

Zadanie 1

Wykaż równości i nierówności dla liczb zespolonych:

$$z_1 = a_1 + ib_1$$

$$z_2 = a_2 + ib_2$$

1.a)

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$$

$$\overline{z_1} \cdot \overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$$

1.b)

$$|z_1 z_2| = |z_1| |z_2|$$

$$\begin{aligned} |z_1 z_2| &= |(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2} = \\ &= \sqrt{a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2a_1 a_2 b_1 b_2 + b_1^2 a_2^2} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \\ &= \sqrt{(a_1^2 + b_1^2)} \cdot \sqrt{(a_2^2 + b_2^2)} = |z_1| |z_2| \end{aligned}$$

1.c)

$$z \cdot \bar{z} = |z|^2$$

$$z \cdot \bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \sqrt{a^2 + b^2}^2 = |z|^2$$

1.d)

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} \cdot \bar{z}_2 = \frac{\bar{z}_1}{\bar{z}_2} \cdot \bar{z}_2$$

$$\frac{\bar{z}_1}{\bar{z}_2} \cdot \bar{z}_2 = \bar{z}_1$$

1.e)

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Dodawanie wektorów, nierówność trójkąta.

1.f)

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

$$\begin{aligned}|z_1| &= |z_2 + (z_1 - z_2)| \leq |z_2| + |z_1 - z_2| \\ |z_1| - |z_2| &\leq |z_1 - z_2|\end{aligned}$$

Analogicznie

$$\begin{aligned}|z_2| - |z_1| &\leq |z_2 - z_1| = |z_1 - z_2| \\ |z_1| - |z_2| &\geq -|z_1 - z_2|\end{aligned}$$

Czyli

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2| \Leftrightarrow ||z_1| - |z_2|| \leq |z_1 - z_2|$$

Zadanie 2

Oblicz:

2.a)

$$\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i-3i^2}{1-i^2} = \frac{2+i+3}{1+1} = \frac{5+i}{2}$$

2.b)

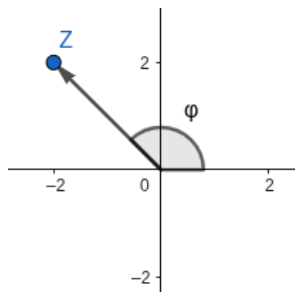
$$\begin{aligned}\frac{(i+\sqrt{3})(-1-i\sqrt{3})}{1+2i} &= \frac{(-i-i^2\sqrt{3}-\sqrt{3}-3i)(1-2i)}{(1+2i)(1-2i)} = \frac{(-4i+\sqrt{3}-\sqrt{3})(1-2i)}{1-4i^2} = \\ &= \frac{-4i+8i^2}{1+4} = \frac{-8-4i}{5}\end{aligned}$$

2.c)

$$|3-4i| = \sqrt{3^2+4^2} = 5$$

2.d)

$$\arg(-2+2i)$$



$$\arg(-2 + 2i) = \varphi = \frac{3\pi}{4}$$

2.e)

$$\frac{(1+i)^n}{(1-i)^{n-2}}, \text{ dla } n \in \mathbb{N}$$

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i$$

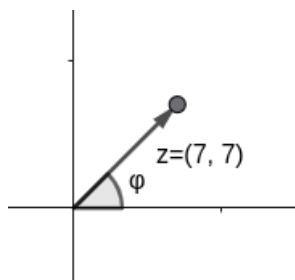
$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^{n-2}}{(1-i)^{n-2}} \cdot (1+i)^2 = i^{n-2} \cdot 2i = 2i^{n-1}$$

Zadanie 3

Przedstaw podane liczby zespolone w postaci trygonometrycznej:

3.a)

$$z = 7 + 7i$$



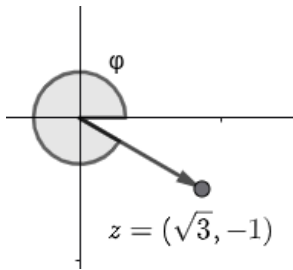
$$\varphi = \frac{\pi}{4}$$

$$|z| = \sqrt{7^2 + 7^2} = 7\sqrt{2}$$

$$z = 7\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

3.b)

$$z = \sqrt{3} - i$$



$$\varphi = \frac{11\pi}{6}$$

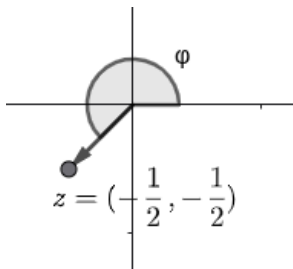
$$|z| = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$$

