

The big formula  $\det A = \sum (\det P, \alpha_{1,j_1} \alpha_{2,j_2} \dots \alpha_{n,j_n})$  is a direct definition of the determinant.

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$$

When  $n=4$ ,

there are  $4! = 24$  terms.

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}$$

$$\begin{aligned}
 &= \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32} \\
 &\quad - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{13}\alpha_{22}\alpha_{31} \\
 &= \alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}) + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}) \\
 &= \begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} = C_{11} \\
 &= \alpha_{11}C_{11} + \alpha_{12}C_{12} + \alpha_{13}C_{13}
 \end{aligned}$$

$$C_{12} = - \begin{vmatrix} \alpha_{21} & \alpha_{23} \\ \alpha_{31} & \alpha_{33} \end{vmatrix} \quad \text{"factors"}$$

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ A_{31} & 0 & A_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & A_{13} \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & 0 \end{vmatrix}$$

Choose one entry  
from each row and each column

Associated with  $A_{ij}$ , we cross out row 1 and column  $j$  from  $A$

$$A = \begin{bmatrix} A_{11} & \cancel{A_{1j}} & A_{1n} \\ A_{21} & \cdots & \cancel{A_{2j}} & \cdots & A_{2n} \\ \vdots & & & & \\ A_{n1} & \cdots & \cancel{A_{nj}} & \cdots & A_{nn} \end{bmatrix}$$

to get a submatrix  $M_{1j}$  of size  $n-1$

$$\text{The cofactor } C_{1j} = (-1)^{1+j} \det M_{1j}$$

$$\det A = A_{11}C_{11} + A_{12}C_{12} + \dots + A_{1n}C_{1n}$$

Cofactor expansion.

Each  $A_{ij}$  has its cofactor  $|C_{ij} = (-1)^{i+j} \det M_{ij}|$

$$\det A = A_{11}\det M_{11} - A_{12}\det M_{12} + \dots + (-1)^{1+n} A_{1n}\det M_{1n}$$

The determinant of order  $n$

The cofactor formula along row  $i$  is  $\det A = A_{i1}C_{i1} + A_{i2}C_{i2} + \dots + A_{in}C_{in}$  is a combination of  $n$  determinants of order  $n-1$ .

$$\det A = \det A^T \quad \text{The cofactor formula along column } j \text{ is } \det A = A_{1j}C_{1j} + A_{2j}C_{2j} + \dots + A_{nj}C_{nj}$$

Example

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} d = d$$

$$C_{12} = (-1)^{1+2} C = -C$$

$$C_{21} = (-1)^{1+2} b = -b$$

$$C_{22} = (-1)^{2+2} a = a$$

$$\left| \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right|$$

$$= a_{11} C_{11} + a_{12} C_{12}$$

$$= 2(-1)^{1+1} \left| \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right| + (-1)^{1+2} \left| \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right|$$

$$= 2 \left[ 2(-1)^{1+1} \left| \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right| + (-1)^{1+2} \left| \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right| \right] + (1)(-1)(-1)^{1+1} \left| \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right|$$

$$= 2[2 \times 3 + (-2)] - 3 = 5$$

### 3.3 Cramer's rule, inverses and volumes

$$\det(AB) = \det A \det B.$$

Cramer's rule, formula for the solution to  $A\vec{x} = \vec{b}$ )

Suppose  $\vec{x}$  is a solution to  $A\vec{x} = \vec{b}$ .  $I = [\vec{e}_1 \vec{e}_2 \vec{e}_3]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = A[\vec{x} \vec{e}_2 \vec{e}_3] = [\vec{b} A\vec{e}_2 A\vec{e}_3] = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1$$

Then  $\det A \vec{x} = \det B_1$ . If  $\det A \neq 0$ ,  $x_1 = \frac{\det B_1}{\det A}$  is the first component of the solution to  $A\vec{x} = \vec{b}$ .

Similarly,  $A \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} = B_2$  gives  $\det A x_2 = \det B_2$ . So  $x_2 = \frac{\det B_2}{\det A}$

Example. Solve  $3x_1 + 4x_2 = 2$  by Cramer's rule

$$5x_1 + 6x_2 = 4$$

$$\text{The solution is } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}} = \frac{-4}{-2} = 2, x_2 = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}} = \frac{2}{-2} = -1$$

Theorem. If  $\det A \neq 0$ ,  $A\vec{x} = \vec{b}$  is solved by determinants as

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \quad \dots, \quad x_n = \frac{\det B_n}{\det A}$$

Each matrix  $B_i, i=1, 2, \dots, n$  has the  $i$ -th column of  $A$  replaced by  $\vec{b}$ .

Example.  $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ , and we assume  $\det A \neq 0$ . We write  $A^{-1} = [\vec{x} \quad \vec{y}] = \begin{bmatrix} |A\vec{x}| & |A\vec{y}| \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = A A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

We have  $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$   
 $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}} = A\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

By Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\det A} = \frac{d}{\det A} \quad x_2 = \frac{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}{\det A} = \frac{-c}{\det A}$$

$$y_1 = \frac{\begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix}}{\det A} = \frac{-b}{\det A} \quad y_2 = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{\det A} = \frac{a}{\det A}$$

Each entry in  $A^{-1}$  is a cofactor divided by  $\det A$ .

(Cramer's rule)  
evaluates  $n+1$  determinants.