Zadanie 1

Wykaż równości i nierówności dla liczb zespolonych:

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

1.a)

$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

$\overline{z_1 z_2} = \overline{(a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)} = (a_1a_2 - b_1b_2) - i(a_1b_2 + b_1a_2)$

$\overline{z_1} \cdot \overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1a_2 - b_1b_2) - i(a_1b_2 + b_1a_2)$

1.b)

$|z_1 z_2| = |z_1||z_2|$

$|z_1 z_2| = |(a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)| = \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2} =$
$\sqrt{a_1^2a_2^2 - \cancel{2a_1a_2b_1b_2} + b_1^2b_2^2 + a_1^2b_2^2 + \cancel{2a_1a_2b_1b_2} + b_1^2a_2^2} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} =$
$\sqrt{(a_1^2 + b_1^2)} \cdot \sqrt{(a_2^2 + b_2^2)} = |z_1||z_2|$

1.c)

$z \cdot \overline{z} = |z|^2$

$z \cdot \overline{z} = (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2 = \sqrt{a^2 + b^2}^2 = |z|$

1.d)

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} \cdot \overline{z_2} = \frac{\overline{z_1}}{\overline{z_2}} \cdot \overline{z_2}$$

$$\overline{\frac{z_1}{z_2} \cdot z_2} = \overline{z_1}$$

1.e)

$|z_1 + z_2| \leq |z_1| + |z_2|$

Dodawanie wektorów, nierówność trójkąta.

1.f)

$||z_1| - |z_2|| \leq |z_1 - z_2|$

$$|z_1| = |z_2 + (z_1 - z_2)| \leq |z_2| + |z_1 - z_2|$$
$$|z_1| - |z_2| \leq |z_1 - z_2|$$

Analogicznie

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$$
$$|z_1| - |z_2| \geq -|z_1 - z_2|$$

Czyli

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2| \Leftrightarrow ||z_1| - |z_2|| \leq |z_1 - z_2|$$

Zadanie 2

Oblicz:

2.a)

$$\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i-3i^2}{1-i^2} = \frac{2+i+3}{1+1} = \frac{5+i}{2}$$

2.b)

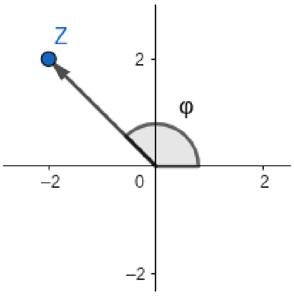
$$\frac{\left(i+\sqrt{3}\right)\left(-1-i\sqrt{3}\right)}{1+2i} = \frac{\left(-i-i^2\sqrt{3}-\sqrt{3}-3i\right)(1-2i)}{(1+2i)(1-2i)} = \frac{\left(-4i+\sqrt{3}-\sqrt{3}\right)(1-2i)}{1-4i^2} =$$
$$\frac{-4i+8i^2}{1+4} = \frac{-8-4i}{5}$$

2.c)

$$|3-4i| = \sqrt{3^2+4^2} = 5$$

2.d)

$\arg(-2+2i)$



$\arg(-2+2i) = \varphi = \frac{3\pi}{4}$

2.e)

$\frac{(1+i)^n}{(1-i)^{n-2}}$, dla $n \in \mathbb{N}$

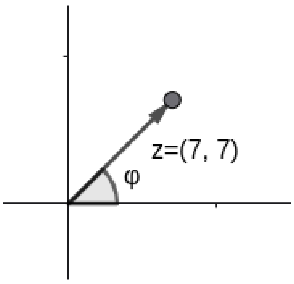
$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^n(1+i)^{n-2}}{(1-i)^{n-2}(1+i)^{n-2}} = \frac{(1+i)^{2n-2}}{(1-i^2)^{n-2}} = \frac{(1+i)^{2n-2}}{2^{n-2}}$$

Zadanie 3

Przedstaw podane liczby zespolone w postaci trygonometrycznej:

3.a)

$z = 7+7i$



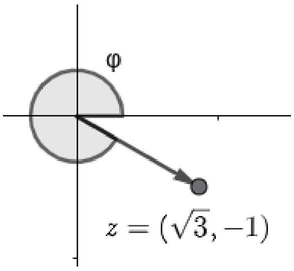
$\varphi = \frac{\pi}{4}$

$|z| = \sqrt{7^2+7^2} = 7\sqrt{2}$

$z = 7\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

3.b)

$z = \sqrt{3} - i$



$\varphi = \frac{11\pi}{6}$

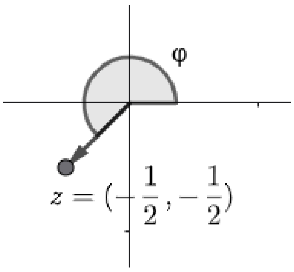
$|z| = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$

