Zadanie 1

Wykaż równości i nierówności dla liczb zespolonych:

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

1.a)

$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

$\overline{z_1 z_2} = \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$

$\overline{z_1} \cdot \overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$

1.b)

$|z_1 z_2| = |z_1||z_2|$

$|z_1 z_2| = |(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2} =$
$\sqrt{a_1^2 a_2^2 - \cancel{2a_1 a_2 b_1 b_2} + b_1^2 b_2^2 + a_1^2 b_2^2 + \cancel{2a_1 a_2 b_1 b_2} + b_1^2 a_2^2} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} =$
$\sqrt{(a_1^2 + b_1^2)} \cdot \sqrt{(a_2^2 + b_2^2)} = |z_1||z_2|$

1.c)

$z \cdot \overline{z} = |z|^2$

$z \cdot \overline{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \sqrt{a^2 + b^2}^2 = |z|$

1.d)

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} \cdot \overline{z_2} = \frac{\overline{z_1}}{\overline{z_2}} \cdot \overline{z_2}$$

$$\overline{\frac{z_1}{z_2} \cdot z_2} = \overline{z_1}$$

1.e)

$|z_1 + z_2| \leq |z_1| + |z_2|$

Dodawanie wektorów, nierówność trójkąta.

1.f)

$||z_1| - |z_2|| \leq |z_1 - z_2|$

$$|z_1| = |z_2 + (z_1 - z_2)| \leq |z_2| + |z_1 - z_2|$$
$$|z_1| - |z_2| \leq |z_1 - z_2|$$

Analogicznie

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$$
$$|z_1| - |z_2| \geq -|z_1 - z_2|$$

Czyli

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2| \Leftrightarrow ||z_1| - |z_2|| \leq |z_1 - z_2|$$

Zadanie 2

Oblicz:

2.a)

$$\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i-3i^2}{1-i^2} = \frac{2+i+3}{1+1} = \frac{5+i}{2}$$

2.b)

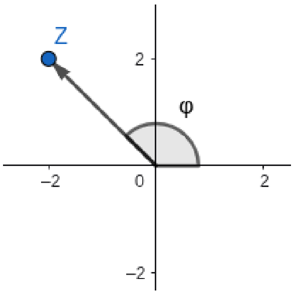
$$\frac{\left(i+\sqrt{3}\right)\left(-1-i\sqrt{3}\right)}{1+2i} = \frac{\left(-i-i^2\sqrt{3}-\sqrt{3}-3i\right)(1-2i)}{(1+2i)(1-2i)} = \frac{\left(-4i+\sqrt{3}-\sqrt{3}\right)(1-2i)}{1-4i^2} =$$
$$\frac{-4i+8i^2}{1+4} = \frac{-8-4i}{5}$$

2.c)

$$|3-4i| = \sqrt{3^2+4^2} = 5$$

2.d)

$\arg(-2+2i)$



$\arg(-2+2i) = \varphi = \frac{3\pi}{4}$

2.e)

$\frac{(1+i)^n}{(1-i)^{n-2}}$, dla $n \in \mathbb{N}$

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i$$

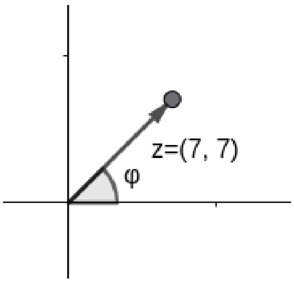
$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^{n-2}}{(1-i)^{n-2}} \cdot (1+i)^2 = i^{n-2} \cdot 2i = 2i^{n-1}$$

Zadanie 3

Przedstaw podane liczby zespolone w postaci trygonometrycznej:

3.a)

$z = 7+7i$



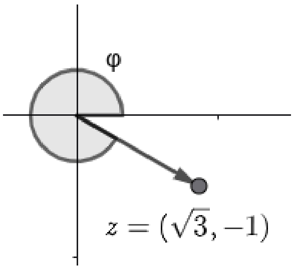
$\varphi = \frac{\pi}{4}$

$|z| = \sqrt{7^2+7^2} = 7\sqrt{2}$

$z = 7\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

3.b)

