Zadanie 1

Zbadaj rzędy następujących macierzy:

1.a)

$$A = \begin{bmatrix} 2 & 6 & -1 & 4 & 3 \\ 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 3 & 10 & 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 2 & 6 & -1 & 4 & 3 \\ 0 & -2 & -5 & 6 & 3 \\ 3 & 10 & 1 & 3 & 3 \end{bmatrix}\begin{matrix} \\ -2w_1 \\ \\ -3w_1 \end{matrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & -2 & -5 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & -2 & -5 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 2$

1.b)

$$B = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ -3 & -1 & -1 & 4 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 3 & 2 & 2 & 1 & 8 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix}\begin{matrix} \\ +w_1 \\ \\ -w_1 \\ \\ \end{matrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 & 4 \end{bmatrix}\begin{matrix} \\ \\ \\ -w_2 \\ -w_2 \end{matrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 \end{bmatrix}\begin{matrix} \\ \\ \\ \\ -w_3 \end{matrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -1 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(B) = 4$

1.c)

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \end{bmatrix}\begin{matrix} \\ -w_1 \\ -w_1 \\ -w_1 \\ -w_1 \\ -2w_1 \end{matrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}\begin{matrix} \\ \\ \\ \\ -2w_2 \\ +w_2 \end{matrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix}\begin{matrix} \\ \\ \\ \\ +2w_3 \\ -3w_3 \end{matrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$r(C) = 4$

1.d)

$$D = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 4 & -3 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ -3w_1 \\ -2w_1 \\ -4w_1 \end{array} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \\ 0 & -5 & -5 & 3 \\ 0 & -15 & -15 & 9 \end{bmatrix} \begin{array}{l} \\ -2w_3 \\ \\ -3w_3 \end{array} =$$

$$\begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(D) = 2$

Zadanie 2

Wyznacz rzędy następujących macierzy w zależności od parametru rzeczywistego p:

2.a)

$$A = \begin{bmatrix} 1-p & 2 & 1 & p \\ 1 & 2-p & 1 & 0 \\ 1 & 2 & 1-p & p \end{bmatrix} \begin{array}{l} \\ -w_1 \\ -w_1 \end{array} = \begin{bmatrix} 1-p & 2 & 1 & p \\ p & -p & 0 & -p \\ p & 0 & -p & 0 \end{bmatrix}$$

Szukamy niezerowego minora o największym wymiarze.

$$W = \begin{vmatrix} 1-p & 2 & 1 \\ p & -p & 0 \\ p & 0 & -p \end{vmatrix} = p^2 \begin{vmatrix} 1-p & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \overset{\text{z tw. Laplace'a}}{=}$$

$$p^2 \left((1-p)(-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \right) =$$

$$p^2((1-p) \cdot 1 - 2 \cdot (-1) + 1) = p^2(4-p)$$

Jeżeli $p \notin \{0, 4\}$ to $W \neq 0$, czyli znaleźlismy niezerowego minora, więc $r(A) = 3$.
Jeżeli $p = 0 \Rightarrow$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \Rightarrow r(A) = 1$$

Jeżeli $p = 4 \Rightarrow$

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & -2 & 1 & 0 \\ -3 & 2 & 1 & 4 \end{bmatrix} \begin{matrix} \\ -w_1 \\ +3w_1 \end{matrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 8 & -8 & 16 \end{bmatrix} \begin{matrix} \\ \\ +2w_2 \end{matrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$r(A) = 3$

Ostatecznie $r(A) = \begin{cases} 3, \text{ gdy } p \neq 0 \\ 1, \text{ gdy } p = 0 \end{cases}$

2.b)

$$B = \begin{bmatrix} p-1 & p-1 & 1 & 1 \\ 1 & p^2-1 & 1 & p-1 \\ 1 & p-1 & p-1 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} p-1 & 1 & 1 \\ 1 & 1 & p-1 \\ 1 & p-1 & 1 \end{vmatrix} = 3(p-1) - 1 - (p-1)^3 - 1$$

