ±45° plies (Plies 2, 3, 6 and 7) in the shear mode, $F.I.s = 0.81$. So, with the applied tensile load $N_1 = 294$ N/mm, no subsequent ply failure has yet occurred in this reduced stiffness laminate. Therefore, the load to cause the second-ply failure, by the maximum stress theory, in the $\pm45^o$ plies in the shear mode is

$$N_1 = 294/0.81 = 363 \text{ N/mm}$$

Third-ply-failure

So far then, a longitudinal tensile load $N_1 = 294$ N/mm causes the FPFs in the two 90^o plies in the TT mode; and second-ply failures occur in the two $+45^o$ and two -45^o plies in the shear mode at a load level of $N_1 = 363$ N/mm. We are thus left with two 0^o plies which have not yet failed. Using the complete ply failure approach, we now assume that all the elastic values E_x, E_y and E_s for the failed $\pm45^o$ plies are set to zero (in addition to all the elastic values of the already failed 90^o plies which are zero to zero); the transformed reduced stiffnesses for all these failed plies will, therefore, be zero. The new assumed transformed reduced stiffness terms for all the plies in the laminate are given in Table 8.8.

Table 8.8

$\bar{Q}_{ij}$ Values (kN/mm^2): $[0/45/-45/90]_s$: Plies Completely Failed: 90^o, $\pm45^o$

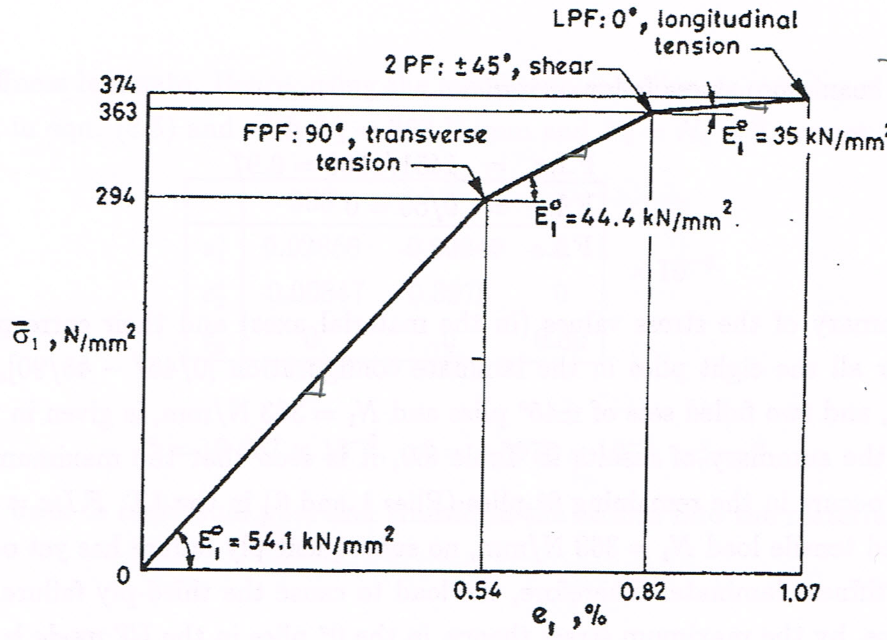
Ply	θ^o	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{33}$	$\bar{Q}_{12}$	$\bar{Q}_{13}$	$\bar{Q}_{23}$
1	0	140.9	10.1	5.0	3.0	0	0
2	45	0	0	0	0	0	0
3	-45	0	0	0	0	0	0
4	90	0	0	0	0	0	0
5	90	0	0	0	0	0	0
6	-45	0	0	0	0	0	0
7	45	0	0	0	0	0	0
8	0	140.9	10.1	5.0	3.0	0	0