$z = \sqrt{3} - i$



$\varphi = \frac{11\pi}{6}$

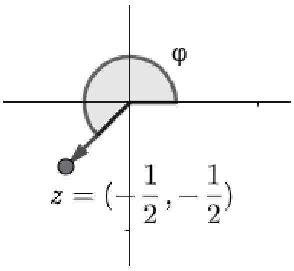
$|z| = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$

$z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

3.c)

$z = \frac{1}{i} \cdot \frac{1}{1+i}$

$$\frac{1}{i} \cdot \frac{1}{1+i} = \frac{i}{i^2} \cdot \frac{1-i}{1-i^2} = \frac{i+1}{-1(1+1)} = \frac{i+1}{-2} = -\frac{1}{2} - \frac{1}{2}i$$



$\varphi = (5\frac{\pi}{4})$

$|z| = \frac{1}{2} \cdot \sqrt{2}$

$z = \frac{\sqrt{2}}{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

3.d)

$z = 1 + i\,\mathrm{tg}\,\alpha$

$$|z| = \sqrt{1+\mathrm{tg}^2\,\alpha} = \sqrt{1+\frac{\sin^2\alpha}{\cos^2\alpha}} = \sqrt{1+\frac{1-\cos^2\alpha}{\cos^2\alpha}} = \sqrt{\frac{1}{\cos^2\alpha}} = \frac{1}{|\cos\alpha|} = \pm\frac{1}{\cos\alpha}$$

$$\cos\varphi = \frac{1}{|z|} = |\cos\alpha| = \pm\cos\alpha$$

$$\sin\varphi = \frac{\mathrm{tg}\,\alpha}{|z|} = \mathrm{tg}\,\alpha \cdot |\cos\alpha| = \frac{\sin\alpha}{\cos\alpha} \cdot |\cos\alpha| = \pm\sin\alpha$$

$$z = \pm\frac{1}{\cos\alpha}(\pm\cos\alpha \pm i\sin\alpha) = \frac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha)$$

3.e)

$1+\cos\alpha+i\sin\alpha, \alpha\in\left(0,\frac{\pi}{2}\right)$

$\alpha\in\left(0,\frac{\pi}{2}\right)\Rightarrow\sin\alpha,\cos\alpha>0$

$$|a|=\sqrt{(1+\cos\alpha)^2+\sin^2\alpha}=\sqrt{(1+\cos\alpha)^2+1-\cos^2\alpha}=$$
$$\sqrt{(1+\cos\alpha)^2+(1+\cos\alpha)(1-\cos\alpha)}=\sqrt{(1+\cos\alpha)(1+\cos\alpha+1-\cos\alpha)}=$$
$$\sqrt{2(1+\cos\alpha)}=\sqrt{2\left(1+2\cos^2\frac{\alpha}{2}-1\right)}=\sqrt{4\cos^2\frac{\alpha}{2}}=2\cos\frac{\alpha}{2}$$

$$\cos\varphi=\frac{1+\cos\alpha}{2\cos\frac{\alpha}{2}}=\frac{1+2\cos^2\frac{\alpha}{2}-1}{2\cos\frac{\alpha}{2}}=\cos\frac{\alpha}{2}$$

$$\sin\varphi=\frac{\sin\alpha}{2\cos\frac{\alpha}{2}}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}}=\sin\frac{\alpha}{2}$$

$$z=2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)$$

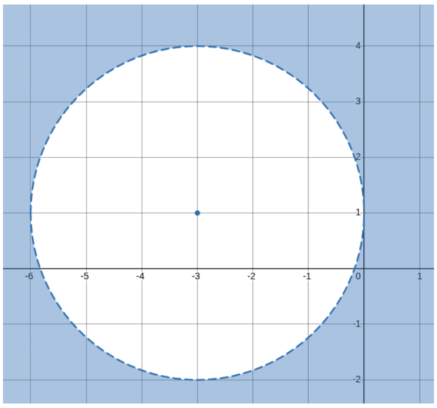
Zadanie 4

Zilustruj na płaszczyźnie zespolonej następujące zbiory:

4.a)

$\{z\in\mathbb{C}:|z-i+3|>3\}$

$|z-i+3|=|z-(-3+i)|>3\Leftrightarrow$ odległość z od punktu $(-3,1)>3$

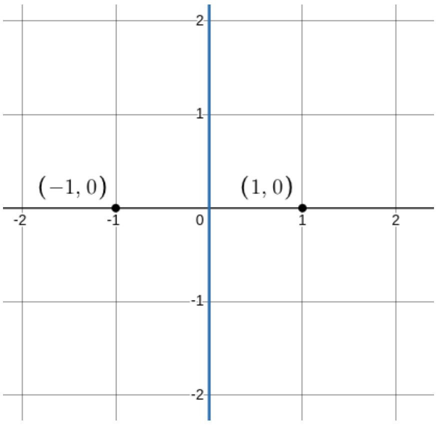


4.b)

$\{z\in\mathbb{C}:|z-1|=|z+1|\}$

$|z-1|=|z+1|\Leftrightarrow|z-(1+0i)|=|z-(-1+0i)|$

Odległości z od $(1, 0)$ i $(-1, 0)$ są równe.



4.c)

$\left\{z \in \mathbb{C} : \frac{|z-2i|}{|z+3|} < 1\right\}$

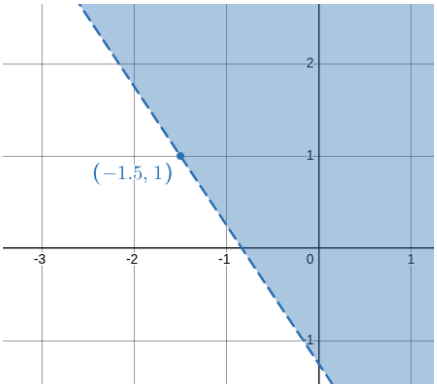
$\frac{|z-2i|}{|z+3|} < 1 \Leftrightarrow |z - (0 + 2i)| < |z - (-3 + 0i)|$

Odległość z od $(0, 2)$ jest mniejsza niż od $(-3, 0)$.

Granicą jest symetralna odcinka między tymi punktami. Ma nachylenie $-\frac{3}{2}$ i przechodzi przez punkt $\left(-\frac{3}{2}, 1\right)$.

$1 = \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right) + b \Rightarrow b = 1 - \frac{9}{4} = -\frac{5}{4}$

$y = -\frac{3}{2}x - \frac{5}{4}$



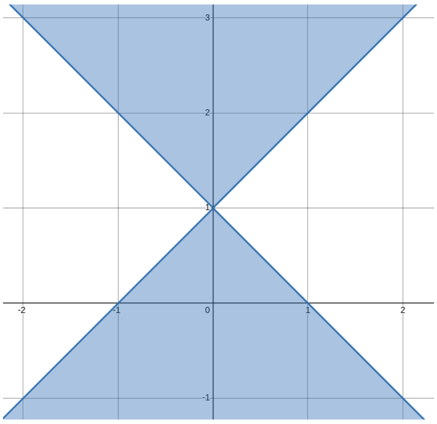
4.d)

$\{z \in \mathbb{C} : \mathfrak{R}(z - i)^2 \leq 0\}$

$\arg(z - i) = \varphi \Rightarrow \arg\left((z - i)^2\right) = 2\varphi \bmod 2\pi$

$2\varphi \bmod 2\pi \in \left\langle\frac{\pi}{2}, \frac{3\pi}{2}\right\rangle \Leftrightarrow \varphi \bmod \pi \in \left\langle\frac{\pi}{4}, \frac{3\pi}{4}\right\rangle$

