- Some terms and concepts are directly cited from the book "Statistics".

- Important probability concepts you need to know throughout this semester.
  - probability properties
  - addition rule
  - multiplication rule
  - conditional probability
  - Independence

- Notations to help you take notes.

- usually we write $P(A)$ to represent the probability of A event. for example, P(toss a coin and get a head) means the probability of tossing a coin and get a head.

- $A \cap B$ means the event that A and B both happen. For example, (A student is a math major) $\cap$ (a student takes STAT 301) means the event that a student who is math major and also takes STAT 301.

- $A \cup B$ means the event that A happens or B happens or they both happen.
  For example, roll a die. (getting a multiple of 2) $\cup$ (getting a multiple of 3) = (getting 2, 3, 4, or 6)

▸ A^c means the opposite of event A. For example, (rolling a die, and get a multiple of 3)^c means (rolling a die and NOT to get a multiple of 3), is equivalent to (getting a 1 or 2 or 4 or 5)

- Probability Properties
  - The chance of something gives the percentage of time it is expected to happen, when the basic process is done over and over again, independently and under the same condition.

    : If something is impossible, its probability is 0%
    If something is sure to happen, its probability is 100%

    : Probabilities / Chances are between 0% and 100%
    OR in above notation. $0\% \le P(A) \le 100\%$, A is an event.

  - The chance of something equals 100% minus the chance of opposite thing. In previous notation, $P(A^c) = 100\% - P(A)$, for A is an event.

■ Equally likely outcomes and counting in probability.

: Equally likely events are events that have the same theoretical probability of occuring.

Example: Each numeral on a die is equally likely to occur when a die is tossed.

Example: For a fair coin, you have equal chance of getting a head or tail.

$$P(\text{outcome}) = \frac{1}{\text{number of outcomes}}$$

$$P(\text{event}) = \frac{\text{number of outcome in event}}{\text{number of outcomes}}$$

Example:

You roll one fair six-sided die, with the number {1,2,3,4,5,6} on its faces. Let the event E = rolling a number that is at least five, then the probability of it is

$$\frac{2}{6} \quad \begin{matrix} \{5, 6\} \\ \{1, 2, 3, 4, 5, 6\} \end{matrix}$$

Example:

A pair of fair six-sided dice are rolled and the pair of numbers on the uppermost face is observed. Let E be the event that the numbers observed added to 7.

All the possible outcomes:

$$\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$$
$$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$$
$$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$$
$$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$$
$$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6) \} \Rightarrow 36 \text{ outcomes}$$

and all the outcomes added to 7:

$$\{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \} \Rightarrow 6 \text{ outcomes}$$

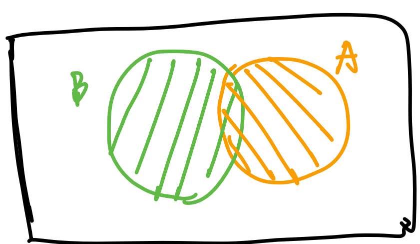
So the probability of getting a pair added to 7 is $\frac{6}{36}$

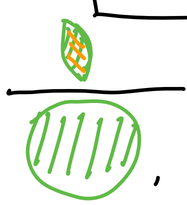
• Conditional probability

- Given two events A and B, the conditional probability of A given B, with the notation $P(A|B)$, is the probability of A occuring if B has or is assumed to have happened.

  By mathematical formulation, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

  By graphing,



$P(A|B) =$  , by conditioning, we shrink our sight to the space of B happening, and we are asking under this condition, the chance that A happens.

- If $P(A|B) = P(A)$ / $P(B|A) = P(B)$ / $P(A \cap B) = P(A) \cdot P(B)$ then A and B events are independent.

Example:

	Have pets	Do not have pets	Total
Biology major	0.41	0.08	0.49
Non-Bio major	0.45	0.06	0.51
Total	0.86	0.14	1

the above table means the probability that someone is a biology major AND has pets is 0.41.