Niech $t = p - 1$. Wtedy mamy:

$$\begin{aligned} 3t - 1 - t^3 - 1 &= 0 \\ t^3 - 3t + 2 &= 0 \\ (t-1)(t^2 + t - 2) &= 0 \\ (t-1)^2(t+2) &= 0 \\ t = 1 \lor t &= -2 \\ p = 2 \lor p &= -1 \end{aligned}$$

$$p = 2 \Rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow r(B) = 2$$

$$p = -1 \Rightarrow B = \begin{bmatrix} -2 & -2 & 1 & 1 \\ 1 & 0 & 1 & -2 \\ 1 & -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ -2 & -2 & 1 & 1 \\ 1 & -2 & -2 & 1 \end{bmatrix} \begin{matrix} \\ +2w_1 \\ -w_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -2 & 3 & -3 \\ 0 & -2 & -3 & -3 \end{bmatrix} \begin{matrix} \\ \\ -w_2 \end{matrix} =$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & -6 & 0 \end{bmatrix} \Rightarrow r(B) = 3$$

Ostatecznie $r(B) = \begin{cases} 3, \text{ gdy } p \neq 2 \\ 2, \text{ gdy } p = 2 \end{cases}$

Zadanie 3

Dla jakich wartości parametru $a \in \mathbb{R}$, rząd macierzy jest najmniejszy, a dla jakich największy?

3.a)

$$A = \begin{bmatrix} -2 & -1-a & 1 \\ a & 0 & -a \\ -1 & a+a^2 & 1 \end{bmatrix}$$

$$W = (-1-a)(-1)^{1+2} \begin{vmatrix} a & -a \\ -1 & 1 \end{vmatrix} + 0 + (a+a^2)(-1)^{3+2} \begin{vmatrix} -2 & 1 \\ a & -a \end{vmatrix} =$$
$$(1+a)(a-(-a)(-1)) - (a+a^2)(-2(-a)-a) =$$
$$(1+a)\cdot 0 - (a+a^2)\cdot a = -a^2(a+1)$$

$a \notin \{-1, 0\} \Rightarrow$ wyznacznik jest niezerowy, więc rząd to 3.

$$a = 0 \Rightarrow A = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \Rightarrow \text{rząd to 2}$$

$$a = -1 \Rightarrow A = \begin{bmatrix} -2 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$W = \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = -2 - (-1) = -1 \neq 0 \Rightarrow \text{rząd to 2}$$

Ostatecznie $r(A) = \begin{cases} 3, \text{ gdy } a \notin \{-1,0\} \\ 2, \text{ gdy } a \in \{-1,0\} \end{cases}$

3.b)

$$B = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$$

$$W = \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \begin{matrix} +w_2+w_3+w_4 \\ \\ \\ \\ \end{matrix} = \begin{vmatrix} a+3 & a+3 & a+3 & a+3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \begin{matrix} \\ -w_1 \\ -w_1 \\ -w_1 \end{matrix} =$$
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 1 \\ 0 & 0 & 0 & a-1 \end{vmatrix} = (a-1)^3$$

Jeżeli $a \neq 1$ to całe B jest minorem niezerowym stopnia 4 czyli rząd B to 4.

$$a = 1 \Rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow r(B) = 1$$

Ostatecznie $r(B) = \begin{cases} 4, \text{ gdy } a \neq 1 \\ 1, \text{ gdy } a = 1 \end{cases}$

Zadanie 4

Oblicz wyznacznik macierzy i jeśli jest ona nieosobliwa, znajdź macierz do niej odwrotną:

4.a)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = \left|\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{matrix}\right|\begin{matrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{matrix} = 1 + 0 + 1 - 0 - 2 - 0 = 0$$

