

# Chapter 3 Determinants

## 3.1 The properties of determinants

The determinant of a square matrix  $A$  is a single number. It is written as  $\det A$  or  $|A|$ .

The determinant of  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Three basic properties for the determinant. (can be apply to any  $n$  by  $n$  formula)

(1) The  $n$  by  $n$  identity matrix  $I$  has  $\det I = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1$   $\begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1 \times 1 - 0 \times c = 1$ .

(2, Sign reversal) The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc) = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\det(A+B) \neq \det A + \det B$$

$$\det(2I) = 2^n$$

The determinant of any permutation matrix  $\det P = 1$  or  $-1$ .

$$\det P_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \stackrel{\text{rule 2}}{=} -1 \quad \text{In general } \det I_n = 1 \quad \xrightarrow{\text{exchange two rows}} \det P_1 = -1 \quad \xrightarrow{\text{exchange two more rows}} \det P_2 = 1$$

(3) (linearity) The determinant is linear with respect to every row separately.

$$13-1) \begin{vmatrix} l a & l b \\ c & d \end{vmatrix} = l a \times d - l b \times c = l (ad - bc) = l \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \text{ for any number } l.$$

$$13-2) \begin{vmatrix} a+e & b+f \\ \boxed{c} & \boxed{d} \end{vmatrix} = (a+e)d - (b+f)c = (ad - bc) + (ed - fc) = \begin{vmatrix} a & b \\ \boxed{c} & \boxed{d} \end{vmatrix} + \begin{vmatrix} e & f \\ \boxed{c} & \boxed{d} \end{vmatrix}$$

Note, this rule only applies when other rows stay fixed. It is a linear combination in one row.

All these three rules completely determine  $\det A$  for any square matrix.

Example 1 For  $A = \begin{bmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$   $\det A = \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow{\text{rule}} 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4 \left( \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \right)$

Example 2  $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4 \times 1 = 4$   $S = 1^2 = 1$   $S = (2)^2 = 4$

(4) If two rows of  $A$  are equal, then  $\det A = 0$ . eg.  $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = a \times b - b \times a = 0$ .

Proof: When we exchange the two equal rows, by rule 2, the determinant is supposed to change sign.

So  $\det A = -\det A$  forces  $\det A = 0$ .

(5) Subtracting a multiple of one row from another row leaves  $\det A$  unchanged.

$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} \xrightarrow{\text{rule 3}} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\xrightarrow{\text{rule 3}} -l \begin{vmatrix} a & b \\ a & b \end{vmatrix} \xrightarrow{\text{rule 4}} (-l) \times 0 = 0$

(6) A matrix  $A$  with a row of zeros has  $\det A = 0$ . eg.  $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$

Proof: We can add the other row to this zero row, by rule 5, the determinant is unchanged.

Now we have two equal rows.

By rule 4,  $\det A = 0$ .

In the elimination,  $A \xrightarrow{\text{row operations}} U$  By rule 5,  $\det A = \det U$ .

7. If  $A$  is triangular, upper triangular, lower triangular, or even diagonal, then  $\det A = a_{11} a_{22} \dots a_{nn}$ .

eg.  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad$ .

Proof: Suppose all diagonal entries are nonzero.  $A \xrightarrow{\text{elimination}} U$

By rule 5,  $\det A = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{nn} \end{vmatrix} \xrightarrow{\text{rule 3}} = a_{11} a_{22} \dots a_{nn}$

If a diagonal entry  $a_{ii} = 0$ .

$A = \begin{bmatrix} a_{11} & \dots & \times \\ & \ddots & \times \\ a_{ii} & \dots & \times \\ & \ddots & \times \\ & & a_{nn} \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} a_{11} & \dots & \times \\ & \ddots & \times \\ 0 & \dots & 0 \\ & \ddots & \times \\ & & a_{nn} \end{bmatrix}$

By rule 6,  $\det A = 0 = a_{11} \dots a_{nn}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{r_2 - \frac{c}{a} r_1} U = \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}$

$a \neq 0$

$\det A \xrightarrow{\text{rule 5}} \det U \xrightarrow{\text{rule 7}} a(d - \frac{c}{a} b)$

With no row exchange, the product of the pivots is the determinant.

$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \rightarrow \begin{bmatrix} c & d \\ 0 & b - \frac{a}{c} d \end{bmatrix}$

$\det(PA) \xrightarrow{\text{rules 4 \& 7}} c(b - \frac{a}{c} d)$   
 $= cb - ad = -(ad - bc)$   
 $= -\det A = (\det P)(\det A)$

(8) The determinant of  $AB$  is  $\det A$  times  $\det B$   $|AB| = |A||B|$

Factorization  $PA = LU$

$$\det P, \det A = \det(PA) = \det(LU) = \det L, \det U = \det U \quad \text{So } \det A = \pm \det U$$

$\det P = \pm 1$

When  $A$  is invertible, the determinant of  $A^{-1}$  is  $\frac{1}{\det A}$

Proof:  $I = AA^{-1}$ , then  $1 = \det I = \det(AA^{-1}) \stackrel{\text{rule 8}}{=} (\det A)(\det A^{-1})$ .