Introduction

- If you know the distribution of a variable, then under specific condition, you can know how likely or unlikely that this variable is of certain value.

- Normal distribution is one of the commonly-used distribution. You can use normal distribution if you know that certain variable, with some scientific knowledge, follows normal distribution.
  OR
  If you do not know the population distribution, by central limit theorem, which we would talk about in later chapter, you are able to use normal distribution to approximate certain statistical quantities.

- Things to know in this chapter
  - features of normal curve
  - How to check normal curve table
  - How to standardize a quantity

• Features of Standard Normal Curve

Feature 1:

Normal curve is symmetric around the mean / average. Standard normal curve is a normal curve with average 0 and standard deviation 1, thus, standard normal distribution is symmetric around 0

Feature 2:

The area under normal curve has the interpretation of probability, this is the same as the interpretation of area of density histogram. As a result, the whole area under normal curve equals 100%.

Feature 3:

Some rules of thumb to remember:

- the area under standard normal curve between -1 and +1 is about 68%.
- the area under standard normal curve between -2 and +2 is about 95%.
- the area under standard normal curve between -3 and +3 is about 99.7%

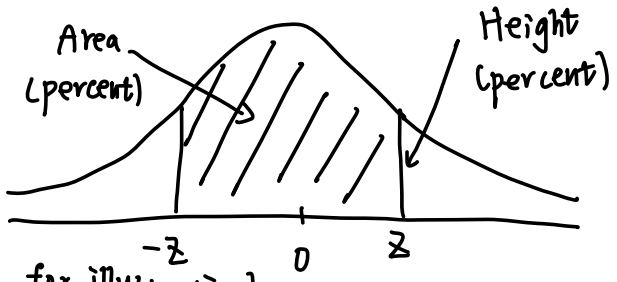
Standardization

- A value is converted to standard units by seeing how many standard deviations it is below or above the average. The value you get after standardization is called z-score.

Check the normal curve table

- the important techniques are related to the features of normal curve. Usually it is recommended to graph your desired area first and check the table. It's important to know that you might see different versions of tables. Pay attention to the shaded area being labeled.

version 1 from the textbook



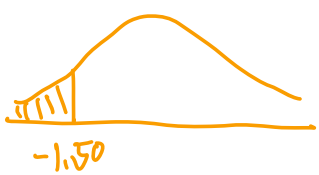
(Not complete, for illustration)

z	Area	z	Area	z	Area
0.00	0	1.50*	86.64	3.00	99.73
0.05	3.99	1.55	87.89	3.05	99.771
0.10	7.97	1.60	89.04	3.10	99.806
0.15	11.92	1.65	90.11	3.15	99.837
0.20	15.85	1.70	91.09	3.20	99.863
0.25	19.74	1.75	91.99	3.25	99.885
0.30	23.58	1.80	92.81	3.30	99.903
0.35	27.37	1.85	93.57	3.35	99.919
0.40	31.08	1.90	94.26	3.40	99.933
0.45	34.73	1.95	94.88	3.45	99.944
...	...	...	...	...	...

Question:

Based on the above table, find the area under the curve below -1.5.

Solution: see *, our desired area is

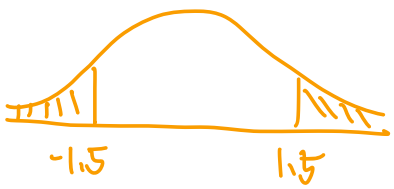


and the table gives that the area

is 86.64%

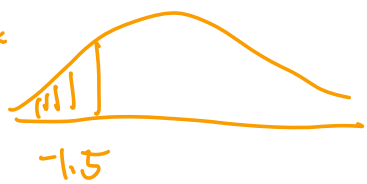
We know that the whole area under the curve is 100%,

So, $100\% - 86.64\% = 13.36\%$



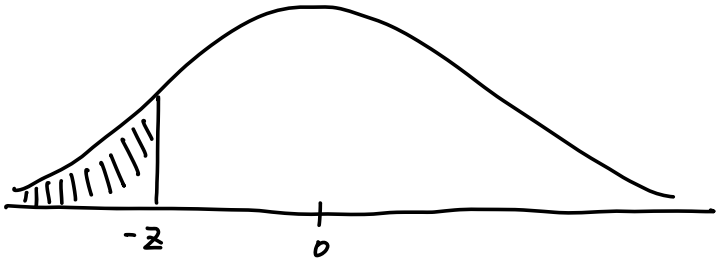
is 13.36%.