4.b)

$$B = \begin{bmatrix} 1+i & -1 \\ 0 & 2 \end{bmatrix}$$

$$W = \begin{vmatrix} 1+i & -1 \\ 0 & 2 \end{vmatrix} = 2(1+i) - 0 = 2 + 2i \neq 0$$

Macierz nie jest osobliwa, więc możemy znaleźć jej macierz odwrotną.

$$\left[\begin{matrix} 1+i & -1 \\ 0 & 2 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \begin{matrix} \cdot \frac{1}{1+i} \\ \cdot \frac{1}{2} \end{matrix} = \left[\begin{matrix} 1 & \frac{-1}{1+i} \\ 0 & 1 \end{matrix}\middle|\begin{matrix} \frac{1}{1+i} & 0 \\ 0 & \frac{1}{2} \end{matrix}\right] + \frac{1}{1+i} w_2 = \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} \frac{1}{1+i} & \frac{1}{2+2i} \\ 0 & \frac{1}{2} \end{matrix}\right]$$

$$B^{-1} = \begin{bmatrix} \frac{1}{1+i} & \frac{1}{2+2i} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} & \frac{2-2i}{4+4} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$$

4.c)

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\det C = \left|\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{matrix}\right| \begin{matrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{matrix} = 0 + 1 + 0 - 0 + 1 + 1 = 3$$

Metoda Gaussa:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -w_1 \\ -w_1 \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ -2w_2 \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -2 & 1 \end{array}\right] \begin{matrix} \\ \cdot(-1) \\ \cdot\left(-\frac{1}{3}\right) \end{matrix} =$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \begin{matrix} \\ +w_3 \\ \\ \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \begin{matrix} -w_2 \\ \\ \\ \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

Metoda dopełnień algebraicznych:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1 \cdot (0 - (-1)) = 1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 1) = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 0) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) \cdot (-1 - 1) = 2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1 - 0) = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (-1) \cdot (1 - 0) = -1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1$$

$$C^{-1} = (\det C)^{-1}(C^D)^T = 3^{-1}\begin{bmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 1 & -1 & -1\end{bmatrix}^T = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ -1 & 2 & -1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

Metoda układem równań:

$$C = \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & -1\end{bmatrix} \longrightarrow \begin{cases} x_1 + x_2 + 0 = y_1\\ x_1 + 0 + x_3 = y_2\\ x_1 - x_2 - x_3 = y_3\end{cases}$$

$$3x_1 = y_1 + y_2 + y_3$$

$$x_1 = \frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3$$

$$x_2 = y_1 - x_1 = \frac{2}{3}y_1 - \frac{1}{3}y_2 - \frac{1}{3}y_3$$

$$x_3 = y_2 - x_1 = -\frac{1}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3$$

$$\Rightarrow C^{-1} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

4.d)

$$D = \begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\\ 13 & 14 & 15 & 16\end{bmatrix}$$

$$\det D = \begin{vmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\\ 13 & 14 & 15 & 16\end{vmatrix}\begin{matrix} \\ -w_1\\ -w_2\\ \end{matrix} = \begin{vmatrix}1 & 2 & 3 & 4\\ 4 & 4 & 4 & 4\\ 4 & 4 & 4 & 4\\ 13 & 14 & 15 & 16\end{vmatrix} = 0$$

bo są dwa takie same wiersze

4.e)

$$E = \begin{bmatrix}4 & 2 & 1 & 1\\ 1 & -1 & 0 & 2\\ 3 & 0 & 1 & 3\\ 2 & 2 & 0 & 3\end{bmatrix}$$

$$\det E = 1\cdot(-1)^{1+3}M_{13} + 0 + 1\cdot(-1)^{3+3}M_{33} + 0$$

$$M_{13} = \left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 3 & 0 & 3 & 3 & 0 \\ 2 & 2 & 3 & 2 & 2 \end{array}\right. = 0 - 6 + 12 - 0 - 6 + 9 = 9$$