$z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

3.c)

$z = \frac{1}{i} \cdot \frac{1}{1+i}$

$$\frac{1}{i} \cdot \frac{1}{1+i} = \frac{i}{i^2} \cdot \frac{1-i}{1-i^2} = \frac{i+1}{-1(1+1)} = \frac{i+1}{-2} = -\frac{1}{2} - \frac{1}{2}i$$



$\varphi = \left(5\frac{\pi}{4}\right)$

$|z| = \frac{1}{2} \cdot \sqrt{2}$

$z = \frac{\sqrt{2}}{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

3.d)

$z = 1 + i \operatorname{tg} \alpha$

$$|z| = \sqrt{1 + \operatorname{tg}^2 \alpha} = \sqrt{1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}} = \sqrt{1 + \frac{1 - \cos^2 \alpha}{\cos^2 \alpha}} = \sqrt{\frac{1}{\cos^2 \alpha}} = \frac{1}{|\cos \alpha|} = \pm \frac{1}{\cos \alpha}$$

$$\cos \varphi = \frac{1}{|z|} = |\cos \alpha| = \pm \cos \alpha$$

$$\sin \varphi = \frac{\operatorname{tg} \alpha}{|z|} = \operatorname{tg} \alpha \cdot |\cos \alpha| = \frac{\sin \alpha}{\cos \alpha} \cdot |\cos \alpha| = \pm \sin \alpha$$

$$z = \pm \frac{1}{\cos \alpha} (\pm \cos \alpha \pm i \sin \alpha) = \frac{1}{\cos \alpha} (\cos \alpha + i \sin \alpha)$$

3.e)

$1 + \cos \alpha + i \sin \alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$

$\alpha \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin \alpha, \cos \alpha > 0$

$$|a| = \sqrt{(1 + \cos \alpha)^2 + \sin^2 \alpha} = \sqrt{(1 + \cos \alpha)^2 + 1 - \cos^2 \alpha} =$$

$$\sqrt{(1 + \cos \alpha)^2 + (1 + \cos \alpha)(1 - \cos \alpha)} = \sqrt{(1 + \cos \alpha)(1 + \cos \alpha + 1 - \cos \alpha)} =$$

$$\sqrt{2(1 + \cos \alpha)} = \sqrt{2\left(1 + 2\cos^2 \frac{\alpha}{2} - 1\right)} = \sqrt{4 \cos^2 \frac{\alpha}{2}} = 2 \cos \frac{\alpha}{2}$$

$$\cos \varphi = \frac{1 + \cos \alpha}{2 \cos \frac{\alpha}{2}} = \frac{1 + 2\cos^2 \frac{\alpha}{2} - 1}{2 \cos \frac{\alpha}{2}} = \cos \frac{\alpha}{2}$$

$$\sin \varphi = \frac{\sin \alpha}{2 \cos \frac{\alpha}{2}} = \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos \frac{\alpha}{2}} = \sin \frac{\alpha}{2}$$

$$z = 2 \cos \frac{\alpha}{2} \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2}\right)$$

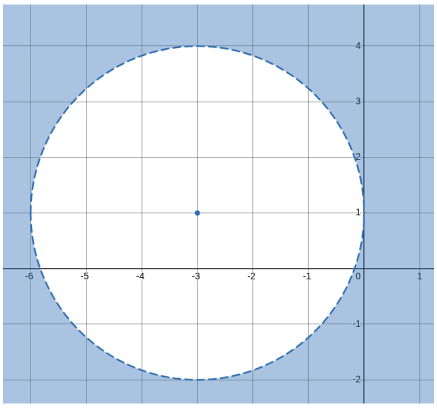
Zadanie 4

Zilustruj na płaszczyźnie zespolonej następujące zbiory:

4.a)

$\{z \in \mathbb{C} : |z - i + 3| > 3\}$

$|z - i + 3| = |z - (-3 + i)| > 3 \Leftrightarrow$ odległość z od punktu $(-3, 1) > 3$

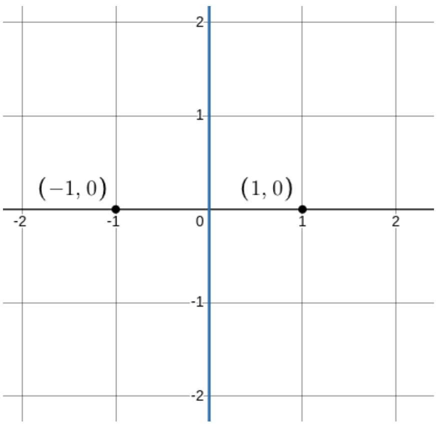


4.b)

$\{z \in \mathbb{C} : |z - 1| = |z + 1|\}$

$|z - 1| = |z + 1| \Leftrightarrow |z - (1 + 0i)| = |z - (-1 + 0i)|$

Odległości z od $(1, 0)$ i $(-1, 0)$ są równe.