Also, we know that normal curve is symmetric around 0, so the left and right shaded areas are in the same size.



$= 13.36\% \div 2 = 6.68\%$

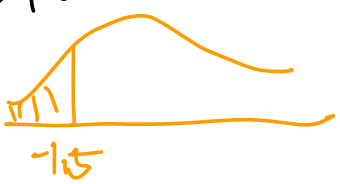
version 2 : Table for Negative Z scores (Not complete, for illustration)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427*	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537*	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668*	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823

Question: Find the area under the curve below -1.5.

Solution: see *, based on this table



is 0.0668, which is 6.68%, and we get the answer directly. Notice our answer is the same as previously we use version 1 table.

Question: Find the area under the curve between -1.72 and 1.61.

Solution: the desired area is 

there are several methods to find it.

Method 1: 

For 

, It is equivalent to

= 

By symmetry

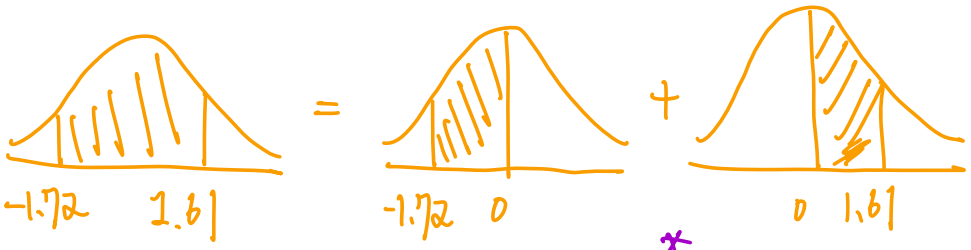
$= 100\% - 5.37\% = 94.63\%$

For 

, directly find to be 4.27%

so the answer is $94.63\% - 4.27\% = \underline{90.36\%}$

Method 2:



For

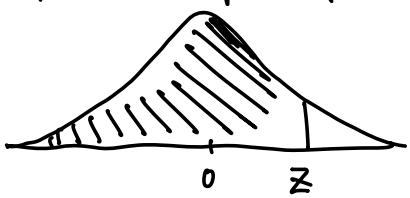
$= 50\% - 4.27\% = 45.73\%$

For

$= 50\% - 5.37\% = 44.63\%$

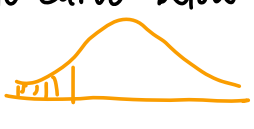
thus the final answer is $45.73\% + 44.63\% = 90.36\%$

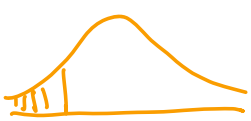
Version 3: Table for positive Z score (Not complete)



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224

Question: what is the area under the curve below -0.5 ?

Solution: the desired area is  ,

Method 1: 

= 

By symmetry

= 

- 

$= 100\% - 69.15\%$

$= 30.85\%$

• Exercises with Solutions

■ On a certain exam, the average of the score was 50 and the standard deviation was 10.

(a) Convert each of the following scores to standard units.
60, 45, 75.

Solution:

- $\frac{60-50}{10} = 1$. Thus, the standard unit of 60 is 1, it means 60 is 1 standard deviation more than the average.
- $\frac{45-50}{10} = -0.5$. Thus the standard unit of 45 is -0.5, it means 45 is 0.5 SD less than the average.
- $\frac{75-50}{10} = 2.5$, the standard unit of 75 is 2.5.

(b) Find the scores which in standard units are: 0, +1.5, -2.8

Solution:

- Standard unit being 0 means you are the average, thus the raw score is 50.
- the raw score of standard unit +1.5 is $50 + 1.5 \times 10 = 65$

- The raw score of standard unit -2.8 is
  $50 - 2.8 \times 10 = 22$.

