Chiral Magnetic Skrymions

Li Wang

1 Introduction

1.1 Background and Energy

A chiral magnetic skrymion is a special kind of magnetic vortex with non-trivial magnetization configuration, which has been observed in magnetic metals [9]. Chiral magnetic skyrmions were first predicted theoretically [3, 4], and then later observed experimentally.

In micromagnetics, a planar magnet is described by the magnetization vector $\vec{m} = (m_1, m_2, m_3)$, normalized as $|\vec{m}| = 1$. Consider a micromagnetic energy including the following interactions:

- exchange (isotropic): $\mathcal{E}_e = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{m}|^2 dx$
- anisotropy (easy axis along $\hat{k}$): $\mathcal{E}_a = \frac{1}{2}\int_{\mathbb{R}^2}(1 - m_3^2)dx$
- Zeeman (constant external field in $\hat{k}$ direction): $\mathcal{E}_z = \int_{\mathbb{R}^2}(1 - m_3)dx$
- Dzyaloshinskii-Moriya (spin-orbit coupling): $\mathcal{E}_{DM} = \int_{\mathbb{R}^2} \vec{m}\cdot(\nabla\times\vec{m})dx$

The DM interaction is an antisymmetric spin coupling produced in magnetic crystals [4, 10].

The statics configurations of the magnet are critical points. The total energy of "micromagnetic" is [3, 4]

$$\mathcal{E} = \mathcal{E}_e + a\mathcal{E}_a + b\mathcal{E}_z + \kappa\mathcal{E}_{DM}$$

The magnetic configurations of skrymions are classified by the integer skrymion number

$$Q(\vec{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \vec{m}\cdot\partial_1\vec{m}\times\partial_2\vec{m}\ dx$$

It has been shown that magnetic vortices can be stabilized in magnetic materials by DM interactions [7]. Essentially, the Zeeman and anisotropy can be reduced by dilation, while exchange energy stays invariant. Therefore, the chiral term plays a central role in stabilizing the vortex-type structure [7].

- $\lambda > 0, x \mapsto \frac{1}{\lambda}x, \implies \mathcal{E}_e \mapsto\ \mathcal{E}_e, \quad \mathcal{E}_a \mapsto \lambda^2\mathcal{E}_a, \quad \mathcal{E}_z \mapsto \lambda^2\mathcal{E}_z, \quad \mathcal{E}_{DM} \mapsto \lambda\mathcal{E}_{DM}$

The presence of an additional balancing term limits the energy gain under dilation. However, if the total energy incorporates only the exchange and DM energy, the compactness of the energy fails, because the DM energy is not bounded below. It was found that skyrmions exist even in high fields without collapse, and remain stable in zero and negative applied field with $\kappa < 1$ [9]. Results about the relation between external field and DM interaction for existence of the isolated skrymions has been studied in [5].

The sign of DM parameter κ determines the direction of rotation of skrymion. The radial stability of isolated chiral skrymion has been studied by [4]. Numerical experiments show that solutions are stable against small radial perturbation as long as the applied magnetic field is nonnegative[4]. The dynamics of skrymions with skrymion number Q = 1, and Q = 0, with easy-axis anisotropy and the effect of an applied

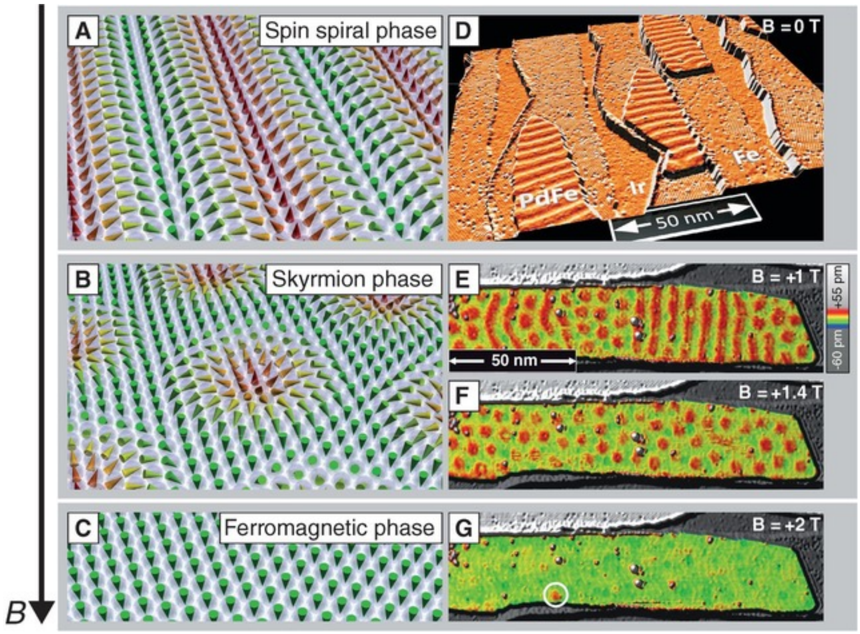

Figure 1: Romming et. al., *Science* 09 Aug 2013: Vol. 341

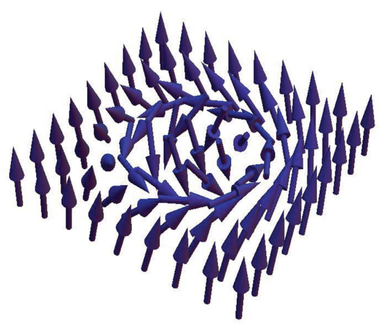

Figure 2: [1] A chiral skyrmion of degree $Q = 1$ with sense of rotation as preferred for $\kappa > 0$

magnetic field, has been studied by [8]. Under spin-transfer torque, Q = 1 skrymion was found to converge to a steady state with constant velocity at an angle to the current flow [8]. Without DM interaction, the vortex structures are shown to be unstable [4]. Under negative applied magnetic field, large core vortices are shown to be unstable because the smallest eigenvalue of the second variation operator is negative [5].

Without the symmetry assumption, [1] proves that the least energy among fields with $Q = 1$ topological charge is attained by studying the range of existence of isolated skyrmions as energy minimizer in a simplified ferromagnetic model including Zeeman effect and DM interaction. By comparing the lower bound of energy for different skyrmion numbers, Melcher obtained compactness from concentration-compactness argument, ruling out the cases of Vanishing and Dichotomy. The case of vanishing means the corresponding energy of minimizing sequence goes to zero locally in space. The case of dichotomy means the existence of minimizing sequences where the support of their energy is in two region separated far enough. Concentration(Skrymion collapse) is excluded by energy upper bound. The topological degree $Q = 1$ was shown to give the lowest possible energy among the non-zero degrees, due to the direction of Zeeman field pointing upward along z axis.

Theorem 1. *[1] Suppose $\kappa \neq 0$, and $h > \kappa^2$. Then the minimum of*

$$E(\vec{m}) = \int_{\mathbb{R}^2} \big(\frac{1}{2}|\nabla\vec{m}|^2 + \frac{h^2}{2}|\vec{m} - \hat{e_3}|^2 + \kappa\vec{m}\cdot(\nabla\times m)\big)dx$$

among all admissible fields with $Q(\vec{m}) = -1$ is attained by a smooth field $\vec{m}$ in this homotopy class. If $h \geq 2\kappa^2$, then the least energy in all other non-trivial homotopy classes is strictly larger.

1.2 Landau-Lifshitz

The dynamics of the magnetization is given by [15] *Landau-Lifschitz* equations with

$$\frac{\partial\vec{m}(x)}{\partial t} = \alpha\vec{m}\times\mathcal{E}'(\vec{m}) - \beta\mathcal{E}'(\vec{m}), \quad \beta \geq 0$$

$$\mathcal{E}(\vec{m}) = \mathcal{E}_e + a\mathcal{E}_a + b\mathcal{E}_z + \kappa\mathcal{E}_{DM}$$

$$\begin{aligned}
\mathcal{E}'_e(\vec{m}) &= -P_{T_{\vec{m}}\mathbb{S}^2}\Delta\vec{m} = -\Delta\vec{m} - |\nabla\vec{m}|^2\vec{m}\\
\mathcal{E}'_a(\vec{m}) &= -P_{T_{\vec{m}}\mathbb{S}^2}m_3\hat{k} = m_3(m_3\vec{m} - \hat{k})\\
\mathcal{E}'_z(\vec{m}) &= -P_{T_{\vec{m}}\mathbb{S}^2}\hat{k} = m_3\vec{m} - \hat{k}\\
\mathcal{E}'_{DM}(\vec{m}) &= -2P_{T_{\vec{m}}\mathbb{S}^2}\nabla\times\vec{m} = 2(\nabla\times\vec{m} - (\vec{m}\cdot(\nabla\times\vec{m}))\vec{m})
\end{aligned}$$

The Euler-Lagrange Equation for critical points of $\mathcal{E}$ (statics solutions) is

$$\begin{aligned}
0 = \mathcal{E}'(\vec{m}) &= P_{T_{\vec{m}}\mathbb{S}^2}\left(-\Delta\vec{m} - 2\kappa\nabla\times\vec{m} - \beta\hat{k} - \alpha m_3\hat{k}\right)\\
&= -\Delta\vec{m} - |\nabla\vec{m}|^2\vec{m} - 2\kappa\left(\nabla\times\vec{m} - (\vec{m}\cdot(\nabla\times\vec{m}))\vec{m}\right) + (b + am_3)(m_3\vec{m} - \hat{k}),
\end{aligned}$$

We can rewrite the *Landau-Lifshitz* equation as

$$\partial_t\vec{m} = \alpha J\mathcal{E}'(\vec{m}) - \beta\mathcal{E}'(\vec{m}),$$

where $J\vec{\xi} := \vec{m}\times\vec{\xi}$ for $\xi \in T_{\vec{m}}\mathbb{S}^2$, J acting on ξ is a $\frac{\pi}{2}$ rotation on $T_{\vec{m}}\mathbb{S}^2$.

