

Zadanie 1

Zbadaj rzędy następujących macierzy:

1.a)

$$A = \begin{bmatrix} 2 & 6 & -1 & 4 & 3 \\ 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 3 & 10 & 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 2 & 6 & -1 & 4 & 3 \\ 0 & -2 & -5 & 6 & 3 \\ 3 & 10 & 1 & 3 & 3 \end{bmatrix} - 2w_1 = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & -2 & -5 & 6 & 3 \end{bmatrix} - 3w_1 = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r(A) = 2$$

1.b)

$$B = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ -3 & -1 & -1 & 4 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 3 & 2 & 2 & 1 & 8 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} + w_1 = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} - w_1 = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 \end{bmatrix} - w_2 = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 \end{bmatrix} - w_3 = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r(B) = 4$$

1.c)

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \end{bmatrix} - w_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} - 2w_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} + w_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix} + 2w_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix} - 3w_3$$

$$r(C) = 4$$

1.d)

$$D = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 4 & -3 & 1 & 1 \end{bmatrix} - 3w_1 = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \\ 0 & -5 & -5 & 3 \\ 0 & -15 & -15 & 9 \end{bmatrix} - 2w_3 =$$

$$\begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r(D) = 2$$

Zadanie 2

Wyznacz rzędy następujących macierzy w zależności od parametru rzeczywistego p :

2.a)

$$A = \begin{bmatrix} 1-p & 2 & 1 & p \\ 1 & 2-p & 1 & 0 \\ 1 & 2 & 1-p & p \end{bmatrix} - w_1 = \begin{bmatrix} 1-p & 2 & 1 & p \\ p & -p & 0 & -p \\ p & 0 & -p & 0 \end{bmatrix} - w_1$$

Szukamy niezerowego minora o największym wymiarze.

$$W = \begin{vmatrix} 1-p & 2 & 1 \\ p & -p & 0 \\ p & 0 & -p \end{vmatrix} = p^2 \begin{vmatrix} 1-p & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \xrightarrow{\text{z tw. Laplace'a}} =$$

$$p^2 \left((1-p)(-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \right) =$$

$$p^2((1-p) \cdot 1 - 2 \cdot (-1) + 1) = p^2(4-p)$$

Jeżeli $p \notin \{0, 4\}$ to $W \neq 0$, czyli znaleźliśmy niezerowego minora, więc $r(A) = 3$.

Jeżeli $p = 0 \Rightarrow$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \Rightarrow r(A) = 1$$

Jeżeli $p = 4 \Rightarrow$

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & -2 & 1 & 0 \\ -3 & 2 & 1 & 4 \end{bmatrix} - w_1 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 8 & -8 & 16 \end{bmatrix} + 3w_1 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix} + 2w_2$$

$$r(A) = 3$$

Ostatecznie $r(A) = \begin{cases} 3, \text{ gdy } p \neq 0 \\ 1, \text{ gdy } p = 0 \end{cases}$

2.b)

$$B = \begin{bmatrix} p-1 & p-1 & 1 & 1 \\ 1 & p^2-1 & 1 & p-1 \\ 1 & p-1 & p-1 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} p-1 & 1 & 1 \\ 1 & 1 & p-1 \\ 1 & p-1 & 1 \end{vmatrix} = 3(p-1) - 1 - (p-1)^3 - 1$$

Niech $t = p - 1$. Wtedy mamy:

$$\begin{aligned} 3t - 1 - t^3 - 1 &= 0 \\ t^3 - 3t + 2 &= 0 \\ (t-1)(t^2+t-2) &= 0 \\ (t-1)^2(t+2) &= 0 \\ t = 1 \vee t &= -2 \\ p = 2 \vee p &= -1 \end{aligned}$$

$$p = 2 \Rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow r(B) = 2$$

$$\begin{aligned} p = -1 \Rightarrow B &= \begin{bmatrix} -2 & -2 & 1 & 1 \\ 1 & 0 & 1 & -2 \\ 1 & -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ -2 & -2 & 1 & 1 \\ 1 & -2 & -2 & 1 \end{bmatrix} + 2w_1 = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -2 & 3 & -3 \\ 0 & -2 & -3 & -3 \end{bmatrix} - w_1 = \\ &\quad \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & -6 & 0 \end{bmatrix} \Rightarrow r(B) = 3 \end{aligned}$$

