Fundamentals of Electric Circuits

CHAPTER 8 AC Power Analysis

Lingling Cao, PhD, Associate Professor

Email: caolingling@hit.edu.cn

CHAPTER 8 Sinusoids and Phasors

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8.2 Instantaneous and average power

Instantaneous Power

- Consider the generalized case where the voltage and current at the terminals of a circuit are:

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad i(t) = I_m \cos(\omega t + \theta_i)$$

- The instantaneous power $p(t)$ is the power at any instant of time.
- The $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ and the instantaneous current $i(t)$.

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

Instantaneous Power

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

- Note that the power has two parts.
  - One is constant, depending on the phase difference between the voltage and current
  - The second is a sinusoidal function whose frequency 2ω, which is twice the frequency of the voltage or current.
- A sketch of the possible instantaneous power is below.

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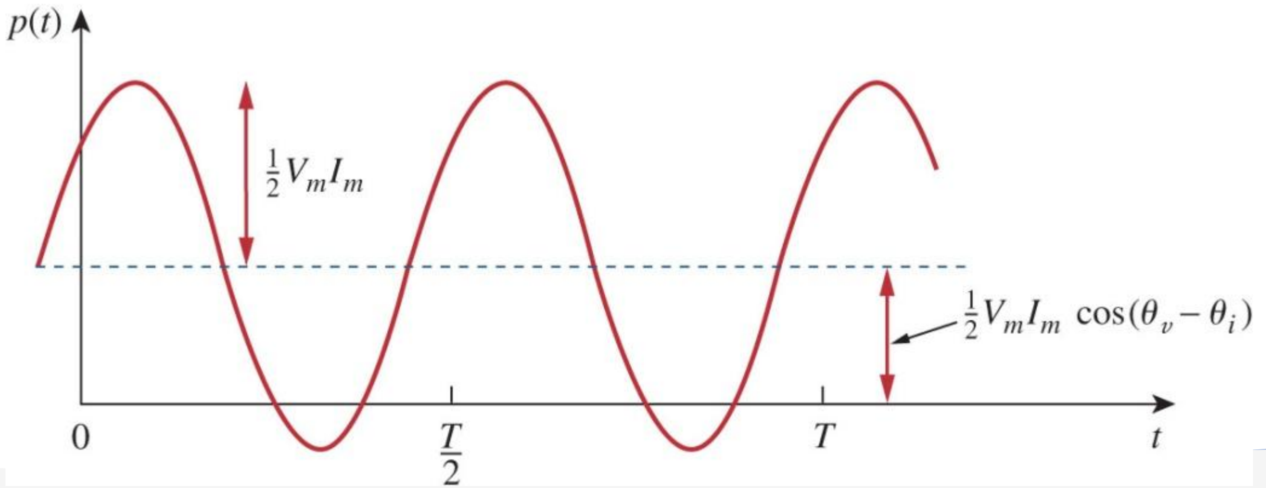
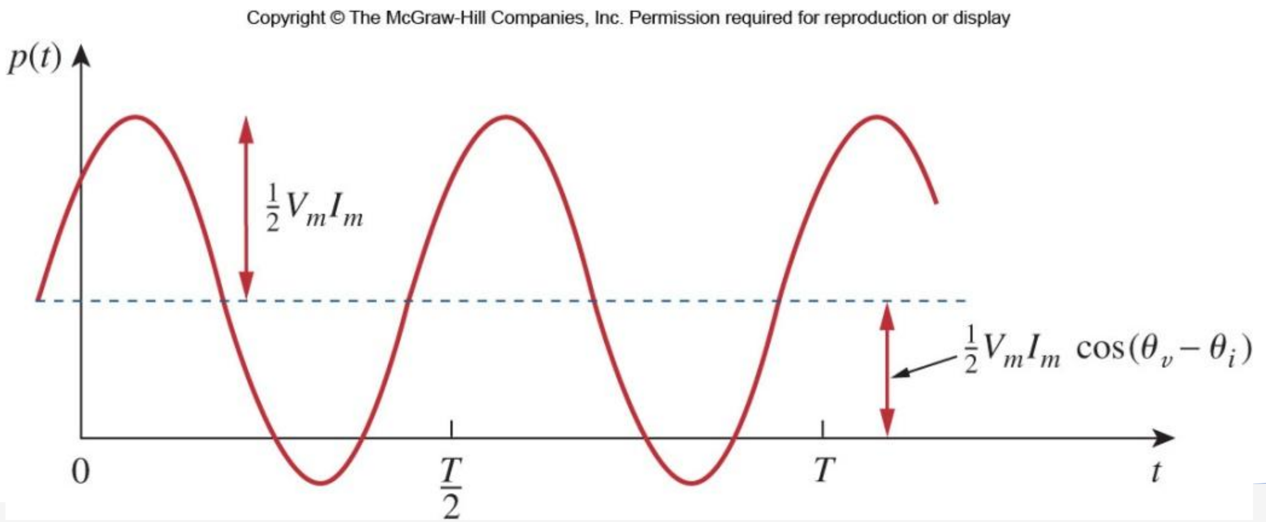