Studies on the following *Landau-Lifshitz equations* incorporating only the "exchange energy" have been made[12, 13].

$$\frac{\partial\vec{m}}{\partial t} = -\alpha\vec{m}\times\Delta\vec{m} + \beta(\Delta\vec{m} + |\nabla\vec{m}|^2\vec{m}) \tag{1}$$

In space dimension 2, the energy of is invariant under scaling. The energy of our case is not because of the anisotropy and DM term. The case with $\alpha = 0$, $\beta \neq 0$ corresponds to the harmonic map heat flow. On the other hand, the case with $\beta = 0$, $\alpha \neq 0$, (i.e no dissipation term) corresponds to the Schr̆odinger maps. The questions of local well-posedness and global results of equivarant solutions with energy near the harmonic map energy have been studied in [11, 12, 13, 14].

We are interested in properties of skrymion solutions such as: symmetry, stability, and nearby dynamics. Our first result in this direction is the existence of minimizer in a symmetry subclass: see Theorem 2. Our configuration is pinned at the origin, while [1] needs to consider the translation invariance because the center of vortex is not fixed. Thus our minimizing problem is somehow simpler.

2 Symmetries

2.1 Energy invariance

Let

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e^{\alpha R} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

that is, $e^{\alpha R}$ is a rotation in $\mathbb{R}^3$ by angle α about $\hat{k}$, and R is the generator of these rotations. Similarly, denote $2D$ rotations:

$$\tilde{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{\alpha \tilde{R}} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

We will call a map $\vec{m} : \mathbb{R}^2 \mapsto \mathbb{S}^2 \subset \mathbb{R}^3$ **equivariant** if

$$e^{-\alpha R}\vec{m}(e^{\alpha \tilde{R}}x) = \vec{m}(x) \text{ for all } x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}.$$

Note that by applying $e^{\alpha R}$ to both sides, this is equivalent to $\vec{m}(e^{\alpha \tilde{R}}x) = e^{\alpha R}\vec{m}(x)$ (a spatial rotation through angle α produces a target-space ($\mathbb{S}^2$) rotation through α).

Consider the following transformations:

- scaling: $\vec{m}(x) \mapsto \vec{m}(\lambda x)$
- spatial translation: $\vec{m}(x) \mapsto \vec{m}(x + y), \quad y \in \mathbb{R}^2$
- spatial rotations: $\vec{m}(x) \mapsto \vec{m}(e^{\alpha \tilde{R}}x)$
- target rotations: $\vec{m}(x) \mapsto e^{\alpha R}\vec{m}(x)$
- compensating spatial and target rotations: $\quad \vec{m} \mapsto e^{-\alpha R}\vec{m}(e^{\alpha \tilde{R}}\cdot)$

E_e is invariant under scaling.
All the energy term are invariant under spatial translation.
Each of the energy terms E_e, E_a, and E_z are separately invariant under spatial rotations and target rotations. This is immediate for E_a and E_z, because they do not involve derivatives (and so do not see the spatial rotation), and moreover only involve m_3 (which is unchanged by target rotation). The invariance of the Dirichlet energy E_e under spatial rotation is a classical fact, and easily checked, as is its invariance under target rotations.
This is *not* true for the DM energy. But if we *combine* spatial and target rotations, we *do* have invariance:

Lemma 1. *For any* $\alpha \in \mathbb{R}$,

$$E_{DM}\left(e^{-\alpha R}\vec{m}(e^{\alpha \tilde{R}}\cdot)\right) = E_{DM}(\vec{m}) \tag{2}$$

2.2 Preservation of equivariance by the Landau-Lifshitz dynamics

First, let's observe that if an energy functional E is *invariant* under a unitary group of symmetries, then its variational derivative E' is *covariant*:

Lemma 2. *If* $e^{\alpha T}$ *is a one-parameter group of transformations of maps from (in our case)* $\mathbb{R}^2 \mapsto \mathbb{S}^2 \subset \mathbb{R}^3$ *which acts unitarily on tangent vector fields: denoting as usual* $\langle \vec{f},\, \vec{g} \rangle = \int_{\mathbb{R}^2} \vec{f}(x) \cdot \vec{g}(x)\, dx$,

$$\vec{\xi}(x), \vec{\eta}(x) \in T_{\vec{m}(x)}\mathbb{S}^2 \quad \Longrightarrow \quad \langle \vec{\xi},\, e^{\alpha T}\vec{\eta} \rangle := \langle e^{-\alpha T}\vec{\xi},\, \vec{\eta} \rangle$$

such that $E(e^{\alpha T}\vec{m}) = E(\vec{m})$ *for all* $\alpha \in \mathbb{R}$, *then*

$$E'(e^{\alpha T}\vec{m}) = e^{\alpha T}E'(\vec{m}) \tag{3}$$

For the micromagnetic energy E, we'd like to apply this to the transformation group

$$\vec{m}(x) \mapsto \left(e^{\alpha T}\vec{m}\right)(x) = e^{-\alpha R}\vec{m}\left(e^{\alpha\tilde{R}}x\right)$$

which leaves E invariant. Note that $e^{\alpha T}$ acts unitarily, because each of the target and spatial rotation operators are separately act unitarily (easy to check). If $\vec{m}$ is equivariant, then by (3),

$$e^{\alpha T}\vec{m} = \vec{m} \implies E'(\vec{m}) = E'(e^{\alpha T}\vec{m}) = e^{\alpha T}E'(\vec{m})$$

and so we see $E'(\vec{m})$ is also equivariant. Thus we expect that the solution of the gradient flow $\vec{m}_t = -E'(\vec{m})$ with equivariant initial data should remain equivariant.

We can be more precise. The generating operator T can be found simply by differentiating:

$$T\vec{m}(x) = \partial_\alpha \left(e^{\alpha T}\vec{m}\right)(x)|_{\alpha=0} = -R\vec{m}(x) + (\tilde{R}x \cdot \nabla)\vec{m}(x) = (\partial_\theta - R)\vec{m}(x)$$

so $T = \partial_\theta - R$, just the sum of the generators of the target and spatial rotations. (The fact that $e^{\alpha T}$ is unitary is reflected in the fact that its generator T is skew-adjoint: $T^* = -T$.) By differentiating with respect to α at $\alpha = 0$, we find an equivalent condition for $\vec{m}$ to be equivariant:

$$e^{\alpha T}\vec{m} = \vec{m} \quad\implies\quad 0 = T\vec{m} = (\partial_\theta - R)\vec{m} \;\; (\in T_{\vec{m}(x)}\mathbb{S}^2),$$

and a similar condition at the level of tangent vector fields which we should differentiate *covariantly* ($D_\mu\vec{\xi} = P_{T_{\vec{m}}\mathbb{S}^2}\partial_\mu\xi$):

$$\vec{\xi}(x) \in T_{\vec{m}(x)}\mathbb{S}^2, \;\; e^{\alpha T}\vec{\xi} = \vec{\xi} \quad\implies\quad 0 = PT\vec{\xi} = (D_\theta - PR)\vec{\xi} \;\; (\in T_{\vec{m}(x)}\mathbb{S}^2). \tag{4}$$

Suppose $\vec{m}(x,t)$ solves the Landau-Lifshitz equation

$$\partial_t\vec{m} = aJE'(\vec{m}) - bE'(\vec{m}),$$

where recall $J\vec{\xi} := \vec{m} \times \vec{\xi}$ for $\xi \in T_{\vec{m}}\mathbb{S}^2$. Then

$$D_tT\vec{m} = D_t(\partial_\theta - R)\vec{m} = (D_\theta - PR)\partial_t\vec{m} = (D_\theta - PR)(aJE'(\vec{m}) - bE'(\vec{m}))$$

where we used the fact (easily checked) that usual and covariant derivatives commute: $D_\mu\partial_\nu\vec{m} = D_\nu\partial_\mu\vec{m}$. Using (4), together with the (straightforward to check) facts that J commutes with covariant derivatives, and with PR, we see that is $\vec{m}$ is (initially) equivariant, then

$$D_tT\vec{m} = 0$$

and so we expect the equivariance condition $T\vec{m} = 0$ to be preserved by the dynamics (though this is not yet a proof).