Ostatecznie $r(B) = \begin{cases} 3, \text{ gdy } p \neq 2 \\ 2, \text{ gdy } p = 2 \end{cases}$

Zadanie 3

Dla jakich wartości parametru $a \in \mathbb{R}$,rząd macierzy jest najmniejszy, a dla jakich największy?

3.a)

$$A = \begin{bmatrix} -2 & -1-a & 1 \\ a & 0 & -a \\ -1 & a+a^2 & 1 \end{bmatrix}$$

$$\begin{aligned} W &= (-1-a)(-1)^{1+2} \begin{vmatrix} a & -a \\ -1 & 1 \end{vmatrix} + 0 + (a+a^2)(-1)^{3+2} \begin{vmatrix} -2 & 1 \\ a & -a \end{vmatrix} = \\ &= (1+a)(a-(-a)(-1)) - (a+a^2)(-2(-a)-a) = \\ &= (1+a) \cdot 0 - (a+a^2) \cdot a = -a^2(a+1) \end{aligned}$$

$a \notin \{-1, 0\} \Rightarrow$ wyznacznik jest niezerowy, więc rząd to 3.

$$a = 0 \Rightarrow A = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \Rightarrow \text{rząd to 2}$$

$$a = -1 \Rightarrow A = \begin{bmatrix} -2 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = -2 - (-1) = -1 \neq 0 \Rightarrow \text{rząd to 2}$$

Ostatecznie $r(A) = \begin{cases} 3, \text{ gdy } a \notin \{-1, 0\} \\ 2, \text{ gdy } a \in \{-1, 0\} \end{cases}$

3.b)

$$B = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$$

$$\begin{aligned} W &= \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} + w_2 + w_3 + w_4 = \begin{vmatrix} a+3 & a+3 & a+3 & a+3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} - w_1 = \\ &\quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 1 \\ 0 & 0 & 0 & a-1 \end{vmatrix} = (a-1)^3 \end{aligned}$$

Jeżeli $a \neq 1$ to całe B jest minorem niezerowym stopnia 4 czyli rząd B to 4.

$$a = 1 \Rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow r(B) = 1$$

Ostatecznie $r(B) = \begin{cases} 4, \text{ gdy } a \neq 1 \\ 1, \text{ gdy } a=1 \end{cases}$

Zadanie 4

Oblicz wyznacznik macierzy i jeśli jest ona nieosobliwa, znajdź macierz do niej odwrotną:

4.a)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = \left| \begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right| = 1 + 0 + 1 - 0 - 2 - 0 = 0$$

4.b)

$$B = \begin{bmatrix} 1+i & -1 \\ 0 & 2 \end{bmatrix}$$

$$W = \begin{vmatrix} 1+i & -1 \\ 0 & 2 \end{vmatrix} = 2(1+i) - 0 = 2 + 2i \neq 0$$

Macierz nie jest osobliwa, więc możemy znaleźć jej macierz odwrotną.

$$\left[\begin{array}{cc|cc} 1+i & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array} \right] \cdot \frac{1}{1+i} \cdot \frac{1}{2} = \left[\begin{array}{cc|cc} 1 & \frac{-1}{1+i} & \frac{1}{1+i} & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right] + \frac{1}{1+i} w_2 = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{1+i} & \frac{1}{2+2i} \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right]$$

$$B^{-1} = \begin{bmatrix} \frac{1}{1+i} & \frac{1}{2+2i} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} & \frac{2-2i}{4+4} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$$

4.c)

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\det C = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{vmatrix} = 0 + 1 + 0 - 0 + 1 + 1 = 3$$

Metoda Gaussa:

$$\begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array} \right] -w_1 = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array} \right] -2w_2 = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -2 & 1 \end{array} \right] \cdot (-1) = \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] +w_3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] -w_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] \end{array}$$

