Elaine D'mello

Be-IT

VANDERMONDE MATRIX

- ## What is a Vandermonde matrix.

In linear algebra, a **Vandermonde matrix**, named after **Alexandre-Théophile Vandermonde**, is a matrix with the terms of a geometric progression in each row, i.e., an $m \times n$ matrix.

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix},$$

or

$$V_{i,j} = \alpha_i^{j-1}$$

for all indices i and j. The **determinant** of a square Vandermonde matrix (where $m = n$) can be expressed as

$$\det(V) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i).$$

This is called the **Vandermonde determinant** or **Vandermonde polynomial.** If all the numbers α_i are distinct, then it is non-zero.

The Vandermonde determinant is sometimes called the discriminant, although many sources, including this article, refer to the discriminant as the square of this determinant. Note that the Vandermonde determinant is *alternating* in the entries, meaning that permuting the α_i by an **odd permutation** changes the sign, while permuting them by an **even permutation** does not change the value of the determinant. It thus depends on the order, while its square (the discriminant) does not depend on the order.

When two or more α_i are equal, the corresponding **polynomial interpolation** problem (see below) is **underdetermined**. In that case one may use a generalization called **confluent Vandermonde matrices**, which makes the matrix **non-singular** while retaining most properties. If $\alpha_i = \alpha_{i+1} = \ldots = \alpha_{i+k}$ and $\alpha_i \neq \alpha_{i-1}$, then the $(i+k)$th row is given by

$$V_{i+k,j} = \begin{cases} 0, & \text{if } j \leq k; \\ \frac{(j-1)!}{(j-k-1)!}\alpha_i^{j-k-1}, & \text{if } j > k. \end{cases}$$

A Vandermonde matrix is a type of matrix that arises in the **polynomial least squares fitting**, **Lagrange interpolating polynomials**, and the reconstruction of a **statistical distribution** from the distribution's moments

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

A Vandermonde matrix is sometimes also called an alternant matrix .The solution of an $n \times n$Vandermonde matrix equation requires $O(n^2)$ operations. The determinants of Vandermonde matrices have a particularly simple form.

- **How To solve simple example of a Vandermonde matrix**

We'll find the interpolating polynomial passing through the three points (1,-6), (2,2), (4,12), using the Vandermonde matrix.

For our polynomial, we'll take (1,-6)=(x0,y0),(2,2)=(x1,y1) ,(4,12)=(x2,y2) and .

Since we have 3 points, we can expect degree 2 polynomial.

So define our interpolating polynomial as:

$$p(x) = a_2 x^2 + a_1 x + a_0 .$$

So, to find the coefficients of our polynomial, we solve the system

$p(x_i) = y_i , i \in \{0, 1, 2\} .$

$$\begin{pmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{pmatrix} * \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

In order to solve the system, we will use an augmented matrix based on the Vandermonde matrix, and solve for the coefficients using **Gaussian elimination.** Substituting in our x and y values, our augmented matrix is:

$$\left(\begin{array}{ccc|c} 1^2 & 1 & 1 & -6 \\ 2^2 & 2 & 1 & 2 \\ 4^2 & 4 & 1 & 12 \end{array}\right)$$

Then, using Gaussian elimination,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 4 & 2 & 1 & 2 \\ 16 & 4 & 1 & 12 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 0 & -2 & -3 & 26 \\ 0 & -12 & -15 & 108 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 0 & -2 & -3 & 26 \\ 0 & 0 & 3 & -48 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 2 & 0 & -22 \\ 0 & 0 & 1 & -16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -16 \end{array}\right)$$

Our coefficients are $a_2 = -1$, $a_1 = 11$, and $a_0 = -16$. So, the interpolating polynomial is

$$p(x) = -x^2 + 11x - 16.$$

- **How do we add a point in a Vandermonde matrix Adding a point**

Now we add a point, $(3, -10) = (x_3, y_3)$, to our data set and find a new interpolation polynomial with this method.