2.3 A special case of equivariant maps

Consider now the (discrete) transformation

$$\vec{m}(x) \mapsto (F\vec{m})(x) := \begin{bmatrix} -m_1(x_1,-x_2) \\ m_2(x_1,-x_2) \\ m_3(x_1,-x_2) \end{bmatrix}$$

which is in fact a reflection: $F^2 = I$ (indeed the composition of a reflection on the domain $\mathbb{R}^2$ and a reflection on the target $\mathbb{S}^2 \subset \mathbb{R}^3$). It is easily checked that

$$\left(F\vec{m} \cdot \nabla \times (F\vec{m})\right)(x) = \left(\vec{m} \cdot \nabla \times \vec{m}\right)(x_1,-x_2)$$

and so $E_{DM}(F\vec{m}) = E_{DM}(\vec{m})$ (since the Jacobian of $(x_1, x_2) \mapsto (x_1, -x_2)$ is just 1). Is is also clear the other terms in the energy are invariant under F: $E(F\vec{m}) = E(\vec{m})$. As above this implies

$$E'(F\vec{m}) = FE'(\vec{m}) \text{ and so: } F\vec{m} = \vec{m} \implies FE'(\vec{m}) = E'(\vec{m}).$$

Maps which are both equivariant (as above) and invariant under F are exactly those which we first considered:

$$e^{\alpha T}\vec{m} = \vec{m} \text{ and } F\vec{m} = \vec{m} \quad \implies \quad \vec{m}(r,\theta) = e^{\theta R}\begin{bmatrix} 0 \\ \sin(u(r)) \\ \cos(u(r)) \end{bmatrix} = \begin{bmatrix} -\sin(u(r))\sin(\theta) \\ \sin(u(r))\cos(\theta) \\ \cos(u(r)) \end{bmatrix}.$$

Now for the gradient-flow, $\partial_t \vec{m} = E'(\vec{m})$, we expect the class $F\vec{m} = \vec{m}$ to be preserved, since if this holds initially, then

$$\partial_t(F\vec{m} - \vec{m}) = (F - I)\partial_t\vec{m} = (F - I)E'(\vec{m}) = 0.$$

For the Schrödinger flow $\partial_t\vec{m} = JE'(\vec{m})$, however,

$$\partial_t(F\vec{m} - \vec{m}) = FJE'(\vec{m}) - JE'(\vec{m}),$$

but F and J do *not* commute. In fact, they *anticommute*: for $\vec{\xi} \in T_{\vec{m}}\mathbb{S}^2$,

$$FJ^{\vec{m}}\vec{\xi} = -J^{F\vec{m}}F\vec{\xi}$$

so if (initially) $F\vec{m} = \vec{m}$,

$$\partial_t(F\vec{m} - \vec{m}) = -2J^{\vec{m}}E'(\vec{m}) \neq 0$$

and so this condition is *not* preserved by the Schrödinger term.
Thus, we consider a larger class of equivariant map, which does not have the property $F(\vec{m}) = \vec{m}$.

$$\vec{m} = (-\sin(u(r)\sin(\theta + f(r)), \sin(u(r))\cos(\theta + f(r)), \cos(u(r)))$$

3 Minimize $E(\vec{m})$ in the isotropic case: harmonic maps

Before proceeding, we quickly review the minimization problem in the isotropic case.
Minimize $E_e(\vec{m}) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{m}|^2\, dx \qquad \vec{m} : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$
The true ground state is purely ferromagnetic: $\vec{m}(x) \equiv c \in \mathbb{S}^2$
Consider a local deviation from a pure ferromagnetic state:
$\lim_{|x|\to\infty} \vec{m}(x) = \hat{k} \qquad \implies \qquad \vec{m} : \mathbb{R}^2 \approx \mathbb{S}^2 \to \mathbb{S}^2$
Recall the " Skyrmion number " is:

$$Q = \text{ degree}(\vec{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \vec{m}\cdot(\partial_1\vec{m} \times \partial_2\vec{m}) \in \mathbb{Z}$$

"Bogomolnyi lower bound" is

$$\begin{aligned} E_e(\vec{m}) &= \frac{1}{2}\int |\partial_1\vec{m} \mp J\partial_2\vec{m}|^2 \pm \int \partial_1\vec{m}\cdot J\partial_2\vec{m} \\ &= \frac{1}{2}\int |\partial_1\vec{m} \mp J\partial_2\vec{m}|^2 \pm 4\pi Q \quad \geq 4\pi|Q| \end{aligned}$$

where $\min\{E_e(\vec{m}) \mid deg(\vec{m}) = Q\} = 4\pi|Q|$ is attained if $\partial_1\vec{m} = \pm J\partial_2\vec{m}$.
If we use the stereographic projection $\psi = \frac{m_1 + im_2}{1 + m_3}$

$$\partial_1\vec{m} = \pm J\partial_2\vec{m} \;\leftrightarrow\; (\partial_1 \pm i\partial_2)\psi = 0, \text{ which is the Cauchy-Riemann}$$

ψ is an explicit $4|N|$-dim. family of minimizers (harmonic maps)

4 Minimization in subclass

For a 2D magnetization vector field

$$\vec{m} = (m_1, m_2, m_3) : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3, \quad |\vec{m}(x)|^2 = m_1^2(x) + m_2^2(x) + m_2^2(x) \equiv 1,$$

consider a micromagnetic energy including exchange energy, either an easy-axis anisotropy a Zeeman (external field) energy, and a Dzyaloshinskii-Moriya energy:

$$\mathcal{E}(\vec{m}) = \mathcal{E}_e + \alpha\mathcal{E}_a + \mathcal{E}_z + \mathcal{E}_{DM} = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{m}|^2 dx + \frac{\alpha}{2}\int_{\mathbb{R}^2}(1-m_3^2)dx + h\int_{\mathbb{R}^2}(1-m_3)dx + \kappa\int_{\mathbb{R}^2} m_3(\partial_1 m_2 - \partial_2 m_1)dx$$

where $\kappa \in \mathbb{R}$, $h \geq 0$, (and the other coefficients have been normalized by rescaling).

Consider a subclass of equivariant magnetizations of the form

$$\vec{m}(x) = (-\sin(u(r))\sin(\theta), \sin(u(r))\cos(\theta), \cos(u(r))), \quad u : [0,\infty) \to \mathbb{R} \tag{5}$$

where $(r = |x|, \theta)$ are polar coordinates on $\mathbb{R}^2$.

For magnetization of this form, the energy becomes

$$E(u) := \frac{1}{2\pi}\mathcal{E}(\vec{m}) = E_e(u) + \alpha E_a + hE_z + \kappa E_{DM}(u)$$

where

$$E_e(u) = \frac{1}{2}\int_0^\infty \left(u_r^2 + \frac{1}{r^2}\sin^2(u(r))\right) rdr$$
$$E_a(u) = \frac{1}{2}\int_0^\infty \sin^2(u(r))rdr$$
$$E_z(u) = \int_0^\infty (1-\cos(u(r)))rdr.$$

Note that finiteness of E_e requires (in some sense)

$$\sin(u(r)) \to 0 \text{ as } r \to 0, \infty, \tag{6}$$

while finiteness of E_a and E_z just requires this for $r \to \infty$. Finally, since $\partial_{x_1} m_2 - \partial_{x_2} m_1 = \frac{1}{r}(\sin(u)r)_r$, integration by parts gives

$$E_{DM}(u) = \int_0^\infty \sin^2(u(r))u_r rdr.$$

The radial function $u(r)$ satisfies the following equation if it is a critical point.

$$-u_{rr} - \frac{1}{r}u_r + \frac{\sin(u(r))\cos(u(r))}{r^2} + \alpha\cos(u(r))\sin(u(r)) + h\sin(u(r)) - \kappa\frac{1}{r}\sin^2(u(r)) = 0$$

For positive κ, u(r) is expected to be a decreasing function [5].

Note that since $\sin^2(u) \leq 2(1-\cos(u))$, $\implies E_a(u) \leq E_z(u)$

$$E(u) \geq \begin{cases} E_e(u) + (h+\alpha)E_a(u) + \kappa E_{DM}(u) & \alpha \geq 0 \\ E_e(u) + (h+\alpha)E_z(u) + \kappa E_{DM}(u) & -h \leq \alpha < 0 \end{cases} \tag{7}$$

which, together with the observation that $\{\cos(u) = 1\} \subsetneqq \{\sin(u) = 0\}$, suggests the anisotropy problem may be more difficult than the Zeeman problem in theorem 1 .

4.1 Topology and the variational problem

The boundary conditions consistent with (6) are

$$u(0) = n\pi,\ u(\infty) = j\pi,\ \ n, j \in \mathbb{Z} \quad (\implies \vec{m}(0) = (-1)^n \hat{k},\ \vec{m}(\infty) = (-1)^j \hat{k}).$$

For those boundary conditions, the skyrmion number (topological degree) is

$$\begin{aligned} deg(u) := Q(u) &= \frac{1}{4\pi}\int_{\mathbb{R}^2} \vec{m}\cdot\partial_1\vec{m}\times\partial_2\vec{m} = \frac{1}{2}\int_0^\infty \sin(u(r))u_r dr = \frac{1}{2}[\cos(u(0)) - \cos(u(\infty))] \\ &= \frac{1}{2}[\cos(n\pi) - \cos(j\pi)] = (-1)^n \left\{ \begin{array}{ll} 1 & n-j \text{ odd} \\ 0 & n-j \text{ even} \end{array} \right\}. \end{aligned}$$

We can reduce our study to the cases $j = 0$, $n \geq 0$ by noting some discrete symmetries:

- $u \mapsto u + \pi$ ($\leftrightarrow \vec{m} \mapsto -\vec{m}$): preserves E, flips sign of deg;
- $u \mapsto -u$ ($\leftrightarrow x \mapsto -x$), $\kappa \mapsto -\kappa$: leaves E, deg invariant.

So let us consider E on the space

$$X_n = \{u \in H^1([0,\infty), rdr) \mid (u - n\pi)/r \in L^2([0,1], rdr)\}.$$

Remark 1. *Note the X_n are topologically distinct: the boundary condition at $r = 0$ cannot jump by an integer multiple of π under continuous deformation. This is despite the fact that $deg = 0$ for all even n (and $deg = -1$ for all odd n), so by a classical result in topology, such a configuration can be continuously deformed to a constant map. The point is that an $\vec{m}$ corresponding to $u \in X_2$, for example, can only be continuously deformed to a constant (which would correspond to X_0) by leaving the equivariant class* (5).