- Compute the conditional probability that someone has pets given he or she is a biology major.

$$P(\text{have pets} \mid \text{Biology major})$$

$$= \frac{P(\text{have pets and Bio-major})}{P(\text{Bio-major})}$$

$$= \frac{P(\text{have pets and Bio-major})}{P(\text{have pets and Bio-major}) + P(\text{do not have pets and Bio-major})}$$

$$= \frac{0.41}{0.49}$$

- Compute the conditional probability that someone is not a Biology-major, given he or she has pets.

	Have pets	Do not have pets	Total
Biology major	0.41	0.08	0.49
Non-Bio major	0.45	0.06	0.51
Total	0.86	0.14	1

$$P(\text{not a Bio-major} \mid \text{has pets})$$

$$= \frac{P(\text{not a Bio-major and has pets})}{P(\text{has pets})}$$

$$= \frac{0.45}{0.86}$$

- Being a Bio-major and having pets, are these two independent?

$P(\text{having pets and Bio-major}) = 0.41.$

$P(\text{Bio-major}) = 0.49$

$P(\text{having pets}) = 0.86$

Since $0.41 \neq 0.49 * 0.86$, so not independent.

• Multiplication Rule

▶ The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given the first has happened.

Example:

A deck of cards is shuffled, and two cards are dealt. what is the chance that both are aces?

Solution:

The chance that the first card is an ace equals $\frac{4}{52}$. Given that the first card is an ace, there are 3 aces among the 51 remaining cards. So the chance of a second ace equals $\frac{3}{51}$. By multiplication rule, the chance that both cards are aces equals

$$\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}.$$

• The Addition Rule

- Two things are mutually exclusive when the occurence of one prevents the occurence of the other.

- To find the chance that at least one of two things will happen, check to see if they are mutually exclusive. If they are, add the chance.

Example:

A card is dealt off the top of a well-shuffled deck. There is 1 chance in 4 for it to be a heart. There is 1 chance in 4 for it to be a spade. What is the chance for it to be in a major suit (hearts or spades)?

Solution:

Being a heart and being a spade are mutually exclusive. By addition rule, the probability in a major suit is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Exercises with Solutions

- A coin is tossed 3 times

(a) what is the chance of getting 3 heads?

Solution:

this one tests multiplication rule.

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

(b) what is the chance of not getting 3 heads?

Solution:

this tests probability property.

$$100\% - \frac{1}{8} = 7/8$$

(c) what is the chance of getting at least 1 tail?

Solution:

this tests probability property and multiplication Rule.

the chance of getting at least 1 tail

= 100% − not getting any tail

= 100% − all heads

= 100% − 1/8 = 7/8

■ Two tickets are drawn at random without replacement from the box $\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}$

(a) What is the chance that the second ticket is 4?

Solution:

Since it is drawing without replacement, to have the second ticket being 4, you can only draw $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ for the first and draw $\boxed{4}$ as the second one.

the probability of getting $\boxed{1}$ or $\boxed{2}$ or $\boxed{3}$ for the first ticket is $\frac{3}{4}$, and after you take one ticket out, for the second draw, you have 3 remaining ticket, and one of them is 4. So the answer is

$$\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$$

(b) What is the chance that the second ticket is 4, given the first one is 2?

Solution:

$$P(\text{second is } 4 \mid \text{first one is } 2)$$

$$= \frac{P(\text{second is } 4 \text{ and first is } 2)}{P(\text{first is } 2)}$$

$$= \frac{\frac{1}{4} \cdot \frac{1}{3}}{\frac{1}{4}} = \frac{1}{3}$$

■ What is the probability that in a randomly chosen year your birthday is on a weekend?

Solution:

Your Birthday is equally likely to be on Monday, Tuesday ---- Sunday. Thus the answer is $\frac{2}{7} = 28.6\%$

■ Imagine a bag containing 10 balls: 3 Green, 4 Blue, 2 Yellow, and 1 Red

If we draw 2 balls from the bag, with replacement, what is the probability the first ball is either Red or Blue and the second ball is either Green or Yellow?