$$z = 2\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

3.c)

$$z = \frac{1}{i} \cdot \frac{1}{1+i}$$

$$\frac{1}{i} \cdot \frac{1}{1+i} = \frac{i}{i^2} \cdot \frac{1-i}{1-i^2} = \frac{i+1}{-1(1+1)} = \frac{i+1}{-2} = -\frac{1}{2} - \frac{1}{2}i$$



$$\varphi = \left(5\frac{\pi}{4}\right)$$

$$|z| = \frac{1}{2} \cdot \sqrt{2}$$

$$z = \frac{\sqrt{2}}{2}\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$$

3.d)

$$z = 1 + i \operatorname{tg} \alpha$$

$$|z| = \sqrt{1 + \operatorname{tg}^2 \alpha} = \sqrt{1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}} = \sqrt{1 + \frac{1 - \cos^2 \alpha}{\cos^2 \alpha}} = \sqrt{\frac{1}{\cos^2 \alpha}} = \frac{1}{|\cos \alpha|} = \pm \frac{1}{\cos \alpha}$$

$$\cos \varphi = \frac{1}{|z|} = |\cos \alpha| = \pm \cos \alpha$$

$$\sin \varphi = \frac{\operatorname{tg} \alpha}{|z|} = \operatorname{tg} \alpha \cdot |\cos \alpha| = \frac{\sin \alpha}{\cos \alpha} \cdot |\cos \alpha| = \pm \sin \alpha$$

$$z = \pm \frac{1}{\cos \alpha} (\pm \cos \alpha \pm i \sin \alpha) = \frac{1}{\cos \alpha} (\cos \alpha + i \sin \alpha)$$

3.e)

$$1 + \cos \alpha + i \sin \alpha, \alpha \in (0, \frac{\pi}{2})$$

$$\alpha \in (0, \frac{\pi}{2}) \Rightarrow \sin \alpha, \cos \alpha > 0$$

$$\begin{aligned} |a| &= \sqrt{(1 + \cos \alpha)^2 + \sin^2 \alpha} = \sqrt{(1 + \cos \alpha)^2 + 1 - \cos^2 \alpha} = \\ &= \sqrt{(1 + \cos \alpha)^2 + (1 + \cos \alpha)(1 - \cos \alpha)} = \sqrt{(1 + \cos \alpha)(1 + \cos \alpha + 1 - \cos \alpha)} = \\ &= \sqrt{2(1 + \cos \alpha)} = \sqrt{2\left(1 + 2\cos^2 \frac{\alpha}{2} - 1\right)} = \sqrt{4\cos^2 \frac{\alpha}{2}} = 2\cos \frac{\alpha}{2} \end{aligned}$$

$$\cos \varphi = \frac{1 + \cos \alpha}{2\cos \frac{\alpha}{2}} = \frac{1 + 2\cos^2 \frac{\alpha}{2} - 1}{2\cos \frac{\alpha}{2}} = \cos \frac{\alpha}{2}$$

$$\sin \varphi = \frac{\sin \alpha}{2\cos \frac{\alpha}{2}} = \frac{2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2\cos \frac{\alpha}{2}} = \sin \frac{\alpha}{2}$$

$$z = 2\cos \frac{\alpha}{2} \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right)$$

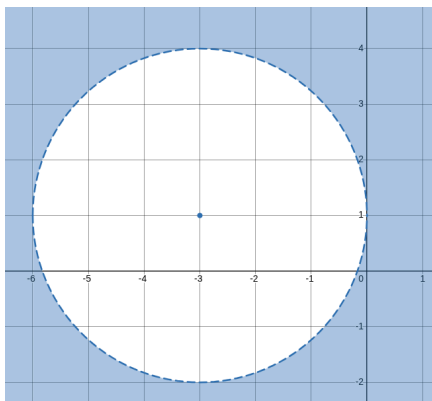
Zadanie 4

Zilustruj na płaszczyźnie zespolonej następujące zbiory:

4.a)

$$\{z \in \mathbb{C} : |z - i + 3| > 3\}$$

$$|z - i + 3| = |z - (-3 + i)| > 3 \Leftrightarrow \text{odległość } z \text{ od punktu } (-3, 1) > 3$$

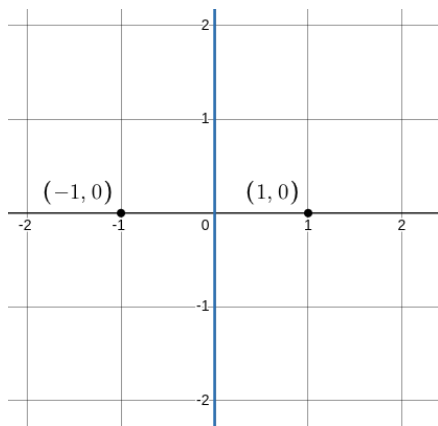


4.b)

$$\{z \in \mathbb{C} : |z - 1| = |z + 1|\}$$

$$|z - 1| = |z + 1| \Leftrightarrow |z - (1 + 0i)| = |z - (-1 + 0i)|$$

Odległości z od $(1, 0)$ i $(-1, 0)$ są równe.



