

## Zadanie 1

Zbadaj rzędy następujących macierzy:

### 1.a)

$$A = \begin{bmatrix} 2 & 6 & -1 & 4 & 3 \\ 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 3 & 10 & 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 2 & 6 & -1 & 4 & 3 \\ 0 & -2 & -5 & 6 & 3 \\ 3 & 10 & 1 & 3 & 3 \end{bmatrix} \begin{matrix} \\ -2w_1 \\ \\ -3w_1 \end{matrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & -2 & -5 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r(A) = 2$$

### 1.b)

$$B = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ -3 & -1 & -1 & 4 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 3 & 2 & 2 & 1 & 8 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} \begin{matrix} \\ +w_1 \\ \\ -w_1 \\ \end{matrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} \begin{matrix} \\ \\ -w_2 \\ -w_2 \end{matrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ -w_3 \end{matrix}$$
$$= \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r(B) = 4$$

### 1.c)

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -w_1 \\ -w_1 \\ -w_1 \\ -w_1 \\ -2w_1 \end{matrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \\ -2w_2 \\ +w_2 \end{matrix} =$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{matrix} \\ \\ +2w_3 \\ -3w_3 \end{matrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$$r(C) = 4$$

1.d)

$$D = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 4 & -3 & 1 & 1 \end{bmatrix} \begin{matrix} -3w_1 \\ -2w_1 \\ -4w_1 \end{matrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \\ 0 & -5 & -5 & 3 \\ 0 & -15 & -15 & 9 \end{bmatrix} \begin{matrix} -2w_3 \\ -3w_3 \end{matrix} =$$

$$\begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r(D) = 2$$

## Zadanie 2

Wyznacz rzędy następujących macierzy w zależności od parametru rzeczywistego  $p$ :

2.a)

$$A = \begin{bmatrix} 1-p & 2 & 1 & p \\ 1 & 2-p & 1 & 0 \\ 1 & 2 & 1-p & p \end{bmatrix} \begin{matrix} -w_1 \\ -w_1 \end{matrix} = \begin{bmatrix} 1-p & 2 & 1 & p \\ p & -p & 0 & -p \\ p & 0 & -p & 0 \end{bmatrix}$$

Szukamy niezerowego minora o największym wymiarze.

$$W = \begin{vmatrix} 1-p & 2 & 1 \\ p & -p & 0 \\ p & 0 & -p \end{vmatrix} = p^2 \begin{vmatrix} 1-p & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \stackrel{\text{z tw. Laplace'a}}{=} p^2 \left( (1-p)(-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \right) =$$

$$p^2((1-p) \cdot 1 - 2 \cdot (-1) + 1) = p^2(4-p)$$

Jeżeli  $p \notin \{0, 4\}$  to  $W \neq 0$ , czyli znaleźliśmy niezerowego minora, więc  $r(A) = 3$ .

Jeżeli  $p = 0 \Rightarrow$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \Rightarrow r(A) = 1$$

Jeżeli  $p = 4 \Rightarrow$

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & -2 & 1 & 0 \\ -3 & 2 & 1 & 4 \end{bmatrix} \xrightarrow{-w_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 8 & -8 & 16 \end{bmatrix} \xrightarrow{+3w_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{+2w_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$r(A) = 3$$

$$\text{Ostatecznie } r(A) = \begin{cases} 3, & \text{gdy } p \neq 0 \\ 1, & \text{gdy } p = 0 \end{cases}$$

## 2.b)

$$B = \begin{bmatrix} p-1 & p-1 & 1 & 1 \\ 1 & p^2-1 & 1 & p-1 \\ 1 & p-1 & p-1 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} p-1 & 1 & 1 \\ 1 & 1 & p-1 \\ 1 & p-1 & 1 \end{vmatrix} = 3(p-1) - 1 - (p-1)^3 - 1$$

Niech  $t = p - 1$ . Wtedy mamy:

$$3t - 1 - t^3 - 1 = 0$$

$$t^3 - 3t + 2 = 0$$

$$(t-1)(t^2 + t - 2) = 0$$

$$(t-1)^2(t+2) = 0$$

$$t = 1 \vee t = -2$$

$$p = 2 \vee p = -1$$

$$p = 2 \Rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow r(B) = 2$$

$$p = -1 \Rightarrow B = \begin{bmatrix} -2 & -2 & 1 & 1 \\ 1 & 0 & 1 & -2 \\ 1 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{-w_1} \begin{bmatrix} 1 & 0 & 1 & -2 \\ -2 & -2 & 1 & 1 \\ 1 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{+2w_1} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -2 & 3 & -3 \\ 0 & -2 & -3 & -3 \end{bmatrix} \xrightarrow{-w_2} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & -6 & 0 \end{bmatrix} \Rightarrow r(B) = 3$$

$$\text{Ostatecznie } r(B) = \begin{cases} 3, & \text{gdy } p \neq 2 \\ 2, & \text{gdy } p = 2 \end{cases}$$

## Zadanie 3

Dla jakich wartości parametru  $a \in \mathbb{R}$ , rząd macierzy jest najmniejszy, a dla jakich największy?