The natural variational problem is to minimize the energy:

$$\boxed{I_n := \inf_{u\in X_n} E(u)} \tag{8}$$

To begin the study of (8), we investigate if the energy is bounded from below. First, we bound the size of E_{DM} by E_e and E_a, or E_e and E_z :

Lemma 3. *For any $u \in H^1$,*

$$|\kappa E_{DM}(u)| \leq \left\{ \begin{array}{ll} \min\left(\frac{\kappa^2}{h+\alpha}E_e(u) + (h+\alpha)E_a(u), (h+\alpha)E_e(u) + \frac{\kappa^2}{h+\alpha}E_a(u)\right) & \alpha \geq 0 \\ \min\left(\frac{\kappa^2}{h+\alpha}E_e(u) + (h+\alpha)E_z(u), (h+\alpha)E_e(u) + \frac{\kappa^2}{h+\alpha}E_z(u)\right), & -h < \alpha < 0 \end{array} \right. \tag{9}$$

Proof. This follows immediately from the elementary inequality

$$\kappa E_{DM} \leq \int_0^\infty |\sin(u)||\kappa||u_r| rdr \leq \left\{ \begin{array}{l} \frac{\kappa^2}{2(h+\alpha)}\int_0^\infty u_r^2\, rdr + \frac{h+\alpha}{2}\int_0^\infty \sin^2(u) rdr \\ \frac{h+\alpha}{2}\int_0^\infty u_r^2\, rdr + \frac{\kappa^2}{2(h+\alpha)}\int_0^\infty \sin^2(u) rdr \end{array} \right.$$

along with (7). □

Next, we recall a classical lower bound on the exchange energy, which, in this equivariant setting, can be refined to provide a lower bound on intervals:

Lemma 4. *For any $u \in \dot{H}^1_{loc}$, $[a\ b] \subset [0,\infty)$,*

$$E_e^{[a\ b]}(u) := \frac{1}{2}\int_a^b \left(u_r^2 + \frac{1}{r^2}\sin^2(u(r))\right) rdr \geq |\cos(b) - \cos(a)| \tag{10}$$

with equality if and only if u agrees on $[a\ b]$ with the 'bubble'

$$Q(r) = \pi - 2\tan^{-1}(r), \quad Q'(r) = -\frac{1}{r}\sin(Q(r)), \quad E_e(Q) = E_e(Q^s) = \min_X E_e = 2, \tag{11}$$

one of its scalings $Q^s(r) = Q(r/s)$, $s > 0$, or an inversions $(-Q^s)$ or shift $(\pm Q^s + \pi\mathbb{Z})$ of one of these.

Proof. By the simple pointwise inequality $|u_r \ \sin(u)\ \frac{1}{r}| \leq \frac{1}{2}(u_r^2 + \frac{\sin^2(u)}{r^2})$,

$$E_e^{[a\ b]} \geq \left| \int_a^b -u_r \sin(u(r))\ dr \right| = \left| \int_a^b [\cos(u(r))]_r\ dr \right| = |\cos(u(b)) - \cos(u(a))|.$$

Moreover, equality holds if and only if $u_r = \pm\frac{1}{r}\sin(u)$ holds (almost everywhere) on $(a\ b)$, and $\cos(u(r))$ is monotone. It can be shown that the only such solutions to this ode are the ones listed above. $\square$

As a simple corollary, we have a lower bound for the full exchange energy on X_n:

Lemma 5.

$$u \in X_n \implies E_e(u) \geq 2|n|. \tag{12}$$

Proof. Suppose $n \geq 0$ (the $n < 0$ argument is almost identical). By continuity, there are $r_0 = 0 < r_1 < r_2 \cdots < r_n = \infty \in [0,\infty]$ such that $u(r_0) = n\pi$, $u(r_1) = (n-1)\pi$, $\ldots u(r_n) = 0$, and so applying (10) on each of the n intervals $[r_0\ r_1]$, $[r_1\ r_2]$, $\ldots$, $[r_{n-1}\ r_n]$ we arrive at (12). $\square$

This topological lower bound on the exchange energy has the following immediate consequence for the full energy, combining (9) and (12):

Lemma 6.

$$\kappa^2 \leq h + \alpha \implies I_n \geq 2|n|(1 - \frac{\kappa^2}{h+\alpha}). \tag{13}$$

Proof.

$$\alpha E_a(u) + hE_z(u) = \frac{\alpha}{2}\int_0^\infty \sin^2(u)rdr + h\int_0^\infty (1-\cos(u))rdr \geq \begin{cases} \frac{1}{2}(h+\alpha)\int_0^\infty \sin^2(u)rdr = (h+\alpha)E_a(u) & \alpha \geq 0 \\ (h+\alpha)\int_0^\infty (1-\cos(u))rdr = (h+\alpha)E_z(u) & -h \leq \alpha \leq 0 \end{cases}$$

Combining (9). The inequality follows immediately. $\square$

It will be useful to have a variant of (10) which also exploits the anisotropy/Zeeman, in order to get a decay rate as $r \to \infty$ (and hence some compactness) – this is similar to the classical Strauss lemma [16] giving compact embedding of H^1 into a Lebesgue space for radial functions:

Lemma 7. *For $u \in H^1_{rdr}$,*

$$1 - \cos(u(r)) + \sin^2(u(r)) \leq \frac{6}{r}E_a(u)^{1/2}E_e(u)^{1/2} \leq \frac{6}{r}E_z(u)^{1/2}E_e(u)^{1/2}. \tag{14}$$

Proof. Since $1-\cos(u(r)) \to 0$ as $r \to \infty$, by the fundamental theorem of calculus, and Cauchy-Schwarz,

$$1-\cos(u(r)) = \int_r^\infty \frac{1}{s}\sin(u(s))u_r(s)\; sds \le \frac{1}{r}\|\sin(u)\|_{L^2_{rdr}}\|u_r\|_{L^2_{rdr}} \le \frac{2}{r}E_a^{1/2}E_e^{1/2}.$$

And similarly for $\sin^2(u)$:

$$\sin^2(u(r)) = -2\int_r^\infty \frac{1}{s}\sin(u(s))\cos(u(s))u_r(s)\; sds \le \frac{2}{r}\|\sin(u)\|_{L^2_{rdr}}\|u_r\|_{L^2_{rdr}} \le \frac{4}{r}E_a^{1/2}E_e^{1/2}.$$

The last inequality in (14) follows from (7). □

As a partial converse to the lower bound (13), we have:

Proposition 1. $\kappa^2 > 9(\alpha+h) \implies I_n = -\infty$.

Proof. Take $\kappa > 0$ (a similar construction should work for $\kappa < 0$). Consider, for positive, even (for simplicity) integers $n \ll t \ll M$, a piecewise linear test function of the form

$$u(r) = \left\{ \begin{array}{cc} n\pi + tr & 0 \le r \le \frac{M-n}{t}\pi \\ (M+\frac{M-n}{t})\pi - r & \frac{M-n}{t}\pi \le r \le (\frac{M-n}{t}+M)\pi \\ 0 & r \ge (\frac{M-n}{t}+M)\pi \end{array} \right\} \in X_n.$$

Begin with the exchange energy:

$$\begin{aligned} \frac{1}{2}\int_0^\infty u_r^2\; rdr &= \frac{1}{2}t^2\frac{1}{2}\frac{(M-n)^2}{t^2}\pi^2 + \frac{1}{2}\frac{1}{2}\pi^2\left((\frac{M-n}{t}+M)^2 - \frac{(M-n)^2}{t^2}\right) \\ &= \frac{M^2\pi^2}{4}\left((1-n/M)^2 + 1 + \frac{2}{t}(1-n/M)\right) = \frac{M^2\pi^2}{2}\left(1+O(\frac{1}{t}+\frac{n}{M})\right). \end{aligned}$$

The other part of the exchange is (mainly) bounded by the anisotropy:

$$\begin{aligned} \frac{1}{2}\int_0^\infty \frac{\sin^2(u)}{r^2}\; rdr &\le \frac{1}{2}\int_0^{1/t} t^2\; rdr + \frac{1}{2}\int_{1/t}^{M/t}\frac{dr}{r} + \frac{1}{2}\frac{t^2}{M^2}\int_{M/t}^\infty \sin^2(u)\; rdr \\ &= O\left(1+\log M + \frac{t^2}{M^2}E_a(u)\right), \end{aligned}$$

and combining these gives

$$E_e(u) = \frac{M^2\pi^2}{2}\left(1+O(\frac{1}{t}+\frac{n}{M}+\frac{t^2}{M^4}E_a(u))\right).$$

Next the anisotropy: setting $S(y) = y/2 - \sin(2y)/4$, $\hat{S}(y) = y^2/4 - y\sin(2y)/4 + \sin^2(y)/4$, so that $S' = \sin^2(y)$ and $\hat{S}' = y\sin^2(y)$,

$$\begin{aligned} E_a(u) = \frac{1}{2}\int_0^\infty \sin^2(u(r))\; rdr &= \frac{1}{2}\int_0^{\frac{M-n}{t}\pi}\sin^2(tr)\; rdr + \frac{1}{2}\int_{\frac{M-n}{t}\pi}^{(\frac{M-n}{t}+M)\pi}\sin^2(r-\frac{M-n}{t}\pi)\; rdr \\ &= \frac{1}{2t^2}\hat{S}((M-n)\pi) + \frac{1}{2}\hat{S}(M\pi) + \frac{(M-n)\pi}{2t}S(M\pi) = \frac{M^2\pi^2}{8}\left(1+O(\frac{1}{t})\right). \end{aligned}$$

The Zeeman term is similar: set $\tilde{S}(y) = y^2/2 - y\sin(y) - \cos(y)$ so that $\tilde{S}' = y(1-\cos(y))$,

$$\begin{aligned} E_z(u) = \int_0^\infty (1-\cos(u(r)))\; rdr &= \int_0^{\frac{M-n}{t}\pi}(1-\cos(tr))\; rdr + \int_{\frac{M-n}{t}\pi}^{(\frac{M-n}{t}+M)\pi}(1-\cos(r-\frac{M-n}{t}\pi)\; rdr \\ &= \frac{1}{t^2}\tilde{S}((M-n)\pi) + \tilde{S}(M\pi) + \frac{(M-n)\pi}{t}M\pi = \frac{M^2\pi^2}{4}\left(1+O(\frac{1}{t})\right). \end{aligned}$$