Solution:

Notice that each color is mutually exclusive.
By addition rule, the chance of first ball being either Red or Blue is $\frac{1+4}{10} = \frac{5}{10}$.

By addition rule, the chance of second ball being either Green or Yellow is $\frac{3+2}{10} = \frac{5}{10}$.

By multiplication rule, the first being Red or Blue and the second being Green or Yellow is $\frac{5}{10} \times \frac{5}{10} = \frac{1}{4}$

■ A deck is shuffled and two cards are dealt.

(a) Find the chance that the second card is a heart given the first card is a heart.

solution:

If the first one is a heart, then remaining deck has $(13-1)=12$ hearts. Also, after the first card is dealt, we only have $(52-1)=51$ remaining cards, thus the answer is $\frac{12}{51}$

(b) Find the chance that the first card is heart and the second one is a heart.

By multiplication rule, it is

$$\frac{13}{52} \times \frac{12}{51} = \frac{1}{17} \approx 0.0588$$

■ A deck is shuffled and three cards are dealt.

(a) Find the chance that the first card is a king.

Solution:

Out of 52 cards, there are 4 kings.
Assuming that the chance of getting each card is equally likely.
the answer is $\frac{4}{52}$.

(b) Find the chance that the first card is a king, the second is a queen, and the third is a jack.

Solution:

$$\frac{4}{52} \times \frac{4}{51} \times \frac{4}{50}$$

■ A coin is tossed six times. Two possible sequences of results are

(i) H T T H T H (ii) H H H H H H

H=Heads, T=Tails, which sequence is more likely?

Solution:

the probabilities of (i) and (ii) are both $\left(\frac{1}{2}\right)^6$, thus they are the same.

- Which of the two options is better, or are they the same?

(i) You toss a coin 100 times. On each toss, if the coin lands heads, you win \$1, If it lands tails, you lose \$1.

(ii) You draw 100 times at random with replacement from $\boxed{\boxed{1}\ \boxed{0}}$, On each draw, you are paid the number on the ticket.

Solution:

For (i), on each toss, you have 50% chance of getting a head, and 50% chance of getting a tail. Therefore, for each toss, your estimated gain of money is $50\% \times 1 + 50\% \times (-1) = 0$, thus for 100 tosses you are estimated to gain $0 \times 100 = 0$ dollars.

For (ii), Each draw you have 50% of winning 1 dollar and 50% of getting 0 dollar. This gives you an estimated gain of $50\% \times 1 + 50\% \times 0 = 0.5$.

The estimated gain for 100 tosses would be $0.5 \times 100 = 50$ dollars. (ii) is better.

[redacted] Three cards are dealt from a well-shuffled deck.

(a) Find the chance that all of the cards are diamonds.

Solution: $\frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} = 0.0129 = 1.29\%$

(b) Find the chance that none of the cards are diamonds

Solution: $\frac{39}{52} \times \frac{38}{51} \times \frac{37}{50} = 0.4135 = 41.35\%$

(c) Find the chance that the cards are not all diamonds.

$$
\begin{aligned}
P(\text{Not all Diamonds}) &= 100\% - P(\text{All Diamonds}) \\
&= 100\% - 1.29\% \\
&= 98.71\%
\end{aligned}
$$

■ Four draws are going to be made at random with replacement from the box $\boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{3}$

Find the chance that $\boxed{2}$ is drawn at least once.

Solution:

P($\boxed{2}$ drawn at least once) = 100% − P($\boxed{2}$ never drawn)

and P($\boxed{2}$ never drawn) = Four draws only contain $\boxed{1}$ or $\boxed{3}$.

thus, P($\boxed{2}$ never drawn) $= \left(\frac{3}{5}\right)^4$

and the answer is $1-\left(\frac{3}{5}\right)^4 = 87\%$

■ One ticket will be drawn at random from each of the two boxes shown below:

(A) $\boxed{1}\boxed{2}\boxed{3}$ (B) $\boxed{1}\boxed{2}\boxed{3}\boxed{4}$

Find the chance that the number drawn from A is larger than that from B.