4.c)

$$\{z \in \mathbb{C} : \frac{|z-2i|}{|z+3|} < 1\}$$

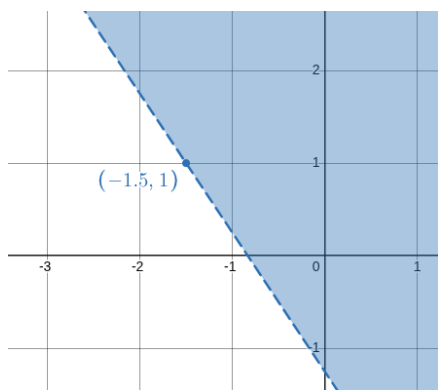
$$\frac{|z-2i|}{|z+3|} < 1 \Leftrightarrow |z - (0 + 2i)| < |z - (-3 + 0i)|$$

Odległość z od $(0, 2)$ jest mniejsza niż od $(-3, 0)$.

Granica jest symetralna odcinka między tymi punktami. Ma nachylenie $-\frac{3}{2}$ i przechodzi przez punkt $(-\frac{3}{2}, 1)$.

$$1 = (-\frac{3}{2})(-\frac{3}{2}) + b \Rightarrow b = 1 - \frac{9}{4} = -\frac{5}{4}$$

$$y = -\frac{3}{2}x - \frac{5}{4}$$



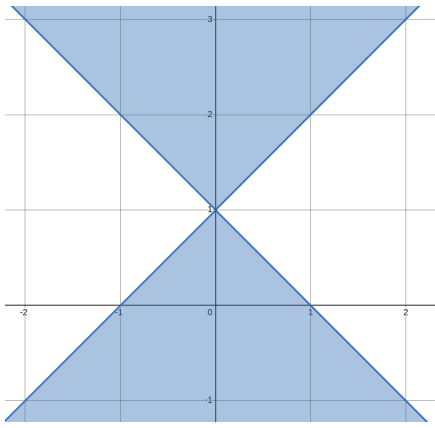
4.d)

$$\{z \in \mathbb{C} : \Re(z-i)^2 \leq 0\}$$

$$\arg(z-i) = \varphi \Rightarrow \arg((z-i)^2) = 2\varphi \bmod 2\pi$$

$$2\varphi \bmod 2\pi \in \langle \frac{\pi}{2}, \frac{3\pi}{2} \rangle \Leftrightarrow \varphi \bmod \pi \in \langle \frac{\pi}{4}, \frac{3\pi}{4} \rangle$$

$$z-i \xrightarrow{T_{[0,1]}} z$$

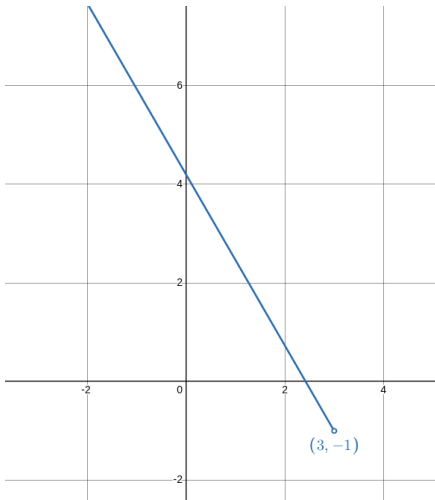


4.e)

$$\{z \in \mathbb{C} : \arg(z - 3 + i) = \frac{2\pi}{3}\}$$

Kąt $\frac{2\pi}{3}$ bez punktu $(0, 0)$, bo $\arg(0) = 0$

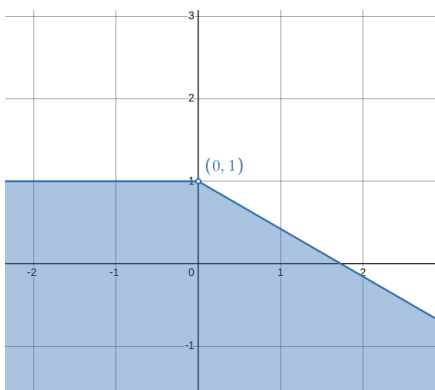
$$z - 3 + i \xrightarrow{T_{[3, -1]}} z$$



