Let $\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

- For $\boldsymbol{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, find two numbers x and y so that the linear combination $x\boldsymbol{v} + y\boldsymbol{w}$ equals $\boldsymbol{b}$.
- Find a unit vector $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $\boldsymbol{x} \cdot \boldsymbol{v} = 3$ and $\boldsymbol{x} \cdot \boldsymbol{w} = 0$.

Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$x\vec{v} + y\vec{w} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Matrix form $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$2x - y = 3$

$-x + 2y = 0$

A rectangular array of numbers is called a matrix, and the numbers are called the entries of the matrix

If a matrix A has size $\underline{m \times n}$, it has m rows and n columns

The (i,j)-entry a_{ij} of the matrix A lies in row i and column j.

Let $\vec{a}_j$ denote column j of A, $j = 1, 2, \ldots, n$. Each $\vec{a}_j$ is an m-dimensional vector.

$$A = [\,\vec{a}_1\ \ \vec{a}_2\ \ \cdots\ \ \vec{a}_n\,]$$

if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n-dimensional vector, $A\vec{x} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$, is an m-dimensional vector

(column way)

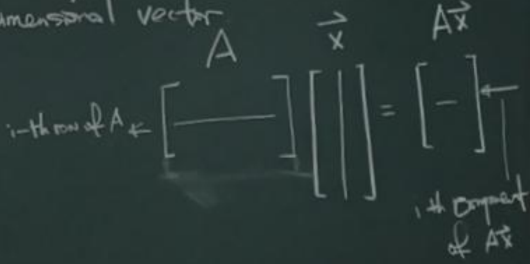
When A is of size m by $\underset{\Delta}{n}$, the product $A\vec{x}$ is only defined if $\vec{x}$ is an $\underset{\Delta}{n}$-dimensional vector

Let A be a 3 × 4 matrix, and $\vec{x}$ be a 4 dimensional vector

$$A\vec{x} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + x_4\begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

(row way)

$$= \begin{bmatrix} \underline{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{bmatrix} = \begin{bmatrix} (a_{11}, a_{12}, a_{13}, a_{14}) \cdot (x_1, x_2, x_3, x_4) \\ (a_{21}, a_{22}, a_{23}, a_{24}) \cdot (x_1, x_2, x_3, x_4) \\ (a_{31}, a_{32}, a_{33}, a_{34}) \cdot (x_1, x_2, x_3, x_4) \end{bmatrix}$$



Think of $\vec{b}$ as known, and look for $\vec{x}$

$A\vec{x} = \vec{b}$ is a system of linear equations for x_1, x_2, x_3.

$$\begin{aligned} x_1 \quad\quad\quad &= b_1 \\ -x_1 + x_2 \quad &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

It is solved as

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned}$$

For every $\vec{b}$, there is a solution to $A\vec{x} = \vec{b}$

When $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$

And the solution can be written as $\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vec{b}$

$= \underbrace{\boxed{A^{-1}}}_{\text{the inverse of } A} \vec{b}$

· Computing A^{-1} and multiplying $A^{-1}\vec{b}$ is a very slow way to find $\vec{x}$

$(*, *, *) \cdot (b_1, b_2, b_3)$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \boxed{b_1} \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} (1, 0, 0) \cdot (b_1, b_2, b_3) \\ (1, 1, 0) \cdot (b_1, b_2, b_3) \\ (1, 1, 1) \cdot (b_1, b_2, b_3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

For $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ lower triangular matrix

We write b_1, b_2, b_3 for the components of $A\vec{x}$

Given $x_1 = 1$, $x_2 = 4$, $x_3 = 9$. Compute $A\vec{x}$.

(1) The row way: Take the dot product of each row of A with $\vec{x}$

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 0\times 4 + 0\times 9 \\ (-1)\times 1 + 1\times 4 + 0\times 9 \\ 0\times 1 + (-1)\times 4 + 1\times 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

(2) The column way: Take the linear combinations of the columns of A

$$A\vec{x} = (1)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

So $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = A\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

When A is of size n by n

$A\vec{x} = \vec{b}$ gives n equation

For a $\boxed{\text{square}}$ matrix A,

A is called upper triangular

if every entry below the main diagonal is zero.

A is called lower triangular

if every entry above the main diagonal is zero

A is called triangular

if it is either upper or lower triangular.

Example for $C = \left[\begin{array}{ccc} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ \vec{u} & \vec{v} & \vec{w}_* \end{array}\right]$ $\quad \vec{w}_* = -\vec{u} - \vec{v}$

$$C\vec{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Compute $A^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A^{-1}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$

$\boxed{A\vec{x} = \vec{0}}$ has one and only one solution $\vec{x} = \vec{0}$

Not any given $\vec{b}$ can be produced by $C\vec{x}$.

To allow a solution to $C\vec{x} = \vec{b}$, the components of $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ must satisfy $\underline{b_1 + b_2 + b_3 = 0}$.

when $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $C\vec{x} = \vec{b}$ has NO solution

When $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\boxed{C\vec{x} = \vec{0}}$ has infinitely many solutions. eg. $\underline{\vec{x} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}}$. c can be any number.

Given a vector $\vec{b}$, $C\vec{x} = \vec{b}$ either has infinitely many solutions or else no solutions

The plane of $\vec{u}$ and $\vec{v}$ is filled by all their linear combinations $a\vec{u} + b\vec{v}$

It is governed by the equation $\underline{x + y + z = 0}$.

The key difference of A and C is whether the third column $\vec{w}$, and $\vec{w}_*$ is on this plane or not.