2.5 Inverse matrices

If a square matrix A is invertible. the unique inverse of A is denoted by A^{-1}

Then A^{-1} is a square matrix of the same size as A. so that $A^{-1}A = I$ and $AA^{-1} = I$

Ways of verifying that the inverse of a matrix exists:

(1) If a matrix C can be found such that $AC = I$, or $CA = I$, then A is invertible, and C is the inverse of A.

(2) If A is invertible, $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$

(3) If A is invertible, $A \xrightarrow{\text{row operations}} I$ $\quad [A \ I] \longrightarrow [I \ A^{-1}]$

All the elimination steps can be done with matrices.

and they can be inverted with matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

$$= E_{31}E_{21}A$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_{31}(E_{21}A) = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$= E_{32}E_{31}E_{21}A$$

pivots

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad E_{32}(E_{31}E_{21}A) = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$U = E_{32}E_{31}E_{21}A \qquad \text{det } L = \underline{E_{32}E_{31}(E_{21}I)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$[A\ I] \longrightarrow [U\ L]$ agrees with the matrix multiplication.

$U = LA$

The last step contains 2 elementary row operations.

It agrees with the multiplication

$$\underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{D} \begin{bmatrix} 2 & 0 & 0 & \frac{27}{2} & -\frac{11}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 4 & 3 & -1 & 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{27}{4} & -\frac{11}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Since A is invertible with 3 pivots, elimination is equivalent to multiply $[A\ I]$ by A^{-1}

$$[A\ I] \longrightarrow [I\ A^{-1}]$$

agrees with $A^{-1}[A\ I] = [I\ A^{-1}]$.

$$(1)\ [A\ I] = \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3+r_1} \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_3-r_2} \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right].$$

$$(2)\ \xrightarrow{r_2-r_3/4} \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{r_1+r_3/2} \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right] \xrightarrow{r_1-4r_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 27/2 & -11/2 & 3/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right].$$

$$(3)\ \xrightarrow[r_3/4]{r_1/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right].$$

Example 1 (Inverse of an elimination matrix). Let $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For any matrix A with 3 rows, the multiplication EA substracts 2 times row 1 of A from row 2.

To add it back, we multiply $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ times EA such that $E^{-1}(EA) \xrightarrow{?} A$ $E^{-1}(EA) = A$.

We are substracting and adding the same 2 times row 1

Elimination is really a sequence of multiplication:

(1) elimination matrices E's, which produces zeros below and above the pivots.

(2) permutation matrices P's, which exchange rows, if needed

(3) D, which divides the rows by the pivots, to change all the pivots to 1's.

By Gauss-Jordan elimination, a square matrix with a full set of pivots always has a two-sides inverse

In the row reduction $[A \; I] \rightarrow [I \; \vec{x}_1 \; \vec{x}_2 \cdots \vec{x}_n]$ (Solution to $A\vec{x} = \vec{e}_1$)

The change from A to I by $D E \cdot P \cdots E A = I$ (a left inverse of A) And $A[\vec{x}_1 \; \vec{x}_2 \cdots \vec{x}_n] = [\vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n] = I$ (a right inverse of A)

So $D E \cdot P \cdots E = [\vec{x}_1 \; \vec{x}_2 \cdots \vec{x}_n]$

Example For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I$

- A 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad-bc \neq 0$. and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- A triangular matrix is invertible if and only if no diagonal entries are zero, and the inverse is still triangular.

Example Invert $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ by Gauss-Jordan method.

$= E_{32}E_{31}E_{21}$

$$[\,L \;\; I\,] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2+2r_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3-3r_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{r_3+r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [\,I \;\; L^{-1}\,]$$

So
$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$
$= (E_{32}E_{31}E_{21})^{-1}$

- A square matrix is called a diagonal matrix if all the entries off the diagonal are zero.

A diagonal matrix $A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ has an inverse provided no diagonal entries are zero. and $A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$

1. $ad - bc = 1 \times 2 - 2 \times 1 = 0$.

2. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. A has only 1 pivot.

Whenever A is not invertible,
elimination fails to reduce A to I.

3. $A\vec{x} = \vec{0}$ has a nonzero solution $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

[redacted] then A is invertible, and C is the inverse of A.

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$