Finally the D-M term (which is the source of negative energy):

$$\kappa E_{DM}(u) = \kappa \int_0^\infty u_r \sin^2(u)\, r dr = 2\kappa t E_a^{[0,(M-n)\pi/t]} - 2\kappa E_a^{[(M-n)\pi/t,(M+(M-n)/t)\pi]}$$
$$= -\kappa \frac{M^2\pi^2}{4}\left(1 + O(1/t)\right).$$

If we scale $\tilde{u}(r) := u(r\sqrt{a+h})$ So in total

$$E(\tilde{u}) = E_e(\tilde{u}) + \alpha E_a(\tilde{u}) + hE_z(\tilde{u}) + \kappa E_{DM}(\tilde{u}) \leq \frac{M^2\pi^2}{8}(4 + 2 - 2\frac{\kappa}{\sqrt{\alpha+h}} + O(1/t + t^2/M^2)) \to -\infty$$

as $M \gg t \to \infty$, provided $\kappa^2 > 9(\alpha + h)$ $\square$

4.2 Upper bounds

Next, in cases where the energy is bounded below, we'd like to estimate the minimal energy. First:

Lemma 8. *$I_0 \leq 0$, and so $\kappa^2 < h + \alpha \implies I_0 = 0$.*

Proof. This is immediate, since $0 \in X_0$, and $E(0) = 0$. The second statement follows from (13). $\square$

Next:

Proposition 2. *$I_1 \leq 2$. Moreover,*

$$\kappa > 0 \implies I_1 < 2. \tag{15}$$

Proof. Since the bubble Q has infinite E_a and E_z (due to slow spatial decay), let's cut it off: $u_R = Q\phi_R$, where $\phi_R(r) = \phi(r/R)$ and $\phi \geq 0$ is a standard smooth cut-off function with $\phi(r) = 1$ for $r \leq 1$ and $\phi(r) = 0$ for $r \geq 2$. Then $u_R \in X_1$, and as $R \to \infty$ it is easy to check

$$E_e(u_R) = 2 + O(R^{-2}), \quad \left\{ \begin{array}{c} E_a(u_R) \\ E_z(u_R) \end{array} \right\} = O(\log R), \quad \kappa E_{DM} = -A\kappa + o(1),\ A = \int_0^\infty \sin^3(Q(r))\, dr > 0,$$

and so if we scale, $u_R^s(r) := u_R(r/s)$, $s > 0$, we get

$$E(u_R^s) = 2 - A\kappa s + O(R^{-2}) + O(\log R)s^2 + o(1)s$$

so

$$\frac{1}{s}\left[E(u_R^s) - 2\right] = -A\kappa + O(R^{-2})\frac{1}{s} + O(\log R)s + o(1).$$

Then if we take $R \to \infty$ and $s \to 0$ with $s = 1/R$, we see

$$\frac{1}{s}\left[E(u_R^s) - 2\right] \to -A\kappa\ ,$$

which proves both statements of the proposition. $\square$

Remark 2. *If a minimizing sequence in X_1 fails to converge (strongly) due to concentration ('bubbling') at the origin, the minimal energy cost of this is 2 (for the bubble) plus the minimal energy in X_0 (for the weak limit which remains after the bubbling) which by Lemma 8 is 0. If $\kappa > 0$, Proposition 2 shows that the minimizing energy is below 2, and so bubbling should not occur. We make this precise in the next section.*

Conjecture 1. *If $-\sqrt{h+a} \leq \kappa \leq 0$, $I_1 = 2$.*

4.3 Existence of minimizer

Now let's combine our estimates above to prove existence of a minimizer in X_1 if $0 < \kappa < 1$. This is in the spirit of [1], but simpler because of the equivariance. Though replacing Zeeman by anisotropy introduces an extra difficulty.

Theorem 2. *Let $\kappa \in (0, \sqrt{h+\alpha})$. There is $v \in X_1$ with $E(v) = I_1$.*

Proof. By (13) and (15),

$$I_1 \in [2(1 - \frac{\kappa^2}{h+\alpha}), 2) \subset (0, 2). \tag{16}$$

Let $\{u_j\}_{j=1}^{\infty} \subset X_1$ be a minimizing sequence: $E(u_j) \to I_1$. Since $|\kappa| < \sqrt{h+\alpha}$, by (9) (and (7)) we have uniform bounds: $E_e(u_j) \lesssim 1$, $E_a(u_j) \lesssim 1$. We then also have a uniform pointwise bound, $\|u_j\|_\infty \lesssim 1$, due to:

Lemma 9.

$$u \in H^1_{rdr} \implies \|u\|_\infty \le \frac{1}{2} E_e(u) + \pi.$$

Proof. If not, we have $u(R) > \pi + \frac{1}{2}E_e$ for some R. Let $k \in \mathbb{Z}$ be such that $2k\pi \le E_e < 2(k+1)\pi$. Then since $u(R) > (k+1)\pi$, applying (10) on successive subintervals of $[R\ \infty)$ (as in the proof of (12)), we find

$$E_e(u) \ge 2(k+1)\pi > E_e(u),$$

a contradiction. □

It follows from (14) (and continuity) that there is R such that $|u_j(r)| \le \pi/2$ for all $r \ge R$ and j. Then since $|u| \lesssim |\sin(u)|$ when $|u| \le \pi/2$,

$$\int_0^\infty u^2(r)\ rdr = \int_0^R u^2(r)\ rdr + \int_R^\infty u^2(r)\ rdr \lesssim R^2 \|u\|_\infty^2 + E_a(u) \lesssim 1. \tag{17}$$

From here we also get uniform decay as $r \to \infty$, just as in (14):

$$u_j^2(r) = -2\int_r^\infty \frac{1}{s} u_j(s)(u_j)_r(s)\ sds \le \frac{2}{r}\|u_j\|_2 \|(u_j)_s\|_2 \lesssim \frac{1}{r}. \tag{18}$$

Combining (17) with $E_e(u) \lesssim 1$, shows $\|u_j\|_{H^1{}_{rdr}} \lesssim 1$, and so by the standard arguments (weak compactness of bounded sets in H^1 (Alaoglu's theorem), and compactness of the Sobolev embedding of H^1 into L^p on bounded domains (Rellich's lemma)) there is a subsequence (which we still denote $\{u_j\}$) such that

$$\exists\ H^1 \ni v \leftarrow u_j \quad \text{weakly in } H^1_{rdr};\ \text{strongly in } L^p([0,R))_{rdr}\ \forall\ R,\ p < \infty;\ \text{and a.e.} \tag{19}$$

Moreover, since $E_e(u_j) \lesssim 1$, we have $\|\sin(u_j)/r\|_{L^2_{rdr}} \lesssim 1$ and we may assume (passing to a further subsequence if needed) that

$$\frac{\sin(u_j)}{r} \to \frac{\sin(v)}{r} \text{ weakly in } L^2_{rdr}. \tag{20}$$

Additionally, we have uniform convergence away from the origin. This is because on compact intervals $I \subset (0, \infty)$, u_j are uniformly bounded in $H^1(I)_{dr}$ and so the one-dimensional Sobolev embedding gives compactness in $L^\infty(I)$. Combining this observation with the small tail estimate (18):

$$\forall\ R > 0, \quad \lim_{j\to\infty} \left(\sup_{r \ge R} |u_j(r) - v(r)| \right) = 0. \tag{21}$$

As usual, we have weak lower semicontinuity of E_e (since it comes from an inner-product) and E_a and E_z (by Fatou's lemma):

$$E_e(v) \leq \liminf E_e(u_j), \qquad E_a(v) \leq \liminf E_a(u_j), \quad E_z(v) \leq \liminf E_z(u_j). \tag{22}$$

We have full convergence of the DM term, because of the extra $\sin(u)$ factor, and the uniform decay (14):

Lemma 10.

$$E_{DM}(v) = \lim_{j\to\infty} E_{DM}(u_j). \tag{23}$$

Proof. By (14), and Hölder, for any $u \in H^1_{rdr}$,

$$\int_R^\infty \sin^2(u(r))|u_r|\, rdr \leq \sup_{r\geq R} |\sin(u(r)| \left[\int_R^\infty \sin^2 u\, rdr\right]^{1/2} \left[\int_R^\infty u_r^2\, rdr\right]^{1/2} \lesssim \frac{1}{\sqrt{R}} E_a^{3/4} E_e^{3/4}.$$

So given $\epsilon > 0$, choose R such that $\int_R^\infty \sin^2(u_j)|(u_j)_r|\, rdr + \int_R^\infty \sin^2(v)|v_r|\, rdr < \epsilon$. Then

$$\begin{aligned}[E_{DM}(u_j) - E_{DM}(v)] = &\int_0^R (\sin^2(u_j) - \sin^2(v))(u_j)_r\, rdr + \int_0^R \sin^2(v)((u_j)_r - v_r)\, rdr \\ &+ \int_R^\infty \left(\sin^2(u_j)(u_j)_r - \sin^2(v)v_r\right)\, rdr =: I + II + III.\end{aligned}$$

Since $\sin^2$ is a Lipshitz function, and $u_j \to v$ strongly in L^2 on B_R, $\|\sin^2(u_j) - \sin^2(v)\|_{L^2(B_R)} \to 0$, and so by Hölder,

$$|I| \lesssim \|\sin^2(u_j) - \sin^2(v)\|_{L^2(B_R)} E_e^{1/2} \to 0.$$

Weak H^1 convergence of u_j shows $II \to 0$. Finally, by choice of R, $|III| < \epsilon$, which was arbitrary. $\square$

Combining (22) and (23), we have:

$$E(v) \leq \liminf_{j\to\infty} E(u_j) = I_1, \tag{24}$$

and so it remains to show $v \in X_1$. Since $v \in H^1$ and $E_e(v) < \infty$, we have $v \in X_n$ for some n. If $n = 1$, we are finished. So suppose $n \neq 1$.

