Dynamic Programming

CMSC 142: Design and Analysis of Algorithms

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OVERVIEW

Space and Time Tradeoff

Dynamic Programming

Memory Functions

SPACE AND TIME TRADEOFF

In algorithm design, space and time trade-offs are a well-known issue for both theoreticians and practitioners of computing.

► Trading space for time is much more prevalent.

Two principal varieties of trading space for time in algorithm design:

- ▶ input enhancement preprocess the problem's input, in whole or in part, and store the additional information obtained in order to accelerate solving the problem afterward
- prestructuring uses extra space to facilitate a faster and/or more flexible access to the data

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Dynamic Programming

- "invented" by Richard Bellman in 1950s as a general method for optimizing multistage decision processes (e.g. optimization problems)
 Principle of optimality: An optimal solution to any of its instances must be made up of optimal solutions to its subinstances
- general algorithm design technique for solving problems defined by or formulated as a recurrences with overlapping subinstances
- ightharpoonup suggests solving each of the smaller subproblems only once and recording the results in a table from w/c a solution to the original problem can be obtained
- allows solving recursive problems with a highly- overlapping subproblem structure efficiently

Recall: Computing the n^{th} Fibonacci Number

The Fibonacci numbers F(n) are the elements of the sequence above defined by the recurrence: For $n \ge 2$, F(n) = F(n-1) + F(n-2), with F(0) = 0 and F(1) = 1

```
ALGORITHM F(n) //Computes n^{th} Fibonacci number recursively by using its defn //Input: A nonnegative integer n //Output: The n^{th} Fibonacci number if n \le 1 return n else return F(n-1) + F(n-2)
```

Remark. The algorithm is inefficient. Replicated computation is done.

Dynamic Programming: Array-based Methods

```
ALGORITHM Fib(n) //Computes the n^{th} Fibonacci number iteratively using defn //Input: A nonnegative integer n //Output: The n^{th} Fibonacci number F[0] \leftarrow 0; F[1] \leftarrow 1 for i \leftarrow 2 to n do F[i] \leftarrow F[i-1] + F[i-2] return F[n]
```

What is the time efficiency of the above algorithm? $\Theta(n)$ What is the space efficiency? $\Theta(n)$

Dynamic Programming: Array-based Methods

```
ALGORITHM Fib(n) //Computes the n^{th} Fibonacci number iteratively using defn //Input: A nonnegative integer n //Output: The n^{th} Fibonacci number F[0] \leftarrow 0; F[1] \leftarrow 1 for i \leftarrow 2 to n do F[i] \leftarrow F[i-1] + F[i-2] return F[n]
```

What is the time efficiency of the above algorithm? $\Theta(n)$ What is the space efficiency? $\Theta(n)$ (can be reduced to $\Theta(1)$ by storing the new term in F[0] and F[1] simultaneously.)

Dynamic Programming

Main Idea

- ▶ identify the subproblems
- set up a recurrence relating a solution to a larger instance to solutions of smaller instances
- ▶ determine an ordering for the subproblems
- ▶ implement the recurrence by solving the subproblems in order and once; keep results that will be needed at any given point by recording solutions (e.g. in a table)
- extract solution to the initial instance from the table

Computing a Binomial Coefficient

Binomial Coefficient C(n,k) or $\binom{n}{k}$ where $0 \le k \le n$

- \triangleright number of combinations (subsets) of k elements from an n-element set
- coefficients of the binomial formula

Property of Binomial Coefficients:

$$C(n,0) = 1$$

 $C(n,n) = 1$
 $C(n,k) = C(n-1,k-1) + C(n-1,k)$ for $n > k > 0$

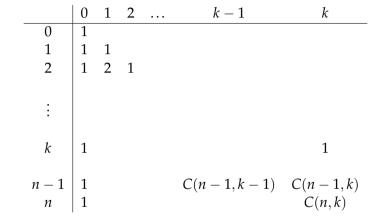


Table. Computing C(n,k) by the dynamic programming algorithm

Computing a Binomial Coefficient

```
ALGORITHM Binomial(n,k) //Computes C(n,k) by the dynamic programming algorithm //Input: A pair of nonnegative integers n \ge k \ge 0 //Output: The value of C(n,k) for i \leftarrow 0 to n do for j \leftarrow 0 to \min(i,k) do if j = 0 or j = i C[i,j] \leftarrow 1 else C[i,j] \leftarrow C[i-1,j-1] + C[i-1,j] return C[n,k]
```

▶ What is the time efficiency of the above algorithm? $\Theta(nk)$

Revisiting the Knapsack Problem

Problem: Given *n* items of known weights

$$w_1, w_2, w_3, \ldots, w_n$$

with corresponding values

$$v_1, v_2, v_3, \ldots, v_n$$

and a knapsack capacity W, find the most valuable subset of the items that fit into the knapsack.

brute force: generate all subsets to determine optimal solution

Revisiting the Knapsack Problem

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with corresponding values

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and a knapsack capacity *W*, find the most valuable subset of the items that fit into the knapsack.

- ▶ brute force: generate all subsets to determine optimal solution
- ► reduction: transform to a linear programming problem

The Knapsack Problem

Dynamic Programming Approach

Consider the subproblem defined by the first i items $(1 \le i \le n)$ with weights w_1, w_2, \ldots, w_i and values v_1, v_2, \ldots, v_i , and a knapsack capacity j $(1 \le j \le W)$.