Metoda dopełnień algebraicznych:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1 \cdot (0 - (-1)) = 1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 1) = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 0) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 1) = 2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1 - 0) = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (-1) \cdot (1 - 0) = -1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1$$

$$C^{-1} = (\det C)^{-1} (C^D)^T = 3^{-1} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Metoda układem równań:

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{cases} x_1 + x_2 + 0 = y_1 \\ x_1 + 0 + x_3 = y_2 \\ x_1 - x_2 - x_3 = y_3 \end{cases}$$

$$3x_1 = y_1 + y_2 + y_3$$

$$x_1 = \frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3$$

$$x_2 = y_1 - x_1 = \frac{2}{3}y_1 - \frac{1}{3}y_2 - \frac{1}{3}y_3$$

$$x_3 = y_2 - x_1 = -\frac{1}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3$$

$$\Rightarrow C^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

4.d)

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\det D = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \begin{array}{l} -w_1 \\ -w_2 \end{array} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0$$

bo są dwa takie same wiersze

4.e)

$$E = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 0 & 1 & 3 \\ 2 & 2 & 0 & 3 \end{bmatrix}$$

$$\det E = 1 \cdot (-1)^{1+3} M_{13} + 0 + 1 \cdot (-1)^{3+3} M_{33} + 0$$

$$M_{13} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 3 \\ 2 & 2 & 3 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 3 & 0 \\ 2 & 2 \end{vmatrix} = 0 - 6 + 12 - 0 - 6 + 9 = 9$$

$$M_{33} = \begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 2 \\ 2 & 2 & 3 \end{vmatrix} \begin{vmatrix} 4 & 2 \\ 1 & -1 \\ 2 & 2 \end{vmatrix} = -12 + 8 + 2 + 2 - 16 - 6 = -22$$

$$\det E = 9 - 22 = -13$$

$$\begin{array}{c} \left[\begin{array}{cccc|ccccc} 4 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 & 0 & 0 & 0 & 1 \end{array} \right] w_1 \leftrightarrow w_2 \\ = \begin{array}{c} \left[\begin{array}{cccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 & 0 & 0 & 0 & 1 \end{array} \right] -4w_1 \\ -3w_1 \\ -2w_1 \end{array} \end{array}$$

$$\begin{array}{c} \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 4 & 0 & -1 & 0 & -2 & 0 & 1 \end{array} \right] -w_3 \\ = \begin{array}{c} \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \end{array} \right] w_2 \leftrightarrow w_4 \\ = \end{array} \end{array}$$

$$\begin{array}{c} \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \end{array} \right] -3w_2 \\ = \begin{array}{c} \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & -9 & 0 & -6 & 4 & -3 \\ 0 & 0 & 7 & -19 & 1 & -10 & 6 & -6 \end{array} \right] 2w_3 - w_4 \\ = \end{array} \end{array}$$

$$\begin{array}{c} \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 7 & -19 & 1 & -10 & 6 & -6 \end{array} \right] -7w_3 \\ = \begin{array}{c} \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -26 & 8 & 4 & -8 & -6 \end{array} \right] 13w_1 \\ 13w_2 \\ 13w_2 \\ \frac{1}{2}w_4 \end{array} \end{array}$$

$$\begin{array}{c} \left[\begin{array}{cccc|ccccc} 13 & -13 & 0 & 26 & 0 & 13 & 0 & 0 \\ 0 & 13 & -13 & 26 & 0 & 13 & -13 & 13 \\ 0 & 0 & 13 & 13 & -13 & -26 & 26 & 0 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] +2w_4 \\ = \begin{array}{c} \left[\begin{array}{cccc|ccccc} 13 & -13 & 0 & 0 & 8 & 17 & -8 & -6 \\ 0 & 13 & -13 & 0 & 8 & 17 & -21 & 7 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] +w_3 \\ +w_3 = \end{array} \end{array}$$

