. A sequence ...

4.4 Independence, basis and dimension
enough, and not more, independent vectors that span the vector space

Given an m by n matrix A, let $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ denote the columns of A

. $\mathbb{R}^m = C(A)$ if and only if for every $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ is solvable.
$= \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$
if and only if the rank of A is $r = m$

That is $N(A) = \{\vec{0}\}$.

. $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ are linearly independent if and only if $\vec{x} = \vec{0} \in \mathbb{R}^n$ is the only solution to $A\vec{x} = \vec{0}$.
if and only if the rank of A is $r = n$

Proof: A matrix A with full column rank $r = n$.
if and only if there are n pivots of A
if and only if there are NO special solutions to $A\vec{x} = \vec{0}$, i.e., $N(A) = \{\vec{0}\}$,
if and only if the columns of A are independent

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Example Test whether $\vec{u} = (1, 2, 1)$, $\vec{v} = (0, 1, 0)$, $\vec{w} = (3, 5, 3)$ are independent or dependent

Sol: $A = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R$

Free column

The rank of A is $r = 2$, and there is $\underset{}{1}$ ($= n - r$) special solution $\vec{s} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ to $A\vec{x} = \vec{0}$ (or $R\vec{x} = \vec{0}$)

It also shows that (1) $(-3)\vec{u} + (1)\vec{v} + (1)\vec{w} = \vec{0}$

(2) $\vec{w} = (3)\vec{u} + (-1)\vec{v}$ (The free column is a combination of the pivot columns)

Definition. The sequence of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is linearly independent when the only combination that gives the zero vector is $0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n$

In other words, $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ only happens when $x_1 = x_2 = \cdots = x_n = 0$

Conversely, if a combination gives $\vec{0}$ when the x's are NOT all zero, the vectors are dependent.

→ If one of them is the zero vector, they must be dependent.

. A sequence of vectors is either dependent or independent. (Which combination of the vectors give zero, with nonzero coefficients or they can't?)

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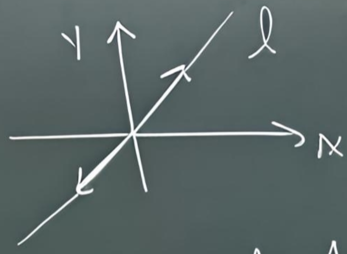
Example (1) (1, 1) and (−1, −1) are dependant



① By taking $x_1 = 1$ and $x_2 = 1$. $x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

② det $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A)$ $A\vec{x} = \vec{0}$ has nonzero solutions

(2) (1, 1) and (0, 0) are dependant.

(3) $\vec{v} = (1, 1)$ and $\vec{w} = (1, 0)$ are independent.

$(1)\vec{u} + (-1)\vec{v} + (-1)\vec{w} = \vec{0}$

It says that $\vec{u}, \vec{v}, \vec{w}$ are dependent

All the combinations of $\vec{v}$ and $\vec{w}$ fill the xy plane. For every $\vec{u} \in \mathbb{R}^2$, $\boxed{\vec{u} = c\vec{v} + d\vec{w}}$ with specific c and d.

Definition A sequence of vectors is dependent if one vector is the combination of the other vectors.

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are independent, none of the vectors is a combination of the other $n-1$ vectors.

Example. In $\mathbb{R}^3$, if $\vec{u}, \vec{v}, \vec{w}$ are in the same plane, they are dependent.

$\vec{w} = 3\vec{u} + (-1)\vec{v}$ It means $\vec{w}$ is on the plane of $\vec{u}$ and $\vec{v}$. They are dependant.

3

Vectors that span a subspace

Definition A set of vectors spans a vector space if their linear combinations fill the space.

The columns of a matrix span its column space. Or the column space is spanned by the columns.

Example The columns of A might be dependent or independent. $C(A)$ might be the whole $\mathbb{R}^m$ or only a subspace

(1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^2$. (2) $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ also span $\mathbb{R}^2$ independent

(3) $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbb{R}^2$. So does $\vec{w}_1$ by itself

2 vectors in $\mathbb{R}^3$ can't span $\mathbb{R}^3$, even if they are independent

3 vectors in $\mathbb{R}^2$, can't be independent, even if they span $\mathbb{R}^2$.

Vectors in $\mathbb{R}^2$ could span $\mathbb{R}^2$ or a line. How about two-dimensional vectors?

The row space of A is the same as $C(A^T)$ or spanned by the rows

Definition The row space of an m by n matrix is the subspace of $\mathbb{R}^n$ containing all combinations of the rows.

Example. Describe the column space and the row space of $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. ① $C(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is a plane in $\mathbb{R}^3$

② $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The row space of A is $\mathbb{R}^2$