Lemma 11. *If $v \in X_n$, $n \neq 1$, then*

$$\liminf_{j\to\infty} E_e(u_j) \geq 2 + E_e(v).$$

Proof. Set $w_j = u_j - v$ and write $u_j = v + w_j$ so that

$$E_e(u_j) = E_e(v) + E_e(w_j) + \int_0^\infty v_r(w_j)_r\, rdr + \int_0^\infty \frac{1}{2r^2}(\sin^2(v + w_j) - \sin^2(v) - \sin^2(w_j))\, rdr. \tag{25}$$

By the weak convergence, the first integral on the right $\to 0$. For the second integral, we first claim that

$$\frac{\sin(w_j)}{r} \to 0 \text{ weakly in } L^2_{rdr}. \tag{26}$$

To see this, note

$$\frac{\sin(w_j)}{r} = \cos(v)\left(\frac{\sin(u_j)}{r} - \frac{\sin(v)}{r}\right) + \frac{\sin(v)}{r}(\cos(v) - \cos(u_j)).$$

The first term tends weakly to 0 by (20), and the fact that $\cos(v)$ is bounded. The second term tends to 0 strongly, since for any $R > 0$,

$$\|\frac{\sin(v)}{r}(\cos(v) - \cos(u_j))\|_2 \leq 2\|\frac{\sin(v)}{r}\|_{L^2(B_R)} + \|\cos(u_j) - \cos(v)\|_{L^\infty(B_R^c)}\|\frac{\sin(v)}{r}\|_2,$$

the first term $\to 0$ as $R \to 0$ while the second $\to 0$ (for any fixed R) as $j \to \infty$ by the uniform convergence (21). By trig identities, the second integral on the right of (25) can be written

$$\int_0^\infty \cos(v)\frac{\sin(v)}{r}\cos(w_j)\frac{\sin(w_j)}{r}\, rdr - \int_0^\infty \frac{\sin^2(v)}{r^2}\sin^2(w_j)\, rdr.$$

The first integral $\to 0$ by (26) (and boundedness of $\cos(w_j)$ and $\cos(v)$, and $\sin(v)/r \in L^2$), while the second integral is bounded by

$$\int_0^R \frac{\sin^2(v)}{r^2}\, rdr + \sup_{r \geq R} \sin^2(w_j(r))\|\sin(v)/r\|_2,$$

with the first term $\to 0$ as $R \to 0$, and the second $\to 0$ (for fixed R) as $j \to \infty$ by (21) (and trig identities). Thus from (25) we see

$$\liminf_{j\to\infty} E_e(u_j) \geq \liminf_{j\to\infty} E_e(w_j).$$

Finally, since $w_j(0) = (1-n)\pi$, the lower bound (12) shows $E_e(w_j) \geq 2|1-n| \geq 2$ (by assumption), and the lemma follows. □

Combining (22), (23), and (24), we get

$$I_1 = \liminf_{j\to\infty} E(u_j) \geq 2,$$

which contradicts (16) and so completes the proof of the theorem. □

5 Monotonicity

If h $= 0$(no Zeeman effect), $\kappa > 0$, $\alpha > 0$, $\delta := \frac{\kappa^2}{\alpha}$
Recall that $u(r)$ satisfies the Euler-Lagrange equation

$$-u_{rr} - \frac{1}{r}u_r + \frac{1}{r^2}\sin(u)\cos(u) - \frac{\kappa}{r}\sin^2(r) + \alpha\sin(u)\cos(u) = 0 \tag{27}$$

with the boundary conditions $u(0) = \pi$, $\lim_{r\to\infty} u(r) = 0$. Since u is a minimizer, $E(u) < 2$
We first need a few following lemmas:

Proposition 3. *If* $\delta < \sqrt{11} - 3$*,* $0 < u(r) \leq \pi$

Proof. We use proof by contradiction here.
If u(r) has a local maximum at r_1, and $u(r_1) > \pi$
By maximal principle, $u_{rr}(r_1) \leq 0$, so

$$(\frac{1}{r^2} + \alpha)\sin(u)\cos(u) - \frac{\kappa}{r}\sin^2(u) \leq 0 \tag{28}$$

Next, we consider two cases, $\sin(u) \geq 0$, and $\sin(u) < 0$

- If $\sin(u(r_1)) \geq 0$, $u \geq 2\pi$.
  By the lower topological bounds (13), $E(u) \geq (1-\delta)[|\cos(u(r_1)) - \cos(u(0)| + 2] \geq 3(1-\delta) > 2$, which is a contradiction.

- If $\sin(u(r_1)) \leq 0$, by(28) $[(\frac{1}{r^2}+\alpha) - \kappa\frac{1}{r}\tan(u)]\cos(u) \geq 0$, For the case $\cos(u(r_1)) \geq 0$, $u(r_1) \geq \frac{3\pi}{2}$. $E(u) \geq 3(1-\delta) > 2$, contradictions.
  For the case $\cos(u(r_1)) < 0$, $[(\frac{1}{r^2}+\alpha) - \kappa\frac{1}{r}\tan(u)] < 0$ implying $u \geq \pi + \arctan(\frac{2}{\sqrt{\delta}})$.Thus, similarly as above, followed from the lower bounds of energy (13), and $\delta < \sqrt{11} - 3$, $E(u) \geq (1-\delta)(|\cos(\pi + \arctan(\frac{2}{\sqrt{\delta}})) - \cos(\pi)| + 2) = (1-\delta)(3 - \frac{\delta}{4+\delta}) \geq 2$, contradicting to $E(u) < 2$.

On the other hand, if u(r) has negative local minimum at $r = r_0$. $u_{rr}(r_0) \geq 0$.
Followed from (28), $\sin(u(r_0))\cos(u(r_0)) \geq 0$. implying $u \leq -\frac{\pi}{2}$, $E(u(r)) \geq 3(1-\delta) > 2$, which is a contradiction. □

Lemma 12. *If u(r) has a local minimum at r_1, $0 \leq u(r_1) \leq \frac{\pi}{2}$*

From Proposition 3, $\sin(u(r_1))\cos(u(r_1)) \geq 0$, $u(r_1)) \in [0,\pi]$, so $0 \leq u(r_1) \leq \frac{\pi}{2}$

Lemma 13. *If u(r) has a local maximum at r_2, then $u(r_2) \geq \frac{\pi}{2}$ or, $\frac{\pi}{2} > u(r_2) \geq \arctan(\frac{2}{\sqrt{\delta}})$*

By Proposition 3. $(\frac{1}{r^2}+\alpha)\sin(u)\cos(u) - \frac{\kappa}{r}\sin^2(u) \leq 0$. If $\cos(u(r_2)) \leq 0, u(r_2) \geq \frac{\pi}{2}$
If $\cos(u) \geq 0$, we have $[(\frac{1}{r_2^2}+\alpha) - \frac{\kappa}{r_2}\tan(u(r_2))] > 0$. Thus $u(r_2) \geq \arctan(\frac{2}{\sqrt{\delta}})$

Lemma 14. *Assume u(r) has a a local minimum at r_1 , $u(r_1)$ followed by a local maximum $u(r_2)$. For small enough δ, $u(r_2) - u(r_1) \lesssim \delta$.*

By the lower topological bounds (13)

$$E(u) \geq (1-\delta)E_e \geq (1-\delta)[1 + \cos(u(r_1)) + 2\cos(u(r_1)) - 2\cos(u(r_2) + 1 - \cos(u(r_1))]$$

so, $(\cos(u(r_1)) - \cos(u(r_2))) < \frac{\delta}{1-\delta}$
If $\arctan(\frac{2}{\sqrt{\delta}}) < u(r_2) < \frac{\pi}{2}$, $\cos(u(r_2)) \leq \sqrt{\frac{\delta}{4+\delta}}, \cos(u(r_1)) \leq \frac{\delta}{1+\delta} + \sqrt{\frac{\delta}{4+\delta}}$,

$$\sin(u(r_1)) = 1 - \frac{1}{8}\delta - \frac{\delta^{\frac{3}{2}}}{2} + \mathcal{O}(\delta^2) \tag{29}$$

$\sin(u(r_1))(u(r_2) - u(r_1)) \leq (\cos(u(r_1)) - \cos(u(r_2))) < \frac{\delta}{1-\delta}$, so we have $(u(r_2) - u(r_1)) < 2\delta$
If $u(r_2) \geq \frac{\pi}{2}, \cos(u(r_2)) \leq 0$
$E \geq (1-\delta)E_e \geq (1-\delta)(2 + 2\cos(u(r_1)))$, then $\cos(u(r_1)) \leq \frac{\delta}{1-\delta}$ implying $\cos(u(r_2)) \geq -\frac{\delta}{1-\delta}$. Thus, by linear approximation. $(u(r_2) - u(r_1)) \leq 4\delta$

Theorem 3. *If $h = 0$, for small enough δ, i.e $\kappa^2 \ll \alpha$, u(r) is monotone.*

Proof. We use proof by contradiction here,
If u(r) has a local minimum at $r = r_1$, multiply the Euler-Lagrange equation (27) by $u_r r$ and integrate with respect to rdr from $[0, r_1]$. We have the following equation,

$$\alpha E_a^{[0,r_1]} + \kappa E_{DM}^{[0,r_1]} = \frac{1}{2}(1 + \alpha r_1^2)\sin(u(r_1))$$

Followed from (29) in Lemma 14, $\sin(u(r_1)) = 1 - \mathcal{O}(\delta)$, and by lemma 29, $\cos(u(r_1)) > 0$

$$\frac{1}{2}(1+\alpha r_1^2)\sin(u(r_1)) \geq \frac{1}{2}\alpha r_1^2(\sin^2(u(r_1)) - \frac{1}{2}) + \frac{1}{2}\sin(u(r_1)) \geq \frac{1}{2}\sin(u(r_1))$$

Using the topological bounds of exchange energy, and lower bounds of total energy, ,

$$\begin{aligned} E(u(r)) = E(u(r))^{[0,r_1]} + E(r)^{[r_1,\infty]} &\geq 1 + \cos(u(r_1)) + \frac{1}{2}\sin(u(r_1)) + (1-\delta)(1-\cos(u(r_1))) \\ &\geq 2 + \frac{1}{2} - \mathcal{O}(\delta) \geq 2 \end{aligned}$$

which is a contradiction. □

6 Further questions

6.1 Energy minimization

Problem 1. *Do we have a minimizer in X_n for $|n| \neq 1$?*

We don't know if $I_n \leq 2n$.

