BOUNDS ON EIGENFUNCTIONS OF HARMONIC OSCILLATORS

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Abstract. We compare upper bounds on eigenfunctions of harmonic oscillators with the general bounds obtained by Koch-Tataru [2], Koch-Tataru-Zworski [3], and Smith-Zworski [4].

1. Introduction

We compare theoretical upper bounds on eigenfuctions of

$$-(h\frac{d^2\psi}{dx^2}) + \omega^2 x^2 = E\psi$$

with numerically computed quantities. We present an approximation to the dependence of L^p - norm on h for a fixed energy.

$$\frac{\|u\|_{L^p}}{\|u\|_{L^2}} \sim Ch^{-\alpha(p)} \qquad \alpha(p) \geq 0$$

We also analysize the behavior of eigenfunctions of the two-dimensional harmonic oscillator for different energy levels generated by matlab.

2. Properties of harmonic oscillator

Our presentation is based on a standard physics point of view – see [1]. For a more mathematical presentation see [5, §6.1].
For one-dimendional, motion of a particle mass m which is attacted to a fixed center by a force proportional to the displacement, this force can be represented by the potential energy $V(x) = \frac{1}{2}kx^2$.
As the potential energy for a linear harmonic oscillator, the time indepentdent Schrödinger eigenvalue equation is

$$-\frac{h^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi(x)$$

This is the author's senior thesis written under supervision of M Zworski.

E is the energy , and the energy spectrum of the linear harmonice oscillator is given by $E_n = (n + \frac{1}{2})\hbar\omega$ .Notice that the energy of ground-state is $\frac{\hbar\omega}{2}$.
Assume $\omega = 1$ and multiply both sides of the equation by $\frac{2}{m}$, we have

$$-\frac{\hbar^2}{m}\frac{d^2\psi(x)}{dx^2} + mx^2\psi = 2E\psi(x)$$

If we let $h = \frac{\hbar}{m}$,and $E \equiv \frac{2E}{m}$. We have a simplier equation as

$$-\hbar^2\frac{d^2\psi(x)}{dx^2} + \psi(x) = E\psi(x)$$

Then our eigenvalues is $E = \frac{2\hbar}{m}(n + \frac{1}{2}) = h(2n + 1)$

Unlike classical mechanics, the energy of E can be any value.In quamtum mechanics, the energy can only be taken as infinite discrete levels. The correpsponding eigenfunctions has to be bounded ,so it physically make sense in quantum mechanics. In one dimension ,the corresponding normalized eigenfuncion of energy E_n is

$$u_n(x) = \frac{1}{\sqrt{2^n n!}}(\pi h)^{-\frac{1}{4}}e^{-x^2/2h}H_n(x)$$

In one-dimentional harmonic oscillator, the eigenvalues are non-degenrate .So for each energy E_n, there exists only one eigenfunction. In addition, the set of of normalised eigenfunctions satisfies the orthnormal relation in L^2.

$$\int \psi_m(x)\psi_n(x)dx = \delta_{mn}$$

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

In two dimension, The Schrödinger equation of harmonic oscillator is

$$-\hbar^2\frac{\partial^2\psi}{\partial x^2} + x^2\psi - \hbar^2\frac{\partial^2\psi}{\partial y^2} + y^2\psi = E\psi$$

The energy eigenvalue $E = E_x + E_y = 2h(n + m + 1)$.
So for the energy level k,$E_k = 2(k+1) = 2h(n+m+1)$, the degenerency is k + 1. Because

$$k = n + m,$$

$n, m \in N$ with $n, m = 0, 1, ..., k$.
The eigenfunction corresponding to energy level $E_(k)$ is

$$u(x, y) = \psi_n(x)\psi_m(y)$$

Since the degenrency is k + 1 for energy level k. We can write eignfunction of E_k as the sum of linear combination of k+1 eigenfunctions

$$u_k(x,y) = \sum_{j=0}^{k} c_j \psi_j(x) \psi_{k-j}(y)$$

and c_j are chosen randomly from the sphere $\mathbb{S}^N \subset \mathbb{R}^{N+1}$ (with respect to the uniform measure; see the matlab code to see how it is done). We can generate $\tilde{c}_j$ by using matlab built in function randn $\tilde{c}_j$ is the random value from the standard normal distributaion

