Write down the augmented matrix of the equations

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

And then

(1) Apply elementary matrices to reach a triangular system.
(2) Solve it by back substitution.
(3) What matrix does both steps at once?

Review "augmented matrix"

Elementary operations performed on systems of linear equations:

The following are called elementary row operations on a matrix

(1) interchange two rows.

(2) multiply one row by a nonzero number,

(3) add a multiple of one row to a different row.

△ The elementary row operations on a matrix can be performed by left multiplying by certain matrices

Definition An m by m matrix E is called an elementary matrix if it is obtained from the identity matrix $\underline{I}$ by a single elementary row operation.

The following are called elementary row operations on a matrix

(1) interchange two rows.

(2) multiply one row by a nonzero number,

(3) add a multiple of one row to a different row.

△ The elementary row operations on a matrix can be performed by left multiplying by certain matrices.

Definition An m by m matrix E is called an elementary matrix if it is obtained from the identity matrix I by a single elementary row operation.

Example Perform the elementary row operations (1) interchanging rows 1 and 2, (2) multiplying row 2 by 9, (3) adding 5 times row 2 to row 1. on $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

the corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ Left multiplying A by an elementary matrix has the same effect as doing the corresponding row operation to A

$$E_1 A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} \qquad E_2 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 9a_{21} & 9a_{22} & 9a_{23} \end{bmatrix} \qquad E_3 A = \begin{bmatrix} a_{11}+5a_{21} & a_{12}+5a_{22} & a_{13}+5a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Theorem. Let E be the elementary matrix obtained by performing an elementary row operation on the m by m identity matrix I. For any m by n matrix A, the result of EA is the same as performing that row operation on A.

$$I = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \qquad IA = A.$$

Proof: Let $R_1, R_2, \dots, R_m$ denote the rows of I.

If E is an elementary matrix obtained by adding k times row p of I to row q $(p \neq q)$.

The i-th row of E is $\begin{cases} R_i & \text{if } i \neq q \\ R_q + kR_p & \text{if } i = q \end{cases}$

Each row of EA is equal to the corresponding row of E times A

The i-th row of EA is $\begin{cases} R_i A = (\text{row } i \text{ of } A), & \text{if } i \neq q \\ (R_q + kR_p)A & \text{if } i = q \end{cases}$

$= (\text{row } q \text{ of } A) + k(\text{row } p \text{ of } A)$.

So EA is the result of adding k times row p of A to row q.

... matrix obtained by performing an elementary row operation on the m by m identity matrix I. For any m by n matrix A, the result of EA is the same as performing that row operation on A.

Proof: Let $R_1, R_2, \ldots, R_m$ denote the rows of I. $I = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$ $\quad IA = A$.

If E is an elementary matrix obtained by adding k times row p of I to row q $(p \neq q)$.

The i-th row of E is $\begin{cases} R_i & \text{if } i \neq q \\ R_q + kR_p & \text{if } i = q \end{cases}$

$\because$ Each row of EA is equal to the corresponding row of E times A

The i-th row of EA is $\begin{cases} \underbrace{R_i A}_{} = (\text{row } i \text{ of } A), & \text{if } i \neq q \\ \underbrace{(R_q + kR_p) A}_{= (\text{row } q \text{ of } A) + k(\text{row } p \text{ of } A)} & \text{if } i = q \end{cases}$

So EA is the result of adding k times row p of A to row q.

Sol: The augmented matrix

$$[A\ \vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right] \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{2,1}(4), \text{ then } E[A\ \vec{b}]$$

$r_2 \leftrightarrow r_3$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad P(E[A\ \vec{b}]) = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The upper triangular system $U\vec{x} = \vec{c}$ is

$$\begin{aligned} x + 2y + 2z &= 1 \\ 3y + 2z &= 1 \\ z &= -1 \end{aligned}$$

Sol: The augmented matrix $[A\ \vec{b}] = \begin{bmatrix} 1 & 2 & 2 & \vdots & 1 \\ 4 & 8 & 9 & \vdots & 3 \\ 0 & 3 & 2 & \vdots & 1 \end{bmatrix} \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$

(1) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}(4)$. then $E[A\ \vec{b}]$

$r_2 \leftrightarrow r_3$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{23}$, $P(E[A\ \vec{b}]) = \begin{bmatrix} 1 & 2 & 2 & \vdots & 1 \\ 0 & 3 & 2 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$

The upper triangular system $U\vec{x} = \vec{c}$ is

$$x + 2y + 2z = 1$$
$$3y + 2z = 1$$
$$z = -1$$

(2) By back substitution, firstly $z = -1$, then $3y + 2(-1) = 1$ gives $y = 1$. Finally, $x + 2(1) + 2(-1) = 1$ gives $x = 1$

So $A\vec{x} = \vec{b}$ has the solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(3) By the associative law,

$P(E[A\ \vec{b}]) = (PE)[A\ \vec{b}]$.