$z - i \xrightarrow{T_{[0,1]}} z$

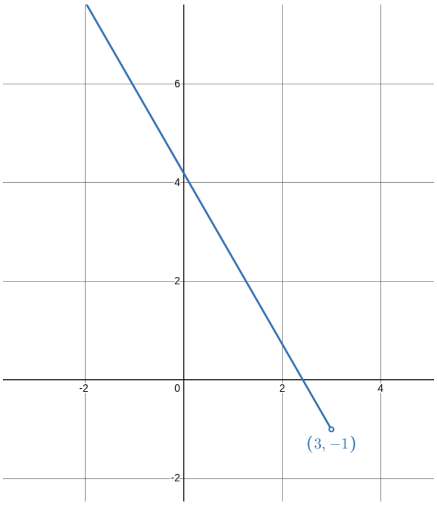


4.e)

$\left\{z \in \mathbb{C} : \arg(z-3+i) = \frac{2\pi}{3}\right\}$

Kąt $\frac{2\pi}{3}$ bez punktu $(0,0)$, bo $\arg(0)=0$

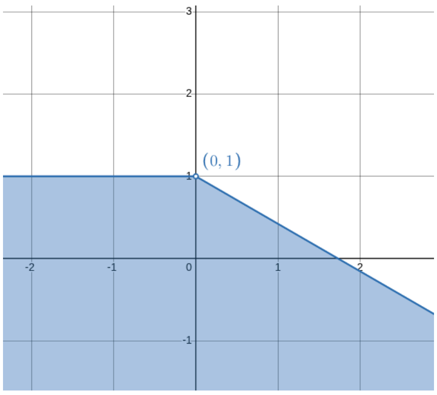
$z-3+i \xrightarrow{T_{[3,-1]}} z$



4.f)

$\left\{z \in \mathbb{C} : \frac{\pi}{6} \leq \arg(\overline{z}+i) \leq \pi\right\}$

$\overline{z}+i \xrightarrow{T_{[0,-1]}} \overline{z} \xrightarrow{S_{OX}} z$



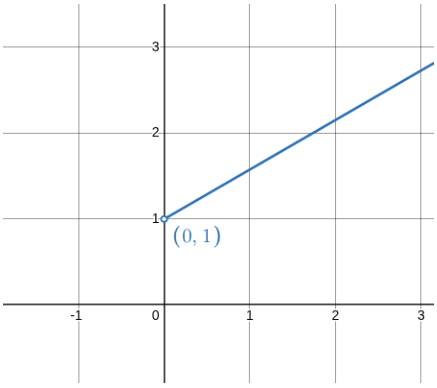
4.g)

$\{z \in \mathbb{C} : \arg(\frac{i}{i-z}) = \frac{4\pi}{3}\}$

Zał: $z \neq i \Rightarrow z \neq (0,1)$

$$\begin{aligned}\arg(i) - \arg(i-z) &\equiv \frac{4\pi}{3} (\bmod 2\pi) \\ \frac{\pi}{2} - \arg(i-z) &\equiv \frac{4\pi}{3} \\ \arg(i-z) &\equiv \frac{\pi}{2} - \frac{4\pi}{3} = \frac{3\pi}{6} - \frac{8\pi}{6} = -\frac{5\pi}{6} \equiv \frac{7\pi}{6}\end{aligned}$$

$i - z \xrightarrow{S_{(0,0)}} z - i \xrightarrow{T_{[0,1]}} z$

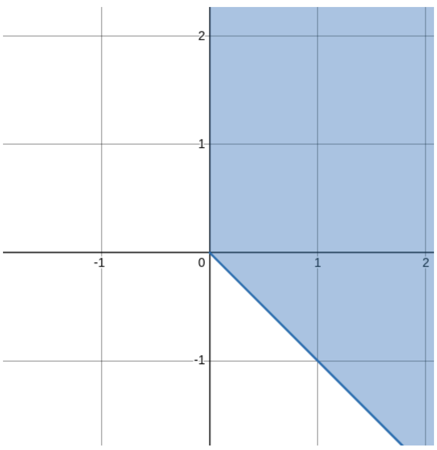


4.h)

