**Definition** The dimension of a vector space is the number of vectors in every basis.

**Example.** The dimension of $\mathbb{R}^n$ is $n$

The line through $\vec{v} = (1, 5, 2)$ has dimension 1

It is a subspace of $\mathbb{R}^3$ with one vector $\vec{v}$ in its basis.

The equation $x + 5y + 2z = 0$ describe a plane in $\mathbb{R}^3$. It is perpendicular to $\vec{v}$.

This plane is also the nullspace of $A = [1\ 5\ 2]$. $A\vec{x} = \vec{0}$ has $2 = 3-1$ special solutions $\vec{s}_1 = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{s}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\vec{s}_1$ and $\vec{s}_2$ are the basis vectors for $N(A)$, and the plane has dimension 2

$\boxed{V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \mid c_i \in \mathbb{R},\ i = 1, 2, \ldots n\}$

Given an $m$ by $n$ matrix $A$, let $\vec{c}_1, \vec{c}_2, \ldots \vec{c}_n$ denote the columns of $A$. Then $C(A) = \operatorname{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ → the dimension is $r$.

let $\vec{s}_1, \vec{s}_2, \ldots \vec{s}_{n-r}$ denote the special solutions to $A\vec{x} = \vec{0}$, $N(A) = \operatorname{span}\{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{n-r}\}$

independent.

the dimension is $n-r$.

Example $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 2 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

In both $A$ and $R$. col 3 = 3 (col 1) + (−1) (col 2)

Both $A\vec{x} = 0$ and $R\vec{x} = \vec{0}$ have $\underset{}{1}$ $^{=n-r}$ special solution $\vec{s} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

Note: $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $C(R) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ are different.

All bases for a vector space contain the same number of vectors

Theorem. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ are both bases for the same vector space, then $m = n$

Proof: Suppose $n > m$. $\vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \dots + a_{m1}\vec{v}_m$

$$W = [\vec{w}_1\ \vec{w}_2\ \cdots\ \vec{w}_n] = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_m] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = VA$$

$A$ is of size $m$ by $n$

Since $n > 0$, $A\vec{x} = \vec{0}$ must have at least 1 special solution $\vec{s}$

$W\vec{s} = VA\vec{s} = V\vec{0} = \vec{0}$

$= s_1\vec{w}_1 + s_2\vec{w}_2 + \dots + s_n\vec{w}_n$

It makes a contradiction.

The only way to avoid a contradiction is to have $n = m$.

The columns of the $n$ by $n$ identity matrix is the "standard basis" for $\mathbb{R}^n$

Example. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ This lower triangular matrix is invertible

① For every $\vec{b} \in \mathbb{R}^3$, $A\vec{x} = \vec{b}$ can always be solved by $\vec{x} = A^{-1}\vec{b}$

In other words, every $\vec{b}$ is a combination of the columns of $A$. The columns span $\mathbb{R}^3$.

② The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. So the columns of $A$ are independent.

By the definition of basis, the columns of $A$ gives a basis for $\mathbb{R}^3$.

Theorem. The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are a basis for $\mathbb{R}^n$ when they are the columns of an $n$ by $n$ invertible matrix.

The columns of every invertible $n$ by $n$ matrix give a basis for $\mathbb{R}^n$.

For a singular (not invertible) matrix $A$, $A\vec{x} = \vec{0}$ has nonzero solution, and the columns of $A$ are dependent.

Given 5 vectors in $\mathbb{R}^7$, how to find a basis for the space they span?

Theorem. The pivot columns of $A$ are a basis for $C(A)$.

We call a set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of vectors linearly independent, or independent,

if it satisfies the condition: if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then $c_1 = c_2 = \cdots = c_n$

Example. The zero vector does not belong to any independent set

$\{\vec{v}\}$ is independent if and only if $\vec{v} \neq \vec{0}$

If $\vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, $\{\vec{v}, \vec{w}\}$ is dependent if and only if $\vec{v}$ and $\vec{w}$ are parallel.

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be nonzero vectors in $\mathbb{R}^3$ where $\{\vec{v}, \vec{w}\}$ is independent

$\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u}$ is not on the plane span$\{\vec{v}, \vec{w}\}$.

$\boxed{V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \mid c_i \in \mathbb{R},\ i = 1, 2, \dots, n\}$.

Given an $m$ by $n$ matrix $A$, let $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ denote the columns of $A$. Then $C(A) = \operatorname{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$

Let $\vec{s}_1, \vec{s}_2, \dots, \vec{s}_k$ denote the special solutions to $A\vec{x} = \vec{0}$, $N(A) = \operatorname{span}\{\vec{s}_1, \vec{s}_2, \dots, \vec{s}_k\}$.

# 4.4 Independence, basis and dimension

① Every vector $\vec{v}$ in the vector space is a combination of the basis vectors.

② The combination that produces $\vec{v}$ is unique

**Basis** for vector space

a set of independent vectors that span the space

Every vector in the space is a unique combination of the basis vectors

**Definition.** A basis for a vector space is a sequence of vectors with two properties:

They are independent and they span the space.

**Theorem** There is one and only one way to write $\vec{v}$ in the vector space as a combination of the basis vectors.

**Proof**: Suppose $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$, and also $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$

By subtraction $\vec{0} = \vec{v} - \vec{v} = (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \cdots + (a_n - b_n)\vec{v}_n$

From the independence of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, $a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$. So there are not two ways to produce $\vec{v}$.

**Example.** The column space of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is $\mathbb{R}^2$ (columns $\vec{i}$, $\vec{j}$)

$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are independent since they are NOT parallel.

And they span $\mathbb{R}^2$ because for every $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, $\vec{x} = x\vec{i} + y\vec{j}$