2.5 Inverse matrices

for a: $a^{-1} = \frac{1}{a}$ (But A^{-1} might not exists)

Suppose A is a square matrix.

We look for an "inverse matrix" A^{-1} of the same size, so that $A^{-1}A = I$

$\vec{x} = \boxed{A^{-1}A}\vec{x} = A^{-1}\vec{b}$

For an invertible matrix A, the inverse is written as A^{-1}. We call it A inverse.

Is A invertible? / Which matrices have inverses?

Definition: The matrix A is invertible if there exists a matrix A^{-1} that "inverts" A.

$A^{-1}A = I$ and $AA^{-1} = I$ (it is a two-sides inverse)

Lemma: For an invertible matrix A, it cannot have two different inverses.

Proof: Suppose $\underbrace{B}_{\text{the left inverse}}A = I$, and $A\underbrace{C}_{\text{the right inverse}} = I$ (To prove $B = C$)

By the associative law, $B = BI = B(AC) = (BA)C = IC = C$. #

For a square matrix, an inverse on one side is automatically the inverse on the other side.

(1) if $AC = I$, then $CA = I$ and $C = A^{-1}$

(2) if A is invertible, the one and only one solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

Example. For every given $\vec{b}$, there is one solution to $A\vec{x} = \vec{b}$. It is consistent with the invertibility of the coefficient matrix A

Elimination solves $A\vec{x} = \vec{b}$ without using A^{-1}. It produce pivots that take us to the solution

(3) The inverse A^{-1} exists when A has a full set of pivots, which are produced by elimination. (Row exchanges are allowed.)

are on the diagonal of the upper triangular U

$A A^{-1} = I$ can be solved by finding each column of A^{-1}

$$A^{-1} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \qquad A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$$

$$I = \begin{bmatrix} \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} & \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} & \underset{\vec{e}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \end{bmatrix}$$

- The solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ of three linear equations

$$\begin{cases} x_1 & & & = & b_1 \\ -x_1 & +x_2 & & = & b_2 \\ & -x_2 & +x_3 & = & b_3 \end{cases}$$

can be write as

$$\boldsymbol{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

PS: this seems to be independent from the other pages, although the teacher displayed it around this time.

Gauss - Jordan elimination.

Suppose A^{-1} exists such that $AA^{-1} = I = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$.

We write $A^{-1} = [\vec{x}_1\ \vec{x}_2\ \vec{x}_3]$. $AA^{-1} = A[\vec{x}_1\ \vec{x}_2\ \vec{x}_3] = [A\vec{x}_1\ A\vec{x}_2\ A\vec{x}_3]$ (the column way to find AB).

$AA^{-1} = I$ is really 3 systems of linear equations $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2$, $A\vec{x}_3 = \vec{e}_3$ $(*)$

They have the same coefficient matrix A, and we solve them together.

Let the 3 solutions $\vec{x}_1, \vec{x}_2, \vec{x}_3$ of the systems $(*)$ go into the columns of a matrix C, such that

$$AC = A[\vec{x}_1\ \vec{x}_2\ \vec{x}_3] = [A\vec{x}_1\ A\vec{x}_2\ A\vec{x}_3] \overset{(*)}{=} [\vec{e}_1\ \vec{e}_2\ \vec{e}_3] = I$$

So C is the right inverse of A, and it is automatically the inverse of A

$[A\ \vec{b}] \xrightarrow{\text{forward elimination}} [U\ \vec{c}]$:

Block matrix

$[A\ \ I] = [A\ \vec{e}_1\ \vec{e}_2\ \vec{e}_3] \to$

Example Invert $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$ by Gauss - Jordan elimination.

Gauss Jordan process

Start from $[A\ I]$, we end with $[I\ A^{-1}]$

Elimination steps on A only have to be done once.

$$[A\ I] = \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cccccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

Augmented matrix $[A\ \vec{c}_1]$

$$\xrightarrow{r_3 + r_1} \left[\begin{array}{cccccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c|cc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$[U\ \vec{c}_1]$

upper triangular U

Method I
$U\vec{x} = \vec{c}_i,\ i = 1, 2, 3$
can be solved by back substitution

· To solve $A\vec{x} = \vec{b}$ without A^{-1},
we deal with one column $\vec{b}$
to find one column $\vec{x}$

Method II
produce zeros above the pivots.

$$\xrightarrow{r_2 - \frac{1}{4}r_3} \left[\begin{array}{cccccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right] \xrightarrow{r_1 + \frac{1}{2}r_3} \left[\begin{array}{cccccc} 2 & 4 & 0 & \frac{5}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right] \xrightarrow{r_1 - 4r_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{27}{2} & -\frac{11}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

diagonal matrix

divide each row by its pivots.

$$\xrightarrow[\frac{1}{4}r_3]{\frac{1}{2}r_1} \left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{27}{4} & -\frac{11}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$$

identity matrix I

The solution to $A\vec{x}_1 = \vec{e}_1$

If there exists a square matrix C such that $AC = I$,
which indicates A is invertible,
then A must have a full set of pivots.

2.5 Inverse matrices

Theorem. When row exchanges are allowed, the inverse A^{-1} exists if and only if A has a full set of pivots.

Example. Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ by Gauss-Jordan elimination.

$$[A \ I] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{r_1 - 3r_2} \begin{bmatrix} 2 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

I

So $A^{-1} = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ -2 & 1 \end{bmatrix}$