The failed plies are, therefore, maintained at their positions, but with altered elastic values. This altered laminate configuration is still symmetric and hence we still only need the membrane stiffnesses and compliances. Using the transformed stiffness values from Table 8.8 and with a ply thickness of 0.125 mm, we get

$$\begin{aligned} A_{11} &= 2\ \{(0.125 \times 140.9)_{\text{Ply1}} + (0.125 \times 0)_{\text{Ply2}} + (0.125 \times 0)_{\text{Ply3}} \\ &\quad + (0.125 \times 0)_{\text{Ply4}}\} = 35.23 \text{ kN/mm} \\ A_{22} &= 2\ \{(0.125 \times 10.1)_{\text{Ply1}} + (0.125 \times 0)_{\text{Ply2}} + (0.125 \times 0)_{\text{Ply3}} \end{aligned}$$

* increase stiffness, place fibers along the loading directions.

progressive failure.



lower slope = lower stiffness

Figure 8.7: Longitudinal tensile strength by complete ply failure method and maximum stress theory: $[0/45/-45/90]_s$. * Remember: $\bar{\sigma}h = N_1 \cdots$

consider the average stress, $\bar{\sigma}_1$, rather than the force intensity N_1; in this way, we can obtain the Young's modulus value E_1^o directly to the force intensity by the laminate thickness: $\bar{\sigma}_1 = N_1/t$, and in this case the laminate thickness is 1.0, so $\bar{\sigma}_1 = N_1$. The corresponding strain at each ply failure is obtained from the relationship $e_1 = a_{11}N_1$ (see eqn. (8.5), with $N_2 = N_6 = 0$) in which the compliance a_{11} value is obtained from the respective calculations given earlier in this example; thus

$$\begin{aligned}
\text{FPF}: \; N_1 &= 294 \text{ N/mm}, \quad e_1 = 0.0185 \times 294 \times 10^{-3} = 0.54\% \\
\text{2PF}: \; N_1 &= 363 \text{ N/mm}, \quad e_1 = 0.0225 \times 363 \times 10^{-3} = 0.82\% \\
\text{LPF}: \; N_1 &= 374 \text{ N/mm}, \quad e_1 = 0.0286 \times 374 \times 10^{-3} = 1.07\%
\end{aligned}$$

Figure 8.7 shows the intact laminate until FPF in the two 90° plies in the TT mode, at which the slope of the graph drops to reflect the laminate reduced stiffness. Second-ply-failure occurs in the ±45° plies in the shear mode with a further reduction in the laminate stiffness, until the LPF occurs in the 0° plies in the LT mode. The first and second-ply-failures are matrix dominated ones, but the third-ply failure and the LPF are caused by the fibre failures; this phenomenon is usually observed in the majority of cases in that a matrix dominated failure does not cause the laminate failure, but a fibre MOF generally causes the laminate failure in that the rest of the plies in the laminate are not able to sustain any more loads.

We now check that the load of $N_1 = -455$ N/mm, which caused the previous ply failures, is still being resisted by the remaining 90° plies in the now further reduced stiffness laminate. Hence, using the compliances, converted into units of N and mm, in eqn. (8.5) and with $N_1 = -455$ N/mm and $N_2 = N_6 = 0$, we get

	-455	0	0
e_1^o	0.4029	-0.0092	0
e_2^o	-0.0092	0.0286	0
e_6^o	0	0	0.7692

$\times 10^{-3}$

giving

$$e_1^o = -183\ 320 \times 10^{-6}, \quad e_2^o = 4186 \times 10^{-6}, \quad e_6^o = 0$$

We then go back to individual plies and transform the strains into the material axes, using eqn. (8.9). We now only need to consider the unfailed 90° plies, Plies 4 and 5.

Plies 4 and 5 at 90°

The strains in the material axes can be obtained by inspection for a ply angle of 90° (see Example 8.1):

$$e_x = 4186 \times 10^{-6}, \quad e_y = -183\ 320 \times 10^{-6}, \quad e_s = 0$$

Next, we make use of eqn. (8.10) to determine the ply stresses in the material axes, having got the ply strains in the material axes. Substituting the values of the ply reduced stiffnesses for the intact plies given in the beginning of this example (and converting the units to N and mm), and the ply strains in the material axes, into eqn. (8.10), we get