Define V[i,j] as value of an optimal solution to this instance, i.e., the value of the most valuable subset of the first i items that fit into the knapsack of capacity j

▶ What is the value of an optimal subset?

The Knapsack Problem: DP Approach

Divide all the subsets of the first *i* items that fit into the knapsack of capacity *j*:

- (i) subsets that do not include the i^{th} item value of an optimal subset is: V[i-1,j]
- (ii) subsets that do include the i^{th} item value of an optimal subset is: $v_i + V[i-1, j-w_i]$

$$V[i,j] = \begin{cases} V[i-1,j], & \text{if } j-w_i < 0, \\ \max\{V[i-1,j], v_i + V[i-1,j-w_i]\}, & \text{if } j-w_i \ge 0, \end{cases}$$

$$V[0,j] = 0 \text{ for } j \ge 0 \text{ and } V[i,0] = 0 \text{ for } i \ge 0.$$

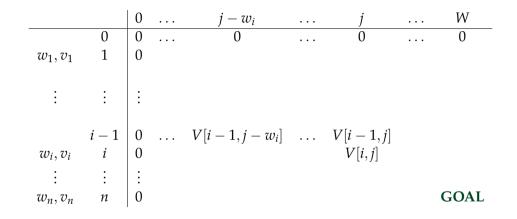


Table. Solving the knapsack problem by the dynamic programming approach

The Knapsack Problem: DP Approach

```
ALGORITHM Knapsack(n, W)
//Input: A pair of nonnegative integers n and K
//Output: Value of the optimal feasible subset of first n items for i \leftarrow 0 to n do

for j \leftarrow 0 to W do

if j = 0 or i = 0 V[i,j] \leftarrow 0

else if j - w_i < 0 V[i,j] \leftarrow V[i-1,j]

else V[i,j] \leftarrow \max\{V[i-1,j], v_i + V[i-1,j-w_i]\}
return V[n, W]
```

- \blacktriangleright What is the time efficiency and space efficiency of the algorithm? $\Theta(nW)$
- \blacktriangleright What is the time needed to find composition of an optimal solution? O(n).

The Knapsack Problem

Example 1

Consider the instance of the Knapsack problem given by the following data:

item	weight	value	
1	2	\$12	
2	1	\$10	capacity $W = 5$
3	3	\$20	
4	2	\$15	

Use the dynamic programming algorithm to solve the knapsack problem.

Answer: The maximal value is V[4,5] = 37.

Example 1: Solution

		capacity j							
	i	0	1	2	3	4	5		
	0	0	0	0	0	0	0		
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12		
$w_2 = 1, v_2 = 10$	2	0	10	12	22	22	22		
$w_3 = 3, v_3 = 20$	3	0	10	12	22	30	32		
$w_4 = 2, v_4 = 15$	4	0	10	15	25	30	37		

To find composition of an optimal subset, backtrace the computations:

- \triangleright V[4,5] > V[3,5] so item 4 has to be included (item left: 3; W: 3u)
- V[3,3] = V[2,3] so item 3 is not included (item left: 2; W: 3u)
- V[2,3] > V[1,3] so item 2 has to be included (item left: 1; W: 2u)
- ▶ V[1,2] > V[0,2] so item 1 has to be included (item left: 0; W: 0u)

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Memory Functions

We proceed with a method that combines the strength of the top-down and bottomup approaches. Such method exists and is based on using **memory functions**.

Memoization

- general technique that attempts to relieve the potential inefficiency of recursion by using basic idea of dynamic programming
- ▶ adds a table indexed by possible inputs to recursive function
- ▶ *Idea*: checks whether value of function for requested input is already stored: if it is, value is returned; if not, calls function recursively, then add value to the table for future reference

```
ALGORITHM MFKnapsack(i, j)
//Input: A nonnegative integer i and j
//Output: Value of the optimal feasible subset of first i items
//Note: Uses as global variables input arrays Weights[1...n], Values[1...n] and table
//V[0...n, 0...W] initialized with -1's except for row 0 and column 0 with 0's
if V[i, i] < 0
    if i < Weights[i]
         value \leftarrow MFKnapsack(i-1,i)
    else
         value \leftarrow \max\{MFKnapsack(i-1,j), Values[i] + MFKnapsack(i-1,j-Weights[i])\}
    V[i,j] \leftarrow value
return V[i, j]
```

The Knapsack Problem

Example 2

Apply the memory function method to the instance considered in the previous example.

item	weight	value	
1	2	\$12	
2	1	\$10	capacity $W = 5$
3	3	\$20	
4	2	\$15	

Example 2: Solution

- V[4,5] is called first which followed by only 11 out of 20 nontrivial values (i.e., not those in row 0 or in column 0) computations (recursive calls).
- \blacktriangleright One nontrivial entry, V[1,2], is retrieved rather than recomputed.

		capacity j						
	i	0	1	2	3	4	5	
	0	0	0	0	0	0	0	
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12	
$w_2 = 1, v_2 = 10$	2	0	_	12	22	_	22	
$w_3 = 3, v_3 = 20$	3	0	_	_	22	_	32	
$w_4 = 2, v_4 = 15$	4	0	_	_	_	_	37	

Following the same procedure in the previous example, the composition of an optimal subset are items 1, 2, and 4.

End of Lecture