Instantaneous Power

- Note that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. This is possible with elements like inductors or capacitors which can store and release energy.
- The instantaneous power changes with time and is therefore difficult to measure.
- The more common power measured is the average power.



Average Power

- Average power is the average of the instantaneous power over one period.
- It is given by:

$$P = \frac{1}{T}\int_0^T p(t)dt$$

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt$$

- The average power depends on the difference in the phases of the voltage and current.

$$\boxed{P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

Average Power

- To find the instantaneous power, we need to work in the time domain.
- But for average power it is possible to work in frequency domain.

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad i(t) = I_m \cos(\omega t + \theta_i)$$

$$\dot{V} = V_m \angle \theta_v \qquad \dot{I} = I_m \angle \theta_i$$

- In this case, the average power is:

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\dot{V}\dot{I}^*] = \mathrm{Re}[\frac{1}{2}V_m I_m \angle(\theta_v - \theta_i)] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

Two special cases

$$P = \frac{1}{2}\mathrm{Re}[\dot{V}\dot{I}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

- When $\boldsymbol{\theta_v - \theta_i = 0}$, the voltage and current are in phase and the circuit is purely **resistive**.

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}V_m^2/R$$

- When $\boldsymbol{\theta_v - \theta_i = \pm 90°}$, the circuit absorbs no power and is purely **reactive.**

$$P = \frac{1}{2}V_m I_m \cos 90° = 0$$

A resistive load absorbs power at all times, while a reactive load absorbs zero average power.

Example

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0^\circ$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0^\circ}{30 - j70} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} = 1.576\angle 66.8^\circ \text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24 \text{ W}$$

Example

Practice Problem 11.2

A current $\mathbf{I} = 33\angle 30^\circ$ A flows through an impedance $\mathbf{Z} = 40\angle -22^\circ\ \Omega$. Find the average power delivered to the impedance.

Answer: 20.19 kW.

Example

Example 11.3

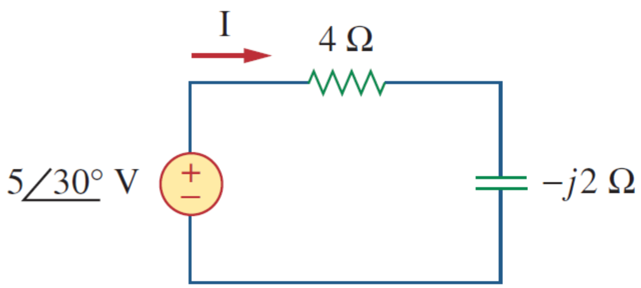


Figure 11.3
For Example 11.3.

