Zadanie 1

Wykaż równości i nierówności dla liczb zespolonych:

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

1.a)

$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

$\overline{z_1 z_2} = \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$

$\overline{z_1} \cdot \overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$

1.b)

$|z_1 z_2| = |z_1||z_2|$

$|z_1 z_2| = |(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2} = \sqrt{a_1^2 a_2^2 - \cancel{2a_1 a_2 b_1 b_2} + b_1^2 b_2^2 + a_1^2 b_2^2 + \cancel{2a_1 a_2 b_1 b_2} + b_1^2 a_2^2} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)} \cdot \sqrt{(a_2^2 + b_2^2)} = |z_1||z_2|$

1.c)

$z \cdot \overline{z} = |z|^2$

$z \cdot \overline{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \sqrt{a^2 + b^2}^2 = |z|$

1.d)

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} \cdot \overline{z_2} = \frac{\overline{z_1}}{\overline{z_2}} \cdot \overline{z_2}$$

$$\overline{\frac{z_1}{z_2} \cdot z_2} = \overline{z_1}$$

1.e)

$|z_1 + z_2| \le |z_1| + |z_2|$

Dodawanie wektorów, nierówność trójkąta.

1.f)

$||z_1| - |z_2|| \leq |z_1 - z_2|$

$$|z_1| = |z_2 + (z_1 - z_2)| \leq |z_2| + |z_1 - z_2|$$
$$|z_1| - |z_2| \leq |z_1 - z_2|$$

Analogicznie

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$$
$$|z_1| - |z_2| \geq -|z_1 - z_2|$$

Czyli

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2| \Leftrightarrow ||z_1| - |z_2|| \leq |z_1 - z_2|$$

Zadanie 2

Oblicz:

2.a)

$$\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i-3i^2}{1-i^2} = \frac{2+i+3}{1+1} = \frac{5+i}{2}$$

2.b)

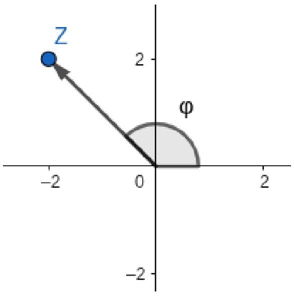
$$\frac{\left(i+\sqrt{3}\right)\left(-1-i\sqrt{3}\right)}{1+2i} = \frac{\left(-i-i^2\sqrt{3}-\sqrt{3}-3i\right)(1-2i)}{(1+2i)(1-2i)} = \frac{\left(-4i+\sqrt{3}-\sqrt{3}\right)(1-2i)}{1-4i^2} =$$
$$\frac{-4i+8i^2}{1+4} = \frac{-8-4i}{5}$$

2.c)

$$|3-4i| = \sqrt{3^2+4^2} = 5$$

2.d)

$\arg(-2+2i)$



$\arg(-2+2i) = \varphi = \frac{3\pi}{4}$

2.e)

$\frac{(1+i)^n}{(1-i)^{n-2}}$, dla $n \in \mathbb{N}$

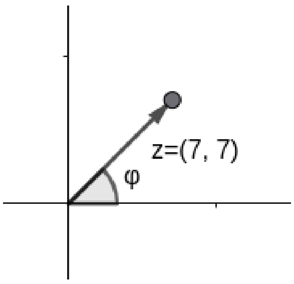
$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^n(1+i)^{n-2}}{(1-i)^{n-2}(1+i)^{n-2}} = \frac{(1+i)^{2n-2}}{(1-i^2)^{n-2}} = \frac{(1+i)^{2n-2}}{2^{n-2}}$$

Zadanie 3

Przedstaw podane liczby zespolone w postaci trygonometrycznej:

3.a)