Since we have 4 points, we will have degree 3 polynomial.

Thus Our polynomial is $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$,

and we get the coefficients by solving the system $p(x_i) = y_i$.

Constructing our augmented matrix as before and using Gaussian elimination, we get:

$$\left(\begin{array}{cccc|c} 1^3 & 1^2 & 1 & 1 & -6 \\ 2^3 & 2^2 & 2 & 1 & 2 \\ 4^3 & 4^2 & 4 & 1 & 12 \\ 3^3 & 3^2 & 3 & 1 & -10 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -6 \\ 0 & -4 & -6 & -7 & 50 \\ 0 & -48 & -60 & -63 & 396 \\ 0 & -18 & -24 & -26 & 152 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -6 \\ 0 & -4 & -6 & -7 & 50 \\ 0 & 0 & 12 & 21 & -204 \\ 0 & 0 & 3 & \frac{11}{12} & -73 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -6 \\ 0 & -4 & -6 & -7 & 50 \\ 0 & 0 & 12 & 21 & -204 \\ 0 & 0 & 0 & \frac{1}{4} & -22 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 82 \\ 0 & -4 & -6 & 0 & -566 \\ 0 & 0 & 12 & 0 & 1644 \\ 0 & 0 & 0 & 1 & -88 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -55 \\ 0 & -4 & 0 & 0 & 256 \\ 0 & 0 & 1 & 0 & 137 \\ 0 & 0 & 0 & 1 & -88 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -64 \\ 0 & 0 & 1 & 0 & 137 \\ 0 & 0 & 0 & 1 & -88 \end{array}\right)$$

Therefore, our polynomial is:

$$p(x) = 9x^3 - 64x^2 + 137x - 88.$$

- **How do we find inverse of a Vandermonde matrix**

ANALYSIS

The Vandermonde matrix A arises as the matrix of coefficients required to

evaluate the coefficients a_i in any polynomial approximation, as, for example,

$$y = a_0 + a_1 x + a_2 x^2 + \cdot \cdot \cdot + a_{n-1} x^{n-1} \tag{1}$$

to a function y given at the n distinct points $x_1, x_2, x_3, \cdot \cdot \cdot, x_n$. The matrix A has the form

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \tag{2}$$

If A^{-1} is known, the value of the coefficients is formally given by the expression

$$\left[a_0, a_1, a_2, \cdot \cdot \cdot, a_{n-1}\right] = A^{-1}\left[y_1, y_2, y_3, \cdot \cdot \cdot, y_n\right] \tag{3}$$

where the brackets denote column matrices and y_i is equal to $y(x_i)$.

A simple form of the inverse matrix A^{-1} is described in terms of the product $U^{-1}L^{-1}$, where U^{-1} is an upper triangular matrix and L^{-1} is a lower triangular matrix.

-- ---

The Vandermonde matrix A has the determinant equal to $\prod_{j>1} (x_j - x_i)$ (ref. 1, p. 9) and is nonsingular if all values of x_i are distinct. It can, therefore, be factored into a lower triangular matrix L and an upper triangular matrix U where A is equal to LU. The factorization is unique if no row or column interchanges are made and if it is specified that the diagonal elements of U are unity.

The upper triangular factor U and the inverse L^{-1} of the lower triangular factor L are developed in reference 2, but the authors were content to depend on the evaluation of the numerical values for the coefficients a_i of equation (1) by the solutions of the equation

$$U\left[a_0, a_1, a_2, \cdot \cdot \cdot, a_{n-1}\right] = L^{-1}\left[y_1, y_2, \cdot \cdot \cdot, y_n\right] \tag{4}$$

in each case.

