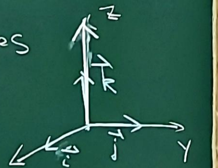
A vector $\vec{u}$ is called a unit vector if $\|\vec{u}\| = 1$

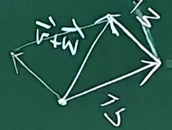
$\vec{v} \neq \vec{0}$ if and only if $\|\vec{v}\| \neq 0$. Then $\dfrac{\vec{v}}{\|\vec{v}\|}$ is the unit vector in the same direction as $\vec{v}$.

The vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors along the positive x, y, z axes



For $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

For nonzero vectors $\vec{v}$ and $\vec{w}$ in 3-dimensional space $\boxed{\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \cos\theta}$, where θ is the angle between $\vec{v}$ and $\vec{w}$.



Since $|\cos\theta| \leq 1$ for any θ, $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|$, holds in n dimensional space.

1 Schwarz inequality If $\vec{v}$ and $\vec{w}$ are n dimensional vectors, $\boxed{|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|}$

$$\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}\|^2 + 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2 \leq \|\vec{v}\|^2 + 2\|\vec{v}\|\,\|\vec{w}\| + \|\vec{w}\|^2 = (\|\vec{v}\|+\|\vec{w}\|)^2$$

2 Triangle inequality $\boxed{\|\vec{v}+\vec{w}\| \le \|\vec{v}\|+\|\vec{w}\|}$ (It is equivalent to $\|\vec{v}+\vec{w}\|^2 \le (\|\vec{v}\|+\|\vec{w}\|)^2$.

Example. Let $\vec{v}=(3,4)$, $\vec{w}=(4,3)$. Find the cosine of the angle between them, and check the above inequalities

Sol. $\vec{v}\cdot\vec{w} = 3\times4+4\times3 = 24$ $\quad \|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{3^2+4^2} = 5 = \|\vec{w}\|$. $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \dfrac{24}{5\times5} = \dfrac{24}{25}$

$|\vec{v}\cdot\vec{w}| = 24 < 25 = \|\vec{v}\|\|\vec{w}\|$

$\vec{v}+\vec{w} = (3+4, 4+3) = (7,7)$. $\|\vec{v}+\vec{w}\|^2 = 7^2+7^2 = 98 < 10^2 = (\|\vec{v}\|+\|\vec{w}\|)^2$

So $\|\vec{v}+\vec{w}\| = 7\sqrt{2} < 10 = \|\vec{v}\|+\|\vec{w}\|$ #

Except ~~zero~~ zero vectors, which kind of $\vec{v}$ and $\vec{w}$ give equalities such that $|\vec{v}\cdot\vec{w}| = \|\vec{v}\|\|\vec{w}\|$ holds if $|\cos\theta|=1$, or $\cos\theta = 1$ or -1

$|\vec{v}\cdot\vec{w}| = \|\vec{v}\|\|\vec{w}\||\cos\theta|$

$\|\vec{v}+\vec{w}\| = \|\vec{v}\|+\|\vec{w}\|$

It means $\theta = 0^\circ$ or 180°

$\vec{w} = c\vec{v}$, $c>0$.

$\vec{w} = c\vec{v}$, $c\neq0$.

$\|\vec{v}+\vec{w}\| = (1+c)\|\vec{v}\|$.

Example. For $\vec{v} = (2, -1)$ and $\vec{w} = (-1, 2)$, find a vector $\vec{x}$ such that $\vec{v} \cdot \vec{x} = 3$ and $\vec{w} \cdot \vec{x} = 0$

$\vec{x} = (x, y)$

The 2 dot products give 2 equations for x and y

$$2x - y = 3$$
$$-x + 2y = 0.$$

"Matrix equation"

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{matrix times vector}} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

1.3 Matrices

We start with $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Their linear combinations are $x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = x_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$ $(*)$

Let $\vec{u}, \vec{v}, \vec{w}$ go into the columns of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Goal: Rewrite $(*)$ using A.

Set $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

How to define this matrix-vector multiplication ?

· Let $[y_1 \ y_2 \ y_3]$ be a row vector with components y_1, y_2, y_3. It is different from the column vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = (y_1, y_2, y_3)$

(1) (The row way) Take the dot product of each row of A with $\vec{x}$. (dot product of a row vector with the column vector).

$$(\text{row } 1 \text{ of } A) \cdot \vec{x} = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (1, 0, 0) \cdot (x_1, x_2, x_3) = x_1$$

$$A\vec{x} = \begin{bmatrix} (1,0,0)\cdot(x_1,x_2,x_3) \\ (-1,1,0)\cdot(x_1,x_2,x_3) \\ (0,-1,1)\cdot(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1+x_2 \\ -x_2+x_3 \end{bmatrix}$$

Each row of A has the same number of components as $\vec{x}$.

(2) (The column way) Use a combination of the columns of A.

$$A\vec{x} = [\vec{u} \ \vec{v} \ \vec{w}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$$

The matrix A times the vector $\vec{x}$ is the same as the linear combination of the columns of A.

For n vectors in m dimensions:

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \vec{v}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ go into the columns of

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The size of the matrix A is m by n or $m \times n$

where m is the number of rows,

n columns

Matrices of size n by n is called square matrix

Matrix notation.

Describe all combinations of the columns of A.

Find $\vec{x}$ produce $\vec{b} = A\vec{x}$.

There are mn entries.

In each entry a_{ij}, $1 \le i \le m$, $1 \le j \le n$.

i gives the row number,

j column

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{11}, a_{22}, a_{33}, \dots$ are called the main diagonal

Key questions:

1. Describe all the combinations of

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n$$

2. Find $x_1, x_2, \dots, x_n$ such that produce a desired $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n$

Example. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, find $A\vec{x}$.

$\overline{I}$ is of size 3 by 2

Sol: (1) By rows. $A\vec{x} = \begin{bmatrix} (1,2)\cdot(7,8) \\ (3,4)\cdot(7,8) \\ (5,6)\cdot(7,8) \end{bmatrix} = \begin{bmatrix} 7+16 \\ 21+32 \\ 35+48 \end{bmatrix}$

(2) By columns. $A\vec{x} = 7\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 7+16 \\ 21+32 \\ 35+48 \end{bmatrix}$

Example. Let $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For every $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \vec{v}$$

3 by 3 identity matrix

$I\vec{v} = \vec{v}$, for every $\vec{v}$

Key questions:

1. Describe all the combinations of

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

2. Find $x_1, x_2, \ldots x_n$ such that produce a desired $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$

Matrix notation.

Describe all combinations

of the columns of A.

Find $\vec{x}$ produce $\vec{b} = A\vec{x}$.