$$\begin{array}{c} \left[\begin{array}{cccc|ccccc} 13 & -13 & 0 & 0 & 8 & 17 & -8 & -6 \\ 0 & 13 & 0 & 0 & -1 & -7 & 1 & 4 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] +w_2 \\ = \begin{array}{c} \left[\begin{array}{cccc|ccccc} 13 & 0 & 0 & 0 & 7 & 10 & -7 & -2 \\ 0 & 13 & 0 & 0 & -1 & -7 & 1 & 4 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] \frac{1}{13}w_1 \\ \frac{1}{13}w_2 \\ \frac{1}{13}w_3 \\ \frac{-1}{13}w_4 \end{array} \end{array}$$

$$E^{-1} = \frac{1}{13} \begin{bmatrix} 7 & 10 & -7 & 2 \\ -1 & -7 & 1 & 4 \\ -9 & -24 & 22 & -3 \\ -4 & -2 & 4 & 3 \end{bmatrix}$$

4.f)

$$F = \begin{bmatrix} 0 & 2 & 3 & 4 & -8 \\ 1 & 1 & 3 & 4 & -8 \\ 1 & 2 & 2 & 4 & -8 \\ 1 & 2 & 3 & 3 & -8 \\ 1 & 2 & 3 & 4 & -9 \end{bmatrix}$$

$$\det F = \begin{vmatrix} 0 & 2 & 3 & 4 & -8 & -w_5 \\ 1 & 1 & 3 & 4 & -8 & -w_5 \\ 1 & 2 & 2 & 4 & -8 & -w_5 \\ 1 & 2 & 3 & 3 & -8 & -w_5 \\ 1 & 2 & 3 & 4 & -9 & \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 2 & 3 & 4 & -9 \end{vmatrix} + w_1 + 2w_2 + 3w_3 + 4w_4$$

$$\begin{vmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{bmatrix} 0 & 2 & 3 & 4 & -8 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 & -8 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 4 & -8 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 3 & -8 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & -9 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 3 & 4 & -9 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + w_1 + 2w_2 + 3w_3 + 4w_4$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -2 & -3 & -4 & 8 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -3 & 8 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -2 & -2 & 8 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 & -9 \end{bmatrix} \cdot (-1)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & -8 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 4 & -8 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 4 & -8 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 3 & 3 & -8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 & -9 \end{bmatrix} \Rightarrow F^{-1} = F$$

4.g)

$$G_n = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix}_{n \times n}$$

$$\det G_n = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{vmatrix} \begin{matrix} -w_n \\ -w_n \\ \vdots \\ 1 \end{matrix} = \begin{vmatrix} -1 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{vmatrix} + w_1 + \dots + w_{n-1} =$$

$$\begin{vmatrix} -1 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & \dots & 0 & n-1 \end{vmatrix} = (-1)^{n-1}(n-1) \neq 0$$

$$\begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{vmatrix} \begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{matrix} \begin{matrix} -w_n \\ -w_n \\ \vdots \\ 1 \end{matrix} = \begin{vmatrix} -1 & 0 & \dots & 1 \\ 0 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{vmatrix} \begin{matrix} 1 & 0 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{matrix} + w_1 + \dots + w_{n-1} =$$

$$\begin{bmatrix} -1 & 0 & \dots & 1 \\ 0 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2-n \end{bmatrix} \cdot (1-n) =$$

$$\begin{bmatrix} n-1 & 0 & \dots & 1-n \\ 0 & n-1 & \dots & 1-n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 \end{bmatrix} \begin{bmatrix} 1-n & 0 & \dots & n-1 \\ 0 & 1-n & \dots & n-1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2-n \end{bmatrix} + w_n =$$

$$\begin{bmatrix} n-1 & 0 & \dots & 0 \\ 0 & n-1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 \end{bmatrix} \begin{bmatrix} 2-n & 1 & \dots & 1 \\ 1 & 2-n & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2-n \end{bmatrix} \cdot \frac{1}{n-1} \cdot \frac{1}{n-1} =$$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \frac{2-n}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{2-n}{n-1} & \dots & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{2-n}{n-1} \end{bmatrix}$$