$z = 7 + 7i$



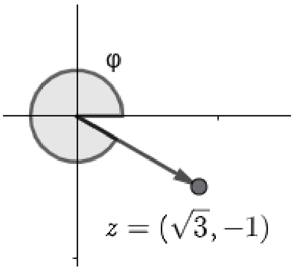
$\varphi = \frac{\pi}{4}$

$|z| = \sqrt{7^2+7^2} = 7\sqrt{2}$

$z = 7\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

3.b)

$z = \sqrt{3} - i$



$\varphi = \frac{11\pi}{6}$

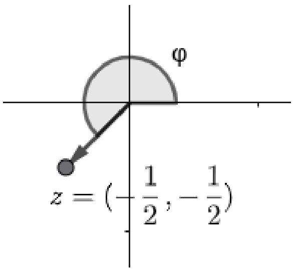
$|z| = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$

$z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

3.c)

$z = \frac{1}{i} \cdot \frac{1}{1+i}$

$$\frac{1}{i} \cdot \frac{1}{1+i} = \frac{i}{i^2} \cdot \frac{1-i}{1-i^2} = \frac{i+1}{-1(1+1)} = \frac{i+1}{-2} = -\frac{1}{2} - \frac{1}{2}i$$



$\varphi = \left(5\frac{\pi}{4}\right)$

$|z| = \frac{1}{2} \cdot \sqrt{2}$

$z = \frac{\sqrt{2}}{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

3.d)

$z = 1 + i\,\mathrm{tg}\,\alpha$

$$|z| = \sqrt{1+\mathrm{tg}^2\,\alpha} = \sqrt{1+\frac{\sin^2\alpha}{\cos^2\alpha}} = \sqrt{1+\frac{1-\cos^2\alpha}{\cos^2\alpha}} = \sqrt{\frac{1}{\cos^2\alpha}} = \frac{1}{|\cos\alpha|} = \pm\frac{1}{\cos\alpha}$$

$$\cos\varphi = \frac{1}{|z|} = |\cos\alpha| = \pm\cos\alpha$$

$$\sin\varphi = \frac{\mathrm{tg}\,\alpha}{|z|} = \mathrm{tg}\,\alpha\cdot|\cos\alpha| = \frac{\sin\alpha}{\cos\alpha}\cdot|\cos\alpha| = \pm\sin\alpha$$

$$z = \pm\frac{1}{\cos\alpha}(\pm\cos\alpha \pm i\sin\alpha) = \frac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha)$$

3.e)

$1+\cos\alpha + i\sin\alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$

$\alpha \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin\alpha, \cos\alpha > 0$

$$|a| = \sqrt{(1+\cos\alpha)^2+\sin^2\alpha} = \sqrt{(1+\cos\alpha)^2+1-\cos^2\alpha} =$$
$$\sqrt{(1+\cos\alpha)^2+(1+\cos\alpha)(1-\cos\alpha)} = \sqrt{(1+\cos\alpha)(1+\cos\alpha+1-\cos\alpha)} =$$
$$\sqrt{2(1+\cos\alpha)} = \sqrt{2\left(1+2\cos^2\frac{\alpha}{2}-1\right)} = \sqrt{4\cos^2\frac{\alpha}{2}} = 2\cos\frac{\alpha}{2}$$

$$\cos\varphi = \frac{1+\cos\alpha}{2\cos\frac{\alpha}{2}} = \frac{1+2\cos^2\frac{\alpha}{2}-1}{2\cos\frac{\alpha}{2}} = \cos\frac{\alpha}{2}$$

$$\sin\varphi = \frac{\sin\alpha}{2\cos\frac{\alpha}{2}} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}} = \sin\frac{\alpha}{2}$$

$$z = 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

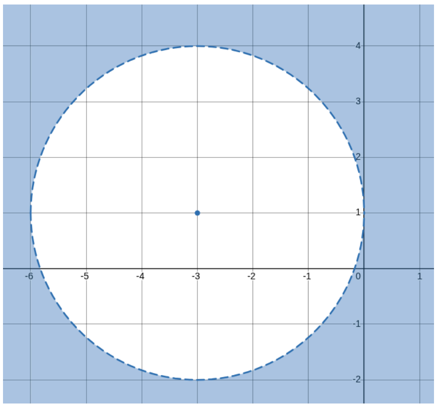
Zadanie 4

Zilustruj na płaszczyźnie zespolonej następujące zbiory:

4.a)

$\{z \in \mathbb{C} : |z - i + 3| > 3\}$

$|z - i + 3| = |z - (-3 + i)| > 3 \Leftrightarrow$ odległość z od punktu $(-3, 1) > 3$

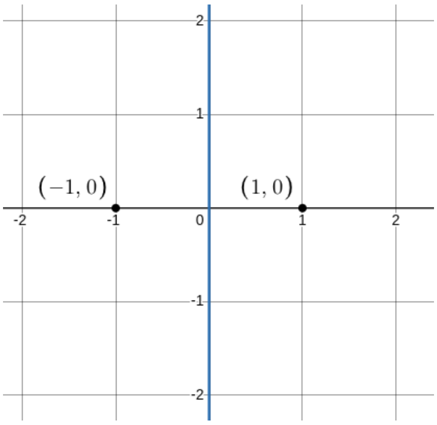


4.b)

$\{z \in \mathbb{C} : |z - 1| = |z + 1|\}$

$|z - 1| = |z + 1| \Leftrightarrow |z - (1 + 0i)| = |z - (-1 + 0i)|$

Odległości z od $(1, 0)$ i $(-1, 0)$ są równe.



4.c)

$\left\{z \in \mathbb{C} : \frac{|z-2i|}{|z+3|} < 1\right\}$

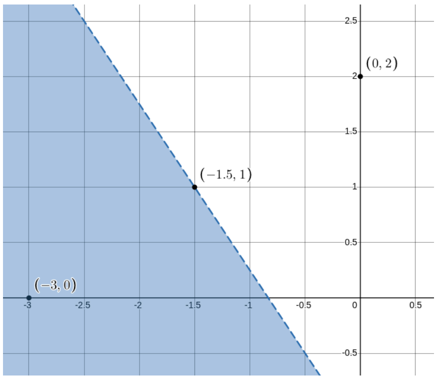
$\frac{|z-2i|}{|z+3|} < 1 \Leftrightarrow |z-(0+2i)| < |z-(-3+0i)|$

Odległość z od $(0, 2)$ jest mniejsza niż od $(-3, 0)$.

Granicą jest symetralna odcinka między tymi punktami. Ma nachylenie $-\frac{3}{2}$ i przechodzi przez punkt $\left(-\frac{3}{2}, 1\right)$.

$1 = \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right) + b \Rightarrow b = 1 - \frac{9}{4} = -\frac{5}{4}$

$y = -\frac{3}{2}x - \frac{5}{4}$



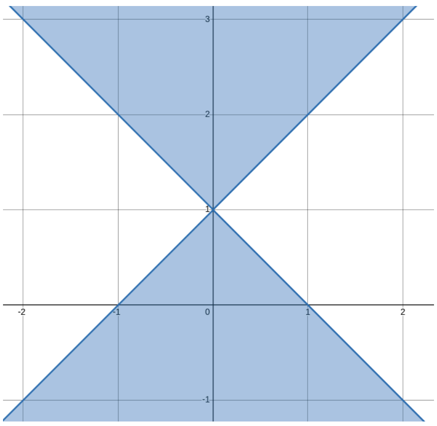
4.d)

$\{z \in \mathbb{C} : \Re(z-i)^2 \leq 0\}$

$z = x + yi \Rightarrow z - i = x + (y-1)i$

$\Re(z-i)^2 = x^2 - (y-1)^2 = (x-y+1)(x+y-1) \leq 0$

$$\begin{cases} y \geq x + 1 \land y \leq -x + 1 \\ y \leq x + 1 \land y \geq -x + 1 \end{cases}$$



