Formulas for determinant of an n by n matrix

- The pivot formula $PA = LU$
- The big formula (derived from a_{ij}) $\underline{n! \text{ terms}}$ Every term use each row and column once.

When $n=2$, the column numbers can (1,2), (2,1)

$\det A = a_{11}a_{22} + (-1)a_{12}a_{21}$

Using the basic properties of determinant, show $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \overset{=0}{\begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix}} + \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} + \overset{=0}{\begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix}} \quad (4 = 2^2 \text{ determinants})$$

$$= a_{11}a_{22}\begin{vmatrix} 1 & \\ & 1 \end{vmatrix} + a_{12}a_{21}\begin{vmatrix} & 1 \\ 1 & \end{vmatrix} = +a_{11}a_{22} + (-1)a_{12}a_{21} \quad (2 = 2! \text{ determinants})$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

, 3^3 determinants,

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

If $a_{11} = 0$.

If $a_{11} \neq 0$, we can eliminate a_{21} to obtain a row of zeros

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

, $3!$ determinants,

Pick out one entry from each row.

Focus on enties from different columns.

$$= a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\underbrace{\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix}}_{=(-1)^2} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$+ a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} I$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - \underset{(-1)^1}{a_{13}}a_{22}a_{31}$$

$$(2, 3, 1) \longrightarrow (3, 2, 1) \longrightarrow (1, 2, 3)$$

Rows 1, 2, 3 and Columns 1, 2, 3 appear once in each term.

Let the row order be 1, 2, 3. There are $3! = 3 \times 2 \times 1 = 6$ ways to order the columns

Column number $= \underbrace{(1\ 2\ 3)}_{\text{identity permutation}}, \boxed{(2, 3, 1)}, (3, 1, 2), (1, 3, 2), (2, 1, 3), (3, 2, 1) \longrightarrow (1, 2, 3)$

$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

And

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}^{-1}\begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

$PP^T = I$

There are $n!$ ways to choose one entry from every row and column.

Let the row order be $1, 2, \ldots, n$ and let the n columns go in each possible order $j_1, j_2, \ldots, j_n$.

There are $n!$ orderings of columns.

$$\boxed{\det A = \cdots + (\pm)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n} + \cdots = \sum (\det P)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}}$$

is the sum of $n!$ terms with $\pm$ signs.

Here $\sum$ indicates the column order $(j_1, j_2, \ldots, j_n)$ running through the set of all $n!$ permutations.

In each column permutation $(j_1, j_2, \ldots, j_n)$, $\det P = 1$ or -1. The ± 1 signs coincide with the even or odd number of exchanges from $j_1, j_2, \ldots, j_n$ to $1, 2, \ldots, n$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

This pattern is INVALID when $n=4$.

$4! = 4\times3\times2\times1 = 24.$

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Example 1. Find the determinant of $A = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{bmatrix}$

$$\begin{vmatrix} a & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{vmatrix} =$$

$j_1 = 2$ $\quad j_4 = 3$

$\det A = \cdots (\det P)\, a_{1j_1} a_{2j_2} a_{3j_3} a_{4j_4} + \cdots$ (with $j_2 = 1$, $j_3 = 4$)

(The only choices for j_2 and j_3 are 1 and 4).

$= \underbrace{(\det P)}_{=(-1)^2} a_{12} a_{21} a_{34} a_{43} = + a_{12} a_{21} a_{34} a_{43}$

$(2, 1, 4, 3) \rightarrow (1, 2, 4, 3) \rightarrow (1, 2, 3, 4)$

$$\det A = \begin{vmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{vmatrix} = a_{12} a_{21} a_{34} a_{43} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \xrightarrow[r_3 \leftrightarrow r_4]{r_1 \leftrightarrow r_2} I$

$= 1 = (-1)^2$

Example. Find the determinant of $Z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{bmatrix}$

Method II. If $c = 0$, $\det Z = 0$

if $c \neq 0$ $\quad Z \xrightarrow{\text{elimination}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & 1 \end{bmatrix}$

$\det Z = \sum (\det P)\, a_{1j_1} a_{2j_2} a_{3j_3} a_{4j_4}$ (with $j_1 = 1$, $j_2 = 2$, $j_3 = 3$, $j_4 = 1$)

$= \underbrace{(\det P)}_{=1} a_{11} a_{22} a_{33} a_{44} = 1 \times 1 \times c \times 1 = c.$