Theorem when $\vec{v}$ and $\vec{w}$ are perpendicular, $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2$ 勾股定理

Proof. Geometrically, for 2 and 3 dimensional vectors, it is Pythagoras formula $a^2 + b^2 = c^2$.

Method 2. $\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$, where $\vec{v} - \vec{w} = (v_1 - w_1, v_2 - w_2)$

$$= (v_1 - w_1)^2 + (v_2 - w_2)^2 = v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2$$

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\underbrace{\vec{v} \cdot \vec{w}}_{=0}$$

$\vec{v}$ is perpendicular to $\vec{w}$

$= \|\vec{v}\|^2 + \|\vec{w}\|^2$ #

(Method 2 can be applied to n dimensions)

1.2 Lengths and angles from dot products

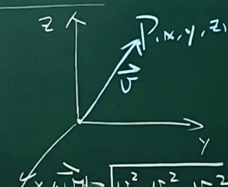
Given $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2$, is a number.

For $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$. $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$

Properties
- $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- $\vec{v} \cdot \vec{0} = 0 = \vec{0} \cdot \vec{v}$
- $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$ for all scalars c.
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v}$
- $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ are perpendicular
- $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

In 2 and 3 dimensions, the angle between $\vec{v}$ and $\vec{w}$ is $90°$.

let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ with $x = v_1$, $y = v_2$, $z = v_3$



$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$\|c\vec{v}\| = |c|\,\|\vec{v}\|$ for all c

$\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$

Definition

Example

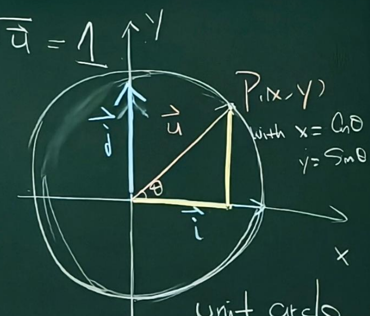
Find the

$\|\vec{u}$

when θ

Example

Definition A unit vector $\vec{u}$ is a vector whose length equals 1. $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = 1$



unit circle with r = 1

Example In a xy plane, the standard unit vectors along the x and y axes are

$\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$.

Find the unit vector $\vec{u}$ that makes an angle θ with the x-axis. $\vec{u} = (\cos\theta, \sin\theta)$

$\|\vec{u}\|^2 = \boxed{\cos^2\theta + \sin^2\theta = 1}$ trigonometric identity

when $\theta = 0^\circ$, it is $\vec{i}$, when $\theta = 90^\circ$, it is $\vec{j}$.

Example. For $\vec{v} = (2, 2, 1)$, $\|\vec{v}\| = \sqrt{2^2 + 2^2 + 1} = 3$.

Find the unit vector $\vec{u}$ in the same direction as $\vec{v}$. $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ Check $\|\vec{u}\| = 1$.

For $\vec{v} = (1, 1, 1, 1)$, the unit vector in the same direction as $\vec{v}$ is $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$\|\vec{v}\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$

For a nonzero vector $\vec{v}$, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is the unit vector in the same direction as $\vec{v}$

For $\vec{v} = (-1, 2)$, $\vec{w} = (4, 2)$ $\quad \vec{v} \cdot \vec{w} = 0 \Rightarrow$ the angle between $\vec{v}$ and $\vec{w}$ is $90°$.

Let $\vec{v}_* = (1, 2)$ $\quad \vec{v}_* \cdot \vec{w} \neq 0 \Rightarrow$ the angle between $\vec{v}$ and $\vec{w}$ is NOT $90°$. Q: What is the angle?

For unit vectors $\vec{i} = (1, 0)$, $\vec{u} = (\cos\theta, \sin\theta)$, the angle between them is θ. $\quad \vec{i} \cdot \vec{u} = \cos\theta$.

After rotation through any angle α, they are still unit vectors.

$\vec{i}$ rotates to $\vec{v} = (\cos\alpha, \sin\alpha)$, $\vec{u}$ rotates to $\vec{w} = (\cos\beta, \sin\beta)$, $\beta = \theta + \alpha$.

$$\vec{v} \cdot \vec{w} = \underbrace{\boxed{\cos\alpha \cos\beta + \sin\alpha \sin\beta \overset{?}{=} \cos(\alpha - \beta)}}_{\text{trigonometric identity}} = \boxed{\cos(-\theta) = \cos\theta}$$

For unit vectors $\vec{v}$ and $\vec{w}$,

the angle θ between them has $\cos\theta = \vec{v} \cdot \vec{w}$. (Cosine formular for unit vectors, holds ture in n dimensions)

$-1 \le \cos\theta \le 1$ $\quad$ when $\vec{v}$ is perpendicular to $\vec{w}$, $\vec{v} \cdot \vec{w} = 0 = \cos 90°$ $\quad$ what if $\vec{v} \cdot \vec{w} = 1$

-1

when $\vec{v} \cdot \vec{w} \neq 0$ $\quad 1 \ge \vec{v} \cdot \vec{w} > 0$, then $\theta < 90°$

$-1 \le \vec{v} \cdot \vec{w} < 0$, then $\theta > 90°$

Find the angle between nonzero vectors $\vec{v}$ and $\vec{w}$.