$$c_j = \frac{\tilde{c_j}}{(\sum_{j=0}^{k} \tilde{c_j}^2)^{\frac{1}{2}}}$$

So we have

$$\sum_{j=0}^{k} c_j^2 = 1$$

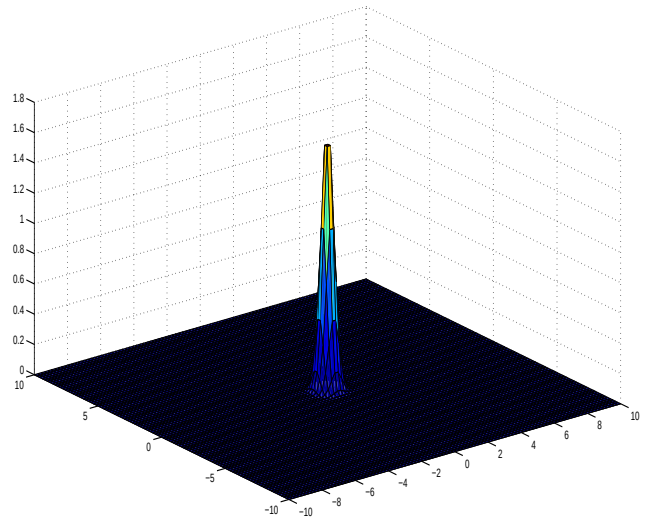
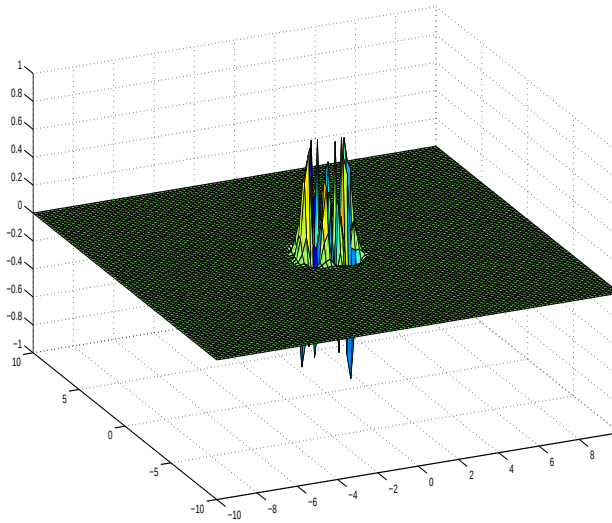
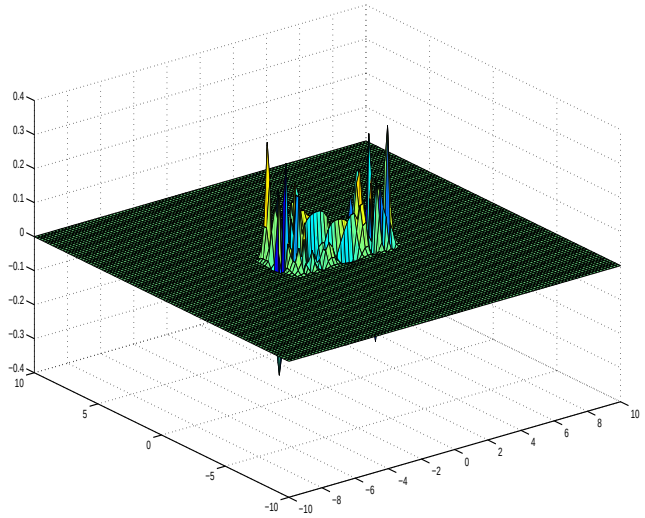
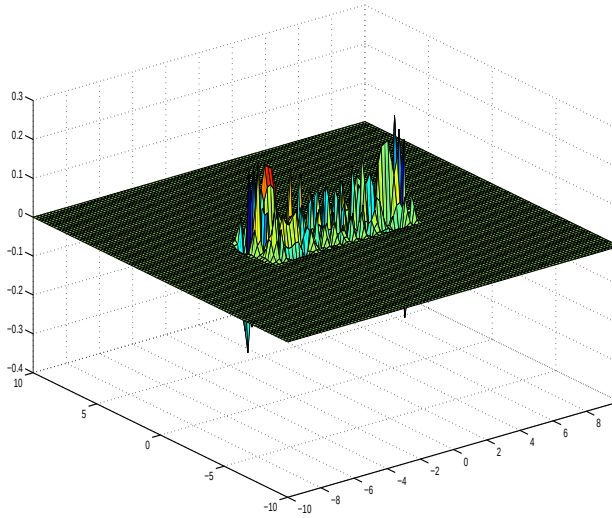
Now

$$\int u_k u_k = \int \sum_{j=0}^{k} c_j \psi_j(x) \psi_{k-j}(y) \sum_{j=0}^{k} c_j \psi_j(x) \psi_{k-j}(y) dxdy$$

Since the eigenfunctions are orthonormal, the integral is

$$\begin{aligned} \int u_k u_k &= \int \sum_{j=0}^{k} c_j^2 \psi_j(x)^2 \psi_{k-j}(y)^2 dxdy \\ &= \sum_{j=0}^{k} c_j^2 \int \psi_j^2(x) dx \int \psi_{k-j}^2(y) dy \\ &= 1 \end{aligned}$$

The following figures are the random eigenfunctions of two-dimensional harmonic oscillator generated by matlab.

The first figure is the eigenfunction of the ground state with N = 0 , and the second, third and last figure are the random eigenfunctions with energy level N = 10 ,50, 100. And we observe that as the energy level goes up, the region where the eigenfunction is not exponentially small spread out.