### 3.a)

$$A = \begin{bmatrix} -2 & -1-a & 1 \\ a & 0 & -a \\ -1 & a+a^2 & 1 \end{bmatrix}$$

$$\begin{aligned} W &= (-1-a)(-1)^{1+2} \begin{vmatrix} a & -a \\ -1 & 1 \end{vmatrix} + 0 + (a+a^2)(-1)^{3+2} \begin{vmatrix} -2 & 1 \\ a & -a \end{vmatrix} = \\ &= (1+a)(a - (-a)(-1)) - (a+a^2)(-2(-a) - a) = \\ &= (1+a) \cdot 0 - (a+a^2) \cdot a = -a^2(a+1) \end{aligned}$$

$a \notin \{-1, 0\} \Rightarrow$  wyznacznik jest niezerowy, więc rząd to 3.

$$a = 0 \Rightarrow A = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \Rightarrow \text{rząd to } 2$$

$$a = -1 \Rightarrow A = \begin{bmatrix} -2 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = -2 - (-1) = -1 \neq 0 \Rightarrow \text{rząd to } 2$$

Ostatecznie  $r(A) = \begin{cases} 3, & \text{gdy } a \notin \{-1, 0\} \\ 2, & \text{gdy } a \in \{-1, 0\} \end{cases}$

### 3.b)

$$B = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$$

$$\begin{aligned} W &= \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} + w_2 + w_3 + w_4 = \begin{vmatrix} a+3 & a+3 & a+3 & a+3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} - w_1 = \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 1 \\ 0 & 0 & 0 & a-1 \end{vmatrix} = (a-1)^3 \end{aligned}$$

Jeżeli  $a \neq 1$  to całe  $B$  jest minorem niezerowym stopnia 4 czyli rząd  $B$  to 4.

$$a = 1 \Rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow r(B) = 1$$

$$\text{Ostatecznie } r(B) = \begin{cases} 4, & \text{gdy } a \neq 1 \\ 1, & \text{gdy } a = 1 \end{cases}$$

#### Zadanie 4

Oblicz wyznacznik macierzy i jeśli jest ona nieosobliwa, znajdź macierz do niej odwrotną:

4.a)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = \begin{vmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{vmatrix} = 1 + 0 + 1 - 0 - 2 - 0 = 0$$

4.b)

$$B = \begin{bmatrix} 1+i & -1 \\ 0 & 2 \end{bmatrix}$$

$$W = \begin{vmatrix} 1+i & -1 \\ 0 & 2 \end{vmatrix} = 2(1+i) - 0 = 2 + 2i \neq 0$$

Macierz nie jest osobliwa, więc możemy znaleźć jej macierz odwrotną.

$$\begin{bmatrix} 1+i & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \cdot \begin{matrix} \frac{1}{1+i} \\ \frac{1}{2} \end{matrix} = \begin{bmatrix} 1 & \frac{-1}{1+i} & \frac{1}{1+i} & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{bmatrix} + \frac{1}{1+i} w_2 = \begin{bmatrix} 1 & 0 & \frac{1}{1+i} & \frac{1}{2+2i} \\ 0 & 1 & 0 & \frac{1}{2} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{1}{1+i} & \frac{1}{2+2i} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} & \frac{2-2i}{4+4} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$$

4.c)

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\det C = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{vmatrix} = 0 + 1 + 0 - 0 + 1 + 1 = 3$$

Metoda Gaussa:

$$\begin{aligned} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array} \right] & \xrightarrow{-w_1} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array} \right] & \xrightarrow{-2w_2} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -2 & 1 \end{array} \right] \cdot \begin{pmatrix} -1 \\ -1 \\ -\frac{1}{3} \end{pmatrix} = \\ \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] & \xrightarrow{+w_3} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] & \xrightarrow{-w_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] \end{aligned}$$

Metoda dopełnień algebraicznych:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1 \cdot (0 - (-1)) = 1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 1) = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 0) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 1) = 2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1 - 0) = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (-1) \cdot (1 - 0) = -1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1$$

$$C^{-1} = (\det C)^{-1}(C^D)^T = 3^{-1} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Metoda układem równań:

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{cases} x_1 + x_2 + 0 = y_1 \\ x_1 + 0 + x_3 = y_2 \\ x_1 - x_2 - x_3 = y_3 \end{cases}$$

$$3x_1 = y_1 + y_2 + y_3$$

$$x_1 = \frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3$$

$$x_2 = y_1 - x_1 = \frac{2}{3}y_1 - \frac{1}{3}y_2 - \frac{1}{3}y_3$$

$$x_3 = y_2 - x_1 = -\frac{1}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3$$

$$\Rightarrow C^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

4.d)

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\det D = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \begin{matrix} -w_1 \\ -w_2 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0$$

bo są dwa takie same wiersze

4.e)

$$E = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 0 & 1 & 3 \\ 2 & 2 & 0 & 3 \end{bmatrix}$$

$$\det E = 1 \cdot (-1)^{1+3}M_{13} + 0 + 1 \cdot (-1)^{3+3}M_{33} + 0$$

$$M_{13} = \left| \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & -1 & \\ 3 & 0 & 3 & 3 & 0 & \\ 2 & 2 & 3 & 2 & 2 & \end{array} \right| = 0 - 6 + 12 - 0 - 6 + 9 = 9$$

$$M_{33} = \left| \begin{array}{ccc|ccc} 4 & 2 & 1 & 4 & 2 & \\ 1 & -1 & 2 & 1 & -1 & \\ 2 & 2 & 3 & 2 & 2 & \end{array} \right| = -12 + 8 + 2 + 2 - 16 - 6 = -22$$

$$\det E = 9 - 22 = -13$$

$$\left[ \begin{array}{cccc|cccc} 4 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 & 0 & 0 & 0 & 1 \end{array} \right] w_1 \leftrightarrow w_2 = \left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \begin{matrix} -4w_1 \\ -3w_1 \\ -2w_1 \end{matrix} =$$

$$\left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 4 & 0 & -1 & 0 & -2 & 0 & 1 \end{array} \right] -w_3 = \left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \end{array} \right] w_2 \leftrightarrow w_4 =$$

$$\left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \end{array} \right] \begin{matrix} -3w_2 \\ -6w_2 \end{matrix} = \left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & -9 & 0 & -6 & 4 & -3 \\ 0 & 0 & 7 & -19 & 1 & -10 & 6 & -6 \end{array} \right] \begin{matrix} 2w_3 - w_4 \\ \end{matrix} =$$

$$\left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 7 & -19 & 1 & -10 & 6 & -6 \end{array} \right] -7w_3 = \left[ \begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -26 & 8 & 4 & -8 & -6 \end{array} \right] \begin{matrix} 13w_1 \\ 13w_2 \\ 13w_2 \\ \frac{1}{2}w_4 \end{matrix} =$$

$$\left[ \begin{array}{cccc|cccc} 13 & -13 & 0 & 26 & 0 & 13 & 0 & 0 \\ 0 & 13 & -13 & 26 & 0 & 13 & -13 & 13 \\ 0 & 0 & 13 & 13 & -13 & -26 & 26 & 0 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] \begin{matrix} +2w_4 \\ +2w_4 \\ +w_4 \end{matrix} = \left[ \begin{array}{cccc|cccc} 13 & -13 & 0 & 0 & 8 & 17 & -8 & -6 \\ 0 & 13 & -13 & 0 & 8 & 17 & -21 & 7 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] +w_3 =$$

$$\left[ \begin{array}{cccc|cccc} 13 & -13 & 0 & 0 & 8 & 17 & -8 & -6 \\ 0 & 13 & 0 & 0 & -1 & -7 & 1 & 4 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] +w_2 = \left[ \begin{array}{cccc|cccc} 13 & 0 & 0 & 0 & 7 & 10 & -7 & -2 \\ 0 & 13 & 0 & 0 & -1 & -7 & 1 & 4 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array} \right] \begin{matrix} \frac{1}{13}w_1 \\ \frac{1}{13}w_2 \\ \frac{1}{13}w_3 \\ \frac{-1}{13}w_4 \end{matrix}$$

$$E^{-1} = \frac{1}{13} \begin{bmatrix} 7 & 10 & -7 & 2 \\ -1 & -7 & 1 & 4 \\ -9 & -24 & 22 & -3 \\ -4 & -2 & 4 & 3 \end{bmatrix}$$

