EXAMPLE 2 **Writing a Quadratic Function in Vertex Form**

Given $f(x) = 3x^2 + 12x + 5$,

a. Write the function in vertex form: $f(x) = a(x - h)^2 + k$.
b. Identify the vertex.
c. Identify the x-intercept(s).
d. Identify the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of f.
h. Write the domain and range in interval notation.

Solution:

a. $f(x) = 3x^2 + 12x + 5$

$= 3(x^2 + 4x \qquad) + 5$

Factor out the leading coefficient of the x^2 term from the two terms containing x. The leading term within parentheses now has a coefficient of 1.

$= 3(x^2 + 4x + 4 - 4) + 5$

Complete the square within parentheses. Add and subtract $[\frac{1}{2}(4)]^2 = 4$ within parentheses.

$= 3(x^2 + 4x + 4) + 3(-4) + 5$

Remove -4 from within parentheses, along with a factor of 3.

$= 3(x + 2)^2 - 7$ (vertex form)

b. The vertex is $(-2, -7)$.

c. $f(x) = 3x^2 + 12x + 5$

$0 = 3x^2 + 12x + 5$

To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(3)(5)}}{2(3)}$$

The right side is not factorable. Apply the quadratic formula.

$$= \frac{-12 \pm \sqrt{84}}{6}$$

$$= \frac{-12 \pm 2\sqrt{21}}{6}$$

$$= \frac{-6 \pm \sqrt{21}}{3} \begin{cases} x \approx -0.47 \\ x \approx -3.53 \end{cases}$$

The x-intercepts are $\left(\frac{-6 + \sqrt{21}}{3}, 0\right)$ and $\left(\frac{-6 - \sqrt{21}}{3}, 0\right)$ or approximately $(-0.47, 0)$ and $(-3.53, 0)$.

d. $f(0) = 3(0)^2 + 12(0) + 5$

$= 5$

To find the y-intercept, evaluate $f(0)$. The y-intercept is $(0, 5)$.

e. The graph of f is shown in Figure 3-3.

f. The axis of symmetry is $x = -2$.

g. The minimum value is -7.

h. The domain is $(-\infty, \infty)$.

The range is $[-7, \infty)$.

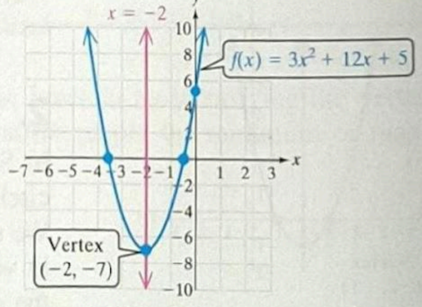


Figure 3-3

Skill Practice 2 Repeat Example 2 with $f(x) = 3x^2 - 6x + 1$.

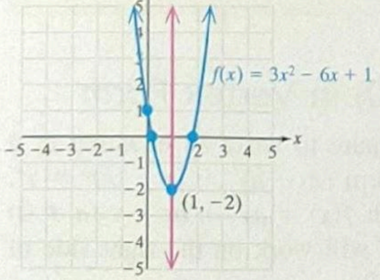
Answers

2. a. $f(x) = 3(x - 1)^2 - 2$ b. $(1, -2)$

c. $\left(\frac{3 \pm \sqrt{6}}{3}, 0\right)$

d. $(0, 1)$

e.



f. $x = 1$

g. The minimum value is -2.

h. The domain is $(-\infty, \infty)$.
The range is $[-2, \infty)$.

In Examples 6 and 7, we demonstrate the process of graphing a polynomial function.

EXAMPLE 6 Graphing a Polynomial Function

Graph $f(x) = x^3 - 9x$.

Solution:

$f(x) = x^3 - 9x$

1. The leading term is x^3. The end behavior is down to the left and up to the right.

The exponent on the leading term is odd and the leading coefficient is positive.

2. $f(0) = (0)^3 - 9(0) = 0$
   The y-intercept is $(0, 0)$.

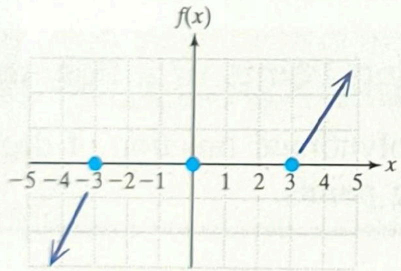
Determine the y-intercept by evaluating $f(0)$.

3. $0 = x^3 - 9x$
   $0 = x(x^2 - 9)$
   $0 = x(x - 3)(x + 3)$

Find the real zeros of f by solving for the real solutions to the equation $f(x) = 0$.

The zeros of the function are 0, 3, and -3, and each has a multiplicity of 1.

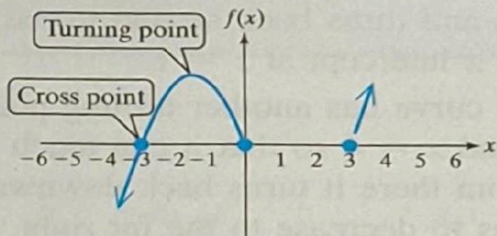
The zeros are real numbers and correspond to x-intercepts on the graph. Since the multiplicity of each zero is an odd number, the graph will cross the x-axis at the zeros.

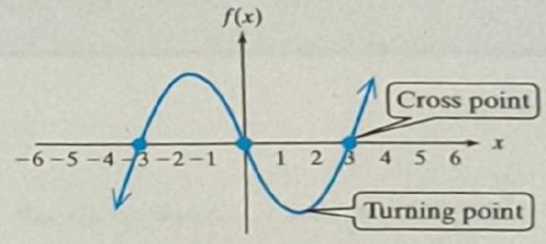
4. 

Plot the x- and y-intercepts and sketch the end behavior.

5. Moving from left to right, the curve increases from the far left and then crosses the x-axis at -3. The graph must have a turning point between $x = -3$ and $x = 0$ so that the curve can pass through the next x-intercept of (0, 0).

The graph crosses the x-axis at $x = 0$. The graph must then have another turning point between $x = 0$ and $x = 3$ so that the curve can pass through the next x-intercept of (3, 0). Finally, the graph crosses the x-axis at $x = 3$ and continues to increase to the far right.





6. $f(x) = x^3 - 9x$

$$f(-x) = (-x)^3 - 9(-x) = -x^3 + 9x \qquad -f(x) = -(x^3 - 9x) = -x^3 + 9x$$

$f(-x) = -f(x)$ (same)

Testing for symmetry, we see that $f(-x) = -f(x)$. Therefore, f is an odd function and is symmetric with respect to the origin.

TIP Techniques of calculus can be used to find the exact coordinates of the turning points of the polynomial function in Example 6.

7. If more accuracy is desired, plot additional points. In this case, since f is symmetric to the origin, if a point (x, y) is on the graph, then so is $(-x, -y)$. The graph of f is shown in Figure 3-13.

x	$f(x)$
1	-8
2	-10
4	28

Use symmetry.

x	$f(x)$
-1	8
-2	10
-4	-28

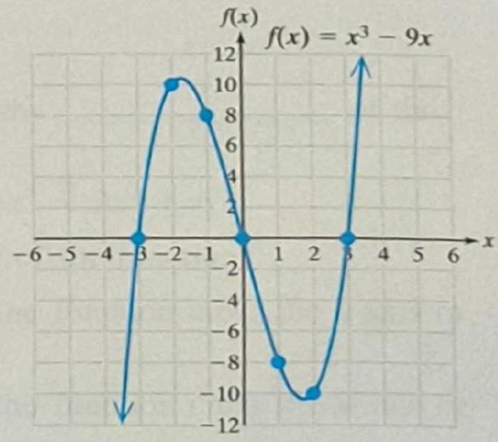


Figure 3-13

Skill Practice 6 Graph $g(x) = -x^3 + 4x$.

Division Algorithm

Suppose that $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either the degree of $r(x)$ is less than $d(x)$, or $r(x)$ is the zero polynomial.

Note: The polynomial $f(x)$ is the **dividend,** $d(x)$ is the **divisor,** $q(x)$ is the **quotient,** and $r(x)$ is the **remainder.**

EXAMPLE 2 Dividing Polynomials Using Long Division

Use long division to divide $(-5 + x + 4x^2 + 2x^3 + 3x^4) \div (x^2 + 2)$.

Solution:

Write the dividend and divisor in descending order and insert place holders for missing powers of x. $(3x^4 + 2x^3 + 4x^2 + x - 5) \div (x^2 + 0x + 2)$

$$\begin{array}{r l}
 & 3x^2 + 2x - 2 \\
x^2 + 0x + 2 & \overline{)\,3x^4 + 2x^3 + 4x^2 + x - 5} \\
 & \underline{-(3x^4 + 0x^3 + 6x^2)} \\
 & \quad 2x^3 - 2x^2 + x \\
 & \quad \underline{-(2x^3 + 0x^2 + 4x)} \\
 & \qquad -2x^2 - 3x - 5 \\
 & \qquad \underline{-(-2x^2 + 0x - 4)} \\
 & \qquad\qquad -3x - 1
\end{array}$$

To begin, divide the leading term in the dividend by the leading term in the divisor.

$$\frac{3x^4}{x^2} = 3x^2$$

Multiply the divisor by $3x^2$ and subtract the result.

Bring down the next term from the dividend and repeat the process.

The process is complete when the remainder is either 0 or has degree less than the degree of the divisor.

The result is $3x^2 + 2x - 2 + \dfrac{-3x - 1}{x^2 + 2}$.

Answer

1. $2x^2 + 5x + 1 + \dfrac{8}{2x - 5}$

Check by using the division algorithm.

$$3x^4 + 2x^3 + 4x^2 + x - 5 \stackrel{?}{=} (x^2 + 2)(3x^2 + 2x - 2) + (-3x - 1)$$
$$\stackrel{?}{=} 3x^4 + 2x^3 - 2x^2 + 6x^2 + 4x - 4 + (-3x - 1)$$
$$\stackrel{?}{=} 3x^4 + 2x^3 + 4x^2 + x - 5 \checkmark$$

Skill Practice 2 Use long division to divide.

$(1 - 7x + 5x^2 - 3x^3 + 2x^4) \div (x^2 + 3)$

EXAMPLE 5 Dividing Polynomials Using Synthetic Division

Use synthetic division to divide. $(-2x + 4x^3 + 18 + x^4) \div (x + 2)$

Solution:

> **TIP** Given a divisor of the form $(x - c)$, we can determine the value of c by setting the divisor equal to zero and solving for x. In Example 5, we have $x + 2 = 0$, which implies that $x = -2$. The value of c is -2.

Write the dividend and divisor in descending order and insert place holders for missing powers of x. $(x^4 + 4x^3 + 0x^2 - 2x + 18) \div (x + 2)$

To use synthetic division, the divisor must be of the form $x - c$. In this case, we have $x + 2 = x - (-2)$. Therefore, $c = -2$.

Value of c

Coefficients from the dividend

$$\begin{array}{r|rrrrr} -2 & 1 & 4 & 0 & -2 & 18 \\ & & -2 & -4 & 8 & -12 \\ \hline & \underbrace{1 \quad 2}_{} & & -4 & 6 & \boxed{6} \end{array}$$

Draw a horizontal line. Bring down the first coefficient.

Remainder

Coefficients of quotient

The dividend is a fourth-degree polynomial and the divisor is a first-degree polynomial. Therefore, the quotient is a third-degree polynomial. The coefficients of the quotient are found below the line: 1, 2, -4, 6. The quotient is $x^3 + 2x^2 - 4x + 6$, and the remainder is 6.

$$\frac{x^4 + 4x^3 - 2x + 18}{x + 2} = x^3 + 2x^2 - 4x + 6 + \frac{6}{x + 2}$$

TIP Polynomials with complex coefficients include polynomials with real coefficients and with imaginary coefficients. The following are complex polynomials.

$f(x) = (2 + 3i)x^2 + 4i$

$g(x) = \sqrt{2}x^2 + 3x + 4i$

$h(x) = 2x^2 + 3x + 4$

The division algorithm and remainder theorem can be extended over the set of complex numbers. The definition of a polynomial was given in Section R.4.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers. We now extend our discussion to **complex polynomials.** These are polynomials with complex coefficients.

We will also evaluate polynomials over the set of complex numbers rather than restricting x to the set of real numbers. A complex number $a + bi$ is a zero of a polynomial $f(x)$ if $f(a + bi) = 0$. For example, given $f(x) = x - (5 + 2i)$, we see that the imaginary number $5 + 2i$ is a zero of $f(x)$.

EXAMPLE 7 Using the Remainder Theorem to Identify Zeros of a Polynomial

Use the remainder theorem to determine if the given number c is a zero of the polynomial.

a. $f(x) = 2x^3 - 4x^2 - 13x - 9$; $c = 4$

b. $f(x) = x^3 + x^2 - 3x - 3$; $c = \sqrt{3}$

c. $f(x) = x^3 + x + 10$; $c = 1 + 2i$

Solution:

In each case, divide $f(x)$ by $x - c$ to determine the remainder. If the remainder is 0, then the value c is a zero of the polynomial.

a. Divide $f(x) = 2x^3 - 4x^2 - 13x - 9$ by $x - 4$.

$$\begin{array}{r|rrrr} 4 & 2 & -4 & -13 & -9 \\ & & 8 & 16 & 12 \\ \hline & 2 & 4 & 3 & \boxed{3} \end{array}$$

By the remainder theorem, $f(4) = 3$. Since $f(4) \neq 0$, 4 is not a zero of $f(x)$.

b. Divide $f(x) = x^3 + x^2 - 3x - 3$ by $x - \sqrt{3}$.

$$\begin{array}{r|rrrr} \sqrt{3} & 1 & 1 & -3 & -3 \\ & & \sqrt{3} & 3 + \sqrt{3} & 3 \\ \hline & 1 & 1 + \sqrt{3} & \sqrt{3} & \boxed{0} \end{array}$$

By the remainder theorem, $f(\sqrt{3}) = 0$. Therefore, $\sqrt{3}$ is a zero of $f(x)$.

c. Divide $f(x) = x^3 + x + 10$ by $x - (1 + 2i)$

$$\begin{array}{r|rrrr} 1 + 2i & 1 & 0 & 1 & 10 \\ & & 1 + 2i & -3 + 4i & -10 \\ \hline & 1 & 1 + 2i & -2 + 4i & \boxed{0} \end{array}$$

Note that $(1 + 2i)(1 + 2i)$
$= 1 + 2i + 2i + 4i^2$
$= 1 + 4i + 4(-1)$
Recall that $i^2 = -1$.
$= -3 + 4i$

Note that $(1 + 2i)(-2 + 4i)$
$= -2 + 4i - 4i + 8i^2$
$= -2 - 8$
$= -10$

By the remainder theorem, $f(1 + 2i) = 0$.
Therefore, $1 + 2i$ is a zero of $f(x)$.