3. Upper bounds on eigenfunctions

3.1. **Upper bound of dimension two.** First we take $\omega_1 = \omega_2 = 1$. We denote by $\Pi(h, N)$ the projection onto the space of eigenfuction of

$$-(h\partial_{x_1})^2 - (h\partial_{x_2})^2 + x_1^2 + x_2^2$$

with eigenvalue $E = h(2 + N)$. We want to keep E fixed and change h.

$$u_n(z) = \frac{1}{\sqrt{2^n n!}}(\pi h)^{-\frac{1}{4}} e^{-z^2/2h} H_n(z)$$

are normalized eigenfuctions of the one dimensional harmonic oscillator. Let $W_0....W_N$ be orthornormal basis of the eigenfunction of the energy level N in the two - dimensional harmonic osciallator.

$$W_k(y_1, y_2) = u_n(k)(y_1)u_{N-k}(y_2)$$

In Hilbert Space, the inner product is defined as

$$\langle f, W_k\rangle \equiv \int f(y_1, y_2)W_k(y_1, y_2)dy_1dy_2$$

By the definiton of projection, let $f(x) \in L^2(\mathbb{R}^2)$ be a random function

$$\Pi(h, N)f(x) = \sum_{k=0}^{N}\langle f, W_k\rangle W_k \quad = \sum_{k=0}^{N}\int_{R^2} u_k(x_1)u_{N-k}(x_2)u_k(y_1)u_{N-k}(y_2)f(y_1, y_2)dy_1dy_2$$

$$\Pi(h, N)f(x) = \int_{\mathbb{R}^2}\sum_{k=0}^{N} u_k(x_1)u_{N-k}(x_2)u_k(y_1)u_{N-k}(y_2)f(y_1, y_2)dy_1dy_2, \quad x \in \mathbb{R}^2$$

We recall that

$$L^p(\mathbb{R}^2) = \{u : \|u\|_p < \infty\}, \quad \|u\|_p := \left(\int_{\mathbb{R}^2} |u(x)|^p\right)^{\frac{1}{p}},$$

(for L^∞ we just take the supremum). and that the *norm* of an operator from L^p L^∞ is

$$\|A\|_{L^p\to L^\infty} = \sup_{u\in L^p, u\neq 0}\sup_{x\in\mathbb{R}^2}\frac{|Au(x)|}{\|u\|_p}.$$

We also recall the Hölder inequality,

$$|\int fg| \leq \|f\|_q\|f\|_p, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In fact,

$$\|\Pi(h, N)\|_{L^2\to L^\infty} = \sup_{x\in\mathbb{R}^2}\left(\sum_{k=0}^{N}|u_k(x_1)u_{N-k}(x_2)|^2\right)^{\frac{1}{2}}.$$

$$\|\Pi(h,N)\|_{L^2\to L^\infty} = \sup_{f\in L^2, f\neq 0}\ \sup_{((x_1,x_2)\in\mathbb{R}^2} \frac{|\Pi f(y_1,y_2)|}{\|f(x_1,x_2)\|_2}$$

and ,

$$\leq \frac{\|\sum_{k=0}^N u_k(x_1)u_{N-k}(x_2)u_k(y_1)u_{N-k}(y_2)\|_2\|f(y_1,y_2)\|_2}{\|f(x_1,x_2)\|_2}$$

since $u_k(x)$are normalized eigenfunctions and the eigenfunctions are orthonormal, we have

$$\begin{aligned}\|\sum_{k=0}^N u_k(x_1)u_{N-k}(x_2)u_k(y_1)u_{N-k}(y_2)\|_2 &= (\int |\sum_{k=0}^N u_k(x_1)u_{N-k}(x_2)u_k(y_1)u_{N-k}(y_2)|^2 dy_1dy_2)^{\frac{1}{2}}\\ &= (\sum_{k=0}^N \int |u_k(x_1)u_{N-k}(x_2)u_k(y_1)u_{N-k}(y_2)|^2 dy_1dy_2)^{\frac{1}{2}}\\ &= (\sum_{k=0}^N |u_k(x_1)u_{N-k}(x_2)|^2 \int |u_k(y_1)u_{N-k}(y_2)|^2 dy_1dy_2)^{\frac{1}{2}}\\ &= (\sum_{k=0}^N |u_k(x_1)u_{N-k}(x_2)|^2)^{\frac{1}{2}}\end{aligned}$$

If we take

$$f(x_1,x_2) = \frac{\sum_{k=0}^N u_k(x_1)u_{N-k}(x_2)u_k(\tilde{x}_1)u_{N-k}(\tilde{x}_2)}{(\sum_{k=0}^N (u_k(\tilde{x}_1)u_{N-k}(\tilde{x}_2))^2)^{\frac{1}{2}}}$$

$$\|f(x_1,x_2)\|_2 = 1$$

Now,

$$\begin{aligned}\Pi_(h,N)f(x_1,x_2) &= \sum_{k=0}^N u_k(x_1)u_{N-k}(x_2)\int_{\mathbb{R}^2} u_k(y_1)u_{N-k}(y_2)f(y_1,y_2)dy_1dy_2.\\ &= \frac{\sum_{k=0}^N u_k(x_1)u_{N-k}(x_2)u_k(\tilde{x}_1)u_{N-k}(\tilde{x}_2)}{(\sum_{k=0}^N (u_k(\tilde{x}_1)u_{N-k}(\tilde{x}_2))^2)^{\frac{1}{2}}}\end{aligned}$$

So

$$\Pi_(h,N)f(\tilde{x_1},\tilde{x_2}) = (\sum_{k=0}^N |u_k(x_1)u_{N-k}(x_2)|^2)^{\frac{1}{2}}$$