■ Find the area under the normal curve

(a) To the right of 1.25

10.56%

(b) To the left of -0.40

34.46%

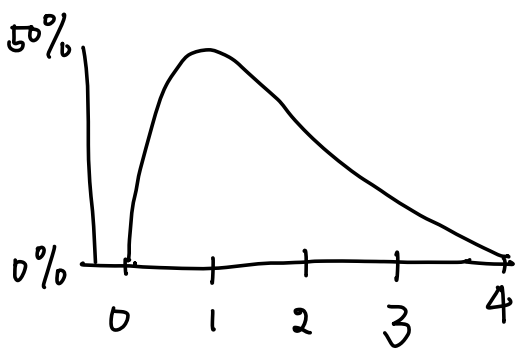
(c) To the left of 0.80

78.81%

(d) between 0.40 and 1.30

24.78%

■ A certain curve (not the normal) is sketched below. The total area under it is 100%, and the area between 0 and 1 is 39%

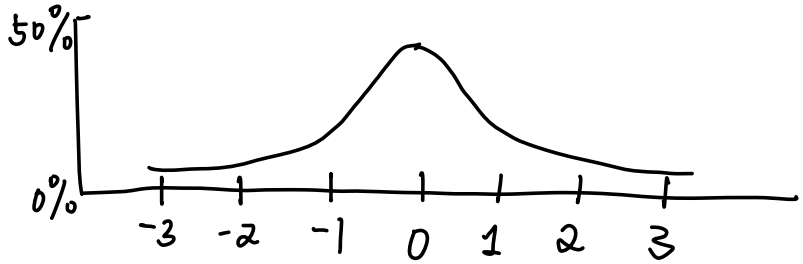


(a) If possible, find the area to the right of 1.

Solution: 100% - 39% = 61%

(b) If possible, find the area between 0 and 0.5 Not possible

■ A certain curve (not the normal) is sketched below. It is symmetric around 0, and the total area under it is 100%. The area between -1 and 1 is 58%



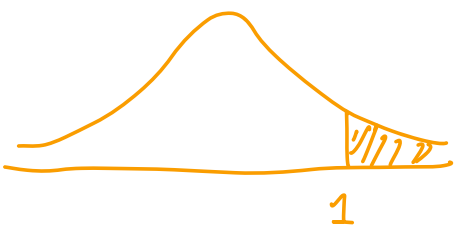
(a) If possible, find the area between 0 and 1.

Solution: 58% ÷ 2 = 29%

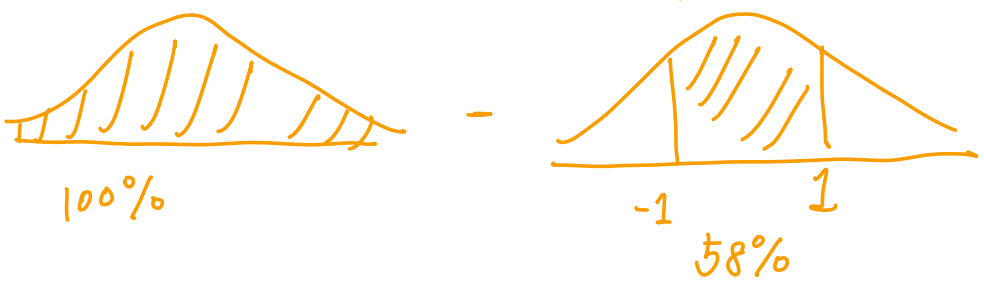
(b) If possible, find the area to the right of 1.

Solution:

the desired area is



and that is the same as one half of



thus is

(100% - 58%) ÷ 2 = 21%

(c) If possible, find the area to the right of 2.

Not possible

- Assume the height of sixteen-year-old witches in Great Britain are normally distributed with a mean of 66 inches and a standard deviation of 2.5 inches.

  - The first witch has a height of exactly 66 inches. Calculate the z score of her height. What percent of witches do we expect to be taller / shorter than her?

    Solution:

    $\frac{66-66}{2.5} = 0$, therefore, the z score is 0.

    If the z score is 0, then we have 50% of witches to be taller / shorter than her, because standard normal curve is symmetric around 0.

  - The second witch is 62 inches tall. Calculate her z score. What percent of sixteen-year-old witches do we expect to be taller than her?

