Zadanie 1

Sprawdź, które z podanych odwzorowań są liniowe:

1.a)

$L : \mathbb{R}[x] \to \mathbb{R}[x], (Lp)(x) = xp'(x) + p(1);$ ✓

Niech $p, q \in \mathbb{R}[x], \alpha, \beta \in \mathbb{R}$

$$L(\alpha p + \beta q) = x(\alpha p + \beta q)' + (\alpha p + \beta q)(1) = x(\alpha p' + \beta q') + \alpha p(1) + \beta q(1) = \\ \alpha(xp' + p(1)) + \beta(xq' + q(1)) = \alpha L(p) + \beta L(q)$$

1.b)

$L : \mathbb{R}[x] \to \mathbb{R}[x], (Lp)(x) = p(x)p'(x);$ ✗

Niech $p \in \mathbb{R}[x], \alpha \in \mathbb{R}$

$$L(\alpha p) = (\alpha p)(\alpha p)' = \alpha^2 pp' \neq \alpha pp'$$

1.c)

$L : \mathbb{R}^2 \to \mathbb{R}^2, L(x, y) = (3x + 2y - 1, 2x - 3y);$ ✗

Nie ma zachowania zera:

$$L(0, 0) = (-1, 0) \neq (0, 0)$$

1.d)

$L : C(\mathbb{R}) \to C(\mathbb{R}), (Lf)(x) = \sin f(x);$ ✗

Kontrprzykład: $f(x) = \frac{\pi}{2}, \alpha = 2$

$$L(\alpha f) = L\left(2 \cdot \frac{\pi}{2}\right) = \sin \pi = 0 \neq 2 \sin \frac{\pi}{2}$$

1.e)

$L : C([0, 1]) \to C([0, 1]), (Lf)(x) = 2f(\frac{x}{2}).$ ✓

Niech $f, g \in C([0, 1]), \alpha, \beta \in \mathbb{R}$

$$(L(\alpha f + \beta g))(x) = 2(\alpha f + \beta g)\left(\frac{x}{2}\right) = 2\alpha f\left(\frac{x}{2}\right) + 2\beta g\left(\frac{x}{2}\right) = \alpha L(f) + \beta L(g)$$

Zadanie 2

Dane są następujące odwzorowania $f : \mathbb{R}^3 \to \mathbb{R}^2$ i $g : \mathbb{R}^2 \to \mathbb{R}^3$:
$f(x, y, z) = (-x + y + z, x - y + z), g(x, y) = (y, x, x + y)$.

2.a)

Sprawdź, że są to odwzorowania liniowe i podaj $f(1, 2, -1)$, $g(-1, 3)$;

Niech $x_1, y_1, z_1, x_2, y_2, z_2, \alpha, \beta \in \mathbb{R}$

$$f(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) = f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) =$$
$$(-\alpha x_1 - \beta x_2 + \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 + \alpha z_1 + \beta z_2,) =$$
$$\alpha(-x_1 + y_1 + z_1, x_1 - y_1 + z_1) + \beta(-x_2 + y_2 + z_2, x_2 - y_2 + z_2) =$$
$$\alpha f(x_1, y_1, z_1) + \beta f(x_2, y_2, z_2)$$

$f(1, 2, -1) = (-1 + 2 - 1, 1 - 2 - 1) = (0, -2)$

Niech $x_1, y_1, x_2, y_2, \alpha, \beta \in \mathbb{R}$

$$g(\alpha(x_1, y_1) + \beta\left(x_2, y_)\right) = g(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) =$$
$$(\alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2, \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2) =$$
$$\alpha(y_1, x_1, x_1 + y_1) + \beta(y_2, x_2, x_2 + y_2) = \alpha g(x_1, y_1) + \beta g(x_2, y_2)$$

$g(-1, 3) = (3, -1, 2)$

2.b)

Znajdź $\mathrm{Ker} f$, $\mathrm{Im} f$, $\mathrm{Ker} g$, $\mathrm{Im} g$ i podaj ich wymiary;

1) $\mathrm{Ker} f = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = (0, 0)\}$

$$f(x, y, z) = (-x + y + z, x - y + z) = (0, 0) \Leftrightarrow$$
$$\begin{cases} -x + y + z = 0 \\ x - y + z = 0 \end{cases}$$
$$z = 0$$
$$x = y$$

$\mathrm{Ker} f = \{(x, x, 0) \in \mathbb{R}^3\}$, $\dim \mathrm{Ker} f = 1$

2) $\mathrm{Im} f = \{(a, b) \in \mathbb{R}^2 \mid \exists (x, y, z) \in \mathbb{R}^3 : f(x, y, z) = (a, b)$