So for every point $x\in R_2$, there exists a function f such that

$$\|\Pi(h,N)\|_{L^2\to L^\infty} = \sup_{x\in\mathbb{R}^2}\left(\sum_{k=0}^N |u_k(x_1)u_{N-k}(x_2)|^2\right)^{\frac{1}{2}}$$

General results of [2, 3, 4] show that for $h(2+N)=E$, with E constant (or close to a fixed value),

$$\|\Pi(h,N)\|_{L^2\to L^\infty} \le Ch^{-\frac{1}{2}}.$$

We can also look at the largest values taken by the "square eigenfunctions":

$$\sup_{0\le k\le N}\sup_{x\in\mathbb{R}^2}|u_k(x_1)u_{N-k}(x_2)|$$

eigenfunction:

$$u(x)=\sum_{k=0}^{N} c_k u_k(x_1)u_{N-k}(x_2)$$

where

$$\sum_{k=0}^{N} c_k^2 = 1,$$

and c_k are chosen randomly from the sphere $\mathbb{S}^N \subset \mathbb{R}^{N+1}$

3.2. **Airy Functions.** Airy function is the solution to the differential equaion $\dfrac{d^2y}{dx^2} - xy = 0$

Recall that the common method is to approximate the potential around the classical turning point x_c by linear

$$V(x) = V(x_c) + V'(x_c)(x-x_c) + O(x-x_c)^2$$

here $V(x)=x^2$ because we assume the angular frequence $\omega = 1$.
The one dimensional Harmonic Oscillator Schrdinger Equation is.

$$-\frac{d^2\psi}{dx^2} + h^{-2}x^2\psi = h^{-2}E\psi$$

When x is sufficiently close to the turning point,x_c. we can approximate $(x^2-E)\sim(x-x_c)$ we rewrite it as

$$\frac{d^2\psi}{dx^2} + h^{-2}(x-x_c)\psi = 0 \tag{3.1}$$

Let $u = h^\alpha(x-x_c)$,then $\dfrac{du}{dx} = h^\alpha$. so we can replace $\dfrac{d^2\psi}{dx^2}$ by $h^{2\alpha}\dfrac{d^2\psi}{du^2}$ in (3.1) now

$$-h^{2\alpha}\frac{d^2\psi}{du^2} + h^{-2-\alpha}u\psi = 0 \tag{3.2}$$

In order to have the airy function as a solution of (3.1), we need

$$-2-\alpha = 2\alpha$$
$$\alpha = -\frac{2}{3}.$$

So using a new variable $u = (h^{-2})^{1/3}(x - x_c)$, the differential equation (3.2) is simplified to

$$\frac{d^2\psi(u)}{du^2} + u\psi(u) = 0$$

Thus, we can use Airy function Ai(u) to approximate the eigenfunction of 1-Dimension harmonic oscillator.

The Airy function $Ai(u)$ is bounded and has the asymptotic behavior

$$Ai(u) \sim \begin{cases} \pi^{\frac{1}{2}} u^{-\frac{1}{4}} e^{-\frac{2}{3}u^{\frac{3}{2}}} & u \to \infty \\ \pi^{\frac{1}{2}} u^{-\frac{1}{4}} \cos(\frac{2}{3}u^{\frac{3}{2}} - \frac{\pi}{4}) & u \to -\infty \end{cases}$$

Recall that the asymptotic function of $|Ai(u)|$ is "$|Ai(u)| \sim \frac{1}{|u|^{\frac{1}{4}}}$"
For any eigenfunction u_h of the differential equation.

$$(\int |u_h(x)|^p dx)^{\frac{1}{p}} \sim (\int_{-1}^{1} |Ai(h^{-\frac{2}{3}}x)|^p dx)^{\frac{1}{p}}$$

