2.4 Matrix operations

Matrix multiplication agrees with elimination

Each column of EA is equal to E times the corresponding column of A.

Each row of EA is equal to the corresponding row of E times A.

Let A be an m by n matrix, and B be an n by p matrix

We write $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p]$, where $\vec{b}_j$ is column j of B for each j.

The product matrix AB is the m by p matrix defined as follows

$$AB = A[\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p] = [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p].$$

Such product $A\vec{b_j}$ makes sense because A is of size m by n, and $\vec{b_j}$ is an n-dimensional vector.

If B is a column matrix, this definition reduced to that for matrix-vector multiplication.

Compatibility rule: Let A and B be matrices of sizes m by n, and n' by p, respectively.

The product AB can be formed if and only if $n = n'$

Example. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \boxed{\begin{matrix} 9 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 8 \\ 6 \end{matrix}} \end{bmatrix}$ ($\vec{b_1}$, $\vec{b_2}$). Compute AB

① The column way: $A\vec{b_1} = \begin{bmatrix} \cancel{1} & \cancel{4} \\ 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 9+28 \\ 53 \\ 27 \end{bmatrix}$, $A\vec{b_2} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 32 \\ 46 \\ 24 \end{bmatrix}$ So $AB = \begin{bmatrix} 37 & 32 \\ 53 & 46 \\ 27 & 24 \end{bmatrix}$

the dot product of row 1 of A with column 1 of B.

② The dot product way. $AB = \begin{bmatrix} \cancel{1} & \cancel{4} \\ 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 1\times 9 + 4\times 7 & 1\times 8 + 4\times 6 \\ * & * \\ * & * \end{bmatrix}$

Since A has 3 rows, B has 2 columns, there are 3×2 dot products.

② The dot product way

Example. Suppose $A = [1\ 2\ 3]$ is a row vector, which is a 1 by 3 matrix.

$B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ is a column vector, which is a 3 by 1 matrix.

Their product AB is the dot product of a row times a column. $AB = \cancel{[1\ 2\ 3]}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1\times 4 + 2\times 5 + 3\times 6 = 32$

In every case, AB is filled with dot products.

We multiply the matrices A with n columns and B with n rows by taking the dot product of each row of A with each column of B.

$m\times n \quad n\times p$

Column j of B

row i of A

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & (AB)_{ij} & \cdots \\ & \vdots & \end{bmatrix}$$

If we pick out row i of A and column j of B, their dot product goes into row i and column j of AB.

$$(AB)_{ij} = (\text{row } i \text{ of } A)\cdot(\text{column } j \text{ of } B)$$
$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$
$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

The matrix AB has as many rows as A, and as many columns as B.

It is an m by p matrix. $A_{m \times n} B_{n \times p} = (AB)_{m \times p}$

There are $m \times p$ dot products with n multiplications each

Example Square matrices can be multiplied if and only if they have the same size

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\times2+1\times3 & 1\times2+1\times4 \\ 2\times2+(-1)\times3 & 2\times2+(-1)\times4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

A row times a column is an "inner" product
another name for dot product

How about a column times a row?
It is an "outer" product.

Example $\underset{3\times1}{B} \underset{1\times3}{A} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4\times1 & 4\times2 & 4\times3 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$ ← The dot product way
by taking dot products of rows of B with columns of A.

How about multiplying every row of B times A?

$$\begin{bmatrix} 4\times[1\ 2\ 3] \\ 5\times[1\ 2\ 3] \\ 6\times[1\ 2\ 3] \end{bmatrix} = BA$$

$$= \left[\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times 1 \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times 2 \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times 3 \right] = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$ ← The column way
by multiply B times every column of A
1 by p matrix

③ The row way to multiply matrices.

let $A = \begin{bmatrix} \text{row 1 of A} \\ \text{row 2 of A} \\ \vdots \\ \text{row m of A} \end{bmatrix}$ $AB = \begin{bmatrix} \text{row 1 of A} \\ \text{row 2 of A} \\ \vdots \\ \text{row m of A} \end{bmatrix} B = \begin{bmatrix} (\text{row 1 of A}) B \\ (\text{row 2 of A}) B \\ \vdots \\ (\text{row m of A}) B \end{bmatrix}$ is rows of A times B.

Every row of AB is a combination of the rows of B

2.4 Matrix operations

when $m \neq n$, suppose A is of size m by n, B is of size n by m

The product AB is of size m by m

BA n by n.

Matrix multiplication is NOT commutative

In most cases $AB \neq BA$

$$BA = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 11 & -1 \end{bmatrix} \neq AB$$

Even for square matrices, usually $AB \neq BA$.