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

Solution:

The current **I** is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

$$P = -\frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = -2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I}_R = \mathbf{I} = 1.118\angle 56.57^\circ \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

Example

Practice Problem 11.3

In the circuit of Fig. 11.4, calculate the average power absorbed by the resistor and inductor. Find the average power supplied by the voltage source.

Answer: 15.361 kW, 0 W, -15.361 kW.

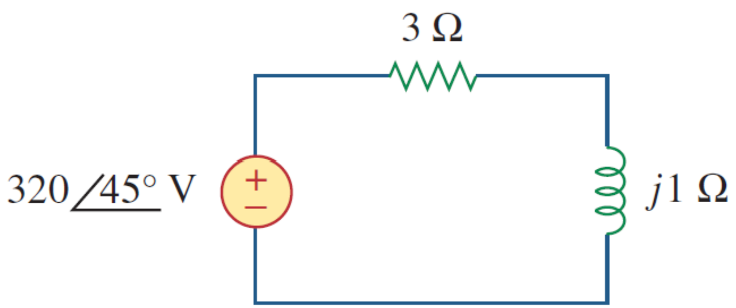
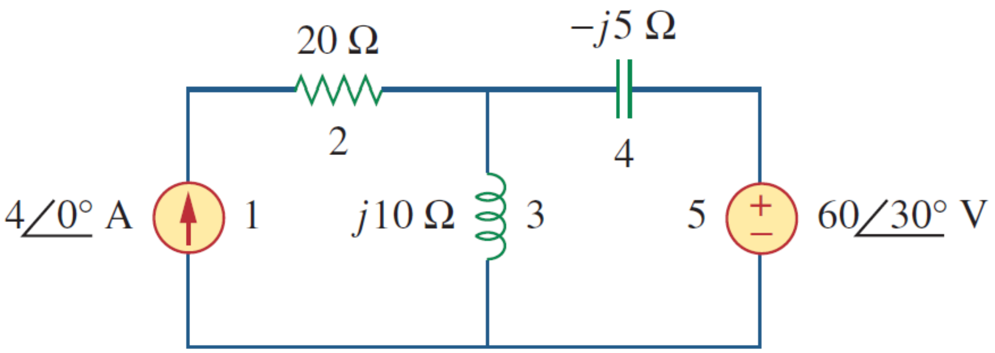


Figure 11.4
For Practice Prob. 11.3.