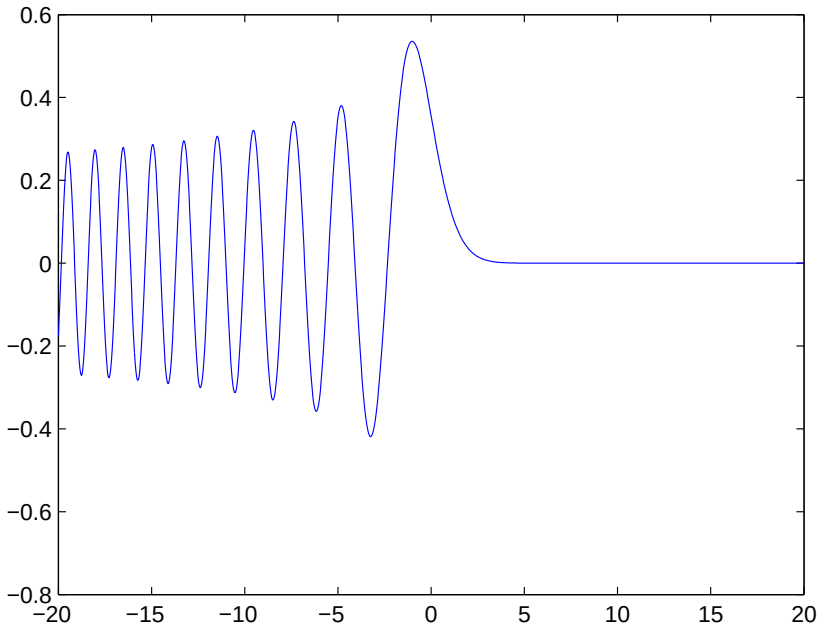


FIGURE 1. this figure shows the plot of Ariy function

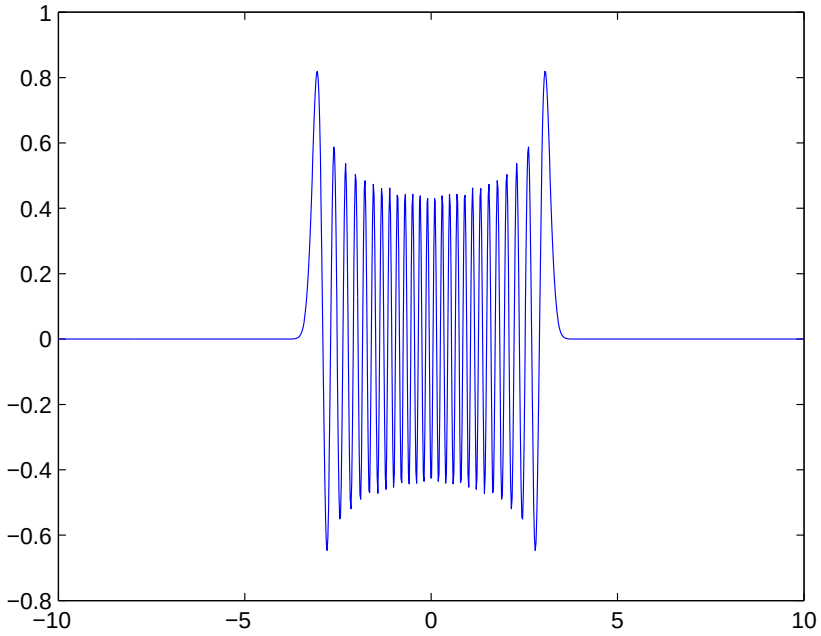


FIGURE 2. this figure shows the plot of the eigenfunction with $N = 50$,and $h = 0.1$

let $t = h^{-\frac{2}{3}}x$

$$\int_{-1}^{1} |Ai(h^{-\frac{2}{3}}x)|^p dx = h^{\frac{2}{3}} \int_{-h^{\frac{2}{3}}}^{h^{\frac{2}{3}}} |Ai(t)|^p dt \sim h^{\frac{2}{3}}(1 + \int_{-h^{\frac{2}{3}}}^{1} \frac{1}{|t|^{\frac{p}{4}}})dt \sim h^{\frac{2}{3}}(1 + h^{\frac{-2}{3}(1-\frac{p}{4})}) = h^{\frac{2}{3}} + h^{\frac{p}{6}}$$

Since we fix energy level E = 1, and $h = \frac{1}{1+2N}, N = 0, 1, 2..., h \leq 1$
When $p \geq 4$, $h^{\frac{2}{3}} \geq h^{\frac{p}{6}}$ the term $h^{\frac{2}{3}}$ dominates
So we have

$$\|u_h\|_p \sim h^{\frac{2}{3p}}$$

When $p \leq 4$ the term $h^{\frac{p}{6}}$ dominates over $h^{\frac{2}{3}}$

$$\int |u_h(x)|^p dx \sim h^{\frac{2}{3}} h^{-\frac{2}{3}(1-\frac{p}{4})} = h^{\frac{p}{6}}$$

If we factor out $h^{\frac{p}{6}}$, we have

$$\|u_h\|_p \sim (h^{\frac{p}{6}}(1 + h^{\frac{2}{3}(1-\frac{p}{4})}))^{\frac{1}{p}}$$

$$\|u_h\|_2 \sim h^{\frac{1}{6}}(1 + h^{\frac{1}{3}})^{\frac{1}{2}}$$

Thus,

$$\frac{\|u_h\|_p}{\|u_h\|_2} \sim \frac{h^{\frac{1}{6}}(1+h^{\frac{2}{3}(1-\frac{p}{4})})^{\frac{1}{p}}}{h^{\frac{1}{6}}(1+h^{\frac{1}{3}})^{\frac{1}{2}}}$$

Use Taylor expansion, we have

$$\frac{\|u_h\|_p}{\|u_h\|_2} \sim 1 - \frac{1}{2}h^{\frac{1}{3}} + \frac{1}{p}h^{\frac{2}{3}(1-\frac{p}{4})}$$

When $p \geq 2$ the term $\frac{1}{p}h^{\frac{2}{3}(1-\frac{p}{4})}$ dominates over $\frac{1}{2}h^{\frac{1}{3}}$

So $\frac{\|u_h\|_p}{\|u_h\|_2} \sim C_1 + C_2 h^{\frac{2}{3}(1-\frac{p}{4})}$,$p \leq 4$, and C_1 ,C_2 are constants depending on p.
Therefore

$$\frac{\|u_h\|_p}{\|u_h\|_2} \sim \begin{cases} C_1 + C_2 h^{\frac{2}{3}(1-\frac{p}{4})} & 2 \leq p \leq 4. \\ C_3 + C_4 h^{\frac{2}{3p}-\frac{1}{6}} & p \geq 4 \end{cases}$$