Skill Practice 7 Use the remainder theorem to determine if the given number, c, is a zero of the function.

a. $f(x) = 2x^4 - 3x^2 + 5x - 11$; $c = 2$

b. $f(x) = 2x^3 + 5x^2 - 14x - 35$; $c = \sqrt{7}$

c. $f(x) = x^3 - 7x^2 + 16x - 10$; $c = 3 + i$

Suppose that we again apply the division algorithm to a dividend of $f(x)$ and a divisor of $x - c$, where c is a complex number.

$$f(x) = (x - c) \cdot q(x) + r$$
$$= (x - c) \cdot q(x) + f(c)$$

By the remainder theorem, $r = f(c)$.

If $f(c) = 0$, then $f(x) = (x - c) \cdot q(x)$

This tells us that if $f(c)$ is a zero of $f(x)$, then $(x - c)$ is a factor of $f(x)$.

Now suppose that $x - c$ is a factor of $f(x)$. Then for some polynomial $q(x)$,

$$f(x) = (x - c) \cdot q(x)$$
$$f(c) = (c - c) \cdot q(x)$$
$$= 0$$

This tells us that if $(x - c)$ is a factor of $f(x)$, then c is a zero of $f(x)$.

These results can be summarized in the factor theorem.

Factor Theorem

Let $f(x)$ be a polynomial.

1. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$.
2. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$.

Answers

7. a. No b. Yes c. Yes

EXAMPLE 8 Identifying Factors of a Polynomial

Use the factor theorem to determine if the given polynomials are factors of $f(x) = x^4 - x^3 - 11x^2 + 11x + 12$.

a. $x - 3$ **b.** $x + 2$

Solution:

a. If $f(3) = 0$, then $x - 3$ is a factor of $f(x)$. Using synthetic division we have:

$$\begin{array}{r|rrrrr} 3 & 1 & -1 & -11 & 11 & 12 \\ & & 3 & 6 & -15 & -12 \\ \hline & 1 & 2 & -5 & -4 & \boxed{0} \end{array}$$

$f(3) = 0$

By the factor theorem, since $f(3) = 0$, $x - 3$ is a factor of $f(x)$.

b. If $f(-2) = 0$, then $x + 2$ is a factor of $f(x)$. Using synthetic division we have:

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -11 & 11 & 12 \\ & & -2 & 6 & 10 & -42 \\ \hline & 1 & -3 & -5 & 21 & \boxed{-30} \end{array}$$

By the factor theorem, since $f(-2) \neq 0$, $x + 2$ is not a factor of $f(x)$.

$f(-2) = -30$

Skill Practice 8 Use the factor theorem to determine if the given polynomials are factors of $f(x) = 2x^4 - 13x^3 + 10x^2 - 25x + 6$.

a. $x - 6$ **b.** $x + 3$

In Example 9, we illustrate the relationship between the zeros of a polynomial and the solutions (roots) of a polynomial equation.

EXAMPLE 9 Factoring a Polynomial Given a Known Zero

a. Factor $f(x) = 3x^3 + 25x^2 + 42x - 40$, given that -5 is a zero of $f(x)$.

b. Solve the equation. $3x^3 + 25x^2 + 42x - 40 = 0$

Solution:

a. The value -5 is a zero of $f(x)$, which means that $f(-5) = 0$. By the factor theorem, $x - (-5)$ or equivalently $x + 5$ is a factor of $f(x)$. Using synthetic division, we have

$$\begin{array}{r|rrrr} -5 & 3 & 25 & 42 & -40 \\ & & -15 & -50 & 40 \\ \hline & 3 & 10 & -8 & \boxed{0} \end{array}$$

This means that $3x^3 + 25x^2 + 42x - 40 = \underbrace{(x + 5)}_{\text{divisor}}\underbrace{(3x^2 + 10x - 8)}_{\text{quotient; factors as } (3x-2)(x+4)} + \underbrace{0}_{\text{remainder}}$

Therefore, $f(x) = (x + 5)(3x - 2)(x + 4)$.

b. $3x^3 + 25x^2 + 42x - 40 = 0$ To solve the equation, set one side equal to zero.

$(x + 5)(3x - 2)(x + 4) = 0$ Factor the left side.

$x = -5, x = \frac{2}{3}, x = -4$ Set each factor equal to zero and solve for x.

The solution set is $\left\{-5, \frac{2}{3}, -4\right\}$.

Skill Practice 9

a. Factor $f(x) = 2x^3 + 7x^2 - 14x - 40$, given that -4 is a zero of f.

b. Solve the equation. $2x^3 + 7x^2 - 14x - 40 = 0$

Answers

8. a. Yes **b.** No

9. a. $f(x) = (x + 4)(x + 2)(2x - 5)$

b. $\left\{-4, -2, \frac{5}{2}\right\}$

EXAMPLE 5 Finding Zeros and Factoring a Polynomial

Given $f(x) = x^4 - 6x^3 + 28x^2 - 18x + 75$, and that $3 - 4i$ is a zero of $f(x)$,

a. Find the remaining zeros.

b. Factor $f(x)$ as a product of linear factors.

c. Solve the equation. $x^4 - 6x^3 + 28x^2 - 18x + 75 = 0$

Solution:

$f(x)$ is a fourth-degree polynomial, so we expect to find four zeros (including multiplicities). Further note that because $f(x)$ has real coefficients and because $3 - 4i$ is a zero, then the conjugate $3 + 4i$ must also be a zero. This leaves only two remaining zeros to find.

$$\begin{array}{r|rrrrr} 3-4i & 1 & -6 & 28 & -18 & 75 \\ & & 3-4i & -25 & 9-12i & -75 \\ \hline & \underbrace{1 \quad -3-4i \quad 3 \quad -9-12i}_{\text{coefficients of the quotient}} & & & & \boxed{0} \end{array}$$

Divide by $[x - (3 - 4i)]$.

One strategy is to use synthetic division twice using the two known zeros.

Note: $(3 - 4i)(-3 - 4i)$
$= -9 - 12i + 12i + 16i^2$
$= -25$

Note: $(3 - 4i)(-9 - 12i)$
$= -27 - 36i + 36i + 48i^2$
$= -75$

Since $3 + 4i$ is a zero of $f(x)$ it must also be a zero of the quotient.

$$\begin{array}{r|rrrr} 3+4i & 1 & -3-4i & 3 & -9-12i \\ & & 3+4i & 0 & 9+12i \\ \hline & 1 & 0 & 3 & \boxed{0} \end{array}$$

Divide by $[x - (3 + 4i)]$.

Divide the quotient by $[x - (3 + 4i)]$.

The resulting quotient is quadratic: $x^2 + 3$.

Now we have $f(x) = [x - (3 - 4i)][x - (3 + 4i)](x^2 + 3)$.
The remaining two zeros are found by solving $x^2 + 3 = 0$.

$$x^2 + 3 = 0$$
$$x^2 = -3$$
$$x = \pm i\sqrt{3}$$

a. The zeros of $f(x)$ are: $3 - 4i$, $3 + 4i$, $i\sqrt{3}$, and $-i\sqrt{3}$.

b. $f(x)$ factors as four linear factors.

$$f(x) = [x - (3 - 4i)][x - (3 + 4i)](x - i\sqrt{3})(x + i\sqrt{3})$$

c. The solution set for $x^4 - 6x^3 + 28x^2 - 18x + 75 = 0$ is $\{3 \pm 4i, \pm i\sqrt{3}\}$.

EXAMPLE 8 Graphing a Rational Function

Graph $f(x) = \dfrac{4x}{x^2 - 4}$.

Solution:

1. Determine the *y*-intercept.

$$f(0) = \frac{4(0)}{(0)^2 - 4} = 0$$ The *y*-intercept is (0, 0).

2. Determine the *x*-intercept(s).

$$\frac{4x}{x^2 - 4} = 0 \quad \text{for } x = 0.$$ The *x*-intercept is (0, 0).

3. Identify the vertical asymptotes.

The zeros of $x^2 - 4$ are 2 and -2. Vertical asymptotes: $x = 2$ and $x = -2$.

4. Determine whether *f* has a horizontal or slant asymptote.

The degree of the numerator is less than the degree of the denominator. The horizontal asymptote is $y = 0$.

5. Determine where f crosses its horizontal asymptote.

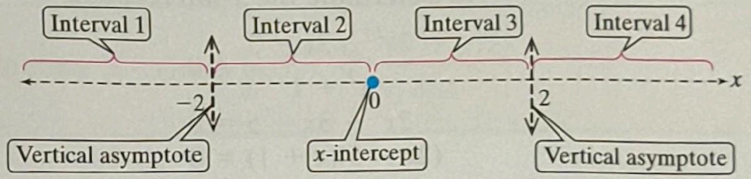
Set $f(x) = 0$. We have $\dfrac{4x}{x^2 - 4} = 0$ for $x = 0$.

Therefore, f crosses its horizontal asymptote at $(0, 0)$.

6. Test for symmetry.

f is an odd function because $f(-x) = \dfrac{4(-x)}{(-x)^2 - 4} = -\dfrac{4x}{x^2 - 4} = -f(x)$.

7. Determine the behavior of f on each interval.



TIP The graph of f is symmetric to the origin because $f(-x) = -f(x)$. Therefore, if $\left(-3, -\frac{12}{5}\right)$ and $\left(-1, \frac{4}{3}\right)$ are points on the graph of f, then $\left(3, \frac{12}{5}\right)$ and $\left(1, -\frac{4}{3}\right)$ are also points on the graph.

Interval	Test Point	Comments
$(-\infty, -2)$	$\left(-3, -\dfrac{12}{5}\right)$	• Since $f(x)$ is negative on this interval, $f(x)$ must approach the horizontal asymptote $y = 0$ from below as $x \to -\infty$. • Since $f(x)$ is negative on this interval, as x approaches the vertical asymptote $x = -2$ from the left, $f(x) \to -\infty$.
$(-2, 0)$	$\left(-1, \dfrac{4}{3}\right)$	• Since $f(x)$ is positive on this interval, as x approaches the vertical asymptote $x = -2$ from the right, $f(x) \to \infty$.
$(0, 2)$	$\left(1, -\dfrac{4}{3}\right)$	• Since $f(x)$ is negative on this interval, as x approaches the vertical asymptote $x = 2$ from the left, $f(x) \to -\infty$.
$(2, \infty)$	$\left(3, \dfrac{12}{5}\right)$	• Since $f(x)$ is positive on this interval, $f(x)$ must approach the horizontal asymptote from above as $x \to \infty$. • Since $f(x)$ is positive on this interval, as x approaches the vertical asymptote $x = 2$ from the right, $f(x) \to \infty$.

8. Sketch the function.

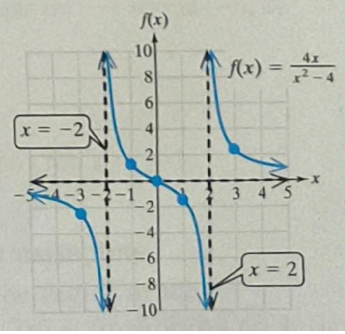
Plot the x- and y-intercept $(0, 0)$.

Graph the asymptotes as dashed lines.

Plot the points.

$\left(-3, -\dfrac{12}{5}\right), \left(-1, \dfrac{4}{3}\right), \left(1, -\dfrac{4}{3}\right)$, and $\left(3, \dfrac{12}{5}\right)$

Sketch the curve.



3. Find the Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the inverse is a function $y = f^{-1}(x)$ that performs the inverse operations in the reverse order. The function given by $f(x) = 100 + 12x$ multiplies x by 12 first, and then adds 100 to the result. Therefore, the inverse function must *subtract* 100 from x first and then *divide* by 12.

$$f^{-1}(x) = \frac{x - 100}{12}$$

To facilitate the process of finding an equation of the inverse of a one-to-one function, we offer the following steps.

Procedure to Find an Equation of an Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows.

Step 1 Replace $f(x)$ by y.
Step 2 Interchange x and y.
Step 3 Solve for y.
Step 4 Replace y by $f^{-1}(x)$.

EXAMPLE 5 **Finding an Equation of an Inverse Function**

Write an equation for the inverse function for $f(x) = 3x - 1$.

Solution:

Function f is a linear function, and its graph is a nonvertical line. Therefore, f is a one-to-one function.

$f(x) = 3x - 1$	
$y = 3x - 1$	**Step 1:** Replace $f(x)$ by y.
$x = 3y - 1$	**Step 2:** Interchange x and y.
$x + 1 = 3y$	**Step 3:** Solve for y. Add 1 to both sides and divide by 3.
$\dfrac{x+1}{3} = y$	
$f^{-1}(x) = \dfrac{x+1}{3}$	**Step 4:** Replace y by $f^{-1}(x)$.

To check the result, verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

$$(f \circ f^{-1})(x) = 3\left(\frac{x+1}{3}\right) - 1 = x \checkmark \quad \text{and} \quad (f^{-1} \circ f)(x) = \frac{(3x-1)+1}{3} = x \checkmark$$

EXAMPLE 1 Graphing Exponential Functions

Graph the functions.

a. $f(x) = 2^x$ **b.** $g(x) = \left(\frac{1}{2}\right)^x$

Solution:

Table 4-4 shows several function values $f(x)$ and $g(x)$ for both positive and negative values of x.