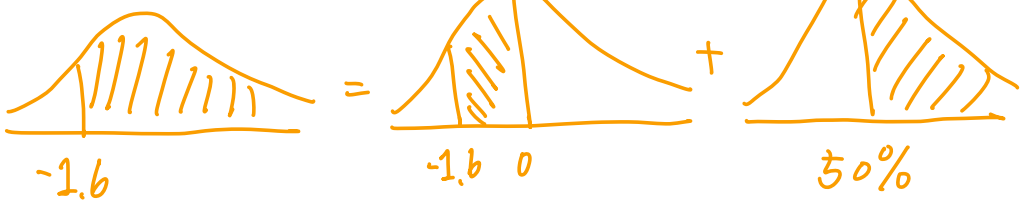
    Solution:

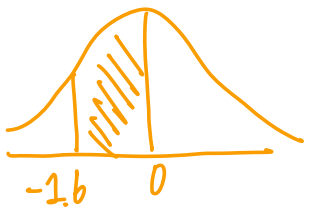
    $\frac{62-66}{2.5} = -1.6$, the z score is -1.6.

    the desired area is 

    and to find the desired area, you can use different version of tables.

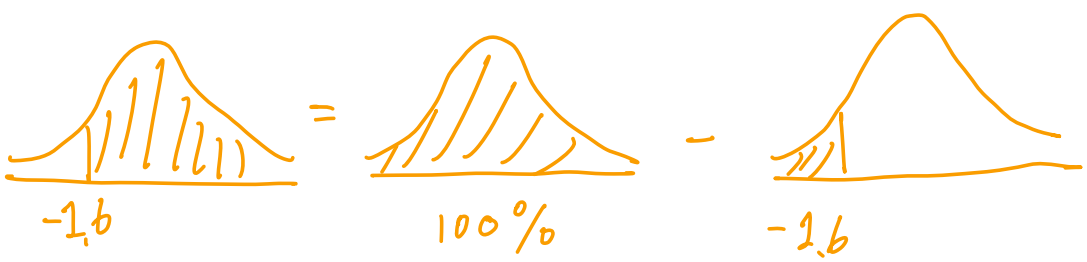
version 1 table method. —



and  is one half of 

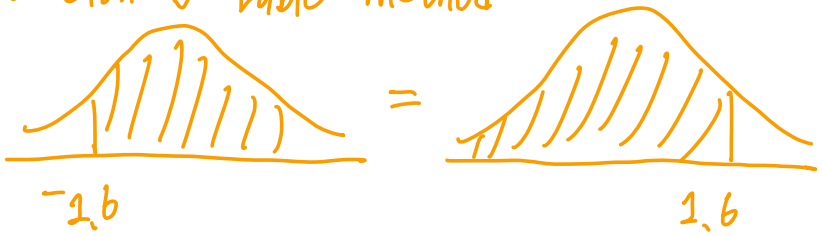
thus the answer is $89.04\% \div 2 + 50\% = 94.52\%$

version 2 table method —



$= 100\% - 5.48\% = 94.52\%$

version 3 table method —



By symmetry

$= 94.52\%$ from the table

- The third witch is 67.5 inches tall. Calculate her z score. What percent do we expect to be shorter than her?

Solution:

$$\frac{67.5-66}{2.5} = 0.6.$$

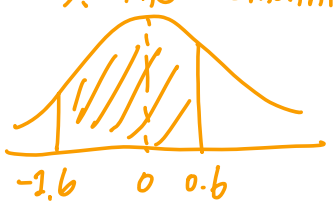
0.6 is her z score.

The way to get the percentage is similar to previous problem, the answer is 72.57%

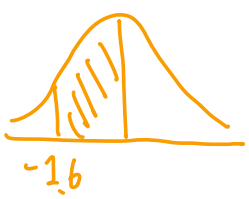
- What percent of sixteen-year-old witches do we expect to be between the second and third witch in height?