4. Numerical experiments

4.1. **One dimensional harmonic oscillator.** The L^p- norm of one dimensional harmonic oscillator is. $\frac{\|u_j\|_p}{\|u_j\|_2} = \|u_j\|_p = (\int_{\mathbb{R}} |u_j|^p)^{\frac{1}{p}}$
We use trapizoid rule to approximate the intergation $(\int |u_j|^p)$ on the interval $[-\sqrt{E}, \sqrt{E}]$
From the theory [2,3,4], we know $\frac{\|u_j\|_p}{\|u_j\|_2} \sim Ch^{-\alpha(p)}$
After taking the logarithm of both sides, we have $\log \frac{\|u_j\|_p}{\|u_j\|_2} \sim \log(C) + \alpha(p)\log(\frac{1}{h})$
We use polyfit function to obtain the values of $\alpha(p)$ by varying p from 2 to 20, and the plot is shown as following
Here, if we set $N = 25, 30, 35, 40$, the blue curve in figure 3 is the plot. If we choose $N = 30, 35, 40, 45$, the black curve is the plot , and the red curve is the theoritical curve. Notice that $\alpha(p)$ is an increasing function of p ,and $\alpha(2) = 0$.

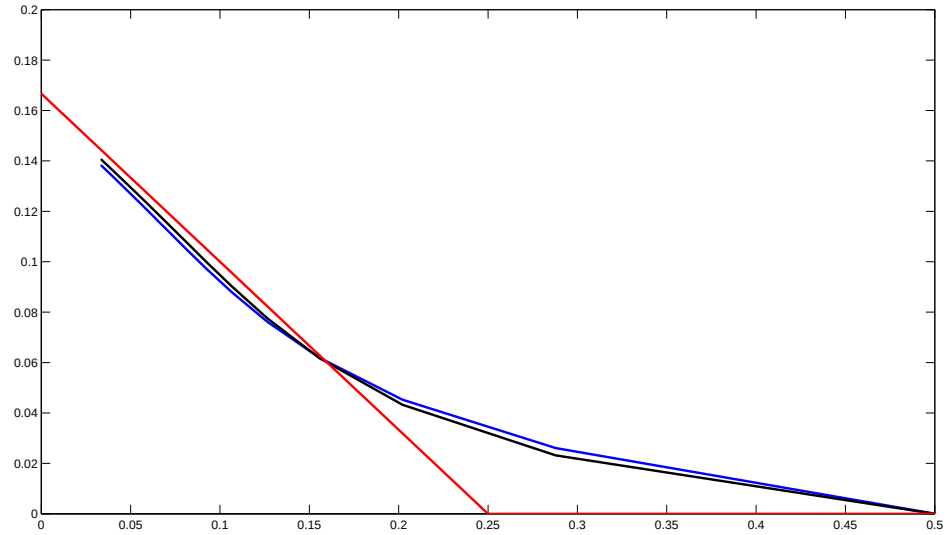


FIGURE 3. this figure shows the plot of $\alpha(p)$ versus $\frac{1}{p}$

The reason of the discrepacy of the actual numerical and the theoretical value of $\alpha(p)$ is that we use polyfit function to obtain the coeffiencts $\alpha(p)$ and $\log(C)$ by plotting different values of h , with fixed energy E from the formula

$$\log(C) + \alpha(p)\log(\frac{1}{h})$$

Since the values of constant $\log(C)$ we got is different by choosing different ranges of h , the values of $\alpha(p)$ obtained from the polyfit function is different from the theory. And as we increase the value of N we chose to plot in the graph, our numerical result is getting closer to the theoritical result.

In addition, our experimental result indicates that the exponential function $\alpha(p)$ has a linear relation with $\frac{1}{p}$.
Recall from the discussion of airy function from the section of 3.1. We expect that

$$\alpha(p) = \frac{2}{3}(1 - \frac{p}{4}) \qquad\qquad 2 \leq p \leq 4$$

Here, we choose $p = 2.5$ and plot the graph(in blue) of L^p - norm versus h , and$h = \frac{1}{2+N}$, where N is from 5 to 150.
Since $p = 2.5$, the expected power $\alpha(p)$ of h from our discussion in section 3.1 is