It is seen in reference 2 that, with a translation to matrix notation, the elements ℓ_{ij} of L^{-1} are given by the relations

$$\left.\begin{aligned} \ell_{ij} &= 0 \qquad i < j \\ \ell_{11} &= 1 \\ \ell_{ij} &= \prod_{\substack{k=1 \\ k \neq j}}^{i} \frac{1}{x_j - x_k} \qquad \text{otherwise} \end{aligned}\right\} \tag{5}$$

The explicit form of L^{-1} for a few rows and columns is

The explicit form of L^{-1} for a few rows and columns is

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \dfrac{1}{x_1 - x_2} & \dfrac{1}{x_2 - x_1} & 0 & \cdots \\ \dfrac{1}{(x_1 - x_2)(x_1 - x_3)} & \dfrac{1}{(x_2 - x_1)(x_2 - x_3)} & \dfrac{1}{(x_3 - x_1)(x_3 - x_2)} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \tag{6}$$

It is asserted and proved herein that the elements u_{ij} of U^{-1} are given by the definition

$$\left.\begin{aligned} u_{ii} &= 1 \\ u_{i1} &= 0 \\ u_{ij} &= u_{i-1,j-1} - u_{i,j-1}x_{j-1} \quad \text{otherwise} \end{aligned}\right\} \tag{7}$$

where

$$u_{0j} = 0$$

The first few rows and columns of the asserted inverse U^{-1} are

3

$$U^{-1} = \begin{pmatrix} 1 & -x_1 & x_1x_2 & -x_1x_2x_3 & \cdots \\ 0 & 1 & -(x_1 + x_2) & x_1x_2 + x_2x_3 + x_3x_1 & \cdots \\ 0 & 0 & 1 & -(x_1 + x_2 + x_3) & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \tag{8}$$

It is noted that the j^{th} column of U^{-1} does not depend on x_j but only on $x_1, x_2, \cdots, x_{j-1}$. A proof that U^{-1} is as described by definition (7) is developed by showing that the product AU^{-1} is lower triangular and, therefore, equal to L.

By direct computation, this is true for the Vandermonde matrix involving the two coordinates x_1 and x_2:

It is noted that the j^{th} column of U^{-1} does not depend on x_j but only on $x_1, x_2, \cdots, x_{j-1}$. A proof that U^{-1} is as described by definition (7) is developed by showing that the product AU^{-1} is lower triangular and, therefore, equal to L.

By direct computation, this is true for the Vandermonde matrix involving the two coordinates x_1 and x_2:

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & x_2 - x_1 \end{pmatrix} \tag{9}$$

At this point, it can be assumed that, for the coordinates $x_1, x_2, x_3, \cdots, x_n$, the product of AU^{-1} is lower triangular, and the effect of adding the ordinate x_{n+1} and the corresponding rows and columns of A and U^{-1} can be considered. It is recalled that the n^{th} column of U^{-1} involves only the ordinates $x_1, x_2, x_3, \cdots, x_{n-1}$, and that the inner products of the rows $(1, x_i, x_i^2, \cdots)$ of A and column n of U^{-1} are zero except for the diagonal element, which has the value $(x_n - x_1)(x_n - x_2)(x_n - x_3)$. Because column $n + 1$ of U^{-1} defined by equation (7) is a linear combination of the eleumn of U^{-1} are all zero. It remains to show only that the element $p_{n,n+1}$ of the product matrix vanishes.

From the recursive definition of columns of U^{-1}, it is seen that the $n+1$ column can be represented as the weighted sum of the two column matrices

$\left[0, u_{1n}, u_{2n}, \cdot \cdot \cdot, u_{nn}\right] - x_n\left[u_{1n}, u_{2n}, u_{3n}, \cdot \cdot \cdot, u_{nn}, 0\right]$. The inner product of the column by the n^{th} row of A, which is $\left[1, x_n, x_n^2, \cdot \cdot \cdot, x_n^n\right]$, produces two terms that

(of U^{-1}) is zero. Therefore, the product AU^{-1} of the augmented matrix is lower triangular, which was to have been proved.