$$M_{33} = \left|\begin{array}{ccc|cc} 4 & 2 & 1 & 4 & 2 \\ 1 & -1 & 2 & 1 & -1 \\ 2 & 2 & 3 & 2 & 2 \end{array}\right. = -12 + 8 + 2 + 2 - 16 - 6 = -22$$

$$\det E = 9 - 22 = -13$$

$$\left[\begin{array}{cccc|cccc} 4 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} w_1 \leftrightarrow w_2 \\ \\ \\ \\ \end{array} = \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -4w_1 \\ -3w_1 \\ -2w_1 \end{array} =$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 4 & 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ -w_3 \end{array} = \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \end{array}\right] w_2 \leftrightarrow w_4 =$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 3 & 1 & -3 & 0 & -3 & 1 & 0 \\ 0 & 6 & 1 & -7 & 1 & -4 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ -3w_2 \\ -6w_2 \end{array} = \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & -9 & 0 & -6 & 4 & -3 \\ 0 & 0 & 7 & -19 & 1 & -10 & 6 & -6 \end{array}\right] \begin{array}{l} \\ \\ 2w_3 - w_4 \\ \\ \end{array} =$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 7 & -19 & 1 & -10 & 6 & -6 \end{array}\right] \begin{array}{l} \\ \\ \\ -7w_3 \end{array} = \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -26 & 8 & 4 & -8 & -6 \end{array}\right] \begin{array}{l} 13w_1 \\ 13w_2 \\ 13w_2 \\ \frac{1}{2}w_4 \end{array} =$$

$$\left[\begin{array}{cccc|cccc} 13 & -13 & 0 & 26 & 0 & 13 & 0 & 0 \\ 0 & 13 & -13 & 26 & 0 & 13 & -13 & 13 \\ 0 & 0 & 13 & 13 & -13 & -26 & 26 & 0 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array}\right] \begin{array}{l} +2w_4 \\ +2w_4 \\ +w_4 \\ \\ \end{array} = \left[\begin{array}{cccc|cccc} 13 & -13 & 0 & 0 & 8 & 17 & -8 & -6 \\ 0 & 13 & -13 & 0 & 8 & 17 & -21 & 7 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array}\right] \begin{array}{l} \\ +w_3 \\ \\ \\ \end{array} =$$

$$\left[\begin{array}{cccc|cccc} 13 & -13 & 0 & 0 & 8 & 17 & -8 & -6 \\ 0 & 13 & 0 & 0 & -1 & -7 & 1 & 4 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array}\right] \begin{array}{l} +w_2 \\ \\ \\ \\ \end{array} = \left[\begin{array}{cccc|cccc} 13 & 0 & 0 & 0 & 7 & 10 & -7 & -2 \\ 0 & 13 & 0 & 0 & -1 & -7 & 1 & 4 \\ 0 & 0 & 13 & 0 & -9 & -24 & 22 & -3 \\ 0 & 0 & 0 & -13 & 4 & 2 & -4 & -3 \end{array}\right] \begin{array}{l} \frac{1}{13}w_1 \\ \frac{1}{13}w_2 \\ \frac{1}{13}w_3 \\ \frac{-1}{13}w_4 \end{array}$$

$$E^{-1} = \frac{1}{13} \begin{bmatrix} 7 & 10 & -7 & 2 \\ -1 & -7 & 1 & 4 \\ -9 & -24 & 22 & -3 \\ -4 & -2 & 4 & 3 \end{bmatrix}$$

4.f)

$$F = \begin{bmatrix} 0 & 2 & 3 & 4 & -8 \\ 1 & 1 & 3 & 4 & -8 \\ 1 & 2 & 2 & 4 & -8 \\ 1 & 2 & 3 & 3 & -8 \\ 1 & 2 & 3 & 4 & -9 \end{bmatrix}$$

4.g)

$$G_n = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix}_{n \times n}$$