4.f)

$$F = \begin{bmatrix} 0 & 2 & 3 & 4 & -8 \\ 1 & 1 & 3 & 4 & -8 \\ 1 & 2 & 2 & 4 & -8 \\ 1 & 2 & 3 & 3 & -8 \\ 1 & 2 & 3 & 4 & -9 \end{bmatrix}$$

$$\det F = \begin{vmatrix} 0 & 2 & 3 & 4 & -8 \\ 1 & 1 & 3 & 4 & -8 \\ 1 & 2 & 2 & 4 & -8 \\ 1 & 2 & 3 & 3 & -8 \\ 1 & 2 & 3 & 4 & -9 \end{vmatrix} \begin{matrix} -w_5 \\ -w_5 \\ -w_5 \\ -w_5 \\ \end{matrix} = \begin{vmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 2 & 3 & 4 & -9 \end{vmatrix} = +w_1 + 2w_2 + 3w_3 + 4w_4$$

$$\begin{vmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{bmatrix} 0 & 2 & 3 & 4 & -8 \\ 1 & 1 & 3 & 4 & -8 \\ 1 & 2 & 2 & 4 & -8 \\ 1 & 2 & 3 & 3 & -8 \\ 1 & 2 & 3 & 4 & -9 \end{bmatrix} \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \begin{matrix} -w_5 \\ -w_5 \\ -w_5 \\ -w_5 \\ \end{matrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 2 & 3 & 4 & -9 \end{bmatrix} \begin{matrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} + w_1 + 2w_2 + 3w_3 + 4w_4 =$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 3 & 4 & -9 \end{matrix} \begin{matrix} -w_5 \\ -w_5 \\ -w_5 \\ -w_5 \\ \end{matrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 0 & -2 & -3 & -4 & 8 \\ -1 & -1 & -3 & -4 & 8 \\ -1 & -2 & -2 & -4 & 8 \\ -1 & -2 & -3 & -3 & 8 \\ 1 & 2 & 3 & 4 & -9 \end{matrix} \begin{matrix} \cdot (-1) \\ \cdot (-1) \\ \cdot (-1) \\ \cdot (-1) \\ \end{matrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 0 & 2 & 3 & 4 & -8 \\ 1 & 1 & 3 & 4 & -8 \\ 1 & 2 & 2 & 4 & -8 \\ 1 & 2 & 3 & 3 & -8 \\ 1 & 2 & 3 & 4 & -9 \end{matrix} \Rightarrow F^{-1} = F$$

4.g)

$$G_n = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix}_{n \times n}$$

$$\det G_n = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{vmatrix} \begin{matrix} -w_n \\ -w_n \\ \vdots \end{matrix} = \begin{vmatrix} -1 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{vmatrix} + w_1 + \dots + w_{n-1} =$$

$$\begin{vmatrix} -1 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & \dots & 0 & n-1 \end{vmatrix} = (-1)^{n-1}(n-1) \neq 0$$

$$\begin{bmatrix} 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{matrix} -w_n \\ -w_n \\ \vdots \end{matrix} = \begin{bmatrix} -1 & 0 & \dots & 1 & 1 & 0 & \dots & -1 \\ 0 & -1 & \dots & 1 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} + w_1 + \dots + w_{n-1} =$$

$$\begin{bmatrix} -1 & 0 & \dots & 1 & 1 & 0 & \dots & -1 \\ 0 & -1 & \dots & 1 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 & 1 & 1 & \dots & 2-n \end{bmatrix} \cdot \begin{matrix} (1-n) \\ (1-n) \\ \vdots \end{matrix} =$$

$$\begin{bmatrix} n-1 & 0 & \dots & 1-n & 1-n & 0 & \dots & n-1 \\ 0 & n-1 & \dots & 1-n & 0 & 1-n & \dots & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 & 1 & 1 & \dots & 2-n \end{bmatrix} \begin{matrix} +w_n \\ +w_n \\ \vdots \end{matrix} =$$

$$\begin{bmatrix} n-1 & 0 & \dots & 0 & 2-n & 1 & \dots & 1 \\ 0 & n-1 & \dots & 0 & 1 & 2-n & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 & 1 & 1 & \dots & 2-n \end{bmatrix} \cdot \begin{matrix} \frac{1}{n-1} \\ \frac{1}{n-1} \\ \vdots \\ \frac{1}{n-1} \end{matrix} =$$

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \frac{2-n}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1} \\ 0 & 1 & \dots & 0 & \frac{1}{n-1} & \frac{2-n}{n-1} & \dots & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{2-n}{n-1} \end{bmatrix}$$