$\{z \in \mathbb{C} : \arg(\frac{i}{z}) \leq \frac{3\pi}{4}\}$

Zał: $z \neq 0 \Rightarrow z \neq (0,0)$

$\frac{i}{z} \xrightarrow{R^{-\frac{\pi}{2}}_{(0,0)}} \frac{i}{z} \cdot \frac{1}{i} = \frac{1}{z} \xrightarrow{S_{\mathfrak{R}}, P^{\frac{1}{|z|}}} z$

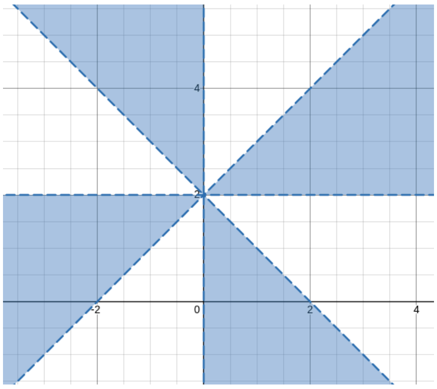


4.i)

$\{z \in \mathbb{C} : \Im((z-2i)^4) > 0\}$

$\arg(z-2i) = \varphi \Rightarrow \arg((z-2i)^4) = 4\varphi \bmod 2\pi \in (0, \pi) \Leftrightarrow \varphi \bmod \frac{\pi}{2} \in (0, \frac{\pi}{4})$

$z - 2i \xrightarrow{T[0,2]} z$



4.j)

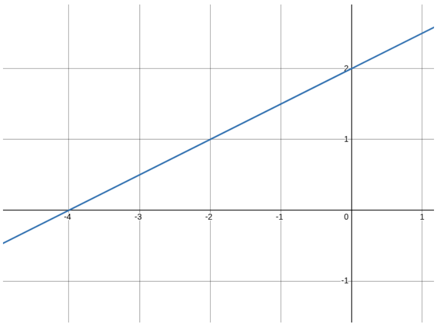
$\{z \in \mathbb{C} : \frac{z+4}{z-2i} \in \mathbb{R}\}$

Zał: $z \neq 2i \Leftrightarrow z \neq (0, 2)$

$\frac{z+4}{z-2i} \in \mathbb{R} \Leftrightarrow \arg(\frac{z+4}{z-2i}) \in \{0, \pi\}$

$z - 4$ i $z - 2i$ są współliniowe

Rozwiązanie to prosta przechodząca przez $(-4, 0)$ i $(0, 2)$ z wyłączeniem $(0, 2)$.



Zadanie 5

Oblicz wartości podanych wyrażeń (wyniki podaj w postaci algebraicznej):

5.a)

$(1-i)^6$

$1 - i = \sqrt{2}e^{i\frac{7\pi}{4}}$

$(1 - i)^6 = \sqrt{2}^6 e^{i6\cdot\frac{7\pi}{4}} = 8e^{i\frac{21\pi}{2}} = 8e^{i10\frac{1}{2}\pi} = 8e^{i\frac{\pi}{2}} = 8i$

5.b)

$\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$

$\frac{1+i\sqrt{3}}{1-i} = \frac{2e^{i\frac{\pi}{3}}}{\sqrt{2}e^{i\frac{7\pi}{4}}} = \frac{2}{\sqrt{2}}e^{i\left(\frac{\pi}{3}-\frac{7\pi}{4}\right)} = \sqrt{2}e^{i\left(\frac{4\pi}{12}-\frac{21\pi}{12}\right)} = \sqrt{2}e^{i\frac{-17\pi}{12}} = \sqrt{2}e^{i\frac{7\pi}{12}}$

