Review $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

matrix form $A\vec{x} = \vec{b}$

Vector equation

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{\text{row way}}{=} \boxed{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}} = \vec{b}$$

(Column way $= x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$

A linear equation $ax+by=c$ is a straight line, if a and b are not both zeros

Prove: the equation $\boxed{ax+by+cz=d}$ is a plane, perpendicular to $\vec{n}=(a,b,c)$

Given a point $P_0 = P_0(x_0, y_0, z_0)$, and a nonzero vector $\vec{n}=(a,b,c)$

there is a unique plane ① going through P_0

② perpendicular to $\vec{n}$.

Proof: A point $P=P(x,y,z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P} = (x-x_0, y-y_0, z-z_0)$ is perpendicular to $\vec{n}$

That is, if and only if $\boxed{0 = \vec{n}\cdot\overrightarrow{P_0P}} = a(x-x_0)+b(y-y_0)+c(z-z_0)$

$= ax+by+cz-(ax_0+by_0+cz_0)$

In other words, a point $P=P(x,y,z)$ lies on this plane if and only if x, y, z satisfies the above equation.

Assume that a, b, c are not all zeros. Let $d = ax_0+by_0+cz_0$, the equation shows that every plane that perpendicular to $\vec{n}=(a,b,c)$ has a linear equation of the form $ax+by+cz=d$. for some constant d.

Example. The plane governed by $x+y+z=0$ goes through the origin $(0,0,0)$. It is perpendicular to $\vec{n}=(1,1,1)$.

By freely choosing $x=a$, $z=b$, where a, b can be any numbers, then $y=-a-b$.

On this plane, every vector starting from the origin is $\begin{bmatrix} a \\ -a-b \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

2.2 Elimination and back substitution

$A\vec{x}=\vec{b}$, the coefficient matrix is square of size n by n. There may be

- no solution
- exactly one solution
- infinitely many solutions

to $A\vec{x}=\vec{b}$

Gaussian elimination.

$$(*)\quad \begin{aligned} 1x - 2y &= 1 \quad ① \\ 3x + 2y &= 11 \quad ② \end{aligned} \qquad A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

first pivot

$3x-6y=3$

② $-\ 3\times$① : $\boxed{3x-3x}+2y-(-6y)=11-3$, and x is eliminated from eq ②. (forward elimination)

The system becomes

$$\begin{aligned} x-2y&=1 \\ 8y&=8 \end{aligned}$$

$U=\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}$: upper triangular matrix $\quad U\vec{x}=\vec{c}$, $\vec{c}=\begin{bmatrix} 1 \\ 8 \end{bmatrix}$

The upper triangular system $U\vec{x}=\vec{c}$ can be solved from the bottom upwards. Firstly, $8y=8$ gives $y=1$.

Going upwards, $x-2=1$ gives $x=3$. (back substitution)

The solution is $\vec{x}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The original $A\vec{x}=\vec{b}$ has the same solution.

Solving (*) by the elimination process. the key step is to eliminate x from the second equation

How do we do this elimination? (How is this number $\overset{= l_{21}}{3}$ found ?).

· The basic elimination step substracts multiplier l_{ij} of equation j from equation i. It leaves a zero in row i of A

Definition. Pivot is the nonzero coefficient in the row of A that does the elimination

$$l_{21} = \frac{\text{the coefficient of } 3x}{\text{the first pivot}} = \frac{3}{1} = 3$$

(We divide the coefficient to be eliminated by the pivot).

$4x - 8y = 4$ The first pivot is 4.

$3x - 2y = 1$

$\longrightarrow$

$② - \underbrace{l_{21}}_{= \frac{4}{3}} \times ①$

Example $2x + 4y - 2z = 2$

$4x + 9y - 3z = 8$

$-2x - 3y + 7z = 10$

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

$$l_{ij} = \frac{\text{entry to eliminate in row } i}{\text{pivot in row } j} = \frac{a_{ij}}{a_{jj}}$$

The first pivot is 2. The first multiplier $l_{21} = \frac{4}{2} = 2$

① In the first step, we subtract $l_{21} = 2$, times equation 1 from equation 2. It leaves $y + z = 4$.

② In the next step. we add equation 1 to equation 3 (substract $\underbrace{(-1)}_{l_{31}} \times$ ① from ③. It leaves $y + 5z = 12$

Each step of elimination produces a zero in the coefficient matrix.

The system becomes

$2x + 4y - 2z = 2$

second pivot

$1y + z = 4$ ②

$y + 5z = 12$ ③

The second pivot is 1.

③ The final step is to eliminate y from the third equation from the second equation.

We subtract $1 \times$ ② from ③, It leaves $4z = 8$

Proof: A point $P = P(x, y, z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is perpendicular to $\vec{n}$

That is, if and only if $\boxed{0 = \vec{n} \cdot \overrightarrow{P_0P}} = a(x - x_0) + b(y - y_0) + c(z - z_0)$

$= ax + by + cz - (ax_0 + by_0 + cz_0)$

In other words, a point $P = P(x, y, z)$ lies on this plane if and only if x, y, z satisfies the above equation.

Assume that a, b, c are not all zeros. Let $d = ax_0 + by_0 + cz_0$, the equation shows that every plane that perpendicular to $\vec{n} = (a, b, c)$ has a linear equation of the form $ax + by + cz = d$ for some constant d.

Example. The plane governed by $x + y + z = 0$ goes through the origin $(0, 0, 0)$. It is perpendicular to $\vec{n} = (1, 1, 1)$

By freely choosing $x = a$, $z = b$, where a, b can be any numbers, then $y = -a - b$.

On this plane, every vector starting from the origin is $\begin{bmatrix} a \\ -a-b \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

The original system $A\vec{x} = \vec{b}$ has been converted into an upper triangular system $U\vec{x} = \vec{c}$ as

$$2x + 4y - 2z = 2$$
$$y + z = 4$$
$$4z = 8$$

$$U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

The forward elimination is complete. The pivots 2, 1, 4 are on the diagonal of the upper triangular U.

$U\vec{x} = \vec{c}$ is ready for back substitution.

$4z = 8$ gives $z = 2$.

Then $y + 2 = 4$ gives $y = 2$.

Finally, $2x + 4\times 2 - 2\times 2 = 2$ gives $x = -1$.

The solution is $(x, y, z) = (-1, 2, 2)$.

It also satisfies the original $A\vec{x} = \vec{b}$.

Idea of Gaussian elimination:

upper triangular matrix

$$A\vec{x} = \vec{b} \xrightarrow{\text{forward elimination}} U\vec{x} = \vec{c}$$

$U\vec{x} = \vec{c}$ is solved by back substitution.

$A\vec{x} = \vec{b}$ and $U\vec{x} = \vec{c}$ have the same set of solutions.