4.e)

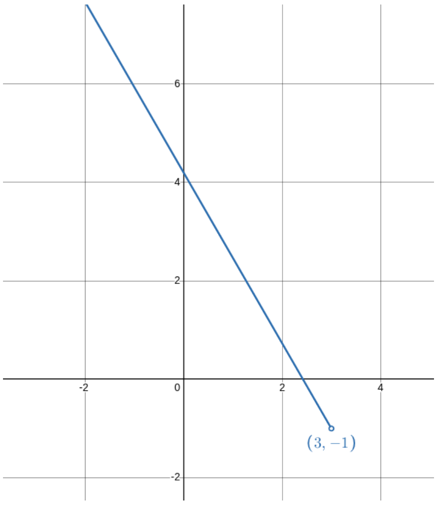
$\left\{z \in \mathbb{C} : \arg(z-3+i) = \frac{2\pi}{3}\right\}$

$z = x + yi \Rightarrow z - 3 + i = (x-3) + (y+1)i$

Szukamy półprostej o początku w $(0,0)$ przechodzącej przez $(x-3, y+1)$ o nachyleniu $\frac{2\pi}{3}$.

$\operatorname{tg} \frac{2\pi}{3} = -\sqrt{3} \Rightarrow (y+1) = -\sqrt{3}(x-3) \Rightarrow y = -\sqrt{3}x + 3\sqrt{3} - 1$

Należy pamiętać, że $\arg(0) = 0$, dlatego $(x-3, y+1) \neq (0,0) \Rightarrow (x,y) \neq (3,-1)$



4.f)

$\left\{z \in \mathbb{C} : \frac{\pi}{6} \leq \arg(\overline{z} + i) \leq \pi\right\}$

$\overline{z} + i \xrightarrow{T_{[0,-1]}} \overline{z} \xrightarrow{S_{OX}} z$

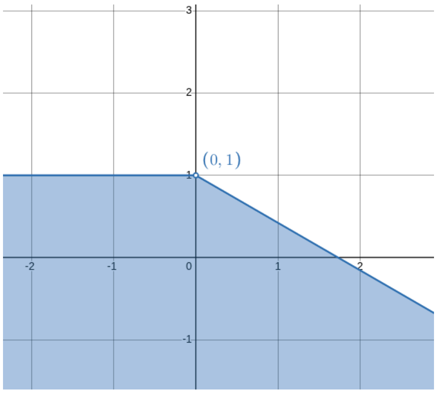
$\operatorname{tg} \frac{\pi}{6} = \frac{1}{\sqrt{3}}, \operatorname{tg} \pi = 0$

$\frac{\pi}{6} \leq \arg(z) \leq \pi \Leftrightarrow$

$$\begin{aligned} y &\geq \frac{1}{\sqrt{3}}x \qquad \wedge y \geq 0 \xrightarrow{T_{[0,-1]}} \\ y &\geq \frac{1}{\sqrt{3}}x - 1 \quad \wedge y \geq 1 \xrightarrow{S_{OX}} \\ y &\leq -\frac{1}{\sqrt{3}}x + 1 \wedge y \leq -1 \end{aligned}$$

$\arg(0) = 0 \Rightarrow \overline{z} + i \neq (0,0)$

$(0,0) \xrightarrow{T_{[0,-1]}} (0,-1) \xrightarrow{S_{OX}} (0,1) \Rightarrow z \neq (0,1)$



4.g)

$\left\{z \in \mathbb{C} : \arg\left(\frac{i}{i-z}\right) = \frac{4\pi}{3}\right\}$

$z = x + yi \Rightarrow$

$$\frac{i}{i-z} = \frac{i}{i-x-yi} = \frac{i}{(1-y)i-x} = \frac{i((1-y)i+x)}{(1-y)^2 i^2 - x^2} = \frac{-1+y+xi}{-1+2y-y^2-x^2}$$