$\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20} = \sqrt{2}^{20}e^{i20\cdot\frac{7\pi}{12}} = 1024e^{i\frac{140\pi}{12}} = 1024e^{i\frac{20\pi}{12}} = 1024e^{i\frac{5\pi}{3}} = 512 - 512\sqrt{3}i$

5.c)

$\frac{(1+i)^{22}}{\left(1-i\sqrt{3}\right)^6}$

$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

$(1 + i)^{22} = \sqrt{2}^{22}e^{i22\cdot\frac{\pi}{4}} = 2048e^{i\frac{\pi}{2}} = -2048i$

$1 - i\sqrt{3} = 2e^{i\frac{5\pi}{3}}$

$(1 - i)^6 = 2^6e^{i6\cdot\frac{5\pi}{3}} = 64e^{i10\pi} = 64e^{i0} = 64$

$\frac{(1+i)^{22}}{\left(1-i\sqrt{3}\right)^6} = \frac{-2048i}{64} = -32i$

5.d)

$\left(-\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)^{14}$

$-\cos\frac{\pi}{7} + i\sin\frac{\pi}{7} = -\cos\left(-\frac{\pi}{7}\right) - \sin\left(-\frac{\pi}{7}\right) = -e^{-i\frac{\pi}{7}}$

$\left(-\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)^{14} = (-1)^{14}e^{-i\frac{14\pi}{7}} = e^{-i2\pi} = e^{i0} = 1$

5.e)

$1 + i + i^2 + ... + i^n, n \in \mathbb{N}$

$1 + i + i^2 + ... + i^n = \frac{(i-1)(1+i+i^2+...+i^n)}{i-1} = \frac{i^n-1}{i-1} = \frac{1-i^n}{1-i} = \frac{(1-i^n)(1+i)}{2} =$

$$= \begin{cases} \frac{(1-1)(1+i)}{2} = 0 & \text{dla } i = 4k \\ \frac{(1-i)(1+i)}{2} = 1 & \text{dla } i = 4k+1 \\ \frac{(1+1)(1+i)}{2} = 1+i & \text{dla } i = 4k+2 \\ \frac{(1+i)(1+i)}{2} = i & \text{dla } i = 4k+3 \end{cases}, k \in \mathbb{Z}$$

Zadanie 6

Znajdź funkcję rzeczywistą taką, że:

Zadanie 7

Oblicz pierwiastki z liczb zespolonych:

7.a)

$\sqrt{-1+\sqrt{3}i}$

$z = -1 + \sqrt{3}i = 2e^{i\frac{2\pi}{3}}$

$z_0 = \sqrt{2}e^{i\frac{2\pi}{3\cdot 2}} = \sqrt{2}e^{i\frac{\pi}{3}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

$z_1 = -z_0 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$

7.b)

$\sqrt[4]{-4}$

$z = -4 = 4e^{i\pi}$

$z_0 = 2e^{i\frac{\pi}{4}} = \sqrt{2} + \sqrt{2}i$

$z_1 = -\sqrt{2} + \sqrt{2}i$

$z_2 = -\sqrt{2} - \sqrt{2}i$

$z_3 = \sqrt{2} - \sqrt{2}i$

7.c)

$\sqrt[6]{-64}$

$z = -64 = 64e^{i\pi}$

$$
\begin{aligned}
z_0 &= 8e^{i\frac{\pi}{6}} &= 4\sqrt{3} + 4i \\
z_1 &= 8e^{i\frac{3\pi}{6}} &= 8i \\
z_2 &= 8e^{i\frac{5\pi}{6}} &= -4\sqrt{3} + 4i \\
z_3 &= 8e^{i\frac{7\pi}{6}} &= -4\sqrt{3} - 4i \\
z_4 &= 8e^{i\frac{9\pi}{6}} &= -8i \\
z_5 &= 8e^{i\frac{11\pi}{6}} &= 4\sqrt{3} - 4i
\end{aligned}
$$