ANOVA III. Application to Categorical Predictors

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Two representations of multiple samples comparison

- Data table may look like:

Factor	Treatment(Trt) 1	Trt 2	$\cdots$	Trt k
Observation	y_{11}	y_{21}	$\cdots$	y_{k1}
	y_{12}	y_{22}	$\cdots$	y_{k2}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	y_{1n_1}	y_{2n_2}	$\cdots$	y_{kn_k}
mean	$\bar{y}_{1\cdot}$	$\bar{y}_{2\cdot}$	$\cdots$	$\bar{y}_{k\cdot}$

- Or more often,

Trt	originally indexed y	re-indexed y'
1	y_{11}	y'_{1}
$\vdots$	$\vdots$	$\vdots$
1	y_{1n_1}	y'_{n_1}
$\vdots$	$\vdots$	$\cdots$
k	y_{k1}	$y'_{n_1+\ldots+n_{k-1}+1}$
$\cdots$	$\cdots$	$\cdots$
k	y_{kn_k}	$y'_{n_1+\ldots+n_{k-1}+n_k}$

Model Formulation

Consider a linear model with a K-level categorical predictor:

$$Y'_n = \beta_1 \mathbb{1}_{Trt1} + \cdots + \beta_k \mathbb{1}_{Trtk} + \varepsilon_n, \quad \text{for } n = 1, \ldots, (n_1 + \ldots + n_k).$$

Equivalently, let

$$Y_{ij} = \mu_i + \varepsilon_{ij},$$

where

- $j = 1, \ldots, n_i$ indexes sample unit and $i = 1, \ldots, k$ indexes treatment levels.
- Y_{ij} is the response variable of the jth unit receiving the ith level of treatment (TrT).
- μ_i is the population mean for the ith treatment level
- ε_{ij} is a random error for the jth sample unit and the ith treatment level. We assume $\varepsilon_{ij} \sim$ iid $N(0, \sigma^2)$.
- The model parameters are $\mu_1, \ldots, \mu_k, \sigma^2$ estimated by $\bar{Y}_{1\cdot}, \ldots, \bar{Y}_{k\cdot}$, and MSE, respectively.

Alternative Model Formulation

Let

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad \text{with constraints} \quad \sum_{i=1}^{k} \alpha_i = 0$$

where

- $j = 1, \ldots, n_i$ indexes sample unit and $i = 1, \ldots, k$ indexes treatment levels.
- μ is the grand population mean $\mu = \frac{1}{k} \sum_{i=1}^{k} \mu_i$.
- $\alpha_i = \mu_i - \mu$ is the difference between the ith treatment mean and the grand mean (note: $\sum_{i=1}^{k} \alpha_i = 0$).
- The random errors are assumed to follow

$$\varepsilon_{ij} \sim \text{iid } N(0, \sigma^2).$$

- The model parameters are $\mu, \alpha_1, \ldots, \alpha_k, \sigma^2$ estimated by $\bar{Y}_{\cdot\cdot}$, $\bar{Y}_{1\cdot} - \bar{Y}_{\cdot\cdot}, \ldots, \bar{Y}_{k\cdot} - \bar{Y}_{\cdot\cdot}$, and MSE, respectively.

Test for Overall Trt Effect

- A one-way ANOVA analysis begins with testing

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k \text{ vs } H_A : \text{not all } \mu_i\text{'s are equal.}$$

- Or equivalently, test

$$H_0 : \alpha_i = 0 \text{ for all } i \text{ vs } H_A : \text{not all } \alpha_i\text{'s are zero.}$$

- The approach is to partition the total sum of squares, construct an ANOVA table, and perform an F test.
- Notation regarding the means:
  - $\bar{Y}_{..}$ is the grand mean,
  - $\bar{Y}_{i.}$ is the ith group (Trt) mean.

Partition of Total Sum of Squares

- Partition the total sum of squares (SS):

$$SST = SSR + SSE,$$

  where
  - Total SS:
    SST $= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{..})^2$ on df $= N - 1$.
    If balanced design: $\sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{..})^2$ on df $= kn - 1$.
  - Between-Trt SS:
    SSE $= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\bar{Y}_{i.} - \bar{Y}_{..})^2$ on df $= k - 1$.
    If balanced design: $n \sum_{i=1}^{k} (\bar{Y}_{i.} - \bar{Y}_{..})^2$ on df $= k - 1$.
  - SS for error:
    SSE $= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2$ on df $= N - k$.
    If balanced design: $\sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{i.})^2$ on df $= k(n - 1)$.
- Recall that MSE is an estimate of the error variance σ^2.