- **What are the applications of Vandermonde matrix**

Because the Vandermonde matrix arises from the basic problem of passing a polynomial function through a given set of data, the results may be thought of as the approximation of a function $y(x)$ by a polynomial $\hat{y}(x)$. To the extent that $\hat{y}(x)$ is a reasonable approximation of $y(x)$, it is possible to approximate linear transformations of $y(x)$ by the corresponding transformation of $\hat{y}(x)$.

Most of the classical formulas for numerical integration, interpolation, or differentiation can be generated directly. Exceptions are cases such as Gaussian and Chebysheff integrations, which require that the ordinates be especially selected by other considerations.

Few examples of standard results will be displayed, but the techniques of their generation should be clear from the special cases. Two cases are considered by which special formulas of high accuracy may be generated.

Formulas for integration of products of functions and other related linear transforms may be developed as follows: Consider an integral or other linear transform of the product $y(x)f(x)$, which is herein designated as $T\left[y(x)f(x)\right]$. The coefficients of the approximation to $y(x)$, namely,

$$\hat{y}(x) = a_0 + a_1x + a_2x^2 + \ldots$$

are given by the relation

$$\left[a_0, a_1, a_2, \ldots\right] = U^{-1}L^{-1}\left[y(x_1), y(x_2), y(x_3), \ldots\right] \tag{12}$$

Then, if it is possible to develop suitably computable expressions for $T(x^nf(x))$, the transform $T(\hat{y}(x)f(x))$ is given by

$$\left(T(f(x)), T(xf(x)), \ldots\right)U^{-1}L^{-1}\left[y_1, y_2, y_3, \ldots\right]$$

and, if $y(x)$ is very nearly a polynomial, $T(y(x)f(x))$ is reasonably approximated by $T(\hat{y}(x)f(x))$.

Because the matrices U^{-1} and L^{-1} exist, an array of Lagrangian coefficients may be computed by the evaluation of

$$\left(T(f(x)), T(xf(x)), T(x^2f(x)), \ldots\right)U^{-1}L^{-1} \tag{13}$$

independently of the actual values of $y(x_i)$.

A second situation arises when $y(x)$ is known, perhaps for analytical reasons, to be expressible by $y(x) = p(x)f(x)$, where $p(x)$ is a polynomial and $f(x)$ is some known function. For example, $f(x)$ and $x^nf(x)$ may be Lesbegue integrable but cannot be accurately approximated by a polynomial. In this case, $y(x)$ is approximated by

$$\hat{y}(x) = \left(a_0 + a_1x + a_2x^2 + \ldots\right)f(x) \tag{14}$$

and the coefficients a_i are computable by the relation

$$\left[a_0,\ a_1,\ a_2,\ \cdot\cdot\cdot\right] = U^{-1}L^{-1}\left[\frac{y(x_1)}{f(x_1)},\ \frac{y(x_2)}{f(x_2)},\ \cdot\cdot\cdot\right] \tag{15}$$

The desired Lagrangian coefficients to compute an approximate transform are generated by evaluating

$$\Big(T(f(x)),\ T(xf(x)),\ \cdot\cdot\cdot\Big)\,U^{-1}L^{-1}\left\{\frac{1}{f(x_1)},\ \frac{1}{f(x_2)},\ \frac{1}{f(x_3)},\ \cdot\cdot\cdot\right\} \tag{16}$$

where the braces denote a diagonal matrix. The resulting matrix of coefficients operates directly on the data

$$\left[y_1,\ y_2,\ y_3,\ \cdot\cdot\cdot\right]$$

REFRENCES :

https://en.wikipedia.org/wiki/**Vandermonde_matrix**
https://proofwiki.org/wiki/**Vandermonde**_Determinan
https://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19660023042.pdf
https://ece.uwaterloo.ca/~dwharder/NumericalAnalysis/05Interpolation/**vandermonde**/