	4186	-183 320	0
σ_x	140.9	3.0	0
σ_y	3.0	10.1	0
σ_s	0	0	5.0

$\times 10^{-6} \times 10^3$ N/mm^2

giving

$$\begin{aligned} \sigma_x &= 40 \text{ N/mm}^2 \\ \sigma_y &= -1839 \text{ N/mm}^2 \\ \sigma_s &= 0 \end{aligned}$$

Using the maximum stress failure criterion:

$$\begin{aligned} \text{F.I.}x &= 40/1500 = 0.03 \\ \text{F.I.}y &= 1839/250 = 7.36 \\ \text{F.I.}s &= 0 \end{aligned}$$

so order is : $N > 0$: tension : $90° \to 45° \to 0°$
$N < 0$: Compression : $0° \to 45° \to 90°$

→ direction of load maters, failure not gradual here

(fiber) FI > 1 means failed.

→ when you fail in 0°, you get a VERY Large loss in stiffness

Considered final failure and you don't continue with progressive approach.

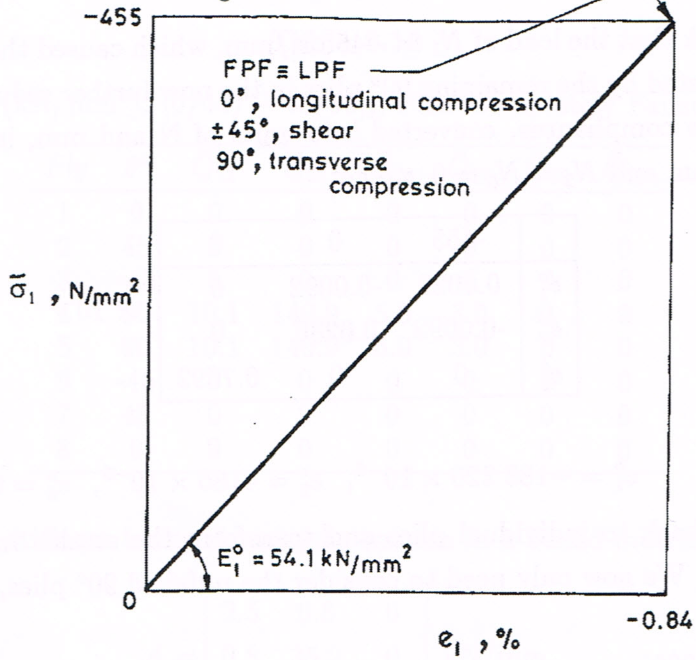


Figure 8.8: Longitudinal compressive strength by complete ply failure method and maximum stress theory: $[0/45/-45/90]_s$.

Failure is predicted in the 90° plies in the transverse direction in the compression mode. So, as expected, the same load level of $N_1 = -455$ N/mm which initiated the FPF and was not sustained by the ±45° plies, is also not sustained by the 90° plies. So the FPF load is effectively the LPF load.

We have thus calculated the LPF load for a longitudinal compressive load case in the symmetric quasi-isotropic laminate $[0/45/-45/90]_s$ by the complete ply failure approach and by using the maximum stress failure criterion. The laminate compressive strength, $\bar{\sigma}_1$, as obtained by the complete ply failure method and the maximum stress theory, is therefore the LPF load divided by the laminate thickness:

$$\bar{\sigma}_1 = N_1/t = -455/1.0 = -455 \text{ N/mm}^2$$

The average stress-strain, $\bar{\sigma}_1$ vs. e_1, graph is shown in Fig. 8.8. The corresponding strain at FPF, and in this case also the LPF, is obtained from the relationship $e_1 = a_{11}N_1$ (see eqn. (8.5), with $N_2 = N_6 = 0$) in which the compliance a_{11} value is obtained from the respective calculations given earlier in this example; thus,

$$\text{FPF} = \text{LPF} : N_1 = -455 \text{ N/mm}; \qquad e_1 = 0.0185 \times -455 \times 10^{-3} = -0.84\%$$

Figure 8.8 shows the intact laminate until FPF in the two 0° plies in the LC mode. The FPF load level is effectively the LPF as the reduced stiffness laminate is unable to sustain