4.f)

$$\{z \in \mathbb{C} : \frac{\pi}{6} \leq \arg(\bar{z} + i) \leq \pi\}$$

$$\bar{z} + i \xrightarrow{T_{[0, -1]}} \bar{z} \xrightarrow{S_{OX}} z$$



4.g)

$$\{z \in \mathbb{C} : \arg\left(\frac{i}{i-z}\right) = \frac{4\pi}{3}\}$$

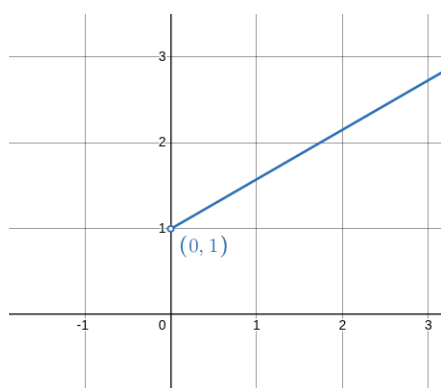
Zał: $z \neq i \Rightarrow z \neq (0, 1)$

$$\arg(i) - \arg(i - z) \equiv \frac{4\pi}{3} \pmod{2\pi}$$

$$\frac{\pi}{2} - \arg(i - z) \equiv \frac{4\pi}{3}$$

$$\arg(i - z) \equiv \frac{\pi}{2} - \frac{4\pi}{3} = \frac{3\pi}{6} - \frac{8\pi}{6} = -\frac{5\pi}{6} \equiv \frac{7\pi}{6}$$

$$i - z \xrightarrow{S_{(0,0)}} z - i \xrightarrow{T_{[0,1]}} z$$

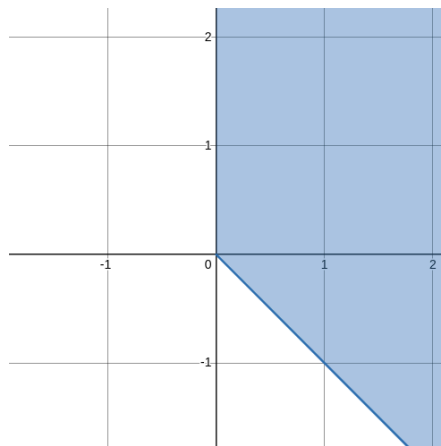


4.h)

$$\{z \in \mathbb{C} : \arg\left(\frac{i}{z}\right) \leq \frac{3\pi}{4}\}$$

Zał: $z \neq 0 \Rightarrow z \neq (0, 0)$

$$\frac{i}{z} \xrightarrow{R_{(0,0)}^{-\frac{\pi}{2}}} \frac{i}{z} \cdot \frac{1}{i} = \frac{1}{z} \xrightarrow{S_{\Re, P^{\frac{1}{|z|}}}} z$$

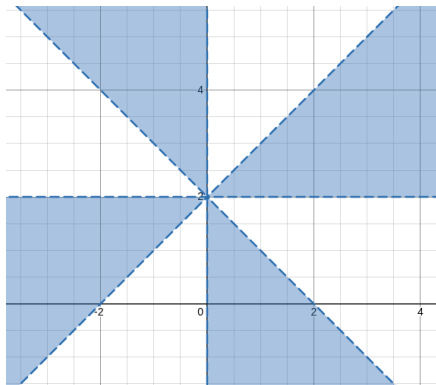


4.i)

$$\{z \in \mathbb{C} : \Im((z - 2i)^4) > 0\}$$

$$\arg(z - 2i) = \varphi \Rightarrow \arg((z - 2i)^4) = 4\varphi \bmod 2\pi \in (0, \pi) \Leftrightarrow \varphi \bmod \frac{\pi}{2} \in (0, \frac{\pi}{4})$$

$$z - 2i \xrightarrow{T[0,2]} z$$



4.j)

$$\{z \in \mathbb{C} : \frac{z+4}{z-2i} \in \mathbb{R}\}$$

$$\text{Zał: } z \neq 2i \Leftrightarrow z \neq (0, 2)$$

$$\frac{z+4}{z-2i} \in \mathbb{R} \Leftrightarrow \arg\left(\frac{z+4}{z-2i}\right) \in \{0, \pi\}$$

$z - 4$ i $z - 2i$ są współliniowe

Rozwiązanie to prosta przechodząca przez $(-4, 0)$ i $(0, 2)$ z wyłączeniem $(0, 2)$.