Solution:

For the number from A larger than B. We could have the following combinations.

$(A,B) = \{(2,1), (3,1), (3,2)\}$

Each combination's probability is $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$, thus the probability of the draw from A larger than B is $3 \times \frac{1}{12} = 25\%$

• Exercises from discussion handout

■ You have a large toy box. Inside, there are 30 action figures, 14 cars, and 10 stuffed animals.

(a) what is the chance of choosing an action figure if drawing with replacement?

Solution:

Suppose each toy is equally likely to be drawn.

then, $\frac{30}{30+14+10} = \frac{30}{54} = 55.6\%$

(b) Suppose you draw two toys with replacement. What is the chance that you get two stuffed animals in a row?

Solution:

multiplication rule, each draw the chance of getting a stuffed animal is $\frac{10}{54}$, so two draws is $\frac{10}{54} \times \frac{10}{54} = 3.43\%$

(c) Suppose you draw three toys without replacement. What is the chance you first get a stuffed animal, then action figure, and finally another stuffed animal?

Solution:

$\frac{10}{54} \times \frac{30}{53} \times \frac{9}{52} = 1.8\%$

- A large airport services only two airlines, A and B. They have kept records of flight delays for a long period of time. Based on past data, if a flight is chosen at random from the flights in a given day. the probabilities of some events are below.

Event	Probability
The flight is from Airline A	40%
The flight is from Airline A and delayed	20%
The flight is from Airline B and not delayed	35%
The flight is delayed	45%

(a) create a two way table to represent the situation.

Solution:

We can first compute the following.

P(Airline A and not delayed) = P(Airline A) − P(Airline A and delayed)

= 40% − 20%

= 20%

P(Airline B) = 100% − P(Airline A) = 60%

$P(\text{Airline B and delayed}) = P(\text{Airline B}) - P(\text{Airline B and not delayed})$
$= 60\% - 35\%$
$= 25\%$

$P(\text{delayed}) = P(\text{Airline A and delayed}) + P(\text{Airline B and delayed})$
$= 20\% + 25\%$
$= 45\%$

$P(\text{not delayed}) = P(\text{Airline A and not delayed}) + P(\text{Airline B and not delayed})$
$= 20\% + 35\%$
$= 55\%$

thus we could have the table

	A	B	Total
delayed	20%	25%	45%
not delayed	20%	35%	55%
total	40%	60%	100%

(b) If you know the flight is from Airline A, what is the probability that it is delayed?

Solution:

$$P(\text{delayed} \mid A) = \frac{P(\text{A and delayed})}{P(A)} = \frac{20\%}{40\%} = 50\%$$

(c) If you know the flight is from Airline B, what is the probability that it is delayed?

Solution:

$$P(\text{delayed} \mid B) = \frac{P(B \text{ and delayed})}{P(B)} = \frac{25\%}{60\%} = 41.7\%$$

(d) If you know the flight is delayed, what is the probability that it is from A?

Solution:

$$P(A \mid \text{delayed}) = \frac{P(A \text{ and delayed})}{P(\text{delayed})}$$

$$= \frac{20\%}{45\%} = 44.4\%$$

(e) Are airline and delay status independent?

Solution 1:

P(Airline A) = 40%

P(A | delayed) = 44.4%

since P(Airline A) $\neq$ P(Airline A | delayed)

Not independent.

Solution.

P(Airline A) = 40%

P(delayed) = 45%

P(Airline A and delayed) = 20%

since P(Airline A and delayed) $\neq$ P(A) X P(delayed)

Not independent.

■ A die is rolled 10 times

(a) the chance of getting 10 sixes

$(\frac{1}{6}) \times (\frac{1}{6}) \times \cdots (\frac{1}{6}) = (\frac{1}{6})^{10}$

(b) the chance of not getting 10 sixes

$1 - (\frac{1}{6})^{10}$

(c) all the rolls showing 5 or fewer spots

Each roll you can get {1, 2, 3, 4, 5}

so for 10 rolls,

$(\frac{5}{6})^{10}$