$$\frac{2}{3}(1 - \frac{2.5}{4}) = \frac{1}{4}$$

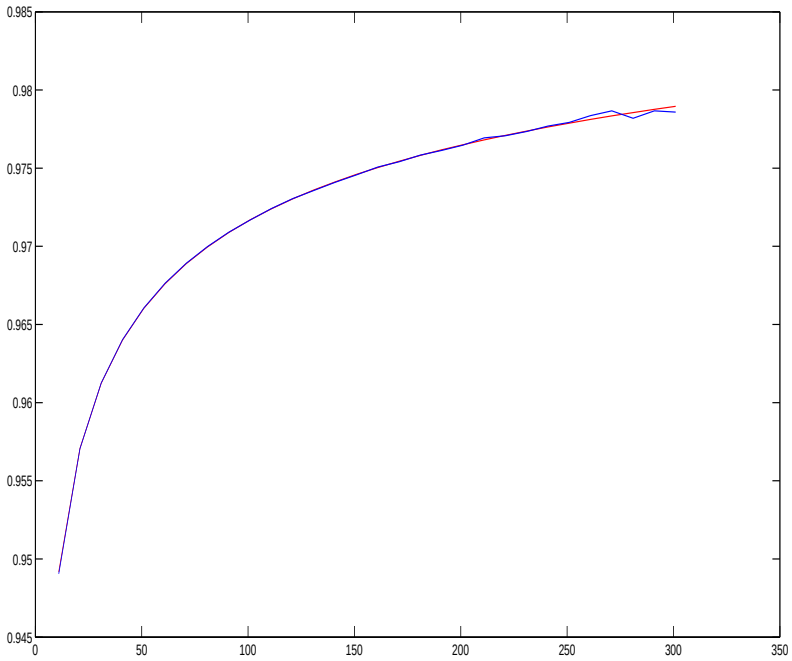


FIGURE 4. this figure shows the $L^{2.5}$ - norm as a function of $\frac{1}{h}$

Thus, we have

$$\frac{\|L\|_{2.5}}{\|L\|_2} \sim C_1 + C_2 h^{\frac{1}{4}}$$

Using the polyfit function in matlab to find the coefficients, we obtain $C_1 = 1.0021$ and $C_2 = -0.0964$.

We plot the function $f(h) = 1.0021 - 0.0964h^{\frac{1}{4}}$ (This is corresponding to the red curve in figure 4). The blue curve in figure 4 is the plot of the actual $L^{2.5}$-norm.

Observe that $f(h)$ does an excellent job approxiamting $L^{2.5}$ - norm,thus it is a valid bound for norm $L^{2.5}$ - norm.

If we plot the values of C_1 as a function of p, and p is from 2.1 to 3.9, we have the plot in figure 5.

Notice that C_1 is decreasing function of p, and C_1 is negative. As $p \to 4$, $C_1 \to -\infty$.

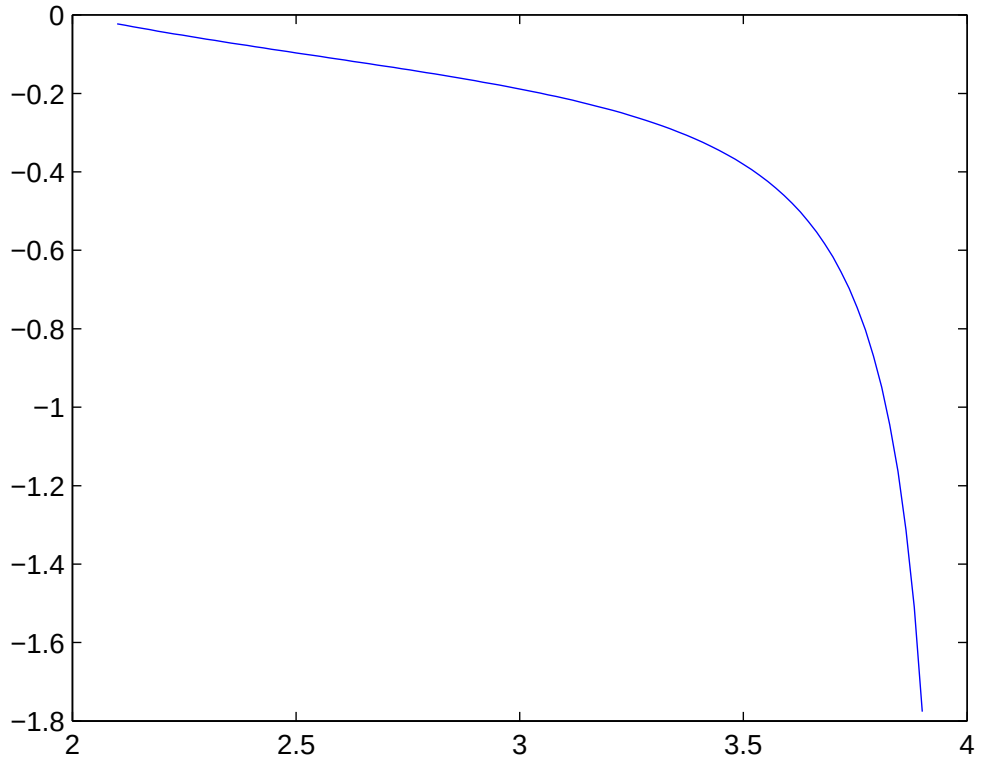


FIGURE 5. this figure shows the plot of C_1 as a function of p

In another case, if we choose p = 5, the expected power of h is $\alpha(p) = \frac{2}{3p} - \frac{1}{6}$.
So we have

$$\frac{\|L\|_5}{\|L\|_2} = \|L\|_5 \sim C_1 + C_2 h^{-\frac{1}{30}}$$

Using similar method as above case, we have

$$\|L\|_5 \sim -0.3050 + 1.1074 h^{-\frac{1}{30}}$$

(corresponding to the red curve of figure 5)
Notice that $-0.3050 + 1.1074 h^{-\frac{1}{30}}$ matches the actual curve(plotted in blue) nicely, so our approximation is a valid bound of the L^5-norm.

