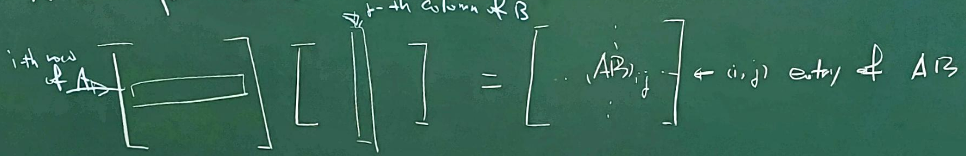
Matrix multiplication. AB : The rows of A must be the same length as the columns of B.

Column number of A = row number of B.

- The column way : A times columns of B
- The row way : rows of A times B
- The dot product way : specify each individual entry

Matrix-vector multiplication. For an m by n matrix A, and an n-dimensional vector $\vec{x}$.

Matrix multiplication: Let A and B be matrices of sizes m by n, and n by p respectively



Laws for matrix multiplication

(1) distributive law from the left $A(B+C) = AB + AC$

(2) distributive law from the right $(A+B)C = AC + BC$

(3) associative law $(AB)C = A(BC)$

When A is a square matrix, and $A=B=C$.

A to the p-th power $A^p = \underbrace{AA \cdots A}_{p \text{ factors}}$

when $p=0$, $A^0 = I$, is the identity matrix

Usually, $AB \neq BA$

$AI = A = IA$.

All square matrices Commute with I.

is of size m by q.

The product $\underset{m\times n}{A}\ \underset{n\times p}{B}\ \underset{p\times q}{C}$

Their sizes must be right for multiplication.

$a^{-1} = \frac{1}{a}$

$A^2 A = AA^2 = A^3$ $A^3 A^4 = A^7$ $(A^3)^4 = A^{12}$

Ordinary laws for exponents.

$$\boxed{A^p A^q = A^{p+q} \qquad (A^p)^q = A^{pq}}$$

These rules still hold when p and q are negative number, provided A has a "-1 power". (inverse matrix)

Block matrices and block multiplications.

Example 1. For an m by n matrix $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$. where $\vec{a}_j$ is column j of A, is a block partition of A

For any n by 1 column matrix $\vec{x}$, if we further cut $\vec{x}$ by rows $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ $\vec{x} = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_n \end{bmatrix}$

Example 2. In the augmented matrix $[A\ \vec{b}]$, it has 2 blocks of different sizes

where $E = [E]$ has only one block. It is a block multiplication

$$E[A\ \vec{b}] = [EA\ \ E\vec{b}]$$

Example $A = \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$

It is cut into smaller matrices. A_1, A_2, A_3, A_4 are called blocks

The blocks give us a 2 by 2 block matrix

A is said to be partitioned into blocks.

If B is also of size 3 by 5, and the block sizes match, we can add A + B a block at a time

· Block multiplication is allowed when the block sizes match correctly.

when the multiplication of A times B is allowed, if further blocks of A can multiply blocks of B,

which means cuts between columns of A match cuts between rows of B,

or the numbers of columns in each block of A equals the number of rows in the correspond block of B.

Example $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ $AB = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$ Mutiply AB using block multiplication.

$= 1 \times [2\ 2] + 1\,[3\ 4] =$

(1) The blocks are allowed to be numbers, and they are 1 by 1 blocks

$$AB = \left[\begin{array}{c|c} 1 & 1 \\ \hline 2 & -1 \end{array}\right] \left[\begin{array}{cc} 2 & 2 \\ \hline 3 & 4 \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_2 \\ A_3B_1 + A_4B_2 \end{bmatrix}$$

(2) $AB = \left[\begin{array}{cc} 1 & 1 \\ \hline 2 & -1 \end{array}\right] \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} A_1B \\ A_2B \end{bmatrix} \rightarrow \underset{1\times 2}{[2\ -1]} \underset{2\times 2}{\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}} = [1\ 0]$

$$= \left[\begin{array}{cc} 1 & 1 \\ \hline 2 & -1 \end{array}\right] \left[\begin{array}{c|c} 2 & 2 \\ 3 & 4 \end{array}\right] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} [B_1\ B_2] = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix} \rightarrow [2\ -1] \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$$

(3) $AB = \left[\begin{array}{c|c} 1 & 1 \\ 2 & -1 \end{array}\right] \left[\begin{array}{cc} 2 & 2 \\ \hline 3 & 4 \end{array}\right] = [A_1\ A_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1 + A_2B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [2\ 2] + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [3\ 4]$

$$= \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix}$$

Example: Elimination by blocks:

Let $A = \left[\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & 9 & -3 \\ -2 & -3 & 7 \end{array}\right] = \begin{bmatrix} 2 & B \\ C & D \end{bmatrix}$ where $B = [4 \ \ -2]$, $C = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, $D = \begin{bmatrix} 9 & -3 \\ -3 & 7 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_{31}(E_{21}A) = \left[\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 0 & 1 & 1 \\ 0 & 1 & 5 \end{array}\right] = (E_{31}E_{21})A$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } E_{32}(E_{31}(E_{21}A)) = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A \;=\; E_{32}(E_{31}(E_{21}A))$$

Let $E = E_{31}E_{21} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & \vec{0} \\ M & I \end{bmatrix}$, where $\vec{0} = [0 \ \ 0]$, $M = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

How E eliminates the 2 entries by blocks?

$$EA = \begin{bmatrix} 1 & \vec{0} \\ M & I \end{bmatrix} \begin{bmatrix} 2 & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 \times 2 + \underbrace{[0, 0]\begin{bmatrix} 4 \\ -2 \end{bmatrix}}_{=0} & 1 \times [4 \ \ -2] + \underbrace{[0 \ \ 0]\begin{bmatrix} 9 & -3 \\ -3 & 7 \end{bmatrix}}_{=[0 \ \ 0]} \\ \underbrace{2\begin{bmatrix} -2 \\ 1 \end{bmatrix} + I\begin{bmatrix} 4 \\ -2 \end{bmatrix}}_{=\begin{bmatrix} 0 \\ 0 \end{bmatrix}} & MB + ID \end{bmatrix}$$

$\Rightarrow$ We complete the elimination of the block matrix $C = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ by using the block multiplication