4.1 Spaces of vectors

A vector space $\mathbb{V}$ consists of a nonempty set of objects, called vectors,
that can be added, multiplied by real numbers, and for which certain axioms hold.

- $\mathbb{V}$ is closed under vector addition and scalar multiplication.

Eg. $\mathbb{R}^n$ The set M_{mn} of all m by n matrices with real entries

<u>Subspace</u>: A vector space inside another vector space.

Example In $\mathbb{R}^3$, a plane going through the origin point.
It is a vector space inside $\mathbb{R}^3$

Definition A subspace of a vector space is a set of "vectors", including $\vec{0}$,
that satisfies the following two requirements: If $\vec{v}$ and $\vec{w}$ are vectors in the subspace, and c is a scalar,
then (1) $\vec{v}+\vec{w}$ is in the subspace
(2) $c\vec{v}$ is in the subspace.

($\vec{v}-\vec{w} = \vec{v}+(-1)\vec{w}$ is also in the subspace)

$\Rightarrow$ A subspace containing $\vec{v}$ and $\vec{w}$ must contain all their linear combinations $c\vec{v}+d\vec{w}$

. Every subspace contains the zero vector.

Q: Is any plane in $\mathbb{R}^3$ a subspace of $\mathbb{R}^3$?

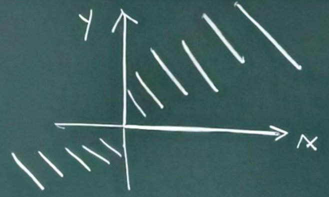
All possible subspaces of $\mathbb{R}^3$:
(1) Planes in $\mathbb{R}^3$ through the origin $(0,0,0)$
(2) line in $\mathbb{R}^3$ through $(0,0,0)$.
(3) $\mathbb{R}^3$. The whole space is a subspace of itself.
(4) The smallest subspace is the space Z that only contains the zero vector.

$$Z = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} = (0,0,0)\}$$

All possible subspaces of $\mathbb{R}^2$?

Q: Is part of a plane or line a subspace of $\mathbb{R}^3$?

Counter example in $\mathbb{R}^2$



(1) Keep only the vectors (x, y) whose components are positive or zero.

Choosing $\vec{v} = (1, 1)$ in this plane. $(-1)\vec{v}$ is NOT in this plane.

∴ The quarter plane is not a subspace of $\mathbb{R}^2$

(2) Now we include the vectors (x, y) whose components are negative

$\vec{v} = (2, 0)$ and $\vec{w} = (-1, -1)$ are in the plane, but $\vec{v} + \vec{w} = (1, -1)$ is not

∴ Two quarter planes also don't make a subspace of $\mathbb{R}^2$

Example. Inside the vector space $\mathbb{M}$ of all 2 by 2 real matrices, verify that

(1) $\mathbb{U}$: the set consisting all upper triangular matrices $\begin{bmatrix} a & b \\ & d \end{bmatrix}$, are two subspaces of $\mathbb{M}$.

(2) $\mathbb{D}$: the set consisting all diagonal matrices $\begin{bmatrix} a & \\ & d \end{bmatrix}$.

$I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ itself is NOT a subspace

And in this case, $\mathbb{D}$ is also a subspace of $\mathbb{U}$

$Z = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{M}$, $\mathbb{U}$, & $\mathbb{D}$. ∵ with one vector

But multiples cI, where c can be any number, form a subspace

Column space of A $C(A)$ is a vector space made up of column vectors.

Definition The column space of an m by n matrix, denoted by $C(A)$, consists of all linear combinations of the columns of A.

$A\vec{x} = \vec{b}$

$C(A)$ can be the whole $\mathbb{R}^m$, or only a subspace of $\mathbb{R}^m$.

- $C(A)$ is a subspace of $\mathbb{R}^m$, and it includes the zero vector in $\mathbb{R}^m$

Example. Describe the column spaces for $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, which are all subspaces of $\mathbb{R}^2$.

(1) $C(I)$ is $\mathbb{R}^2$ for any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, $\vec{b} = b_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(2) $C(A)$ is a line inside $\mathbb{R}^2$ $\ell = \{ \vec{v} = (x, y) \in \mathbb{R}^2 \mid y = 2x \}$.

$A\vec{x} = \vec{b}$ is only solvable when $\vec{b}$ is on the line ℓ.

(3) $C(B)$ is also all of $\mathbb{R}^2$.

$B\vec{x} = \vec{b}$ is always solvable for any given $\vec{b}$

A system of 2 equation but with 3 unknowns

B has the same column space as I, because every vector in $\mathbb{R}^2$ can be written as a combination of the columns of B.

- $A\vec{x} = \vec{b}$ are solvable if and only if $\boxed{\vec{b} \text{ is in } C(A)}$

For $\vec{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\vec{b} = \text{col } 2 + \text{col } 3$

$= 2(\text{col } 1) + \text{col } 3$

Both $\vec{x_1} = (0, 1, 1)$ and $\vec{x_2} = (2, 0, 1)$ are solutions to $B\vec{x} = \vec{b}$

Review linear combination $c\vec{v} + d\vec{w} =$

When $n=3$, what are the pictures of $a\vec{u}$, $a\vec{u}+b\vec{v}$, $a\vec{u}+b\vec{v}+c\vec{w}$?

dot product $\vec{v}\cdot\vec{w} = \qquad = \vec{v}^T\vec{w}$ perpendicular $\Leftrightarrow$

Length $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} =$ unit vector $\vec{u}$? unit vector in the same direction as $\vec{v}$?

θ: angle between nonzero vectors $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$ $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$

is of size m by n $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\|\|\vec{w}\|$

$A\vec{x}$: linear combination of columns of A

dot product of the row of A with $\vec{x}$

$A\vec{x} = \vec{b}$ Column picture

row picture

"pivot" "multiplier"

$A\vec{x} = \vec{b} \xrightarrow{\text{forward elimination}} \underline{U\vec{x} = \vec{c}}$, which can be solved by back substitution

Break down of elimination. EA "$\vec{c}$"

E_{ij}. P_{ij} $[A\ \vec{b}] \longrightarrow [\overset{"}{U}, E\vec{b}]$

$AB \neq BA$ $(AB)C = A(BC)$

Rules for matrix operations

AB
- (i,j) entry of AB
- columns of AB
- rows of AB
- Column times row

A square matrix A is invertible if and only if $AC = I$ or $CA = I$

invertibility of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, triangular, diagonal matrices

$(AB)^{-1} =$

when $n=3$: $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

$A = L\underline{DV}$ $L(U\vec{x}) = A\vec{x} = \vec{b}$

$= L(\vec{c})$