Let θ be the angle between $\vec{v}$ and $\vec{w}$

$$\cos\theta = \frac{\vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{w}}{\|\vec{w}\|}$$

, where θ is also the angle between the unit vectors $\frac{\vec{v}}{\|\vec{v}\|}$ and $\frac{\vec{w}}{\|\vec{w}\|}$.

unit vectors

Cosine formula $\boxed{\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}}$ for nonzero $\vec{v}$ and $\vec{w}$.

$|\cos\theta| \le 1$. when $\vec{v}\cdot\vec{w} > 0$, then $\theta < 90^\circ$

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$=$ $=$

Example. Find the angle θ between $\vec{v} = (1, 0)$ and $\vec{w} = (1, 1)$

Since $\vec{v}\cdot\vec{w} = 1\times 1 + 0\times 1 = 1 > 0$

$\|\vec{v}\| = \sqrt{1^2+0^2} = 1$, $\|\vec{w}\| = \sqrt{1^2+1^2} = \sqrt{2}$

$$\cos\theta = \frac{1}{1\times\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = 45^\circ < 90^\circ$$

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Let θ be the angle between $\vec{v}$ and $\vec{w}$

$$\cos\theta = \frac{\vec{v}}{\|\vec{v}\|}\cdot\frac{\vec{w}}{\|\vec{w}\|}$$

, where θ is also the angle between the unit vectors $\frac{\vec{v}}{\|\vec{v}\|}$ and $\frac{\vec{w}}{\|\vec{w}\|}$.

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Cosine formula $\boxed{\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}}$ for nonzero $\vec{v}$ and $\vec{w}$.

$|\cos\theta| \le 1$. when $\vec{v}\cdot\vec{w} > 0$, then $\theta < 90^\circ$

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$\|\vec{v}\| = \sqrt{1^2+0^2} = 1$, $\|\vec{w}\| = \sqrt{1^2+1^2} = \sqrt{2}$

$$|\vec{v}\cdot\vec{w}| = \left|\|\vec{v}\|\|\vec{w}\|\cos\theta\right| = \|\vec{v}\|\,\|\vec{w}\|\,|\cos\theta| \le \|\vec{v}\|\,\|\vec{w}\|$$

Schwarz inequality $|\vec{v}\cdot\vec{w}| \le \|\vec{v}\|\,\|\vec{w}\|$.

$$\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$$

$$= \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 + 2\,\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 \le \|\vec{v}\|^2 + 2\overbrace{|\vec{v}\cdot\vec{w}|}^{\le\|\vec{v}\|\|\vec{w}\|} + \|\vec{w}\|^2 = (\|\vec{v}\|+\|\vec{w}\|)^2$$

...le inequality $\|\vec{v}+\vec{w}\|$

When $\vec{v}$ is perpendicular to $\vec{w}$

$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v}-\vec{w}\|^2$

For nonzero $\vec{v}$ and $\vec{w}$

$\|\vec{v}\| + \|\vec{w}\| \ ? \ \|\vec{v}+\vec{w}\|$

$$|\vec{v}\cdot\vec{w}| = \big|\,\|\vec{v}\|\|\vec{w}\|\cos\theta\,\big| = \|\vec{v}\|\,\|\vec{w}\|\,|\cos\theta| \le \|\vec{v}\|\,\|\vec{w}\|$$

Schwarz inequality $\boxed{|\vec{v}\cdot\vec{w}| \le \|\vec{v}\|\,\|\vec{w}\|}$

$$\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$$

$$= \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 \le \|\vec{v}\|^2 + 2\overbrace{|\vec{v}\cdot\vec{w}|}^{\le\|\vec{v}\|\|\vec{w}\|} + \|\vec{w}\|^2 = (\|\vec{v}\|+\|\vec{w}\|)^2$$

Triangle inequality $\boxed{\|\vec{v}+\vec{w}\| \le \|\vec{v}\|+\|\vec{w}\|}$

inequalities for n dimensional vectors.

When $\vec{v}$ is perpendicular to $\vec{w}$

$$\|\vec{v}\|^2+\|\vec{w}\|^2 = \|\vec{v}-\vec{w}\|^2$$

For nonzero $\vec{v}$ and $\vec{w}$

$$\|\vec{v}\| + \|\vec{w}\| \quad ? \quad \|\vec{v}+\vec{w}\|$$