Problem 2. *Do we have a minimizer for $\kappa \in (-\sqrt{h+\alpha}, 0)$? $|\kappa| = \sqrt{h+\alpha}$? $\sqrt{h+\alpha} < |\kappa| < 3\sqrt{\alpha+h}$?*

Problem 3. *Is the minimizer found above still a minimizer in the larger class of 'equivariant' magnetizations (See section4.3):*

$$\vec{m}(x) = (-\sin(u(r))\sin[\theta + f(r)], \sin(u(r))\cos[\theta + f(r)], \cos(u(r))), \quad u, f : [0, \infty) \to \mathbb{R}?$$

Problem 4. *Does the minimizer found above in X_n agree with the (non-symmetrically constrained) minimizer of [1] in the Zeeman case?*

6.2 Further properties

Problem 5. *Does the minimizer exhibit exponential decay?*

6.3 Stability (in the energetic sense)

Problem 6. *Is the second variation of the energy about the X_1 minimizer strictly positive along perturbations preserving X_1 (since it is a minimizer, we expect ≥ 0, so we are asking in particular if it is non-degenerate)?*

Problem 7. *Does positivity/non-negativity of the second variation persist when we allow a wider class of perturbations? Or can we find negative-energy directions by breaking the symmetry?*

6.4 Dynamics

Problem 8. *Can we prove dynamical stability of the minimizer for the corresponding Landau-Lifshitz equation (with or without dissipation), eg. by using the energy as Lyapunov function.*

Problem 9. *Numerical computation of the linearized spectrum, if needed.*

Problem 10. *How do the observations we have made match with observations/computations found in the physics literature?*

References

[1] C.Melcher, *Chiral skyrmions in the plane.* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 470 (2014), no. 2172.

[2] L.Dring, and C. Melcher, *"Compactness results for static and dynamic chiral skyrmions near the conformal limit.* Calculus of Variations and Partial Differential Equations 56.3 (2017): 60.

[3] A.Bogdanov, A.Hubert, *Thermodynamically stable Magnetic vortex states in magnetic crystals.* J.Magn.Magn.Mater. **138** (1994), no. 6, 255-269.

[4] A.Bocdanov, A.Hubert, *The propertires of isolated magnetic vortices.* Physica Status Solidi(b). **186** (1994), 527-543

[5] A.Bocdanov,A.Hubert, *The stability of vortex-like structures in uniaxial ferromagnets.* J.Magn.Magn.Mater. **195** (1999),182-192.

[6] A.N.Bogdanov, U.K,Rößler, *Chiral Symmetry Breaking in Magnetic thin Films and Multilayers.* Physical Review Letters **87** (2001), no. 3.

[7] A.Bogdanov, D.A.Yablonskii *Thermodynamically stable vortices in magnetically ordered crystals. The mixed state of magnets.* Sov.Phy.JETP**68** 101-103.

[8] S.Komineas , N.Papanicolaou *Skrymion dynamics in chiral ferromagnets.* Phys.Rev.B **92** (2015).

[9] N.S.Kiselev ,N.A. Bogdanov, Schäfer R, Rößler, *Chiral Skrymions in thin magnetic films:new objects for magnetic storage technologies?* J.Phys.D:Appl.Phys. **44** (2011).

[10] R Kubo, K Tomita *A general theory of magnetic resonance absorption* Journal of the Physical Society of Japan 9.6 (1954): 888-919.

[11] M. Guan, S. Gustafson, K. Kang and T.-P. Tsai, *Global questions for map evolution equations. Singularities in PDE and the calculus of variations* CRM Proc. Lecture Notes, 44, Amer. Math. Soc., Providence, RI, 2008.61–74

[12] S. Gustafson, K. Kang and T.-P. Tsai, *Schrdinger flow near harmonic maps.* Comm. Pure Appl. Math. **60** (2007), pp. 463499

[13] S. Gustafson, K. Kang and T.-P. Tsai, *Asymptotic stability of harmonic maps under the Schrdinger flow.* Duke Math. J. (2008), no. 3, 537583.

[14] S. Gustafson, K. Kang and T.-P. Tsai, *Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrdinger maps on* $\mathbb{R}^2$. Comm. Math. Phys.**300** (2010), no. 1, 205242.

[15] L. A. L. E. Landau, and E. Lifshitz, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies.* Phys. Z. Sowjetunion 8.153 (1935): 101-114.

[16] M. Struwe,*On the evolution of harmonic mappings of Riemannian surfaces* Comment. Math. Helv. **60** (1985) 558-581.

7 Appendix

Working in the original representation, the tangent space to $\mathbb{S}^2$ at $\vec{m} \in \mathbb{S}^2$ is

$$T_{\vec{m}}\mathbb{S}^2 = \vec{m}^{\perp},$$

and the corresponding projection is

$$\mathbb{R}^3 \ni \ \vec{\eta} \mapsto P_{T_{\vec{m}}\mathbb{S}^2}\ \vec{\eta} := \vec{\eta} - (\vec{\eta} \cdot \vec{m})\, \vec{m} \ \in T_{\vec{m}}\mathbb{S}^2.$$

A one-parameter variation

$$\vec{m}^{\epsilon} = \vec{m} + \epsilon\, \vec{\xi} + \frac{1}{2}\epsilon^2\ \vec{\eta} + \cdots \in \mathbb{S}^2$$

of magnetization $\vec{m} \in \mathbb{S}^2$ satisfies

$$1 = |\vec{m}^{\epsilon}|^2 = 1 + \epsilon \left(2\vec{m} \cdot \vec{\xi}\right) + \epsilon^2 \left(|\vec{\xi}|^2 + \vec{m} \cdot \vec{\eta}\right) + \cdots$$

and so

$$\vec{m} \cdot \vec{\xi} = 0 \implies \xi \in T_{\vec{m}}\mathbb{S}^2,$$

and

$$\vec{m} \cdot \vec{\eta} = -|\vec{\xi}|^2 \implies \vec{\eta} = -|\vec{\xi}|^2 \vec{m} + \vec{\zeta}, \quad \vec{\zeta} \in T_{\vec{m}}\mathbb{S}^2,$$

thus

$$\vec{m}^{\epsilon} = \vec{m} + \epsilon\, \vec{\xi} + \frac{1}{2}\epsilon^2 \left(-|\vec{\xi}|^2 \vec{m} + \vec{\zeta}\right) + \cdots, \quad \vec{\xi},\ \vec{\zeta}\ \in T_{\vec{m}}\mathbb{S}^2.$$

- exchange:

$$\begin{aligned} E_e(\vec{m}^{\epsilon}) - E_e(\vec{m}) &= \epsilon \int \nabla\vec{m} \cdot \nabla\vec{\xi} + \frac{1}{2}\epsilon^2 \int \left(|\nabla\vec{\xi}|^2 + \nabla\vec{m} \cdot \nabla(-|\vec{\xi}|^2\vec{m} + \vec{\zeta})\right) + \cdots \\ &= \epsilon\langle -P_{T_{\vec{m}}\mathbb{S}^2}\Delta\vec{m},\ \vec{\xi} + \frac{1}{2}\epsilon\vec{\zeta}\,\rangle + \frac{1}{2}\epsilon^2\langle \vec{\xi},\ -P_{T_{\vec{m}}\mathbb{S}^2}\Delta\vec{\xi} - |\nabla\vec{m}|^2\vec{\xi}\,\rangle + \cdots \end{aligned}$$

  from which

$$\begin{aligned} E_e'(\vec{m}) &= -P_{T_{\vec{m}}\mathbb{S}^2}\Delta\vec{m} = -\Delta\vec{m} - |\nabla\vec{m}|^2\vec{m}, \\ E_e''(\vec{m}) &= -P_{T_{\vec{m}}\mathbb{S}^2}\Delta\ \ - |\nabla\vec{m}|^2. \end{aligned}$$