The matrix does both step is $PE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 1 & 0 \end{bmatrix}$

Assignment for Section 2.3: Elimination using matrices

(1) Write done the 3 by 3 matrices that produce the following elimination steps.

(a) E_{21} subtracts 5 times row 1 from row 2.

(b) P exchanges rows 1 and 2, and then rows 2 and 3.

(2) Multiplies these matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

in the orders EF and FE. Also computed EE and FFF.

(3) Consider

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

(a) Apply elimination to the 2 by 3 augmented matrix $[A \ \boldsymbol{b}]$.

(b) What is the triangular system $U\boldsymbol{x} = \boldsymbol{c}$?

(c) What is the solution $\boldsymbol{x}$?

(4) If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

Sol: The augmented matrix

$$[A\ \vec{b}] = \begin{bmatrix} 1 & 2 & 2 & | & 1 \\ 4 & 8 & 9 & | & 3 \\ 0 & 3 & 2 & | & 1 \end{bmatrix} \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3}$$

(1) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}(4)$, then $E[A\ \vec{b}]$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{23}$, $P\left(E[A\ \vec{b}]\right) = \begin{bmatrix} 1 & 2 & 2 & | & 1 \\ 0 & 3 & 2 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$

The upper triangular system $U\vec{x} = \vec{c}$ is

$$x + 2y + 2z = 1$$
$$3y + 2z = 1$$
$$z = -1$$

(2) By back substitution, firstly $z = -1$, then $3y + 2(-1) = 1$ gives $y = 1$. Finally, $x + 2(1) + 2(-1) = 1$ gives $x = 1$.

So $A\vec{x} = \vec{b}$ has the solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(3) By the associative law,

$P\left(E[A\ \vec{b}]\right) = (PE)[A\ \vec{b}]$.

The matrix does both step is

$PE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 1 & 0 \end{bmatrix}$

2.4 Matrix operations

For an m by n matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

number of rows

When $m=n$, A is a square matrix.

When $n=1$, A is a matrix with only 1 column.

When $m=1$, A 1 row.

$-A = (-1)A = \begin{bmatrix} -1 & -4 \\ -2 & -5 \\ -3 & -0 \end{bmatrix}$

$A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero matrix of size 3 ...

The ent...

a

(1) Matrices can be ...

each entry at a ...

$A + B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} + \ldots$

(2) Matrices ...

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The entry in row i and column j is

a_{ij}. $i = 1, 2, \ldots, m$. $j = 1, 2, \ldots, n$.

...trix.

...ly 1 column.

1 row.

(1) Matrices can be added if their sizes are the same, each entry at a time

$$A+B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1+9 & 4+8 \\ 2+7 & 5+6 \\ 3+0 & 0+5 \end{bmatrix} =$$

(2) Matrices can be multiplied by any Costant c.

$$2A = \begin{bmatrix} 2\times 1 & 2\times 4 \\ 2\times 2 & 2\times 5 \\ 2\times 3 & 2\times 0 \end{bmatrix} =$$

Zero matrix of size 3 by 2

Scanned with CamScanner

Laws for matrix multiplication

(1) Commutative law $A+B = B+A$

(2) distributive law $c(A+B) = cA+cB$

(3) associative law $(A+B)+C = A+(B$

Matrix multiplication. To multiply A times B, if A has n columns, then B mus

Example Find $AB = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}$, $\vec{b}_1$, $\vec{b}_2$. $AB = [A\vec{b}_1 \quad A\vec{b}_2] = [\ [\ 1 \ 4 \ / \ 2 \ 5 \ / \ 3 \ 0$

$= [\ 1\times 9 + 4\times \ / \ 18 + 3 \ / \ 27 + $

The column way to find AB:

Column j of AB is equal to A times column j of B.

If $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p]$ $\quad [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p] = AB = A[\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_p]$

Column picture of matrix multiplication: The product AB equals the matrix A

Each column of AB is a combination of

atim (1) Commutative law $A + B = B + A$

(2) distributive law $c(A+B) = cA + cB$

(3) associative law $(A+B)+C = A+(B+C)$

$A_{m \times n}$ $B_{n \times p}$ p can be any number

multiply A times B, if A has n columns, then B must have n rows.

$$\begin{bmatrix} \cdots \end{bmatrix} \left[\begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} \right] \qquad \vec{b}_1 \quad \vec{b}_2$$

$$AB = [A\vec{b}_1 \quad A\vec{b}_2] = \left[\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} \quad A\vec{b}_2 \right]$$

$$= \begin{bmatrix} 1\times 9 + 4 \times 7 & * \\ 18 + 35 & * \\ 27 + 0 & * \end{bmatrix}$$

times column j of B

$$A\vec{b}_2 \ \cdots \ A\vec{b}_p] = AB = A[\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_p]$$

lication: The product AB equals the matrix A times columns of B.

Each column of AB is a combination of the columns of A.