Solution:

We've got the z-scores. The remaining task is find the area of



version 1 table method —

and 

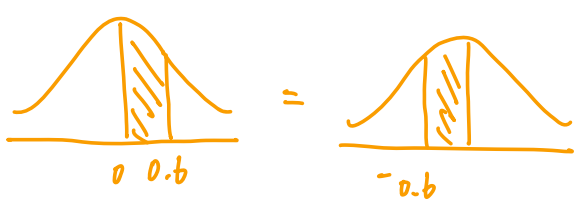
= one half of 

= one half of 

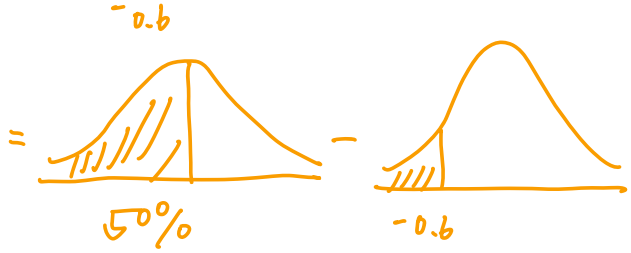
thus, the final answer is $89.04\% \div 2 + 45.15\% \div 2 = 67.095\%$

version 2 table method —

$= 50\% - 5.48\% = 44.52\%$



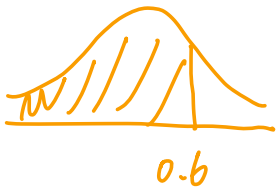
By symmetry



$= 50\% - 27.43\% = 22.57\%$

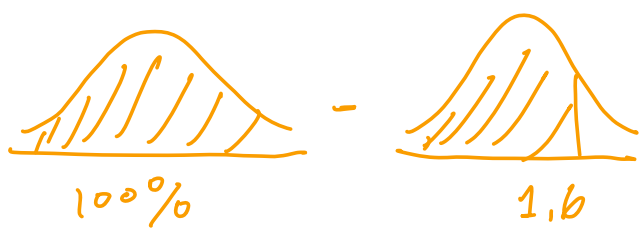
thus, the final answer is $44.52\% + 22.57\% = 67.09\%$

version 3 table method —



= 72.57%

By symmetry



= 100% − 94.52% = 5.48%

thus the final answer is

72.57% − 5.48% = 67.09%

- What percentage of sixteen-year-old witches do we expect to be between 63.5 and 68.5 in height?

Solution:

the z score of raw height 63.5 is

$$\frac{63.5-66}{2.5} = -1.$$

the z score of raw height 68.5 is

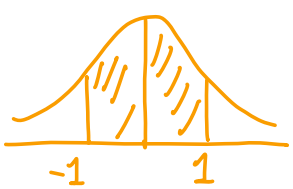
$$\frac{68.5-66}{2.5} = 1.$$

then we need to look for the area 

If you remember from rules of thumb, the area is 68%

the area can be directly found by version 1 table, which is 68.27% precisely.

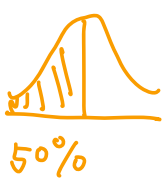
version 2 table method —



= double of 

By Symmetry

and 

= 

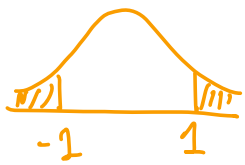
- 

$= 50\% - 15.87\%$

$= 34.13\%$

thus the final answer is $34.13\% \times 2 = 68.26\%$

version 3 table method —

and 

is double of 

$= 100\% - 84.13\%$

$= 15.87\%$

thus, the final answer is

$100\% - 15.87\% \times 2 = \underline{68.26\%}$

■ Similar question as previous one. Final answers provided. Check if you are right.

Assume the wand lengths of all wizards is normally distributed with a mean of 10.4 inches and a standard deviation of 1.1 inches.

- The first student has a wand that is 12.5 inches. Calculate the z-score. What percent of wizards do we expect to have wands that are shorter?
  Solution: z score is 1.91. Percentage is 97.13%

- The second student has a wand that is 9.25 inches long. Calculate the z-score. What percent of wizards do we expect to have wands that are longer?
  Solution: z-score is -1.05, percentage is 85.31%

- The length of the third student's wand is exactly 2.5 standard deviations above the average. What is the actual length of the wand?
  Solution: 13.15 inches

- What percent of wizards do we expect to have wands that are between the second and the third wizards'?
  Solution: 84.7%