Ideas on extending bipoles and bipolars

⇝ First-order systems

Let's consider the first-order Gödel Logic. This case is interesting since although there is only one infinite-valued propositional Gödel logic, different first-order (and quantified propositional) Gödel logics are induced by different infinite sub-sets of truth values over [0, 1]. Let's concentrate on Gödel logic with the full real interval [0, 1].

In this case, sIF is obtained ~~by~~ from first-order IL by adding the axioms

$$(A \supset B) \vee (B \supset A) \qquad (lin)$$

$$\forall x. (A(x) \vee B) \supset (\forall x. A(x) \vee B) \qquad (\vee\forall) \qquad x \text{ does not occur in } B$$

The hypersequent rules for propositional Gödel logic are the ones for IL plus (HG)

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A \qquad G' \mid \Gamma_1, \Gamma_2' \Rightarrow A'}{G \mid G' \mid \Gamma, \Gamma_1' \Rightarrow A \mid \Gamma', \Gamma_1 \Rightarrow A'} \text{ com}$$

For the first-order, we add to HG the following:

$$\frac{G \mid A(t), \Gamma \Rightarrow B}{G \mid \forall x. A(x), \Gamma \Rightarrow B} \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow \forall x. A(x)}$$

$$\frac{G \mid A(a), \Gamma \Rightarrow B}{G \mid \exists x. A(x), \Gamma \Rightarrow B} \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow \exists x. A(x)}.$$

a fresh.

Experiment

⤳ Propositional case We already know that focusing on $\partial_-[(P\supset Q)\vee^+(Q\supset P)]$ gives rise to the α-rule

$$
\dfrac{\dfrac{\dfrac{Q\vdash}{P\vdash}}{\vdots}\qquad\qquad \dfrac{\dfrac{P\vdash}{Q\vdash}}{\vdots}}{\vdash}
$$

This corresponds to the hypersequent rule
(here Φ and Ψ are multisets of formulas)

$$
\dfrac{G \mid \Phi, \Gamma_1 \Rightarrow \Pi_1 \qquad G \mid \Psi, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Psi, \Gamma_1 \Rightarrow \Pi_1 \mid \Phi, \Gamma_2 \Rightarrow \Pi_2}
$$

⤳ In the first-order case: (considering formulas positive) (atomic)

$$
\dfrac{\dfrac{\overline{\Sigma, x : A(x)\vdash}}{\vdots}\qquad \dfrac{\dfrac{A(t)\vdash}{\vdash}}{\vdots}\qquad \dfrac{\dfrac{B\vdash}{\vdash}}{\vdots}}{\Sigma : \quad \vdash}
\qquad\qquad
\dfrac{\dfrac{\overline{\Sigma, x : B\vdash}}{\vdots}\qquad \dfrac{\dfrac{A(t)\vdash}{\vdash}}{\vdots}\qquad \dfrac{\dfrac{B\vdash}{\vdash}}{\vdots}}{\Sigma : \quad \vdash}
$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A(x)\Downarrow}\qquad \overline{\vdash B\Downarrow}}{\vdash A(x)\vee^+ B\Downarrow}}{\vdots}}{\dfrac{\vdash\ \Uparrow A(x)\vee^+ B}{\vdash A(x)\vee^+ B\Uparrow}}\ x\text{ fresh}}{\dfrac{\vdash \forall x.(A(x)\vee^+ B)\Uparrow}{\vdash \forall x.(A(x)\vee^+ B)\Downarrow}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(t)\Uparrow\ \vdash}{\Uparrow A(t)\ \vdash}}{\Downarrow A(t)\ \vdash}}{\Downarrow\forall x.A(x)\ \vdash}}{\vdots}}{\dfrac{\forall x.A(x)\Uparrow\ \vdash}{\Uparrow\forall x.A(x)\vdash}}\qquad \dfrac{B\Uparrow\ \vdash}{\Uparrow B\vdash}}{\Uparrow\forall x\, A(x)\vee^+ B\vdash}}{\Downarrow(\forall x\, A(x))\vee^+ B\vdash}
}{\Downarrow\ \forall x\,(A(x)\vee B)\supset(\forall x.\, A(x)\vee B)\ \vdash}
$$

one thing to think is that in our approach rules are connective free

coming back to the example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x: A(x) \vdash A(x) \qquad \overline{x: B \vdash A(x)}}{x: A(x) \vee B \vdash A(x)}}{x: \forall x.(A(x) \vee B) \vdash A(x)}}{\emptyset: \forall x.(A(x) \vee B) \vdash \forall x.A(x)}}{\emptyset: \forall x.(A(x) \vee B) \vdash \forall x.A(x) \vee B} \qquad \dfrac{\dfrac{\overline{\emptyset: \forall x.(A(x) \vee B), B \vdash B}}{\emptyset: \forall x.(A(x) \vee B) \vdash B}}{\emptyset: \forall x.(A(x) \vee B) \vdash \forall x.A(x) \vee B}}{\emptyset: \forall x.(A(x) \vee B) \vdash \forall x.A(x) \vee B}$$

$$\dfrac{\dfrac{\overline{y: \forall x.(A(x) \vee B), A(y) \vdash A(y)}}{y: \forall x.(A(x) \vee B) \vdash A(y)}}{\emptyset: \forall x.(A(x) \vee B) \vdash \forall x.A(x)}$$

Regarding the hypersequent: