The following are equivalent for an n by n matrix A.

(1) A is invertible

(2) $A\vec{x} = \vec{0}$ has the only solution $\vec{x} = \vec{0}$

(3) $A \xrightarrow{\text{elementary row operations}} I_n$

(4) $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$

(5) There exists an n by n matrix C such that $AC = I$.

Corollary If A and C are square matrices such that $AC = I$, then also $CA = I$

Particulary, both A and C are invertible, $C = A^{-1}$ and $A = C^{-1}$.

Every elementary matrix E is invertible. and E^{-1} is also an elementary matrix of the same type

Example the inverses of $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ are

is self-inverse.

$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

Suppose A is an m by n matrix, and $A \to B$ by a series of k elementary row operations.

let $E_1, E_2, \dots, E_k$ denote the corresponding k elementary matrices

$$A \to E_1 A \to E_2 E_1 A \to \cdots \to E_k E_{k-1} \cdots E_2 E_1 A = B$$

let $E = E_k \cdots E_2 E_1$, then $A \to EA = B$ $\quad E^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

The reduction becomes

$$[A \ I] \to [B \ E]$$

agrees with $E[A \ I]$

E can be computed without find E_i, $i=1,2,\dots,k$.

Inverse of AB.

Theorem If A and B are invertible, so is AB. $(AB)^{-1} = B^{-1}A^{-1}$ (A^{-1} and B^{-1} come in reverse order).

Proof: $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Example. $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract and add the same 2 times row 1 to row 2.

$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$. $F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

Find FE and $(FE)^{-1} = E^{-1}F^{-1}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 - 2r_1} \quad (EI) = E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 - 4r_2} \quad FE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{bmatrix}$$

Row 3 of FE = Row 3 of $I - 4($Row 2 of $E)$

differs from Row 2 of I

$$I \xrightarrow{r_3 + 4r_2} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 + 2r_1} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Row 3 of $E^{-1}F^{-1}$ = Row 3 of F^{-1}

= Row 3 of $I + 4($Row 2 of $I)$.

2.6 Factorization of a matrix $\quad A\vec{x}=\vec{b} \;\rightarrow\; U\vec{x}=\vec{c}$

- How to take U back to A, through matrix multiplication?

Example. Let $A=\begin{bmatrix}2 & 1\\ 6 & 8\end{bmatrix}$ $\quad A \xrightarrow[\text{multiplier } l_{21}=3]{r_2-3r_1} \boxed{\begin{bmatrix}2 & 1\\ 0 & 5\end{bmatrix}=U} = E_{21}A$, where $E_{21}=\begin{bmatrix}1 & 0\\ -l_{21} & 1\end{bmatrix}$ with $l_{21}=3$

To recover A from U, $\quad U \xrightarrow{r_2+3r_1} \begin{bmatrix}2 & 1\\ 6 & 8\end{bmatrix} = A = E_{21}^{-1}U$, where $E_{21}^{-1}=\begin{bmatrix}1 & 0\\ l_{21} & 1\end{bmatrix}$

Let $L=E_{21}^{-1}$, then $A=LU=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 5\end{bmatrix}$: Factorization of A. Both of L and U are triangular.

With no row exchanges, the forward elimination process from A to the upper triangular U is inverted by the lower triangular L such that $A=LU$, where the factors L and U are completely determined by A.

Example 2. Let $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$

$l_{21} = 2 \qquad l_{31} = -1 \qquad l_{32} = 1$

Each elimination step from A to U agrees with the multiplication by an elimination matrix E_{ij}, which is obtained I with $-l_{ij}$ in the (i,j) position.

$$A \to E_{21}A \to E_{31}E_{21}A \to E_{32}E_{31}E_{21}A = U.$$

We see from $[A \ I] \to [U \ E_{32}E_{31}E_{21}]$ that $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$

Then $A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U.$

Let $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$, then $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

Each E_{ij} is reversed by E_{ij}^{-1} by changing the off-diagonal (i,j) entry from $-l_{ij}$ to $+l_{ij}$. Both E_{ij} and E_{ij}^{-1} are lower triangular, and have all 1's on their diagonal.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$[A\ I] = \left[\begin{array}{ccc:ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{ccc:ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3+r_1} \left[\begin{array}{ccc:ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_3-r_2} \left[\begin{array}{ccc:ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right].$$

$$[E_{32}E_{31}E_{21}\ I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2+2r_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3-3r_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{r_3+r_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = [I\ (E_{32}E_{31}E_{21})^{-1}].$$

Without computing

the product $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$,

or using Gauss-Jordon method

to find $(E_{32} E_{31} E_{21})^{-1}$

It follow directly

from $A \rightarrow U$

that $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$