4.c)

$\left\{z \in \mathbb{C} : \frac{|z-2i|}{|z+3|} < 1\right\}$

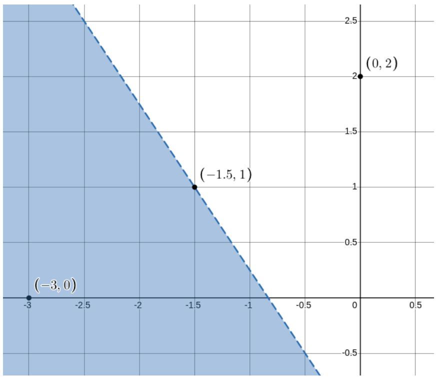
$\frac{|z-2i|}{|z+3|} < 1 \Leftrightarrow |z-(0+2i)| < |z-(-3+0i)|$

Odległość z od $(0,2)$ jest mniejsza niż od $(-3,0)$.

Granicą jest symetralna odcinka między tymi punktami. Ma nachylenie $-\frac{3}{2}$ i przechodzi przez punkt $\left(-\frac{3}{2}, 1\right)$.

$1 = \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right) + b \Rightarrow b = 1 - \frac{9}{4} = -\frac{5}{4}$

$y = -\frac{3}{2}x - \frac{5}{4}$



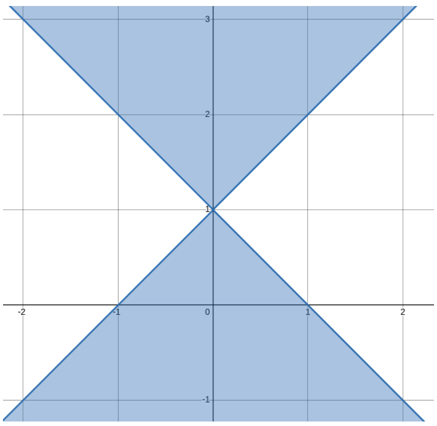
4.d)

$\{z \in \mathbb{C} : \Re(z-i)^2 \le 0\}$

$z = x + yi \Rightarrow z - i = x + (y-1)i$

$\Re(z-i)^2 = x^2 - (y-1)^2 = (x-y+1)(x+y-1) \le 0$

$$\begin{cases} y \ge x+1 \land y \le -x+1 \\ y \le x+1 \land y \ge -x+1 \end{cases}$$



4.e)

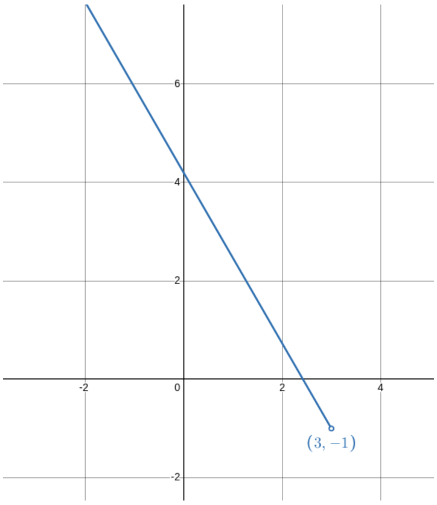
$\left\{ z \in \mathbb{C} : \arg(z - 3 + i) = \frac{2\pi}{3} \right\}$

$z = x + yi \Rightarrow z - 3 + i = (x - 3) + (y + 1)i$

Szukamy półprostej o początku w $(0, 0)$ przechodzącej przez $(x - 3, y + 1)$ o nachyleniu $\frac{2\pi}{3}$.

$\operatorname{tg} \frac{2\pi}{3} = -\sqrt{3} \Rightarrow (y + 1) = -\sqrt{3}(x - 3) \Rightarrow y = -\sqrt{3}x + 3\sqrt{3} - 1$

Należy pamiętać, że $\arg(0) = 0$, dlatego $(x - 3, y + 1) \neq (0, 0) \Rightarrow (x, y) \neq (3, -1)$



4.f)

$\left\{ z \in \mathbb{C} : \frac{\pi}{6} \leq \arg(\overline{z} + i) \leq \pi \right\}$

$\overline{z} + i \xrightarrow{T_{[0,-1]}} \overline{z} \xrightarrow{S_{OX}} z$

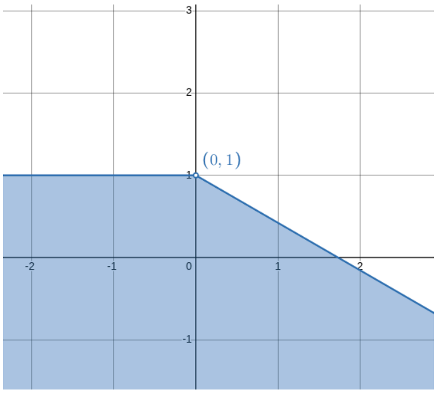
$\operatorname{tg} \frac{\pi}{6} = \frac{1}{\sqrt{3}}, \operatorname{tg} \pi = 0$