Niech $a, b \in \mathbb{R}$

$$f(x,y,z)=(-x+y+z,x-y+z)=(a,b)\Leftrightarrow$$
$$\begin{cases}-x+y+z=a\\ x-y+z=b\end{cases}\Rightarrow z=\frac{a+b}{2}$$
$$x-y+\frac{a+b}{2}=b\Rightarrow x-y=\frac{b-a}{2}$$
$$f\left(\frac{b-a}{2},0,\frac{a+b}{2}\right)=(a,b)$$

$\mathrm{Im} f=\mathbb{R}^2, \dim \mathrm{Im} f=2$

3) $\mathrm{Ker} g=\{(x,y)\in\mathbb{R}^2 : g(x,y)=(0,0,0)\}$

$$g(x,y)=(y,x,x+y)=(0,0,0)\Leftrightarrow x=y=0$$

$\mathrm{Ker} g=\{(0,0)\}, \dim \mathrm{Ker} g=0$

4) $\mathrm{Im} g=\{(x,y,z)\in\mathbb{R}^3 \mid \exists(a,b)\in\mathbb{R}^2 : g(a,b)=(x,y,z)$

Niech $x,y,z\in\mathbb{R}$

$$g(a,b)=(b,a,a+b)=(x,y,z)\Leftrightarrow$$
$$\begin{cases}a=y\\ b=x\\ z=x+y\end{cases}$$

$\mathrm{Im} f=\{(x,y,x+y)\in\mathbb{R}^3\}, \dim \mathrm{Im} f=2$

2.c)

Sprecyzuj $g\circ f$ i $f\circ g$.

$$g\circ f=f(g(x,y))=f(y,x,x+y)=(-y+x+x+y,y-x+x+y)=(2x,2y)$$
$$f\circ g=g(f(x,y,z))=g(-x+y+z,x-y+z)=(x-y+z,-x+y+z,2z)$$

Zadanie 3

Podaj wymiary jąder i obrazów następujących przekształceń liniowych:

3.a)

$L:\mathbb{R}^3\to\mathbb{R}^2, L(x,y,z)=(x+y,y+z);$

$$\begin{cases}x+y=0\\ y+z=0\end{cases}\Rightarrow\begin{cases}y=-x\\ z=x\end{cases}\Rightarrow \mathrm{Ker} L=\{(x,-x,x):x\in\mathbb{R}\}\Rightarrow \dim \mathrm{Ker} L=1$$
$$\dim\mathbb{R}^3=\dim \mathrm{Ker} L+\dim \mathrm{Im} L\Rightarrow \dim \mathrm{Im} L=3-1=2$$

3.b)

$L : \mathbb{R}^3 \to \mathbb{R}^4, L(x, y, z) = (2x - y + z, x + 2y - z, -x + 3y - 2z, 8x + y + z);$

$$\begin{cases} 2x - y + z = 0 \\ x + 2y - z = 0 \\ -x + 3y - 2z = 0 \\ 8x + y + z = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & 3 & -2 \\ 8 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 5 & -3 \\ 0 & -15 & 9 \end{bmatrix}$$

$$y = \frac{3}{5}z$$

$$x = -2y + z = -\frac{7}{5}z$$

$$\mathrm{Ker}L = \{(-7t, 3t, 5t), t \in \mathbb{R}\} \Rightarrow \dim \mathrm{Ker}L = 1$$

$$\dim \mathbb{R}^3 = \dim \mathrm{Ker}L + \dim \mathrm{Im}L \Rightarrow \dim \mathrm{Im}L = 3 - 1 = 2$$

3.c)

$L : R[x]_2 \to R[x]_2, (Lp)(x) = (x^2 + x)p(2) + (3x^2 - x)p(1).$

Niech $p = ax^2 + bx + c$. Wtedy

$$(Lp)(x) = (x^2 + x)(4a + 2b + c) + (3x^2 - x)(a + b + c) = (7a + 3b + 2c)x^2 + (3a + b)x$$

$$\begin{cases} 7a + 3b + 2c = 0 \\ 3a + b = 0 \end{cases} \Rightarrow \begin{cases} b = -3a \\ c = a \end{cases}$$

$$\mathrm{Ker}L = \{(ax^2 - 3ax + a), a \in \mathbb{R}\} \Rightarrow \dim \mathrm{Ker}L = 1$$

$$\dim R[x]_2 = \dim \mathrm{Ker}L + \dim \mathrm{Im}L \Rightarrow \dim \mathrm{Im}L = 3 - 1 = 2$$