Table 4-4

x	$f(x) = 2^x$	$g(x) = \left(\frac{1}{2}\right)^x$
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$

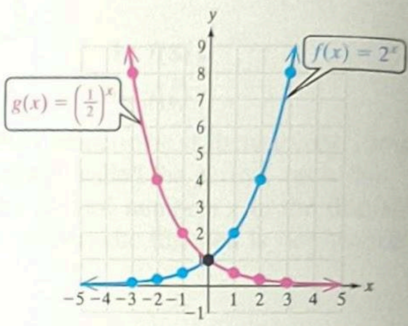


Figure 4-10

TIP The values of $f(x)$ become closer and closer to 0 as $x \to -\infty$. This means that the x-axis is a horizontal asymptote.

Likewise, the values of $g(x)$ become closer to 0 as $x \to \infty$. The x-axis is a horizontal asymptote.

Notice that $g(x) = \left(\frac{1}{2}\right)^x$ is equivalent to $g(x) = 2^{-x}$. Therefore, the graph of $g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$ is the same as the graph of $f(x) = 2^x$ with a reflection across the y-axis (Figure 4-10).

Skill Practice 1 Graph the functions.

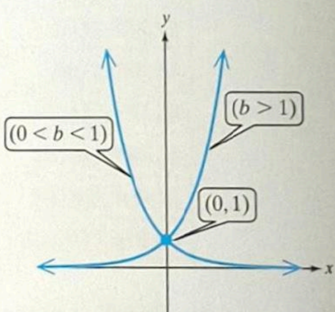
a. $f(x) = 5^x$ **b.** $g(x) = \left(\dfrac{1}{5}\right)^x$

The graphs in Figure 4-10 illustrate several important features of exponential functions.

Graphs of $f(x) = b^x$

The graph of an exponential function defined by $f(x) = b^x$ ($b > 0$ and $b \neq 1$) has the following properties.

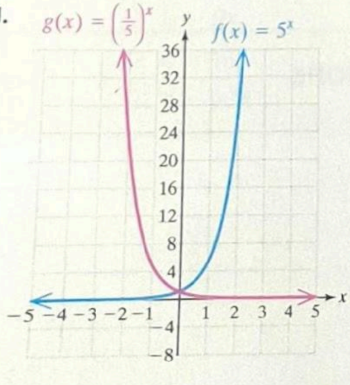
1. If $b > 1$, f is an *increasing* exponential function, sometimes called an **exponential growth function.**
   If $0 < b < 1$, f is a *decreasing* exponential function, sometimes called an **exponential decay function.**
2. The domain is the set of all real numbers, $(-\infty, \infty)$.
3. The range is $(0, \infty)$.
4. The line $y = 0$ (x-axis) is a horizontal asymptote.
5. The function passes through the point $(0, 1)$ because $f(0) = b^0 = 1$.



These properties indicate that the graph of an exponential function is an increasing function if the base is greater than 1. Furthermore, the base affects the rate of increase. Consider the graphs of $f(x) = 2^x$ and $k(x) = 5^x$ (Figure 4-11). For every positive 1-unit change in x, $f(x) = 2^x$ is 2 times as great and $k(x) = 5^x$ is 5 times as great (Table 4-5).

Answer

1.



Definition of a Logarithmic Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function base *b*,** where

$$y = \log_b x \text{ is equivalent to } b^y = x$$

Notes:

- Given $y = \log_b x$, the value y is the exponent to which b must be raised to obtain x.
- The value of y is called the **logarithm,** b is called the **base,** and x is called the **argument.**
- The equations $y = \log_b x$ and $b^y = x$ both define the same relationship between x and y. The expression $y = \log_b x$ is called the **logarithmic form,** and $b^y = x$ is called the **exponential form.**

The logarithmic function base b is defined as the inverse of the exponential function base b.

exponential function	$f(x) = b^x$	First replace $f(x)$ by y.
	$y = b^x$	Next, interchange x and y.
inverse of exponential function	$x = b^y$	This equation provides an implicit relationship between x and y. To solve for y explicitly (that is, to isolate y), we must use logarithmic notation.
logarithmic function	$y = \log_b x$	

To be able to solve equations involving logarithms, it is often advantageous to write a logarithmic expression in its exponential form.

EXAMPLE 1 Writing Logarithmic Form and Exponential Form

Write each equation in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_{10}\left(\dfrac{1}{100}\right) = -2$ **c.** $\log_7 1 = 0$

Solution:

Logarithmic form $y = \log_b x$		**Exponential form $b^y = x$**
a. $\log_2 16 = 4$	$\Leftrightarrow$	$2^4 = 16$
b. $\log_{10}\left(\dfrac{1}{100}\right) = -2$	$\Leftrightarrow$	$10^{-2} = \dfrac{1}{100}$
c. $\log_7 1 = 0$	$\Leftrightarrow$	$7^0 = 1$

The logarithm is the exponent to which the base is raised to obtain x.

Skill Practice 1 Write each equation in exponential form.

a. $\log_3 9 = 2$ **b.** $\log_{10}\left(\dfrac{1}{1000}\right) = -3$ **c.** $\log_6 1 = 0$

EXAMPLE 2 Writing Exponential Form and Logarithmic Form

Write each equation in logarithmic form.

a. $3^4 = 81$ **b.** $10^6 = 1{,}000{,}000$ **c.** $\left(\frac{1}{5}\right)^{-1} = 5$

Solution:

Exponential form $b^y = x$ **Logarithmic form** $\log_b x = y$

logarithm

a. $3^4 = 81 \Leftrightarrow \log_3 81 = 4$

base argument

logarithm (power)

base argument

b. $10^6 = 1{,}000{,}000 \Leftrightarrow \log_{10} 1{,}000{,}000 = 6$

c. $\left(\frac{1}{5}\right)^{-1} = 5 \Leftrightarrow \log_{1/5} 5 = -1$

EXAMPLE 7 Graphing Logarithmic Functions

Graph the functions.

a. $y = \log_2 x$ **b.** $y = \log_{1/4} x$

Solution:

To find points on a logarithmic function, we can interchange the x- and y-coordinates of the ordered pairs on the corresponding exponential function.

a. To graph $y = \log_2 x$, interchange the x- and y-coordinates of the ordered pairs from its inverse function $y = 2^x$. The graph of $y = \log_2 x$ is shown in Figure 4-15.

Exponential Function

x	$y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Logarithmic Function

x	$y = \log_2 x$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Switch x and y.

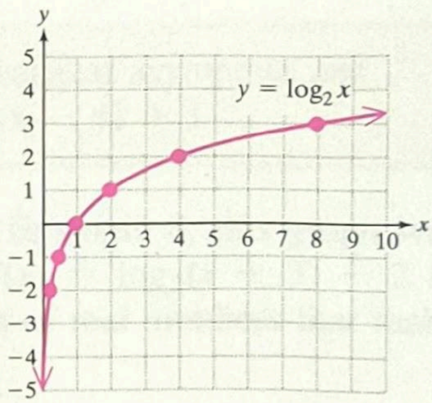


Figure 4-15

b. To graph $y = \log_{1/4} x$, interchange the x- and y-coordinates of the ordered pairs from its inverse function $y = \left(\frac{1}{4}\right)^x$. See Figure 4-16.

Exponential Function

x	$y = \left(\frac{1}{4}\right)^x$
-3	64
-2	16
-1	4
0	1
1	$\frac{1}{4}$
2	$\frac{1}{16}$
3	$\frac{1}{64}$

Logarithmic Function

x	$y = \log_{1/4} x$
64	-3
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2
$\frac{1}{64}$	3

Switch x and y.

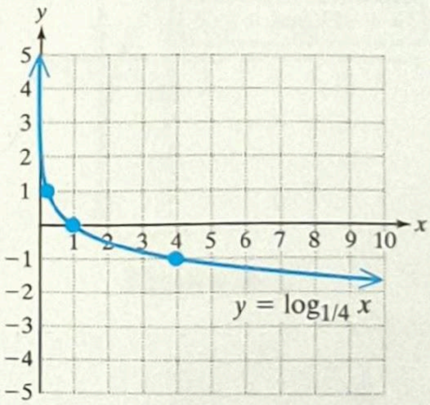


Figure 4-16

EXAMPLE 4 Writing a Logarithmic Expression in Expanded Form

Write the expression as the sum or difference of logarithms.

a. $\log_2\left(\frac{z^3}{xy^5}\right)$ **b.** $\log\sqrt[3]{\frac{(x+y)^2}{10}}$

Solution:

a. $\log_2\left(\frac{z^3}{xy^5}\right) = \log_2 z^3 - \log_2(xy^5)$ Apply the quotient property.

$= \log_2 z^3 - (\log_2 x + \log_2 y^5)$ Apply the product property.

$= \log_2 z^3 - \log_2 x - \log_2 y^5$ Apply the distributive property.

$= 3\log_2 z - \log_2 x - 5\log_2 y$ Apply the power property.

b. $\log\sqrt[3]{\frac{(x+y)^2}{10}} = \log\left[\frac{(x+y)^2}{10}\right]^{1/3}$ Write the radical expression with rational exponents.

$= \frac{1}{3}\log\left[\frac{(x+y)^2}{10}\right]$ Apply the power property.

$= \frac{1}{3}[\log(x+y)^2 - \log 10]$ Apply the quotient property.

$= \frac{1}{3}[2\log(x+y) - 1]$ Apply the power property and simplify: $\log 10 = 1$.

$= \frac{2}{3}\log(x+y) - \frac{1}{3}$ Apply the distributive property.

Avoiding Mistakes

In Example 4(b) do not try to simplify $\log(x+y)$. The argument contains a sum, not a product.

$\underbrace{\log(x+y)}_{\text{sum}}$ cannot be simplified.

Compare to the logarithm of a product which can be simplified.

$\underbrace{\log(xy)}_{\text{product}} = \log x + \log y$

Skill Practice 4 Write the expression as the sum or difference of logarithms.

a. $\ln\left(\frac{a^4 b}{c^9}\right)$ **b.** $\log_5\sqrt[3]{\frac{25}{(a^2+b)^2}}$

3. Write a Logarithmic Expression as a Single Logarithm

In Examples 5 and 6, we demonstrate how to write a sum or difference of logarithms as a single logarithm. We apply Properties 5, 6, and 7 of logarithms in reverse.

EXAMPLE 5 Writing the Sum or Difference of Logarithms as a Single Logarithm

Write the expression as a single logarithm and simplify the result if possible.

$$\log_2 560 - \log_2 7 - \log_2 5$$

Solution:

$\log_2 560 - \log_2 7 - \log_2 5$

$= \log_2 560 - (\log_2 7 + \log_2 5)$ Factor out -1 from the last two terms.

$= \log_2 560 - \log_2(7 \cdot 5)$ Apply the product property.

$= \log_2\left(\frac{560}{7 \cdot 5}\right)$ Apply the quotient property.

$= \log_2 16$ Simplify within the argument.

$= 4$ Simplify. $\log_2 16 = \log_2 2^4 = 4$

Answers

4. a. $4\ln a + \ln b - 9\ln c$

b. $\frac{2}{3} - \frac{2}{3}\log_5(a^2+b)$

This equation is called an **exponential equation** because the equation contains a variable in the exponent. To solve an exponential equation first note that all exponential functions are one-to-one. Therefore, $b^x = b^y$ implies that $x = y$. This is called the equivalence property of exponential expressions.

TIP The equivalence property tells us that if two exponential expressions with the same base are equal, then their exponents must be equal.

Equivalence Property of Exponential Expressions

If b, x, and y are real numbers with $b > 0$ and $b \neq 1$, then

$$b^x = b^y \quad \text{implies that } x = y.$$

EXAMPLE 1 Solving Exponential Equations Using the Equivalence Property

Solve. **a.** $3^{2x-6} = 81$ **b.** $25^{4-t} = \left(\frac{1}{5}\right)^{3t+1}$

Solution:

a.

$3^{2x-6} = 81$

$3^{2x-6} = 3^4$ Write 81 as an exponential expression with a base of 3.

$2x - 6 = 4$ Equate the exponents.

$x = 5$

Check: $3^{2x-6} = 81$

$3^{2(5)-6} \stackrel{?}{=} 81$

$3^4 \stackrel{?}{=} 81$ ✓

The solution set is $\{5\}$.

b.

$25^{4-t} = \left(\frac{1}{5}\right)^{3t+1}$

$(5^2)^{4-t} = (5^{-1})^{3t+1}$ Express both 25 and $\frac{1}{5}$ as integer powers of 5.

$5^{2(4-t)} = 5^{-1(3t+1)}$ Apply the power property of exponents: $(b^m)^n = b^{m \cdot n}$.

$5^{8-2t} = 5^{-3t-1}$ Apply the distributive property within the exponents.

$8 - 2t = -3t - 1$ Equate the exponents.

$t = -9$ The solution checks in the original equation.

The solution set is $\{-9\}$.

Avoiding Mistakes

When writing the expression $(5^2)^{4-t}$ as $5^{2(4-t)}$, it is important to use parentheses around the quantity $(4 - t)$. The exponent of 2 must be multiplied by the entire quantity $(4 - t)$. Likewise, parentheses are used around $(3t + 1)$ in the expression $5^{-1(3t+1)}$.

Skill Practice 1 Solve. **a.** $4^{2x-3} = 64$ **b.** $27^{2w+5} = \left(\frac{1}{3}\right)^{2-5w}$

In Example 4, we have an equation with two exponential expressions involving different bases.

EXAMPLE 4 Solving an Exponential Equation

Solve. $4^{2x-7} = 5^{3x+1}$

Solution:

$4^{2x-7} = 5^{3x+1}$	
$\ln 4^{2x-7} = \ln 5^{3x+1}$	Take a logarithm of the same base on both sides.
$(2x - 7)\ln 4 = (3x + 1)\ln 5$	Apply the power property of logarithms.
$2x \ln 4 - 7 \ln 4 = 3x \ln 5 + \ln 5$	Apply the distributive property.
$2x \ln 4 - 3x \ln 5 = \ln 5 + 7 \ln 4$	Collect x terms on one side of the equation.
$x(2 \ln 4 - 3 \ln 5) = \ln 5 + 7 \ln 4$	Factor out x on the left.
$x = \dfrac{\ln 5 + 7 \ln 4}{2 \ln 4 - 3 \ln 5} \approx -5.5034$	Divide by $(2 \ln 4 - 3 \ln 5)$.

The solution set is $\left\{\dfrac{\ln 5 + 7 \ln 4}{2 \ln 4 - 3 \ln 5}\right\}$. The solution checks in the original equation.

Steps to Solve Exponential Equations by Using Logarithms

1. Isolate the exponential expression on one side of the equation.
2. Take a logarithm of the same base on both sides of the equation.
3. Use the power property of logarithms to "bring down" the exponent.
4. Solve the resulting equation.

EXAMPLE 2 Solving an Exponential Equation Using Logarithms

Solve. $7^x = 60$

Solution:

$7^x = 60$ — The exponential expression 7^x is isolated.

$\log 7^x = \log 60$ — Take a logarithm of the same base on both sides of the equation. In this case, we have chosen base 10.

$x \log 7 = \log 60$ — Apply the power property of logarithms.

This equation is now linear. $x = \dfrac{\log 60}{\log 7} \approx 2.1041$ — Divide both sides by log 7.

It is important to note that the exact solution to this equation is $\dfrac{\log 60}{\log 7}$ or equivalently by the change-of-base formula, $\log_7 60$. The value 2.1041 is merely an approximation.

The solution set is $\left\{\dfrac{\log 60}{\log 7}\right\}$ or $\{\log_7 60\}$.

Avoiding Mistakes

While 2.1041 is only an approximation, it is useful to check the result.

$7^{2.1041} \approx 60$

Skill Practice 2 Solve. $5^x = 83$

To solve the equation from Example 2, we can take a logarithm of any base. For example:

Take the logarithm base 7 on both sides.

$$7^x = 60$$
$$\log_7 7^x = \log_7 60$$
$$x = \log_7 60 \text{ (solution)}$$

Take the natural logarithm on both sides.

$$7^x = 60$$
$$\ln 7^x = \ln 60$$
$$x \ln 7 = \ln 60$$
$$x = \frac{\ln 60}{\ln 7} \text{ (solution)}$$

The values $\log_7 60$, $\dfrac{\log 60}{\log 7}$, and $\dfrac{\ln 60}{\ln 7}$ are all equivalent. However, common logarithms and natural logarithms are often used to express the solution to an exponential equation so that the solution can be approximated on a calculator.

2. Solve Logarithmic Equations

An equation containing a variable within a logarithmic expression is called a **logarithmic equation.** For example:

$\log_2(3x - 4) = \log_2(x + 2)$ and $\ln(x + 4) = 7$ are logarithmic equations.

Given an equation in which two logarithms of the same base are equated, we can apply the equivalence property of logarithms. Since all logarithmic functions are one-to-one, $\log_b x = \log_b y$ implies that $x = y$.

TIP The equivalence property tells us that if two logarithmic expressions with the same base are equal, then their arguments must be equal.

Equivalence Property of Logarithmic Expressions

If b, x, and y are positive real numbers with $b \neq 1$, then

$$\log_b x = \log_b y \quad \text{implies that } x = y.$$

EXAMPLE 6 Solving a Logarithmic Equation Using the Equivalence Property

Solve. $\log_2(3x - 4) = \log_2(x + 2)$

Solution:

$\log_2(3x - 4) = \log_2(x + 2)$ Two logarithms of the same base are equated.

$3x - 4 = x + 2$ Equate the arguments.

$2x = 6$ Solve for x.

$x = 3$ Because the domain of a logarithmic function is restricted, it is mandatory that we check all potential solutions to a logarithmic equation.

Check: $\log_2(3x - 4) = \log_2(x + 2)$

$\log_2[3(3) - 4] \stackrel{?}{=} \log_2[(3) + 2]$

$\log_2 5 \stackrel{?}{=} \log_2 5$ ✓

The solution set is $\{3\}$.

In Example 7, we encounter a logarithmic equation in which one or more solutions does not check.

EXAMPLE 7 Solving a Logarithmic Equation

Solve. $\ln(x-4) = \ln(x+6) - \ln x$

Solution:

$$\ln(x-4) = \ln(x+6) - \ln x$$

$$\ln(x-4) = \ln\left(\frac{x+6}{x}\right)$$ Combine the two logarithmic terms on the right.

$$x - 4 = \frac{x+6}{x}$$ Apply the equivalence property of logarithms.

$$x^2 - 4x = x + 6$$ Clear fractions by multiplying both sides by x.

$$x^2 - 5x - 6 = 0$$ The resulting equation is quadratic.

$$(x-6)(x+1) = 0$$

$$x = 6 \quad \text{or} \quad x = -1$$ The potential solutions are 6 and -1.

<u>Check</u>:

$$\ln(x-4) = \ln(x+6) - \ln x \qquad \ln(x-4) = \ln(x+6) - \ln x$$

$$\ln(6-4) \stackrel{?}{=} \ln(6+6) - \ln 6 \qquad \ln(-1-4) \stackrel{?}{=} \ln(-1+6) - \ln(-1)$$

$$\ln 2 \stackrel{?}{=} \ln 12 - \ln 6 \qquad \ln(-5) \stackrel{?}{=} \ln 5 - \ln(-1)$$

$$\ln 2 \stackrel{?}{=} \ln\left(\frac{12}{6}\right) \checkmark$$

undefined undefined

The only solution that checks is 6.

The solution set is $\{6\}$.

EXAMPLE 8 Solving a Logarithmic Equation

Solve. $4\log_3(2t - 7) = 8$

Solution:

$4\log_3(2t - 7) = 8$	
$\log_3(2t - 7) = 2$	Isolate the logarithm by dividing both sides by 4.
	The equation is in the form $\log_b x = k$, where $x = 2t - 7$.
$2t - 7 = 3^2$	Write the equation in exponential form.
$2t - 7 = 9$	Check: $4\log_3(2t - 7) = 8$
$t = 8$	$4\log_3[2(8) - 7] \stackrel{?}{=} 8$
	$4\log_3 9 \stackrel{?}{=} 8$
	$4 \cdot 2 \stackrel{?}{=} 8$ ✓

The solution set is $\{8\}$.

Skill Practice 8 Solve. $8\log_4(w + 6) = 24$

EXAMPLE 9 Solving a Logarithmic Equation

Solve. $\log(w + 47) = 2.6$

Solution:

$\log(w + 47) = 2.6$	The equation is in the form $\log_b x = k$ where $x = w + 47$ and $b = 10$.
$w + 47 = 10^{2.6}$	Write the equation in exponential form.
$w = 10^{2.6} - 47 \approx 351.1072$	Solve the resulting linear equation.
	Check: $\log(w + 47) = 2.6$
	$\log[(10^{2.6} - 47) + 47] \stackrel{?}{=} 2.6$
	$\log 10^{2.6} \stackrel{?}{=} 2.6$ ✓

The solution set is $\{10^{2.6} - 47\}$.

EXAMPLE 10 Solving a Logarithmic Equation

Solve. $\log_2 x = 3 - \log_2(x - 2)$

Solution:

$\log_2 x = 3 - \log_2(x - 2)$	
$\log_2 x + \log_2(x - 2) = 3$	Isolate the logarithms on one side of the equation.
$\log_2[x(x - 2)] = 3$	Use the product property of logarithms to write a single logarithm.
$x(x - 2) = 2^3$	Write the equation in exponential form.
$x^2 - 2x = 8$	
$x^2 - 2x - 8 = 0$	Set one side equal to zero.
$(x - 4)(x + 2) = 0$	

$x = 4 \quad x = -2$ <u>Check</u>:

$\log_2 x = 3 - \log_2(x - 2)$	$\log_2 x = 3 - \log_2(x - 2)$
$\log_2 4 \stackrel{?}{=} 3 - \log_2(4 - 2)$	$\log_2(-2) \stackrel{?}{=} 3 - \log_2(-2 - 2)$
$\log_2 4 \stackrel{?}{=} 3 - \log_2 2$	$\log_2(\underbrace{-2}_{\text{undefined}}) \stackrel{?}{=} 3 - \log_2(\underbrace{-4}_{\text{undefined}})$
$2 \stackrel{?}{=} 3 - 1$ ✓	

The only solution that checks is $x = 4$.

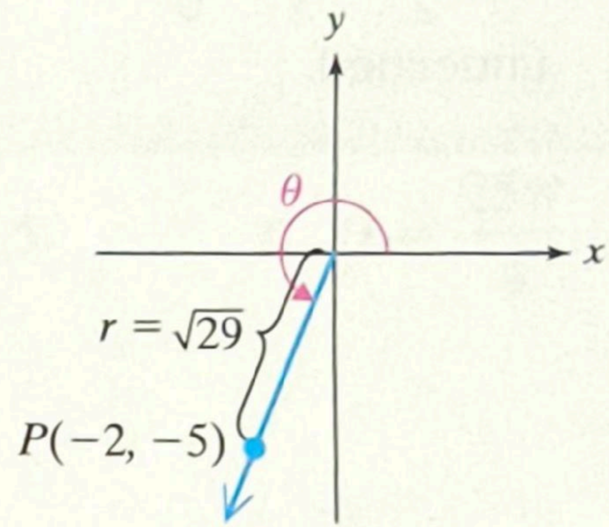
The solution set is $\{4\}$.

EXAMPLE 1 Evaluating Trigonometric Functions

Let $P(-2, -5)$ be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ.

Solution:

We first draw θ in standard position and label point P.
The distance between $P(-2, -5)$ and $(0, 0)$ is



$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}.$$

We have $x = -2$, $y = -5$, and $r = \sqrt{29}$.

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{29}} = \frac{-5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = -\frac{5\sqrt{29}}{29} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{-5} = -\frac{\sqrt{29}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{29}} = \frac{-2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = -\frac{2\sqrt{29}}{29} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{-2} = -\frac{\sqrt{29}}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-2} = \frac{5}{2} \qquad \cot\theta = \frac{x}{y} = \frac{-2}{-5} = \frac{2}{5}$$

2. Determine Reference Angles

For nonacute angles θ, we often find values of the six trigonometric functions by using the related *reference angle.*

Definition of a Reference Angle

Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 5-24 shows the reference angles θ' for angles on the interval $[0, 2\pi)$ drawn in standard position for each of the four quadrants.

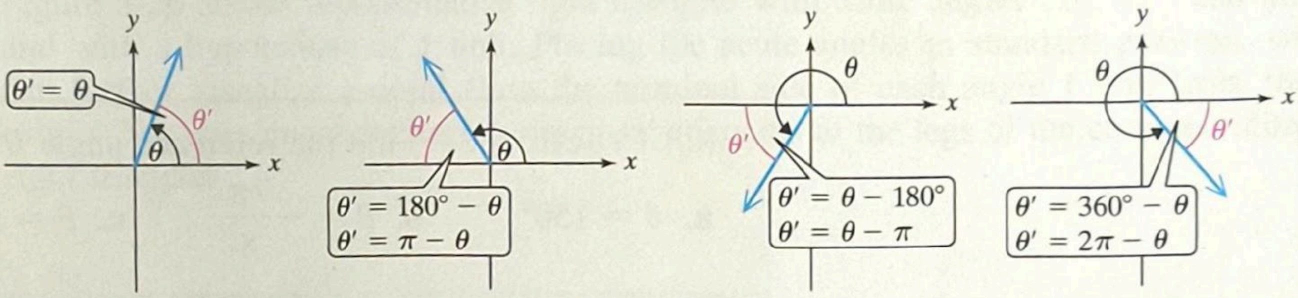


Figure 5-24

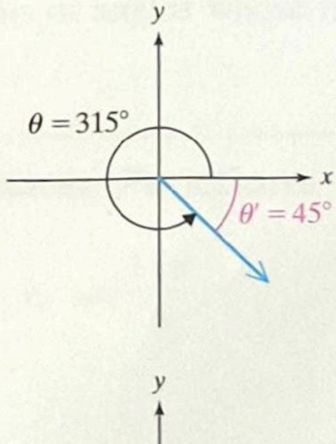
EXAMPLE 3 Finding Reference Angles

Find the reference angle θ'.

a. $\theta = 315°$ **b.** $\theta = -\dfrac{13\pi}{12}$ **c.** $\theta = 3.5$ **d.** $\theta = \dfrac{25\pi}{4}$

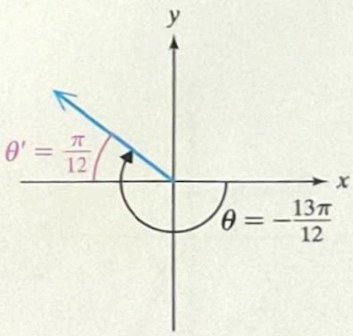
Solution:

a. $\theta = 315°$ is a fourth quadrant angle. The acute angle it makes with the x-axis is $\theta' = 360° - 315° = 45°$.

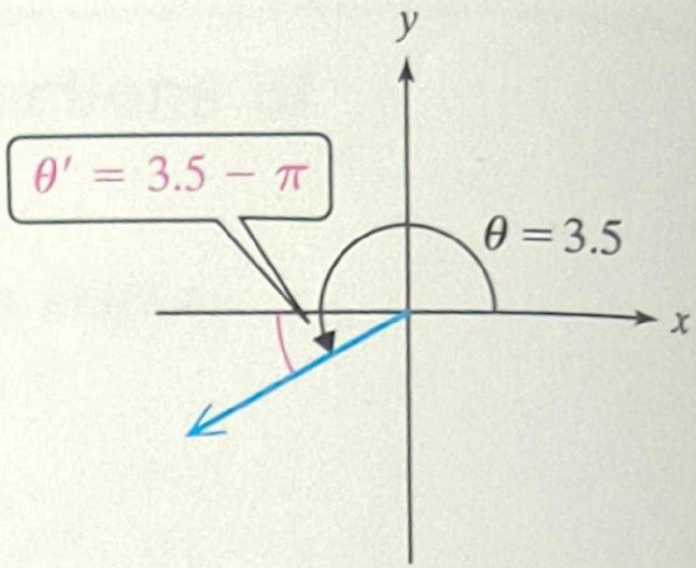


b. $\theta = -\dfrac{13\pi}{12}$ is a second quadrant angle coterminal to $\dfrac{11\pi}{12}$. The actue angle it makes with the x-axis is

$$\theta' = \pi - \frac{11\pi}{12} = \frac{12\pi}{12} - \frac{11\pi}{12} = \frac{\pi}{12}.$$



c. $\theta = 3.5$ is measured in radians. Since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, we know that $\pi < 3.5 < \frac{3\pi}{2}$ implying that 3.5 is in the third quadrant. The reference angle is $\theta' = 3.5 - \pi \approx 0.3584$.



d. $\theta = \frac{25\pi}{4}$ is a first quadrant angle coterminal to $\frac{\pi}{4}$. The angle $\frac{\pi}{4}$ is an acute angle and is its own reference angle. Therefore, $\theta' = \frac{\pi}{4}$.

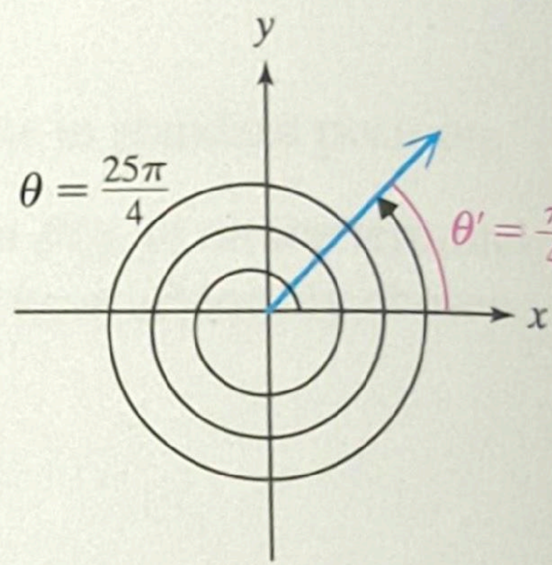


Table 5-6

Trigonometric Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Figure 5-26 shows representative right triangles with acute angles 30°, 45°, and 60° and with a hypotenuse of 1 unit. Placing the acute angles in standard position, we can further visualize a point P on the terminal side of each angle 1 unit from the origin. The coordinates of each point are the lengths of the legs of the corresponding right triangles.

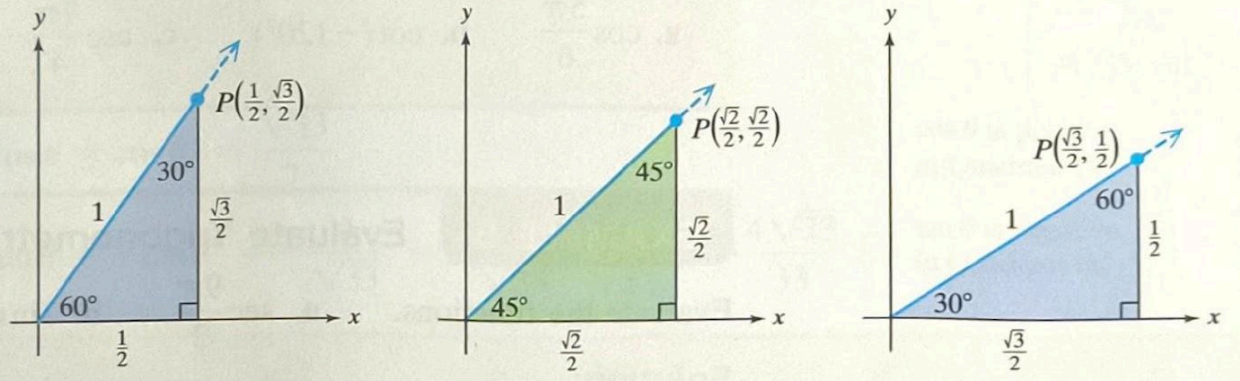


Figure 5-26

In Examples 4 and 5, we evaluate trigonometric functions of angles whose reference angle is one of the special angles from Table 5-6.

EXAMPLE 4 Using Reference Angles to Evaluate Functions

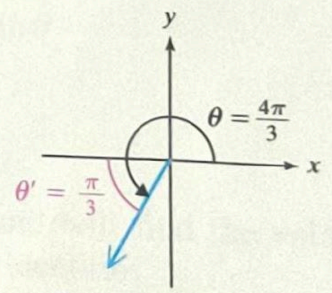
Evaluate the functions. **a.** $\sin\frac{4\pi}{3}$ **b.** $\tan(-225°)$ **c.** $\sec\frac{11\pi}{6}$

Solution:

a. $\theta = \frac{4\pi}{3}$ is in Quadrant III. The reference angle is $\frac{4\pi}{3} - \pi = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}$

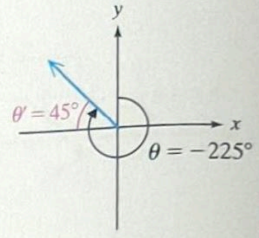
Since $\sin\theta$ is negative in Quadrant III,

$$\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$$



b. $\theta = -225°$ is an angle in Quadrant II, coterminal to $135°$. The reference angle is $180° - 135° = 45°$. Since $\tan\theta$ is negative in Quadrant II,

$$\tan(-225°) = -\tan 45° = -(1) = -1$$

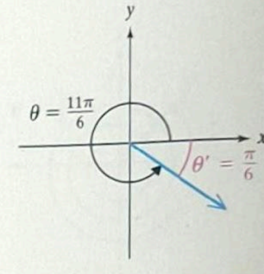


c. $\theta = \dfrac{11\pi}{6}$ is in Quadrant IV. The reference angle is

$$2\pi - \frac{11\pi}{6} = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}$$

Since $\cos\theta$ and its reciprocal $\sec\theta$ are positive in Quadrant IV,

$$\sec\frac{11\pi}{6} = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$$



Skill Practice 4 Evaluate the functions.

a. $\cos\dfrac{5\pi}{6}$ **b.** $\cot(-120°)$ **c.** $\csc\dfrac{7\pi}{4}$

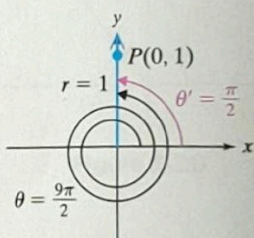
EXAMPLE 5 Evaluate Trigonometric Functions

Evaluate the functions. **a.** $\sec\dfrac{9\pi}{2}$ **b.** $\sin(-510°)$

Solution:

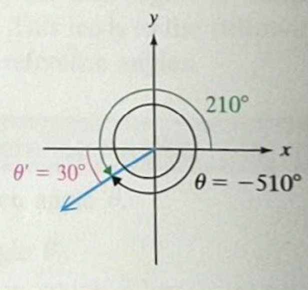
a. $\dfrac{9\pi}{2}$ is coterminal to $\dfrac{\pi}{2}$. The terminal side of $\dfrac{\pi}{2}$ is on the positive y-axis, where we have selected the arbitrary point $(0, 1)$.

$$\sec\frac{9\pi}{2} = \sec\frac{\pi}{2} = \frac{r}{x} \rightarrow \frac{1}{0} \text{ is undefined.}$$



b. $-510°$ is coterminal to $210°$, which is a third quadrant angle. The reference angle is $210° - 180° = 30°$.
Since $\sin\theta$ is negative in Quadrant III,

$$\sin(-510°) = -\sin 30° = -\left(\frac{1}{2}\right) = -\frac{1}{2}.$$



Skill Practice 5 Evaluate the functions.

a. $\csc(11\pi)$ **b.** $\cos(-600°)$

Answers

4. a. $-\dfrac{\sqrt{3}}{2}$ **b.** $\dfrac{\sqrt{3}}{3}$ **c.** $-\sqrt{2}$

5. a. Undefined **b.** $-\dfrac{1}{2}$

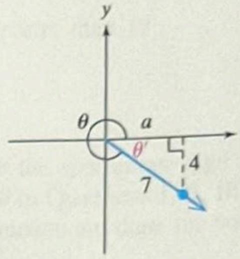
In Example 6, we determine the values of trigonometric functions based on given information about θ and by using reference angles.

EXAMPLE 6 Evaluating Trigonometric Functions

Given $\sin\theta = -\dfrac{4}{7}$ and $\cos\theta > 0$, find $\cos\theta$ and $\tan\theta$.

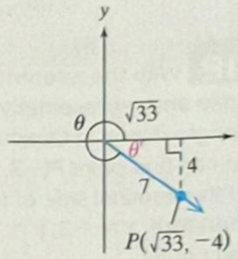
Solution:

First note that $\sin\theta < 0$ and $\cos\theta > 0$ in Quadrant IV. We can label the reference angle θ' and then draw a representative triangle with opposite leg of length 4 units and hypotenuse of length 7 units.



Using the Pythagorean theorem, we can determine the length of the adjacent leg.

$$a^2 + (4)^2 = (7)^2$$
$$a^2 + 16 = 49$$
$$a^2 = 33$$
$$a = \sqrt{33}$$ Choose the positive square root for the length of a side of a triangle.



TIP With the reference angle and representative triangle drawn in Quadrant IV, we can find point $P(\sqrt{33}, -4)$ on the terminal side of θ.

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{33}}{7}$$
$$\tan\theta = \frac{y}{x} = \frac{-4}{\sqrt{33}}$$

$$\cos\theta = \cos\theta' = \frac{\sqrt{33}}{7}$$ $\cos\theta$ is positive in Quadrant IV.

$$\tan\theta = -\tan\theta' = -\frac{4}{\sqrt{33}} = -\frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = -\frac{4\sqrt{33}}{33}$$ $\tan\theta$ is negative in Quadrant IV.

Skill Practice 6 Given $\cos\theta = -\dfrac{3}{8}$ and $\sin\theta < 0$, find $\sin\theta$ and $\tan\theta$.

The identities involving trigonometric functions of acute angles presented in Section 5.2 are also true for trigonometric functions of non-acute angles provided that the functions are well defined at θ.

Pythagorean Identities	Reciprocal Identities	Quotient Identities
$\sin^2\theta + \cos^2\theta = 1$	$\csc\theta = \dfrac{1}{\sin\theta}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
$\tan^2\theta + 1 = \sec^2\theta$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
$1 + \cot^2\theta = \csc^2\theta$	$\cot\theta = \dfrac{1}{\tan\theta}$	

In Example 7, we are given information about an angle θ and will find the values of the trigonometric functions by applying the fundamental identities.

EXAMPLE 7 **Using Fundamental Identities**

Given $\cos\theta = -\frac{3}{5}$ for θ in Quadrant II, find $\sin\theta$ and $\tan\theta$.

Solution:

<u>Using Identities</u>

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1$$

$$\sin^2\theta + \frac{9}{25} = 1$$

$$\sin^2\theta = 1 - \frac{9}{25}$$

$$\sin^2\theta = \frac{25}{25} - \frac{9}{25}$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = \pm\frac{4}{5}$$

In Quadrant II, $\sin\theta > 0$.

Therefore, choose $\sin\theta = \frac{4}{5}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

<u>Alternative Approach</u>

Label the reference angle θ' and then draw a representative triangle with adjacent leg of length 3 units and hypotenuse of length 5 units.

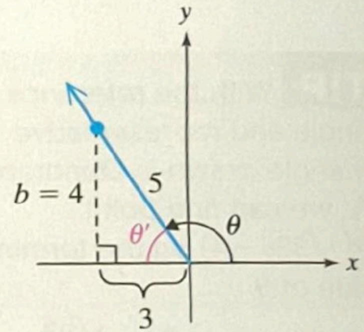
Using the Pythagorean theorem, we can determine the length of the opposite leg.

$$(3)^2 + b^2 = (5)^2$$

$$9 + b^2 = 25$$

$$b^2 = 16$$

$$b = 4$$



In Quadrant II, $\sin\theta > 0$.

Therefore, $\sin\theta = \sin\theta' = \frac{4}{5}$.

In Quadrant II, $\tan\theta < 0$.

Therefore, $\tan\theta = -\tan\theta' = -\frac{4}{3}$.

TIP With the reference angle and representative triangle drawn in Quadrant II, we can find point $P(-3, 4)$ on the terminal side of θ. Therefore, $x = -3$, $y = 4$, and $r = 5$.

$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{-3}$$

The period of the functions $y = \sin x$ and $y = \cos x$ is 2π. That is, the graphs of $y = \sin x$ and $y = \cos x$ show one complete cycle on the interval $0 \le x \le 2\pi$. To determine a comparable interval for one complete cycle of the graphs of $y = \sin Bx$ and $y = \cos Bx$ for $B > 0$, we can solve the following inequality.

$$0 \le Bx \le 2\pi$$

$$\frac{0}{B} \le \frac{Bx}{B} \le \frac{2\pi}{B}$$

"Starting" value for the cycle

$$0 \le x \le \frac{2\pi}{B}$$

"Ending" value for the cycle

This tells us that for $B > 0$, the period of $y = \sin Bx$ and $y = \cos Bx$ is $\frac{2\pi}{B}$.

Amplitude and Period of the Sine and Cosine Functions

For $y = A\sin Bx$ and $y = A\cos Bx$ and $B > 0$, the amplitude and period are

$$\text{Amplitude} = |A| \quad \text{and} \quad \text{Period} = \frac{2\pi}{B}$$

To analyze the period of a sine or cosine function with a negative coefficient on x, we would first rewrite $y = A\sin(-Bx)$ and $y = A\cos(-Bx)$ using the odd and even properties. That is, for $B > 0$,

Rewrite $y = \sin(-Bx)$ as $y = -\sin Bx$ because $y = \sin x$ is an odd function.
Rewrite $y = \cos(-Bx)$ as $y = \cos Bx$ because $y = \cos x$ is an even function.

EXAMPLE 2 Graphing $y = A\sin Bx$

Given $f(x) = 4\sin 3x$,

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Solution:

a. $f(x) = 4\sin 3x$ is in the form $f(x) = A\sin Bx$ with $A = 4$ and $B = 3$.

The amplitude is $|A| = |4| = 4$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{3}$.

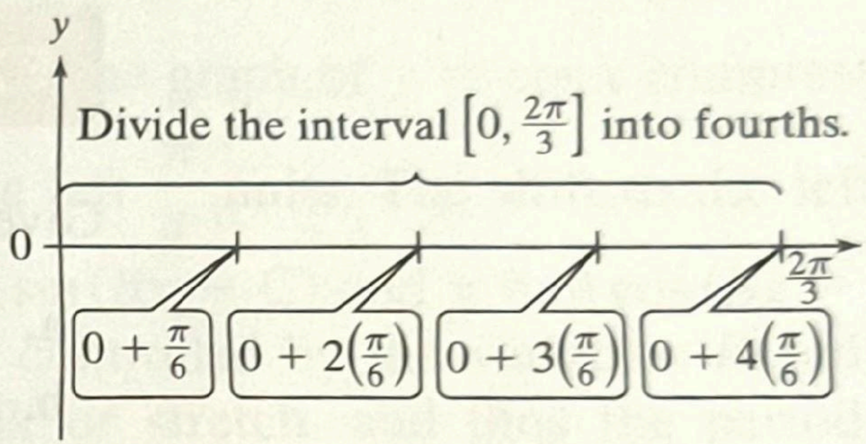
b. The period $\frac{2\pi}{3}$ is shorter than 2π, which tells us that the graph is compressed horizontally. For $y = \sin x$, one complete cycle can be graphed for x on the interval $0 \le x \le 2\pi$. For $y = 4\sin 3x$, one complete cycle can be graphed on the interval defined by $0 \le 3x \le 2\pi$.

$$0 \le 3x \le 2\pi$$

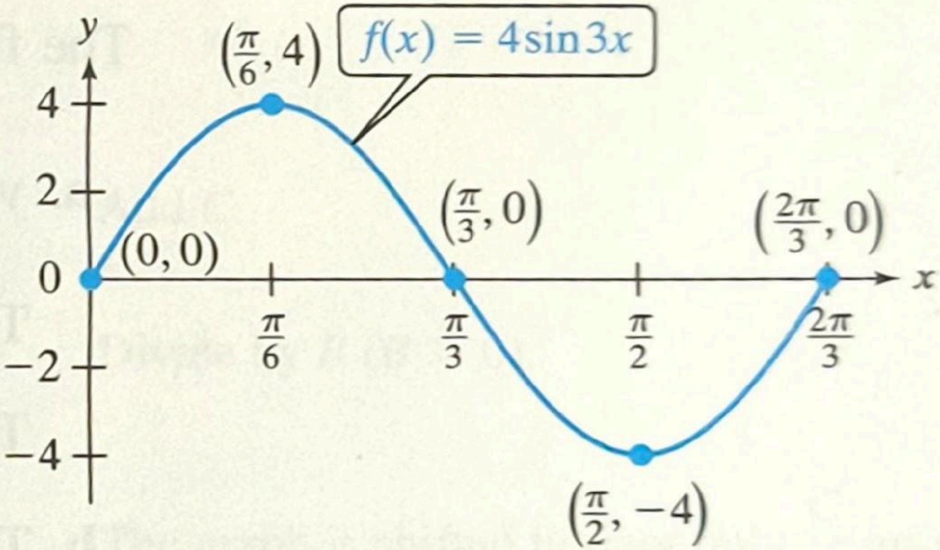
$$\frac{0}{3} \le \frac{3x}{3} \le \frac{2\pi}{3}$$

$$0 \le x \le \frac{2\pi}{3}$$

Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$.



Since the amplitude is 4, the maximum and minimum points have y-coordinates of 4 and -4.



EXAMPLE 3 Graphing $y = A\cos(Bx - C)$

Given $y = \cos\left(2x + \dfrac{\pi}{2}\right)$,

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Solution:

The function is of the form $y = A\cos(Bx - C)$ where $A = 1$, $B = 2$, and $C = -\dfrac{\pi}{2}$.

a. $y = \cos\left(2x + \dfrac{\pi}{2}\right)$ can be written as $y = 1 \cdot \cos\left(2x + \dfrac{\pi}{2}\right)$.

The amplitude is $|A| = |1| = 1$.

The period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{2} = \pi$.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le 2x + \frac{\pi}{2} \le 2\pi$$

$$-\frac{\pi}{2} \le 2x \le 2\pi - \frac{\pi}{2} \qquad \text{Subtract } \frac{\pi}{2}.$$

$$-\frac{\pi}{2} \le 2x \le \frac{4\pi}{2} - \frac{\pi}{2} \qquad \text{Write the right side with a common denominator. Then divide by 2.}$$

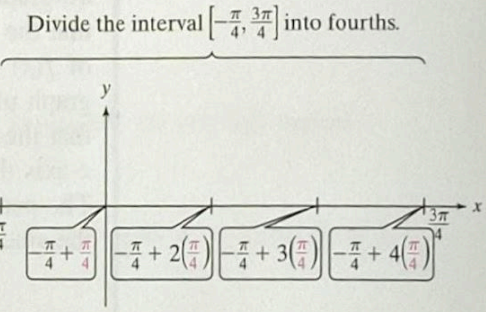
$$-\frac{\pi}{2} \le 2x \le \frac{3\pi}{2}$$

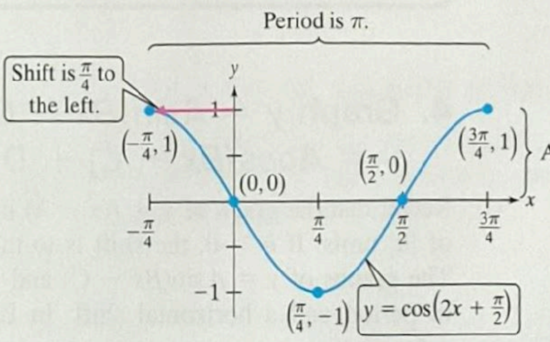
$$-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$

"Starting" value for the cycle

"Ending" value for the cycle

Dividing the period into fourths, we have increments of $\dfrac{1}{4}(\pi) = \dfrac{\pi}{4}$.





TIP Note that the distance between the "ending" point and "starting" point of the cycle shown is

$$\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{4\pi}{4} = \pi$$

which is the period of the function.

Skill Practice 3 Given $y = \cos\left(3x - \dfrac{\pi}{2}\right)$,

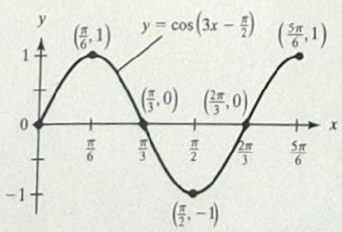
a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Answers

3. a. Amplitude: 1, Period: $\dfrac{2\pi}{3}$

b.



In Example 3, the graph of $y = \cos\left(2x + \frac{\pi}{2}\right)$ is the graph of $y = \cos x$ compressed horizontally by a factor of 2 and shifted to the left $\frac{\pi}{4}$ units. The shift to the left is called the *phase shift*. In general, given $y = A\sin(Bx - C)$ and $y = A\cos(Bx - C)$ for $B > 0$, the horizontal transformations are controlled by the variables B and C. The value of B controls the horizontal shrink or stretch, and thus the period of the function. For $B > 0$, the phase shift can be determined by solving the following inequality.

$$0 \leq Bx - C \leq 2\pi$$

$$C \leq Bx \leq 2\pi + C \qquad \text{Add } C.$$

$$\frac{C}{B} \leq \frac{Bx}{B} \leq \frac{2\pi + C}{B} \qquad \text{Divide by } B\ (B > 0).$$

The phase shift is the left endpoint.

$$\frac{C}{B} \leq x \leq \frac{2\pi + C}{B}$$

The graph is shifted horizontally $\frac{C}{B}$ units. The phase shift is $\frac{C}{B}$.

The phase shift is a horizontal shift of a trigonometric function. To find the vertical shift, recall that the graph of $y = f(x) + D$ is the graph of $y = f(x)$ shifted $|D|$ units upward for $D > 0$ and $|D|$ units downward if $D < 0$. We are now ready to summarize the properties of the graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$.

Properties of the General Sine and Cosine Functions

Consider the graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ with $B > 0$.

1. The amplitude is $|A|$.
2. The period is $\frac{2\pi}{B}$.
3. The phase shift is $\frac{C}{B}$.
4. The vertical shift is D.
5. One full cycle is given on the interval $0 \leq Bx - C \leq 2\pi$.
6. The domain is $-\infty < x < \infty$.
7. The range is $-|A| + D \leq y \leq |A| + D$.

TIP The double arrow symbol ⇔ means that the statements to the left and right of the arrow are logically equivalent. That is, one statement follows from the other and vice versa.

The Inverse Sine Function

The **inverse sine function** (or arcsine) denoted by $\sin^{-1}$ or arcsin is the inverse of the restricted sine function $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Therefore,

$$y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x$$

$$y = \arcsin x \quad \Leftrightarrow \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Avoiding Mistakes

The notation $\sin^{-1} x$ represents the inverse of the sine function, not the reciprocal. That is, $\sin^{-1} x \ne \frac{1}{\sin x}$.

- $y = \sin^{-1} x$ is read as "y equals the inverse sine of x" and $y = \arcsin x$ is read as "y equals the arcsine of x."
- To evaluate $y = \sin^{-1} x$ or $y = \arcsin x$ means to find an angle y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive, whose sine value is x.

EXAMPLE 1 Evaluating the Inverse Sine Function

Find the exact values or state that the expression is undefined.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **b.** $\arcsin \frac{1}{2}$ **c.** $\sin^{-1} 2$

Solution:

Avoiding Mistakes

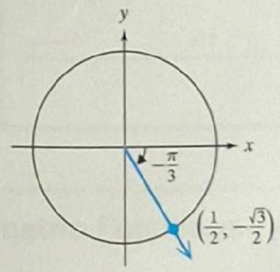
There are infinitely many values x for which $\sin x = -\frac{\sqrt{3}}{2}$, such as $x = \frac{4\pi}{3}$, $\frac{5\pi}{3}$, $-\frac{\pi}{3}$, and $-\frac{2\pi}{3}$ to name a few. However, the inverse sine function requires that x be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

a. Let $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Then $\sin y = -\frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$y = -\frac{\pi}{3}$. Therefore $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

Find an angle y on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = -\frac{\sqrt{3}}{2}$.

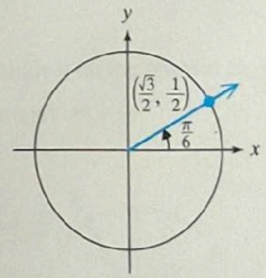


b. Let $y = \arcsin \frac{1}{2}$.

Then $\sin y = \frac{1}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$y = \frac{\pi}{6}$. Therefore $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

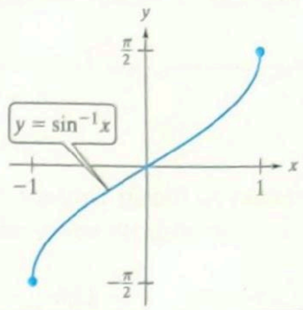
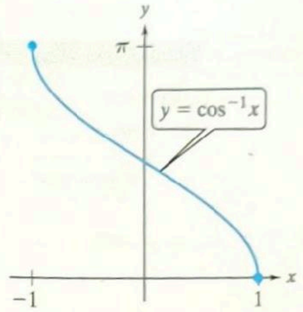
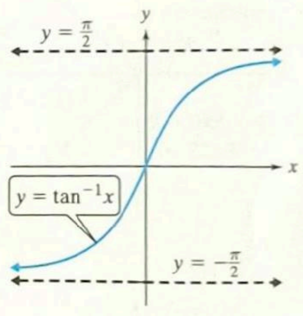
Find an angle y on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \frac{1}{2}$.



c. Let $y = \sin^{-1} 2$.

To evaluate $y = \sin^{-1} 2$ would mean that we find an angle y such that $\sin y = 2$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. However, recall that $-1 \le \sin y \le 1$ for any angle y. Therefore $\sin^{-1} 2$ is undefined.

Inverse Trigonometric Functions

Restricted Function	Inverse Function	Graph
$y = \sin x$ $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $-1 \le y \le 1$	The **inverse sine function** (or arcsine), denoted by $\sin^{-1}$ or arcsin, is defined by $y = \sin^{-1} x \iff \sin y = x$ $y = \arcsin x \iff \sin y = x$ $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	
$y = \cos x$ $0 \le x \le \pi$ $-1 \le y \le 1$	The **inverse cosine function** (or arccosine), denoted by $\cos^{-1}$ or arccos, is defined by $y = \cos^{-1} x \iff \cos y = x$ $y = \arccos x \iff \cos y = x$ $-1 \le x \le 1$ and $0 \le y \le \pi$	
$y = \tan x$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$ $y \in \mathbb{R}$ Vertical asymptotes: $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$	The **inverse tangent function** (or arctangent), denoted by $\tan^{-1}$ or arctan, is defined by $y = \tan^{-1} x \iff \tan y = x$ $y = \arctan x \iff \tan y = x$ $x \in \mathbb{R}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ Horizontal asymptotes: $y = -\frac{\pi}{2}, y = \frac{\pi}{2}$	

EXAMPLE 2 Evaluating Inverse Trigonometric Functions

Find the exact values.

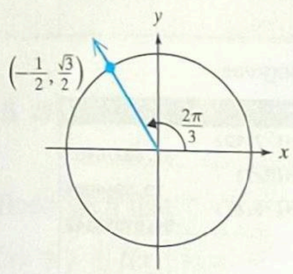
a. $\cos^{-1}\left(-\frac{1}{2}\right)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\arctan(-1)$

Solution:

a. Let $y = \cos^{-1}\left(-\frac{1}{2}\right)$. Find an angle y on the interval $[0, \pi]$ such that $\cos y = -\frac{1}{2}$.

Then $\cos y = -\frac{1}{2}$ for $0 \le y \le \pi$.

$y = \frac{2\pi}{3}$. Therefore, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.



Avoiding Mistakes

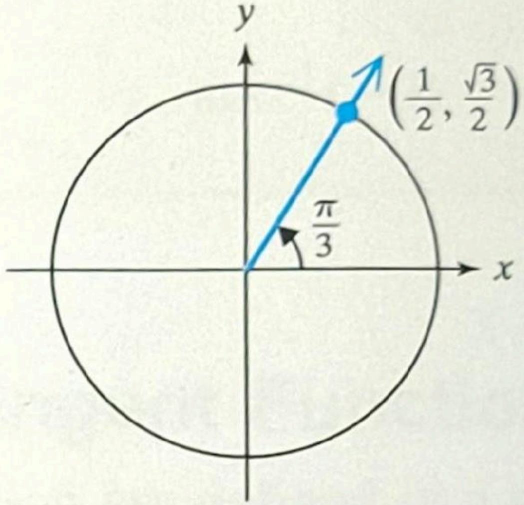
Perhaps the most common error when evaluating the inverse trigonometric functions is to fail to recognize the restrictions on the range. For instance, in Example 2(a), the result of the inverse cosine function must be an angle between 0 and π.

b. Let $y = \tan^{-1}\sqrt{3}$.

Find an angle y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = \sqrt{3}$.

Then $\tan y = \sqrt{3}$ for $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

$y = \dfrac{\pi}{3}$. Therefore, $\tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$.

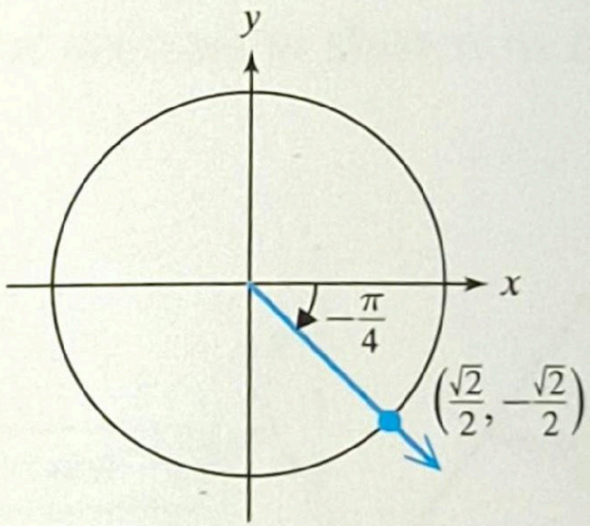


c. Let $y = \arctan(-1)$.

Find an angle y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = -1$.

Then $\tan y = -1$ for $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

$y = -\dfrac{\pi}{4}$. Therefore, $\arctan(-1) = -\dfrac{\pi}{4}$



EXAMPLE 4 Verifying an Identity Containing Negative Arguments

Verify that the equation is an identity. $\dfrac{\cos(-x)\tan(-x)}{\sin x} = -1$

Solution:

$\dfrac{\cos(-x)\tan(-x)}{\sin x} = -1$ The left-hand side (LHS) is the more complicated side.

$= \dfrac{\cos x(-\tan x)}{\sin x}$ To simplify, we require the same arguments for each factor. The cosine function is an even function. Thus, $\cos(-x) = \cos x$. The tangent function is an odd function. Thus, $\tan(-x) = -\tan x$.

$= \dfrac{\cancel{\cos x}\left(-\dfrac{\sin x}{\cancel{\cos x}}\right)}{\sin x}$ Write tangent as the ratio of sine and cosine.

$= \dfrac{-\sin x}{\sin x}$

$= -1$ (RHS) This equals the right-hand side (RHS) of the original equation. Therefore, we have verified the identity.

Skill Practice 4 Verify that the equation is an identity. $\dfrac{\sin(-x)\cot(-x)}{\cos x} = 1$

When we verify an identity, we are trying to show that the expression on one side equals the expression on the other. Therefore, we cannot use the equality of the statement because we have not yet proved it. For this reason, we work with the left or right side of the equation, but not both. In Example 4, we worked only with the left side of the equation.

EXAMPLE 5 Verifying an Identity by Combining Fractions

Verify that the equation is an identity. $\dfrac{1}{1-\cos x} - \dfrac{1}{1+\cos x} = 2\cot x\csc x$

Solution:

$\dfrac{1}{1-\cos x} - \dfrac{1}{1+\cos x}$ The LHS has two terms and the RHS has one term. Therefore, a good strategy is to combine the two terms on the left by adding the fractions.

$= \dfrac{(1+\cos x)}{(1+\cos x)} \cdot \dfrac{1}{(1-\cos x)} - \dfrac{1}{(1+\cos x)} \cdot \dfrac{(1-\cos x)}{(1-\cos x)}$ The LCD is $(1+\cos x)(1-\cos x)$.

$= \dfrac{(1+\cos x)-(1-\cos x)}{(1+\cos x)(1-\cos x)}$ Subtract the fractions.

$= \dfrac{1+\cos x - 1 + \cos x}{1-\cos^2 x}$ Apply the distributive property in the numerator. Multiply conjugates in the denominator.

$= \dfrac{2\cos x}{\sin^2 x}$ In the denominator, from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, we know that $\sin^2 x = 1 - \cos^2 x$.

$= 2 \cdot \dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\sin x}$ The RHS of the original equation has factors of $\cot x$ and $\csc x$. Therefore, we group factors in the numerator and denominator conveniently to reach the desired result.

$= 2\cot x\csc x$ (RHS)

Answer

4. $\dfrac{\sin(-x)\cot(-x)}{\cos x} = \dfrac{-\sin x(-\cot x)}{\cos x}$

$= \dfrac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\cos x} = \dfrac{\cos x}{\cos x} = 1$

Keep in mind that the Pythagorean identities have several alternative forms.

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$
$$1 - \cos^2 x = \sin^2 x \qquad \sec^2 x - 1 = \tan^2 x \qquad \csc^2 x - 1 = \cot^2 x$$
$$1 - \sin^2 x = \cos^2 x \qquad \sec^2 x - \tan^2 x = 1 \qquad \csc^2 x - \cot^2 x = 1$$

These alternative forms each involve a difference of squares that factors as a product of conjugates. This was illustrated in steps 2–5 of Example 5.

$$(1 + \cos x)(1 - \cos x) = 1 - \cos^2 x = \sin^2 x$$

Notice that upon simplification, the expression ultimately results in a single term. This pattern is sometimes helpful when we want to simplify a fraction with a denominator containing two terms as a fraction with a single-term denominator.

If an angle θ is the sum or difference of two angles for which the trigonometric function values are known, we can use the identities from Table 6-2 to find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. For example, consider the angles 15°, 105°, and $\frac{7\pi}{12}$ as a sum or difference of other "special" angles.

$$15° = 45° - 30° \qquad 105° = 135° - 30° \qquad \frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

EXAMPLE 1 **Applying the Addition and Subtraction Formulas**

Find the exact values. **a.** $\cos 15°$ **b.** $\sin\frac{11\pi}{12}$

Solution:

a. $\cos 15° = \cos(45° - 30°)$ — Write 15° as the sum or difference of angles for which the sine and cosine functions are known (that is, integer multiples of 30° or 45°).

$= \cos 45°\cos 30° + \sin 45°\sin 30°$ — Apply $\cos(u - v) = \cos u\cos v + \sin u \sin v$ with $u = 45°$ and $v = 30°$.

$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$ — Substitute the known function values.

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ — Multiply radicals and write the result over the common denominator.

TIP The combination $45° - 30°$ is not the only way to make a sum or difference of 15° from our "special" angles. For example, the same result is obtained from

$\cos(-45° + 60°)$ and $\cos(135° - 120°)$.

b. First note that $\frac{11\pi}{12} = \frac{3\pi}{12} + \frac{8\pi}{12} = \frac{\pi}{4} + \frac{2\pi}{3}$. — Look for a combination of angles that are integer multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

$\sin\frac{11\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$

$= \sin\frac{\pi}{4}\cos\frac{2\pi}{3} + \cos\frac{\pi}{4}\sin\frac{2\pi}{3}$ — Apply $\sin(u + v) = \sin u\cos v + \cos u \sin v$ with $u = \frac{\pi}{4}$ and $v = \frac{2\pi}{3}$.

$= \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$ — Substitute the known function values.

$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ — Add the fractions.

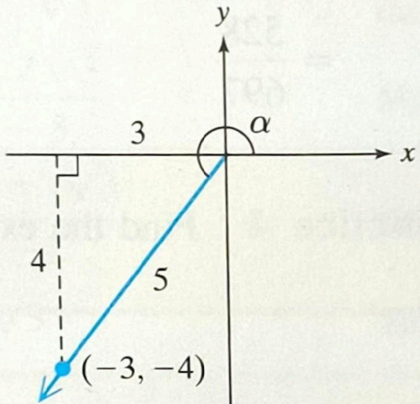
EXAMPLE 3 Applying the Difference Formula for Cosine

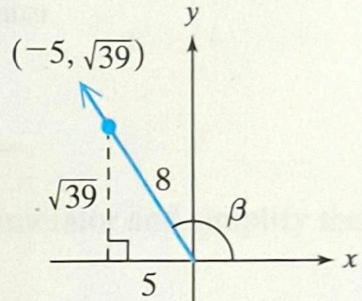
Find the exact value of $\cos(\alpha - \beta)$ given that $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = -\frac{5}{8}$ for α in Quadrant III and β in Quadrant II.

Solution:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

To find the value of $\cos(\alpha - \beta)$, we need to know the values of the factors on the right side of the identity. We can set up a representative triangle for each angle.





$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$	Apply the difference formula for cosine.
$= -\frac{3}{5} \cdot \left(-\frac{5}{8}\right) + \left(-\frac{4}{5}\right)\left(\frac{\sqrt{39}}{8}\right)$	Substitute values for the sine and cosine of α and β.
$= \frac{15 - 4\sqrt{39}}{40}$	Simplify.

EXAMPLE 5 Applying the Subtraction Formula for Tangent

Find the exact value of $\tan 255°$.

Solution:

$$\tan 255° = \tan(300° - 45°)$$

Write 255° as the sum or difference of integer multiples of 30° or 45°.

$$= \frac{\tan 300° - \tan 45°}{1 + \tan 300° \tan 45°}$$

Apply the formula $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$ with $u = 300°$ and $v = 45°$.

$$= \frac{-\sqrt{3} - 1}{1 + (-\sqrt{3})(1)}$$

Substitute known values for $\tan 300°$ and $\tan 45°$.

$$= -\frac{(\sqrt{3} + 1)}{(1 - \sqrt{3})} \cdot \frac{(1 + \sqrt{3})}{(1 + \sqrt{3})}$$

Rationalize the denominator by multiplying numerator and denominator by the conjugate of the denominator.

$$= -\frac{4 + 2\sqrt{3}}{1 - 3}$$

Multiply.

$$= -\frac{2(2 + \sqrt{3})}{-2}$$

Factor the numerator and simplify the denominator.

$$= 2 + \sqrt{3}$$

Simplify common factors.

Sum of $A\sin x$ and $B\cos x$

For the real numbers A, B, and x,

$$A\sin x + B\cos x = k\sin(x + \alpha)$$

where $k = \sqrt{A^2 + B^2}$, and α satisfies $\cos\alpha = \dfrac{A}{k}$ and $\sin\alpha = \dfrac{B}{k}$.

The formula presented here to simplify a sum of $A\sin x$ and $B\cos x$ is also called a **reduction formula** because it "reduces" the sum of two trigonometric functions into one term.

EXAMPLE 8 Writing a Sum of Sine and Cosine as a Single Term

Write $5\sin x - 12\cos x$ in the form $k\sin(x + \alpha)$.

Solution:

$5\sin x - 12\cos x$ is in the form $A\sin x + B\cos x$ with $A = 5$ and $B = -12$.

$$k = \sqrt{A^2 + B^2} = \sqrt{(5)^2 + (-12)^2} = \sqrt{169} = 13$$

$$\cos\alpha = \frac{A}{k} = \frac{5}{13} \text{ and } \sin\alpha = \frac{B}{k} = \frac{-12}{13} = -\frac{12}{13}$$

Since $\cos\alpha > 0$ and $\sin\alpha < 0$, α is in Quadrant IV. Taking the inverse cosine, we have the first quadrant angle

$$\cos^{-1}\left(\frac{5}{13}\right) \approx 1.1760 \text{ or } \approx 67.4°.$$

To find a fourth quadrant angle, subtract from 2π (or 360°).

$$\alpha = 2\pi - \cos^{-1}\frac{5}{13} \approx 5.1072 \text{ or } \alpha = 360° - \cos^{-1}\frac{5}{13} \approx 292.6°$$

Therefore, $5\sin x - 12\cos x \approx 13\sin(x + 5.1072)$ or

$5\sin x - 12\cos x \approx 13\sin(x + 292.6°)$.

Skill Practice 8 Write $-4\sin x + 3\cos x$ in the form $k\sin(x + \alpha)$.

Avoiding Mistakes

You can support your results from Example 8 by using a calculator to graph $y = 5\sin x - 12\cos x$ and $y = 13\sin(x + 5.1072)$. Confirm that the two graphs coincide.

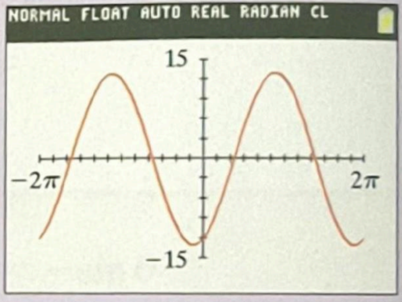


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Table 6-3

Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1$$

EXAMPLE 1 Using the Double-Angle Formulas

Given that $\sin\theta = \frac{2}{3}$ for θ in Quadrant II, find the exact function values.

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$

Solution:

To find $\sin 2\theta = 2\sin\theta\cos\theta$, we need to know the value of $\cos\theta$. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we have $\cos\theta = \pm\sqrt{1-\sin^2\theta}$.

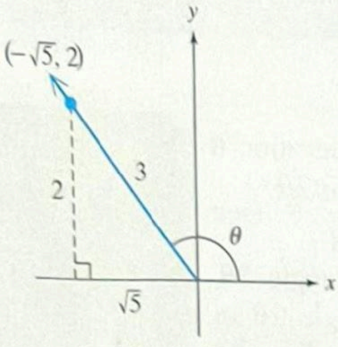


Figure 6-7

In Quadrant II, $\cos\theta < 0$.

