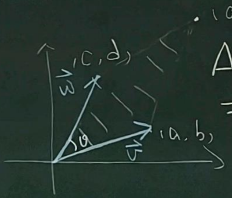
For any square matrix A. the cofactor matrix C: C_{ij} is the (i,j) cofactor of A

the adjugate of A $\quad (\operatorname{adj} A) = C^T$

$$\boxed{A(\operatorname{adj} A) = (\det A) I = (\operatorname{adj} A) A}$$

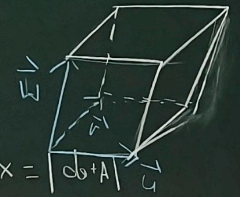
If $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A}(\operatorname{adj} A) = \frac{1}{\det A} C^T$



Area $= \|\vec{v}\|\,\|\vec{w}\| \sin\theta$

The area of the parallelogram with corners $(0,0)$, (a,b), (c,d) and $(a+c, b+d)$ is

the absolute value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$



If the rows, or the columns, of a 3 by 3 matrix give the sides of a box, then the volume of the box $= |\det A|$

The 3 basic rules for determinants are also obeyed by volumns

① The volume of the unit cube is 1 $\quad$ It is $\det I = 1$

② Edge exchanges or row exchanges leave the same box and the same absolute value.

③ When an edge is stretched by l, the volume is multiplied by l

Cross product (a special application for three dimensional vectors) $\vec{u} \times \vec{v}$ $\qquad$ $\vec{u} \cdot \vec{v} =$

Definition. Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$. The cross product of $\vec{u}$ and $\vec{v}$ is a vector

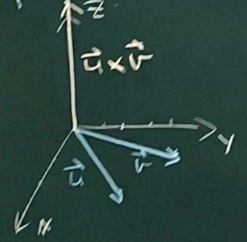
$$\boxed{\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}} = (u_2 v_3 - u_3 v_2)\vec{i} + (u_3 v_1 - u_1 v_3)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

$= (u_2 v_3 - u_3 v_2, 0, 0)$ is the first component of $\vec{u} \times \vec{v}$

where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$ $\quad = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$

Example. Let $\vec{u} = (3, 2, 0)$, $\vec{v} = (1, 4, 0)$. $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = \vec{k}\,(-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$

$= 10\vec{k} = (0, 0, 10)$



Properties of $\vec{u}\times\vec{v}$. ① $\vec{v}\times\vec{u} = -(\vec{u}\times\vec{v})$

② $\vec{u}\times\vec{u} = \vec{0}$

The length of $\vec{u}\times\vec{v}$ equals
the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$

When $\vec{u}$ and $\vec{v}$ are perpendicular, $\vec{u}\cdot\vec{v} = 0$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$.

When $\vec{u}$ and $\vec{v}$ are parallel, $\vec{u}\times\vec{v} = \vec{0}$

$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$

③ $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$

Let $\vec{w} = (w_1, w_2, w_3)$ then $\vec{w}\cdot(\vec{u}\times\vec{v}) = w_1(u_2v_3 - u_3v_2) + w_2(u_3v_1 - u_1v_3) + w_3(u_1v_2 - u_2v_1)$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (\vec{u}\times\vec{v})\cdot\vec{w}$$ the triple product

Proof of ③: For instance, $(\vec{u}\times\vec{v})\cdot\vec{u}$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

The triple product $(\vec{u}\times\vec{v})\cdot\vec{w} = 0$ when $\vec{u}, \vec{v}, \vec{w}$ lie in the same plane.

When the $\vec{u}, \vec{v}, \vec{w}$ box is squashed onto a plane

$0 = \text{Volume} = |\det A| = |(\vec{u}\times\vec{v})\cdot\vec{w}|$

Chapter 4 Vector spaces and subspaces

4.1 Spaces of vectors

$A\vec{x}$: a linear combination of the column vectors of A. AB :

Example $\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2$

$(1, 2, 3, 4) \in \mathbb{R}^4$

$(1+i, 1-i) \in \mathbb{C}^2$

Vector space. Let $\mathbb{R}$ be the set of all real numbers.

Example. The vector space $\mathbb{R}^2$ is the xy-plane. $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1, v_2) \in \mathbb{R}^2$.

The one-dimensional space $\mathbb{R}^1$ is a line, like the x-axis.

$\mathbb{R}^3$ is the xyz space. $\vec{v} = (v_1, v_2, v_3)$

$\mathbb{R}^5$ is a 5-dimensional space, which contains all vectors with 5 components. $\vec{v} = (1, 1, 1, 1, 1) \in \mathbb{R}^5$.

Definition The n-dimensional space $\mathbb{R}^n$ contains all column vectors $\vec{v} = (v_1, v_2, \ldots v_n)$ with n real components.

For vectors in $\mathbb{R}^n$, vector addition and scalar multiplication produce vectors stay in $\mathbb{R}^n$.

Eight conditions required for every vector space

(1) Commutative law $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

(2) associative law $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) associative law for scalars $(c_1 c_2)\vec{v} = c_1(c_2\vec{v})$

(4) distributive law $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

(5) distributive law for scalars $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$

6) There is a unique "zero vector" such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v}$

In $\mathbb{R}^3$, $\vec{0} = (0, 0, 0)$

In $\mathbb{M}$, $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

7) For each $\vec{v}$, there is a unique vector $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = \vec{0}$

8) $1 \times \vec{v} = \vec{v}$

A vector space is a set of "vectors" together with rules for vector addition and scalar multiplication.

Example Denote by $\mathbb{M}$ be the vector space of all real 2 by 2 matrices.

In $\mathbb{M}$, the "vectors" are matrices.

We add matrices to matrices, and multiplying matrices by real numbers. The results remain the 2 by 2 real matrices

And the above 8 rules can also be checked for $\mathbb{M}$

In general, let $\mathbb{V}$ be a vector space

For vectors in $\mathbb{V}$, their linear combinations stay inside the same $\mathbb{V}$.

Particularly, vector addition and scalar multiplication must obey the 8 rules.

Example. The set of all positive vectors $(v_1, v_2, \ldots, v_n)$ with every $v_i > 0$ is NOT a vector space.

A line in $\mathbb{R}^n$ is not a vector space unless it goes through the origin $(0, 0, \ldots, 0)$.

If a line goes through $\vec{0}$, we can multiply points on this line by any number, without leaving the line.

And we can add points on the line, ...