- DM:

$$\begin{aligned} E_{DM}(\vec{m}^{\epsilon}) - E_{DM}(\vec{m}) &= \epsilon \int (\nabla \times \vec{m}) \cdot \vec{\xi} + \frac{1}{2}\epsilon^2 \int \left(\vec{\xi} \cdot (\nabla \times \vec{\xi}) + (\nabla \times \vec{m}) \cdot (-|\vec{\xi}|^2\vec{m} + \vec{\zeta})\right) + \cdots \\ &= \epsilon\langle P_{T_{\vec{m}}\mathbb{S}^2}(\nabla \times \vec{m}),\ \vec{\xi} + \frac{1}{2}\epsilon\vec{\zeta}\,\rangle + \frac{1}{2}\epsilon^2\langle \vec{\xi},\ P_{T_{\vec{m}}\mathbb{S}^2}\nabla \times \vec{\xi} - (\vec{m} \cdot (\nabla \times \vec{m})\vec{\xi}\,\rangle + \cdots \end{aligned}$$

  from which

$$\begin{aligned} E_{DM}'(\vec{m}) &= P_{T_{\vec{m}}\mathbb{S}^2}\nabla \times \vec{m} = \nabla \times \vec{m} - (\vec{m} \cdot (\nabla \times \vec{m}))\vec{m}, \\ E_{DM}''(\vec{m}) &= P_{T_{\vec{m}}\mathbb{S}^2}\nabla \times\ \ - \vec{m} \cdot (\nabla \times \vec{m}). \end{aligned}$$

- Zeeman:

$$\begin{aligned} E_z(\vec{m}^{\epsilon}) - E_z(\vec{m}) &= \epsilon \int (-\xi_3) + \frac{1}{2}\epsilon^2 \int (|\vec{\xi}|^2 m_3 - \zeta_3) + \cdots \\ &= \epsilon\langle -P_{T_{\vec{m}}\mathbb{S}^2}\hat{k},\ \vec{\xi} + \frac{1}{2}\epsilon\vec{\zeta}\,\rangle + \frac{1}{2}\epsilon^2\langle \vec{\xi}, m_3\ \vec{\xi}\,\rangle + \cdots \end{aligned}$$

  from which

$$\begin{aligned} E_z'(\vec{m}) &= -P_{T_{\vec{m}}\mathbb{S}^2}\hat{k} = -\hat{k} + m_3\vec{m}, \\ E_z''(\vec{m}) &= m_3. \end{aligned}$$

- anisotropy:

$$E_a(\vec{m}^\epsilon) - E_a(\vec{m}) = \epsilon \int (-m_3\xi_3) + \frac{1}{2}\epsilon^2 \int \Big(m_3(|\vec{\xi}|^2 m_3 - \zeta_3) - \xi_3^2 \Big) + \cdots$$
$$= \epsilon\langle -m_3 P_{T_{\vec{m}}\mathbb{S}^2}\hat{k},\ \vec{\xi} + \frac{1}{2}\epsilon\vec{\zeta}\ \rangle + \frac{1}{2}\epsilon^2 \langle \vec{\xi}, m_3^2 - P_{T_{\vec{m}}\mathbb{S}^2}\hat{k}\hat{k}\cdot\ \vec{\xi}\ \rangle + \cdots$$

from which

$$E_a'(\vec{m}) = -m_3 P_{T_{\vec{m}}\mathbb{S}^2}\hat{k} = m_3(-\hat{k} + m_3\vec{m}),$$
$$E_a''(\vec{m}) = m_3^2 - P_{T_{\vec{m}}\mathbb{S}^2}\hat{k}\hat{k}\cdot .$$

7.1 Second variation of energy

$$\vec{m} := (-\sin(u(r))\sin(\theta), \sin(u)\cos(\theta), \cos(u(r)))$$
$$\vec{e} := (-\cos(u(r))\sin(\theta), \cos(u(r))\cos(\theta), -\sin(u(r)))$$
$$J\vec{e} := \vec{m}\times\vec{e} = (-\cos(\theta), -\sin(\theta), 0)$$

$\vec{e}$ and $J\vec{e}$ are orthonormal basis on $T_{\vec{m}}\mathbb{S}$ and $m_r = u_r\vec{e}$, $m_\theta = \frac{\sin(u)}{r}J\vec{e}$

Consider $\xi \in T_{\vec{m}}\mathbb{S}^2$

$$\vec{\xi}(r,\theta) = g(r,\theta)\vec{e}(r,\theta) + h(r,\theta)J\vec{e}(r,\theta)$$

Writing them as the vector form with respect to the basis $\vec{e}$ and $J\vec{e}$

$$E_e''(\vec{m})\begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} -\Delta + \frac{1}{r^2}\cos^2(u) - \frac{1}{r^2}\sin^2(u) & \frac{2}{r^2}\cos(u)\partial_\theta \\ -\frac{2}{r^2}\cos(u)\partial_\theta & -\Delta + \frac{1}{r^2}\cos^2(u) - u_r^2 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}$$

$$E_a''(\vec{m})\begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} \cos^2(u) - \sin^2(u) & 0 \\ 0 & \cos^2(u) \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}$$

$$E_z''(\vec{m})\begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} \cos(u) & 0 \\ 0 & \cos(u) \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}$$

$$E_{DM}''(\vec{m})\begin{bmatrix} g \\ h \end{bmatrix} = \kappa\begin{bmatrix} -\frac{2}{r}\sin(u)\cos(u) & -\frac{1}{r}\sin(u)\partial_\theta \\ \frac{1}{r}\sin(u)\partial_\theta & -u_r - \frac{1}{r}\sin(u)\cos(u) \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}$$

Thus the second variation of the energy are self-adjoint operators.
If we write the $\vec{\xi}$ in fourier mode.

$$\vec{\xi}(r,\theta) = \sum_{j=0}^{\infty}(g_j(r)\vec{e} + h_j(r)J\vec{e})e^{ij\theta}$$

$$E''(\vec{m})\xi = \bigoplus_{j=-\infty}^{\infty}[E_j''(\vec{m})\big(g_j(r)\vec{e} + h_j(r)J\vec{e}\big)]e^{ij\theta}$$

$$E_j''(\vec{m}) := E_{ej}''(\vec{m}) + \alpha E_{aj}''(\vec{m}) + hE_{zj}''(\vec{m}) + E_{DMj}''(\vec{m})$$

The jth second variation of energy are shown below.

$$E_{ej}''(\vec{m}) = \begin{bmatrix} -\Delta_r + \frac{1}{r^2}(j^2 + \cos^2(u) - \sin^2(u)) & \frac{2}{r^2}ij\cos(u) \\ -\frac{2}{r^2}ij\cos(u) & -\Delta_r + \frac{1}{r^2}(j^2 + \cos^2(u)) - u_r^2 \end{bmatrix}$$

$$E''_{aj}(\vec{m}) = \begin{bmatrix} \cos^2(u) - \sin^2(u) & 0 \\ 0 & \cos^2(u) \end{bmatrix}$$

$$E''_{zj}(\vec{m}) \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} \cos(u) & 0 \\ 0 & \cos(u) \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix}$$

$$E''_{DMj}(\vec{m}) = \kappa \begin{bmatrix} -\frac{2}{r}\sin(u)\cos(u) & -\frac{ij}{r}\sin(u) \\ \frac{ij}{r}\sin(u) & -u_r - \frac{1}{r}\sin(u)\cos(u) \end{bmatrix}$$

For the j = 1 case.

$$E''_{e1}(\vec{m}) \begin{bmatrix} u_r \\ i\frac{\sin(u)}{r} \end{bmatrix} = \begin{bmatrix} \partial_r(-\Delta_r u + \frac{\sin(2u)}{2r^2}) \\ i\frac{\cos(u)}{r}(-\Delta_r u + \frac{\sin(2u)}{2r^2}) \end{bmatrix}$$

$$E''_{a1}(\vec{m}) \begin{bmatrix} u_r \\ i\frac{\sin(u)}{r} \end{bmatrix} = \begin{bmatrix} \partial_r(\cos(u)\sin(u)) \\ i\frac{\cos(u)}{r}(\cos(u)\sin(u)) \end{bmatrix}$$

$$E''_{z1}(\vec{m}) \begin{bmatrix} u_r \\ i\frac{\sin(u)}{r} \end{bmatrix} = \begin{bmatrix} \partial_r(\sin(u)) \\ i\frac{\cos(u)}{r}\sin(u) \end{bmatrix}$$

$$E''_{DM1}(\vec{m}) \begin{bmatrix} u_r \\ i\frac{\sin(u)}{r} \end{bmatrix} = \kappa \begin{bmatrix} \partial_r(-\frac{1}{r}\sin^2(u)) \\ i\frac{\cos(u)}{r}(-\frac{1}{r}\sin^2(u)) \end{bmatrix}$$

From the Euler-Lagrange equation, $-\Delta_r u + \frac{\sin(2u)}{2r^2} + \alpha\cos(u)\sin(u) + \sin(u) - \kappa\frac{1}{r}\sin^2 u = 0$
As we have , $\partial_1\vec{m} = \cos(\theta)u_r\vec{e} - \frac{\sin(u)\sin(\theta)}{r}J\vec{e}$, and $\partial_2\vec{m} = \sin(\theta)u_r\vec{e} + \frac{\sin(u)\cos(\theta)}{r}J\vec{e}$

$$\partial_1\vec{m} + i\partial_2\vec{m} = (u_r\vec{e} + i\frac{\sin(u)}{r}J\vec{e})e^{i\theta}$$

It is verified that the spatial derivative of $\vec{m}$ is in the kernel of $E''_1(\vec{m})$. This is because of the fact that the total energy is translation invariant. Same as $\partial_2\vec{m} + i\partial_1\vec{m}$ in the kernel of $E''_{-1}(\vec{m})$

If we only consider the radial perturbation. Followed from result of [5], no external fields and positive DM parameter, the smallest eigenvalue of the second variation form is positive, so the solution is stable. However, we don't know about the non-degeneracy of the $E''(\vec{m})$ on arbitrary perturbations .