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(\frac{2}{3}\right)^2} = -\sqrt{1-\frac{4}{9}}$$

$$= -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{3}$$

The cosine function is negative in Quadrant II (Figure 6-7).

a. $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) = -\frac{4\sqrt{5}}{9}$

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2 = \frac{5}{9} - \frac{4}{9} = \frac{1}{9}$

Use any of the three formulas for $\cos 2\theta$.

c. With the values of $\sin 2\theta$ and $\cos 2\theta$ known, the easiest way to find $\tan 2\theta$ is to divide.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{4\sqrt{5}}{9}}{\frac{1}{9}} = -\frac{4\sqrt{5}}{9}\cdot\frac{9}{1} = -4\sqrt{5}$$

Alternatively, from Figure 6-7, $\tan\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$. Therefore,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{2\sqrt{5}}{5}\right)}{1-\left(-\frac{2\sqrt{5}}{5}\right)^2} = \frac{-\frac{4\sqrt{5}}{5}}{1-\frac{4}{5}} = \frac{-\frac{4\sqrt{5}}{5}}{\frac{1}{5}}$$

$$= -\frac{4\sqrt{5}}{5}\cdot\frac{5}{1} = -4\sqrt{5}$$

Skill Practice 1 Given that $\sin\theta = \frac{4}{5}$ for θ in Quadrant II, find the exact function values.

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$

The double-angle formulas can be used with angles other than 2θ. For example:

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta \qquad \cos 6x = \cos^2 3x - \sin^2 3x$$

$$\sin(\alpha - 1) = 2\sin\left(\frac{\alpha-1}{2}\right)\cos\left(\frac{\alpha-1}{2}\right) \qquad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

In Example 2, we apply the sum formula for sine as well as the double-angle formulas for sine and cosine to prove an identity.

Answers

1. a. $-\frac{24}{25}$ b. $-\frac{7}{25}$ c. $\frac{24}{7}$

EXAMPLE 3 Applying the Power-Reducing Formulas

Write $\sin^4 x + \cos^2 x$ in terms of first powers of cosine.

Solution:

$\sin^4 x + \cos^2 x = (\sin^2 x)^2 + \cos^2 x$ — Write each term as a power of $\sin^2 x$ and $\cos^2 x$.

$= \left(\dfrac{1 - \cos 2x}{2}\right)^2 + \dfrac{1 + \cos 2x}{2}$ — Apply the power-reducing formulas.

$= \dfrac{1 - 2\cos 2x + \cos^2 2x}{4} + \dfrac{1 + \cos 2x}{2}$ — Square the first term.

$= \dfrac{1 - 2\cos 2x + \cos^2 2x}{4} + \dfrac{1 + \cos 2x}{2} \cdot \dfrac{2}{2}$ — Write each term with a common denominator.

$= \dfrac{1 - 2\cos 2x + \cos^2 2x}{4} + \dfrac{2 + 2\cos 2x}{4}$

$= \dfrac{1}{4}[1 - 2\cos 2x + \cos^2 2x + 2 + 2\cos 2x]$ — Factor out $\frac{1}{4}$.

$= \dfrac{1}{4}[\cos^2 2x + 3]$ — Combine like terms.

$= \dfrac{1}{4}\left[\left(\dfrac{1 + \cos 4x}{2}\right) + 3\right]$ — Apply the power-reducing formula for cosine $\cos^2 2x = \dfrac{1 + \cos 4x}{2}$.

Notice that the angle on the RHS is doubled.

$= \dfrac{1}{4}\left[\dfrac{1}{2} + \dfrac{\cos 4x}{2} + \dfrac{6}{2}\right]$ — Write each term with a common denominator.

$= \dfrac{1}{4}\left(\dfrac{\cos 4x}{2} + \dfrac{7}{2}\right)$ — Combine like terms.

$= \dfrac{1}{8}\cos 4x + \dfrac{7}{8}$ — Apply the distributive property.

EXAMPLE 6 Using the Half-Angle Formulas

If $\sin\alpha = -\dfrac{4}{5}$ and $\pi < \alpha < \dfrac{3\pi}{2}$, find the exact values of each expression.

a. $\sin\dfrac{\alpha}{2}$ **b.** $\cos\dfrac{\alpha}{2}$ **c.** $\tan\dfrac{\alpha}{2}$

Solution:

To apply the half-angle formulas, we need to know the value of $\cos\alpha$. The angle α is in the third quadrant with $\sin\alpha = -\dfrac{4}{5}$. From Figure 6-8, we have $\cos\alpha = -\dfrac{3}{5}$. Next, since α is on the interval $\left(\pi, \dfrac{3\pi}{2}\right)$, angle $\dfrac{\alpha}{2}$ is on the interval $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right)$. Therefore $\dfrac{\alpha}{2}$ is in the second quadrant.

In Quadrant II, $\sin\frac{\alpha}{2} > 0$.

a. $$\sin\frac{\alpha}{2} = +\sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

In Quadrant II, $\cos\frac{\alpha}{2} < 0$.

b. $$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = -\sqrt{\frac{\frac{2}{5}}{2}} = -\sqrt{\frac{2}{5}\cdot\frac{1}{2}}$$

$$= -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

c. Use any of the three half-angle formulas for tangent.

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = \frac{\frac{8}{5}}{-\frac{4}{5}} = \frac{8}{5}\cdot\left(-\frac{5}{4}\right) = -2$$

TIP Alternatively in Example 6(c),

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{2\sqrt{5}}{5}}{-\frac{\sqrt{5}}{5}}$$

$$= \frac{2\sqrt{5}}{5}\cdot\left(-\frac{5}{\sqrt{5}}\right) = -2$$

EXAMPLE 1 Solving a Trigonometric Equation in Linear Form

Solve $2\tan x = \sqrt{3} - \tan x$

a. Over $[0, 2\pi)$. b. Over the set of real numbers.

Solution:

a. This equation is linear in the variable $\tan x$. Therefore, isolate $\tan x$.

$2\tan x = \sqrt{3} - \tan x$

$3\tan x = \sqrt{3}$ Add $\tan x$ to both sides.

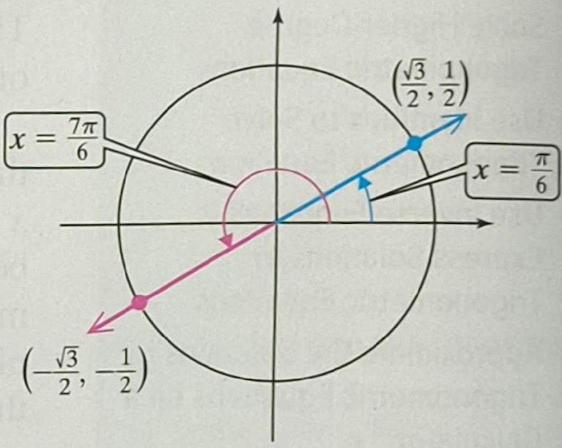
$\tan x = \dfrac{\sqrt{3}}{3}$ Divide by 3.

$x = \dfrac{\pi}{6}, x = \dfrac{7\pi}{6}$ On the interval $[0, 2\pi)$, $\tan x$ equals $\frac{\sqrt{3}}{3}$ for $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.

The solution set is $\left\{\dfrac{\pi}{6}, \dfrac{7\pi}{6}\right\}$. Write the solution set.

b. The period of the tangent function is π. Therefore, adding $n\pi$ to either solution for any integer n also gives a solution.

$$x = \frac{\pi}{6} + n\pi \text{ and } x = \frac{7\pi}{6} + n\pi$$



However, notice that for $n = 1$, $\dfrac{\pi}{6} + (1)\pi = \dfrac{7\pi}{6}$. This means that the formula $x = \dfrac{\pi}{6} + n\pi$ is sufficient to generate all solutions to the equation.

Thus, the general solution is $\left\{x \,\middle|\, x = \dfrac{\pi}{6} + n\pi\right\}$.

EXAMPLE 2 Solving an Equation Containing a Multiple Angle

Given $2\sin 2x - \sqrt{3} = 0$,

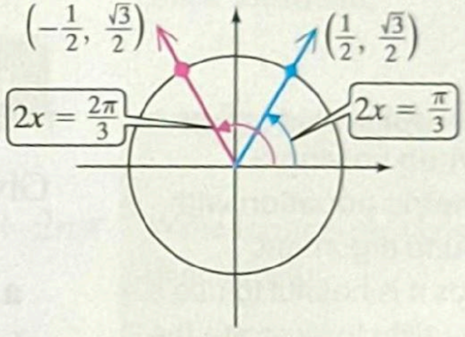
a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

> **TIP** As an alternative approach, we can use a substitution to help solve an equation with a compound argument. In Example 2, letting $u = 2x$, gives the simpler equation
> $2\sin u - \sqrt{3} = 0$.
> The solutions are
> $u = \dfrac{\pi}{3} + 2n\pi$, and
> $u = \dfrac{2\pi}{3} + 2n\pi$.
> Then using back substitution we can solve for x.
> $2x = \dfrac{\pi}{3} + 2n\pi$ and
> $2x = \dfrac{2\pi}{3} + 2n\pi$

Solution:

a. To solve $2\sin 2x - \sqrt{3} = 0$, begin by isolating $\sin 2x$.



$$\sin 2x = \frac{\sqrt{3}}{2}$$

From the unit circle, $\sin 2x = \frac{\sqrt{3}}{2}$ for $2x = \frac{\pi}{3}$ and $2x = \frac{2\pi}{3}$.

$$2x = \frac{\pi}{3} + 2n\pi, \qquad 2x = \frac{2\pi}{3} + 2n\pi$$

$$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot \left(\frac{\pi}{3} + 2n\pi\right), \quad \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot \left(\frac{2\pi}{3} + 2n\pi\right)$$

Solve for x.

$$x = \frac{\pi}{6} + n\pi, \qquad x = \frac{\pi}{3} + n\pi$$

General solution: $\left\{x \,\middle|\, x = \dfrac{\pi}{6} + n\pi, x = \dfrac{\pi}{3} + n\pi\right\}$

b. To find solutions on the interval $[0, 2\pi)$, substitute $n = \ldots, -1, 0, 1, 2, \ldots$

n	$x = \dfrac{\pi}{6} + n\pi$		$x = \dfrac{\pi}{3} + n\pi$	
-1	$x = \dfrac{\pi}{6} + (-1)\pi = -\dfrac{5\pi}{6}$	Not on $[0, 2\pi)$	$x = \dfrac{\pi}{3} + (-1)\pi = -\dfrac{2\pi}{3}$	Not on $[0, 2\pi)$
0	$x = \dfrac{\pi}{6} + (0)\pi = \dfrac{\pi}{6}$		$x = \dfrac{\pi}{3} + (0)\pi = \dfrac{\pi}{3}$	
1	$x = \dfrac{\pi}{6} + (1)\pi = \dfrac{7\pi}{6}$		$x = \dfrac{\pi}{3} + (1)\pi = \dfrac{4\pi}{3}$	
2	$x = \dfrac{\pi}{6} + (2)\pi = \dfrac{13\pi}{6}$	Not on $[0, 2\pi)$	$x = \dfrac{\pi}{3} + (2)\pi = \dfrac{7\pi}{3}$	Not on $[0, 2\pi)$

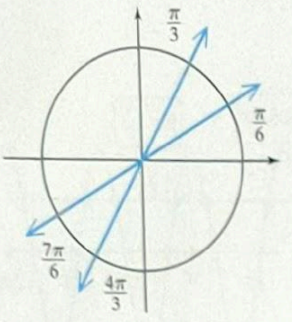


Figure 6-13

The solution set on the interval $[0, 2\pi)$ is $\left\{\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{7\pi}{6}, \dfrac{4\pi}{3}\right\}$ (Figure 6-13).

Skill Practice 2 Solve the equation $1 + \cos 3x = 0$.

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

The solutions to the equation $2\sin 2x - \sqrt{3} = 0$ are the x-coordinates of the points of intersection of the graphs of $y = 2\sin 2x - \sqrt{3}$ and $y = 0$. See Figure 6-14.

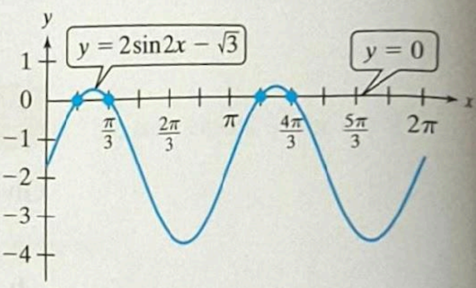


Figure 6-14

EXAMPLE 3 **Solving an Equation Involving a Half-Angle**

Given $-1 + \sin\dfrac{x}{2} = 0$,

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

Solution:

a. $-1 + \sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$ Isolate $\sin\frac{x}{2}$.

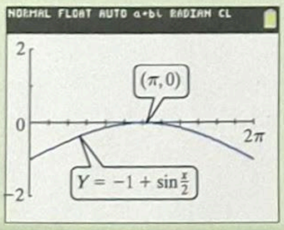
$\dfrac{x}{2} = \dfrac{\pi}{2} + 2n\pi$ From the unit circle, $\sin\frac{x}{2} = 1$ for $\frac{x}{2} = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$.

$2 \cdot \left(\dfrac{x}{2}\right) = 2 \cdot \left(\dfrac{\pi}{2} + 2n\pi\right)$ Solve for x.

$x = \pi + 4n\pi$

The general solution is $\{x \mid x = \pi + 4n\pi\}$.

TIP When solving a trigonometric equation with a compound argument, sometimes it is helpful to use a graphing utility to illustrate the number of solutions on a given interval. In Example 3, we can graph $Y = -1 + \sin\frac{x}{2}$ on the interval $[0, 2\pi)$. From the graph it appears that only one solution exists on the interval.



Answers

2. a. $\left\{x \,\middle|\, x = \dfrac{\pi}{3} + \dfrac{2n\pi}{3}\right\}$ b. $\left\{\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}\right\}$

b. Substituting integer values of n gives $x = \ldots -3\pi, \pi, 5\pi, 9\pi, \ldots.$
The solution set on the interval $[0, 2\pi)$ is $\{\pi\}$.