ANOVA Table

SS: Sum of squares

MS: Mean sum of squares

Within-group SS S_{Within}, Between-group SS S_{Between}

	SS	df	MS	E(MS)
Between Treatment SS (Model)	$S_{Between} = \sum_j n_j(y_{j.} - y_{..})^2$	$K-1$	$S_{Between}/(K-1)$	$\sigma^2 + (K-1)^{-1}\sum_{j=1}^{k} n_j t_j^2$
Within Treatment SS (Error)	$S_{Within} = \sum_j \sum_{k=1}^{n_j}(y_{jk} - y_{j.})^2$	$n-K$	$S_{Within}/(n-K)$	σ^2
Total SS	$S_{Total} = \sum_j \sum_{k=1}^{n_j}(y_{jk} - y_{..})^2$	$n-1$	$S_{Total}/(n-1)$	

where $t_j = \mu_j - \mu$ and $\mu = n^{-1}\sum_j n_j\mu_j$. Q: simplify in a balanced design $n_1 = \ldots = n_K$?

- In a general one-way ANOVA, under the $H_0 : \alpha_i = 0$ for all i,

$$F = \frac{MSR}{MSE} \sim F_{k-1,N-k}.$$

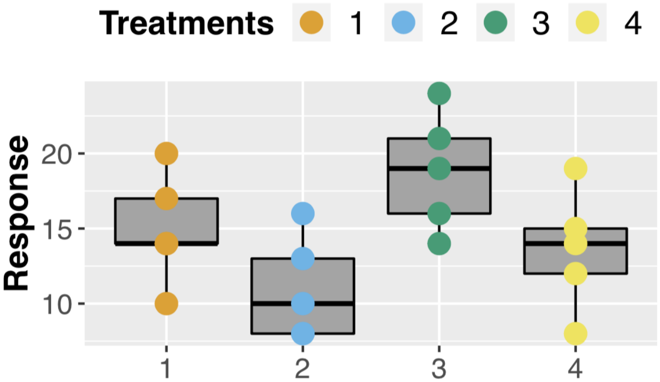
Example

```
library(ggplot2)
y <- c(14, 20, 10, 14, 17, 13, 8, 10, 16, 8, 16, 14, 24, 21, 19, 8, 14, 19, 12, 15)
trt <- factor(rep(1:4, each = 5))
cbind(y, trt)
```

```
##        y trt
##  [1,] 14   1
##  [2,] 20   1
##  [3,] 10   1
##  [4,] 14   1
##  [5,] 17   1
##  [6,] 13   2
##  [7,]  8   2
##  [8,] 10   2
##  [9,] 16   2
## [10,]  8   2
## [11,] 16   3
## [12,] 14   3
## [13,] 24   3
## [14,] 21   3
## [15,] 19   3
## [16,]  8   4
## [17,] 14   4
## [18,] 19   4
## [19,] 12   4
## [20,] 15   4
```

Example

```
ggplot(data.frame(response=y, trt = trt), aes(x = trt, y = response)) +
  geom_boxplot(aes(colour=factor(trt)), fill="#A4A4A4", color="black")+
  geom_point(aes(color = factor(trt)), size = 4)+
  scale_colour_manual(values = c("#E69F00", "#56B4E9", "#009E73", "#F0E442"))+
  scale_fill_manual(values = c("#E69F00", "#56B4E9", "#009E73",  "#F0E442"))+
  labs(colour = "Treatments")+xlab("")+ylab("Response")+
  theme(axis.text=element_text(size=12),
  axis.title=element_text(size=14,face="bold"),
  legend.text=element_text(size=14),
  legend.title=element_text(size=14, face="bold"), legend.position = "top")
```



Example

```r
anova(lm(y~trt))
```

```
## Analysis of Variance Table
##
## Response: y
##           Df Sum Sq Mean Sq F value  Pr(>F)
## trt        3  158.8  52.933  3.6506 0.03533 *
## Residuals 16  232.0  14.500
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Marginal evidence against the null hypothesis.

Note that

```r
1-pf(3.6506 ,3, 16)
```

```
## [1] 0.03532877
```

```r
pf(3.6506 ,3, 16, lower.tail  = FALSE)
```

```
## [1] 0.03532877
```