Confidence Intervel

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Review: Point Estimators

- Most probability distributions are indexed by one or more **parameters**. For example, $N(\mu, \sigma^2)$.
- In hypothesis tests, we have used **point estimators** for parameters. For example, consider an i.i.d. sample $D_1, D_2, \ldots, D_n \sim_{\text{i.i.d}} N(\mu_D, \sigma_D^2)$. Let

$$\bar{D} = \frac{1}{n} \sum_{i=1}^{n} D_n, \quad S_D^2 = \frac{1}{n-1} \sum_{i=1}^{n} (D_n - \bar{D})^2.$$

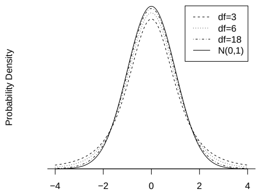
  Then $\bar{D}$ is a point estimator of μ_D and S_D^2 is a point estimator of σ^2.
- We know that $E(\bar{D}) = \mu_D$ and $E(S_D^2) = \sigma_D^2$.
- That is, $\bar{D}$ is an **unbiased estimator** of μ_D and S_D^2 is an unbiased estimator of σ_D^2.

Interval Estimators

- Now we turn to **interval estimators** to give a reasonable interval for parameters.
  - For μ: $[a_1, a_2]$ for some constants a_1, a_2 based on data
  - For σ^2: $[b_1, b_2]$ for some constants b_1, b_2 based on data
- The assumptions are the same as in hypothesis testing, but we do not need a null hypothesis about the parameters (e.g. $\mu_D = \mu_1 - \mu_2 = 0$).

Confidence Interval for μ_D

- Suppose $D_1, D_2, \ldots, D_n$ is an i.i.d. sample from $N(\mu_D, \sigma_D^2)$ and σ_D^2 is unknown.
- Note that
$$\frac{\bar{D} - \mu_D}{S_D/\sqrt{n}} \sim T_{n-1}.$$
- Let $t_{n-1,\alpha/2}$ denote the t critical value such that
$$P(-t_{n-1,\alpha/2} \leq T_{n-1} \leq t_{n-1,\alpha/2}) = 1 - \alpha.$$



Confidence Interval for $\mu_D = \mu_1 - \mu_2$

- Then we have

$$1 - \alpha = P\left(\mu_D \in \left[\bar{D} - t_{n-1,\alpha/2}\frac{S_D}{\sqrt{n}},\ \bar{D} + t_{n-1,\alpha/2}\frac{S_D}{\sqrt{n}}\right]\right)$$

- A $(1-\alpha)$ CI for μ_D is

$$\mu_D \in \left[\bar{d} - t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}},\ \bar{d} + t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}\right]$$

or

$$\bar{d} \pm t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$$

- The half width of this CI is $t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$.
- The width of this CI is $2 \times t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$.

Confidence Intervals for $\mu_D = \mu_1 - \mu_2$

- A $(1-\alpha)$ CI for μ_D is

$$\mu_D \in \left[\bar{d} - t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}},\ \bar{d} + t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}\right]$$

- In the lake clarity 1980 vs. 1990 example, a 95% CI for μ_D is

$$0.497 - 2.080 \times \frac{0.435}{\sqrt{22}} \le \mu_D \le 0.497 + 2.080 \times \frac{0.435}{\sqrt{22}}$$

which is [0.30, 0.69] or 0.497 ± 0.195.

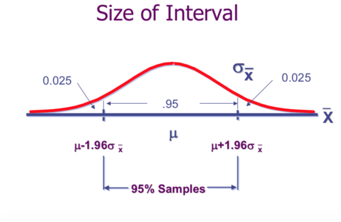
Remarks

- By convention, CIs are two-sided. But one-sided confidence bounds are possible.
- It is not true that $P(0.30 \leq \mu_D \leq 0.69) = 0.95$. Why not? because

  once a sample is observed, there is nothing random.
- The 0.95 probability concerns with the repeated random sampling. It is interpreted as, 95% of the time, the (random) CIs calculated in this way contains (fixed) μ_D.
- For a single case, it is interpreted as ? having 95% confidence that μ_D is between 0.30 m and 0.69 m.
- The interval $[0.30, 0.69]$ (or 0.497 ± 0.195) can be thought of as a plausible range of μ_D.
- What are the assumptions made when we perform a paired T test or construct a corresponding confidence interval for μ?

Two-sided Confidence Interval

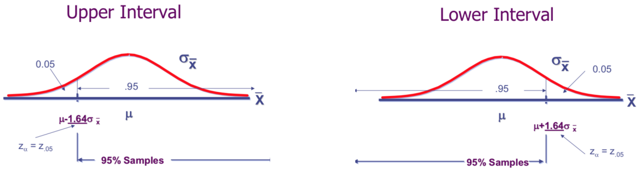
- Based on Z-statistic: $(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$



- Based on T-statistic: $(\bar{X} - t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}},\ \bar{X} + t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}})$

One-sided Confidence Interval

- Based on Z-statistic:
  - Lower interval: $(-\infty,\ \underbrace{\bar{X} + z_{\alpha}\frac{\sigma}{\sqrt{n}}}_{\text{Upper bound}})$
  - Upper interval: $(\underbrace{\bar{X} - z_{\alpha}\frac{\sigma}{\sqrt{n}}}_{\text{Lower bound}},\ \infty)$



- Based on T-statistic:
  - Lower interval: $(-\infty,\ \underbrace{\bar{X} + t_{n-1,\alpha}\frac{\hat{\sigma}}{\sqrt{n}}}_{\text{Upper bound}})$
  - Upper interval: $(\underbrace{\bar{X} - t_{n-1,\alpha}\frac{\hat{\sigma}}{\sqrt{n}}}_{\text{Lower bound}},\ \infty)$

Example

Problem: Suppose the mean of an i.i.d. sample of $n = 100$ is $\bar{x} = 50$ with sample standard deviation 10. Set up an upper 95%-CI estimate for the population mean μ.

Answer: Assume the observation $X_i \sim_{\text{i.i.d.}} N(\mu, \sigma^2)$ for all $i = 1, \ldots, 100$. Since σ is unknown, we consider the T-statistic. Note that $t_{99,0.05} = 1.66$ and $\hat{\sigma} = 10$. So the 95%-CI for μ is

$$(\bar{x} - t_{99,0.05} * \frac{\hat{\sigma}}{\sqrt{n}}, \infty) = (50 - 1.66 * \frac{10}{\sqrt{100}},\ \infty) = (48.34,\ \infty).$$