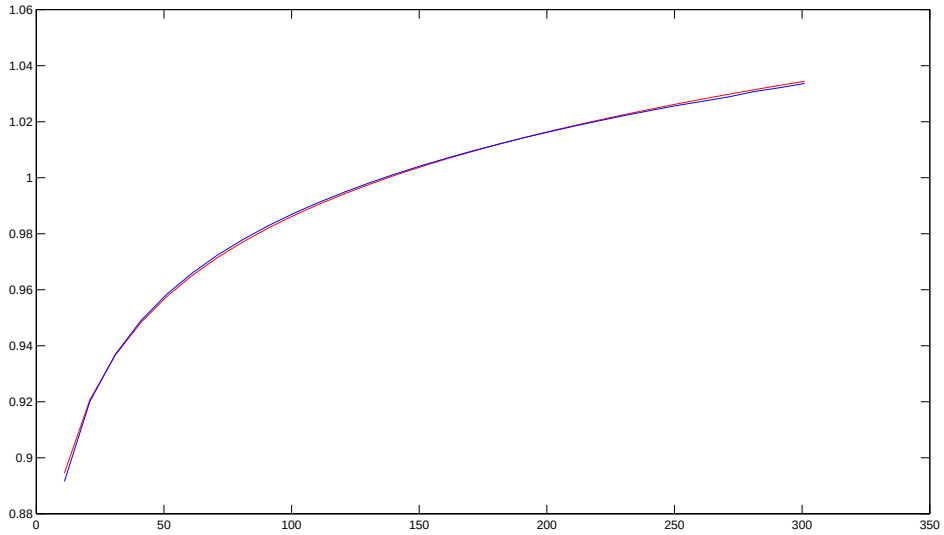


FIGURE 6. this is the plot of L^5 - norm

4.2. **Two dimension harmonic oscillator.** For two dimensional harmonic oscillator

$$E = h(2N + 2)$$

$$\|\pi(h, N)\|_{L^2 \to L^p} = \sup_{f \in H_{E,h}} \frac{\|f\|_p}{\|f\|_2}$$

Fix E as the value of 2, we investigate the dependece of the L^p norm $\|\pi(h, N)\|_{L^2 \to L^p}$ on h. By theory, $\|\pi(h, N)\|_{L^2 \to L^p}$ behaves similarly to a power function of h

$$\sup_{f \in H_{E,h}} \frac{\|f\|_p}{\|f\|_2} \sim C h^{-\alpha(p)}$$

and

$$\alpha(p) = \begin{cases} -\frac{1}{2}\frac{1}{p} + \frac{1}{4} & 2 \le p \le \frac{10}{3} \\ -\frac{4}{3}\frac{1}{p} + \frac{1}{2} & \frac{10}{3} \le p < \infty \end{cases} \tag{4.1}$$

We use polyfit function in matlab to obtain the values of constant $\log(C)$ and $\alpha(p)$ for different values of p , and $h = \frac{E}{2N+2}$

$$\log f(h) = \alpha(p) \log \frac{1}{h} + \log(C)$$

By varying the value of p from 2 to 20, the plot of $\alpha(p)$ is shown in figure 7.
The red curve is the theoritical curve (4.1).

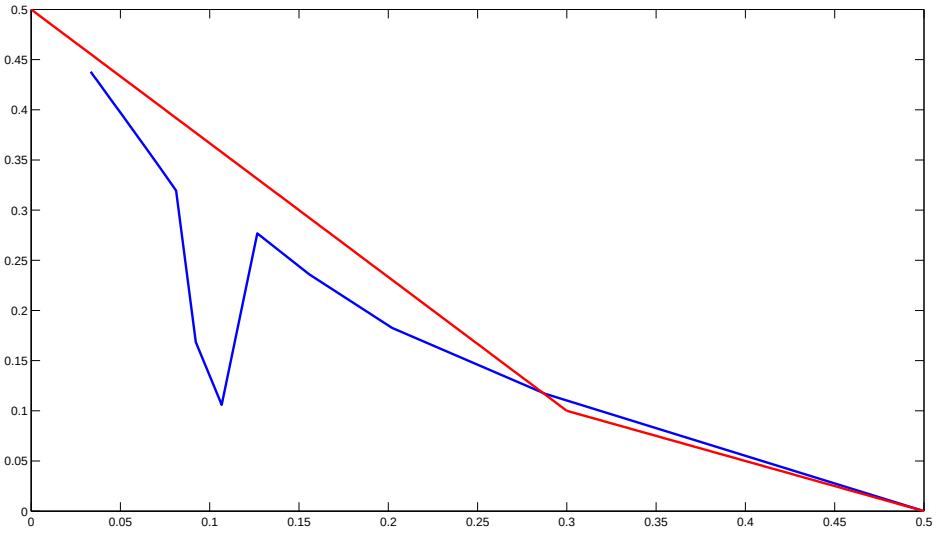


FIGURE 7. this figure shows the plot of $\alpha(p)$ as a function of 1/p

$$\lim_{p\to\infty} \alpha(p) = \frac{1}{2}, \alpha(2) \approx \frac{1}{2}$$

The reason of the discrepacy of the theory and the actual curve is the fact that the extremal eigenfunctions are very special and the fminsearch we used in the mathlab code to find the L^p norm misses them consideratbly.

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