$\frac{\pi}{6} \leq \arg(z) \leq \pi \Leftrightarrow$

$$\begin{aligned} y &\geq \frac{1}{\sqrt{3}}x \qquad &\wedge\, y \geq 0 \xrightarrow{T_{[0,-1]}} \\ y &\geq \frac{1}{\sqrt{3}}x - 1 &\wedge\, y \geq 1 \xrightarrow{S_{OX}} \\ y &\leq -\frac{1}{\sqrt{3}}x + 1 \wedge y \leq -1 \end{aligned}$$

$\arg(0) = 0 \Rightarrow \overline{z} + i \neq (0, 0)$

$(0, 0) \xrightarrow{T_{[0,-1]}} (0, -1) \xrightarrow{S_{OX}} (0, 1) \Rightarrow z \neq (0, 1)$



4.g)

$\{z \in \mathbb{C} : \arg(\frac{i}{i-z}) = \frac{4\pi}{3}\}$

Zał: $z \neq i \Rightarrow z \neq (0,1)$

$z = x + yi \Rightarrow$

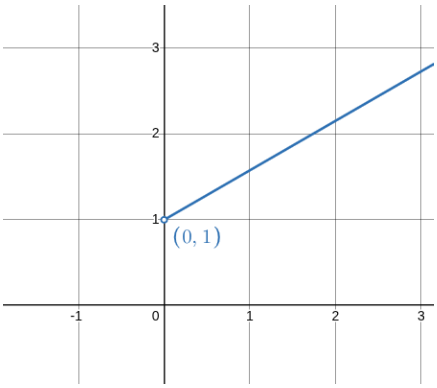
$$\frac{i}{i-z} = \frac{i}{i-x-yi} = \frac{i}{-x+(1-y)i} = \frac{i(-x-(1-y)i)}{x^2-(1-y)i^2} = \frac{-xi-i^2+yi^2}{x^2+(1-y)^2} = \frac{1-y-xi}{x^2+(1-y)^2}$$

$$\operatorname{tg}\frac{4\pi}{3} = \sqrt{3} = \frac{\mathfrak{I}}{\mathfrak{R}} = \frac{\frac{-x}{x^2+(1-y)^2}}{\frac{1-y}{x^2+(1-y)^2}} = \frac{-x}{1-y}$$

$$\begin{aligned} \sqrt{3}-\sqrt{3}y &= -x \\ -\sqrt{3}y &= -x-\sqrt{3} \\ y &= \frac{1}{\sqrt{3}}x+1 \end{aligned}$$

Interesuje nas tylko ta półprosta, która tworzy kąt $4\frac{\pi}{3}$. Jest ona w trzeciej ćwiartce, dlatego $\mathfrak{I}, \mathfrak{R} < 0$

$$\begin{cases} \frac{-x}{x^2+(1-y)^2} < 0 \\ \frac{1-y}{x^2+(1-y)^2} < 0 \end{cases} \Leftrightarrow \begin{cases} -x < 0 \\ 1-y < 0 \end{cases} \Leftrightarrow \begin{cases} x > 0 \\ y > 1 \end{cases}$$



4.h)

$\left\{z \in \mathbb{C} : \arg\left(\frac{i}{z}\right) \leq \frac{3\pi}{4}\right\}$

Zał: $z \neq 0 \Rightarrow z \neq (0,0)$

$$z = x + yi \Rightarrow \frac{i}{z} = \frac{i}{x+yi} = \frac{i(x-yi)}{x^2 - y^2 i^2} = \frac{xi - yi^2}{x^2+y^2} = \frac{y+xi}{x^2+y^2}$$

$$\operatorname{tg}\frac{3\pi}{4} = -1 \geq \frac{\mathfrak{I}}{\mathfrak{R}} = \frac{x}{y}$$

$$\frac{x}{y} + 1 \leq 0$$

$$\frac{x-y}{y} \leq 0$$

$$(x-y)y \leq 0$$

$$\begin{cases} y \geq x \wedge y \leq 0 \\ y \leq x \wedge y \geq 0 \end{cases}$$

$\langle 0, \frac{3\pi}{4}\rangle$ to pierwsza i druga ćwiartka $\Rightarrow \mathfrak{I} \geq 0 \Leftrightarrow \frac{x}{x^2+y^2} \geq 0 \Leftrightarrow x \geq 0$