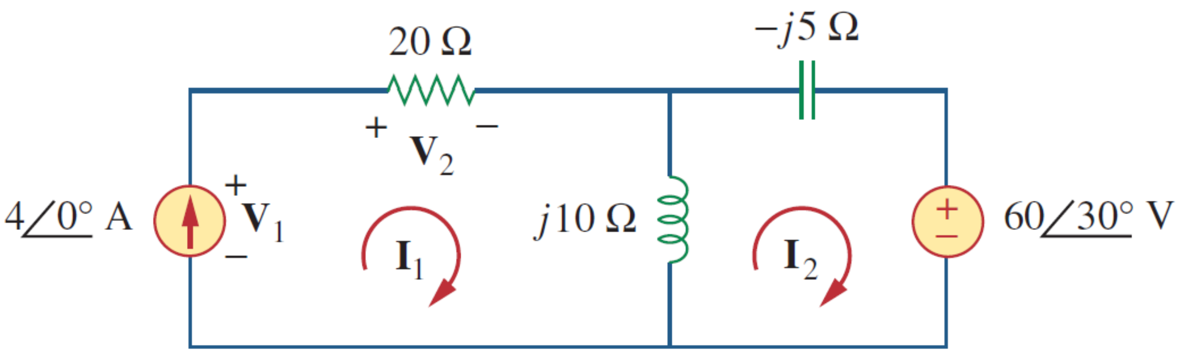
Example

Example 11.4

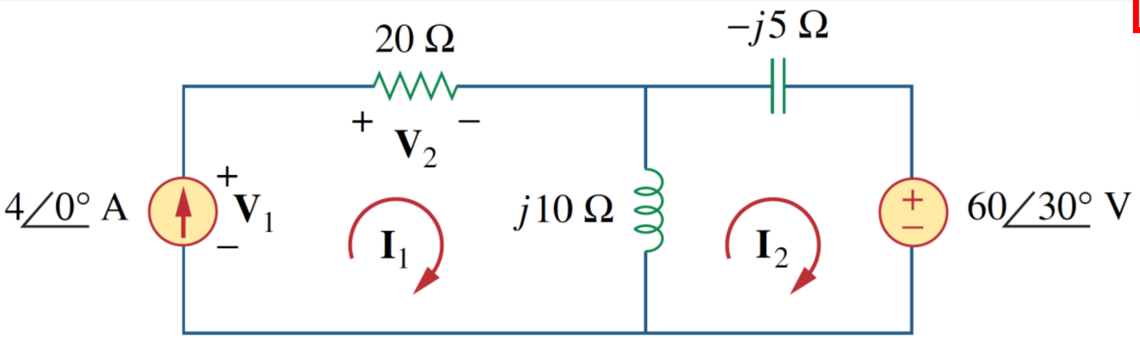
Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).



(a)



(b)



Solution:

We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\underline{/30^\circ} = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\underline{/30^\circ} + j40 \quad \Rightarrow \quad \mathbf{I}_2 = -12\underline{/-60^\circ} + 8 = 10.58\underline{/79.1^\circ} \text{ A}$$

For the voltage source, the current flowing from it is $\mathbf{I}_2 = 10.58\underline{/79.1^\circ}$ A and the voltage across it is $60\underline{/30^\circ}$ V, so that the average power is

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

Following the passive sign convention (see Fig. 1.8), this average power is absorbed by the source, in view of the direction of $\mathbf{I}_2$ and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

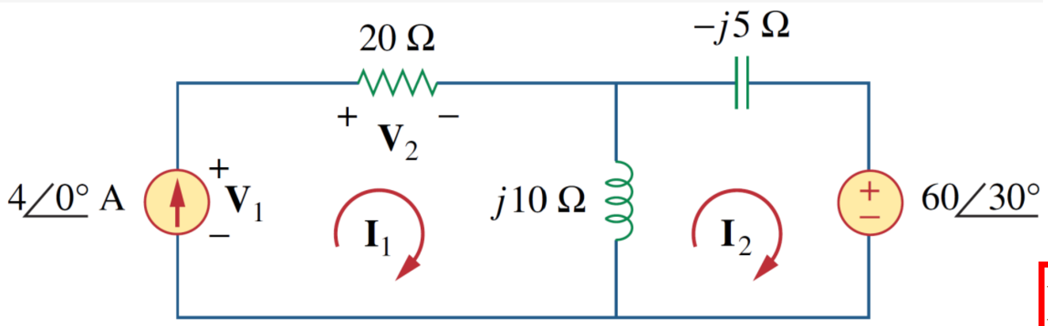
For the current source, the current through it is $\mathbf{I}_1 = 4\underline{/0^\circ}$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\underline{/6.21^\circ} \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.



For the resistor, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is $20\mathbf{I}_1 = 80\angle 0^\circ$, so that the power absorbed by the resistor is

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor, the current through it is $\mathbf{I}_2 = 10.58\angle 79.1^\circ$ and the voltage across it is $-j5\mathbf{I}_2 = (5\angle -90^\circ)(10.58\angle 79.1^\circ) = 52.9\angle 79.1^\circ - 90^\circ$. The average power absorbed by the capacitor is

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor, the current through it is $\mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\angle -79.1^\circ$. The voltage across it is $j10(\mathbf{I}_1 - \mathbf{I}_2) = 10.58\angle -79.1^\circ + 90^\circ$. Hence, the average power absorbed by the inductor is

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or

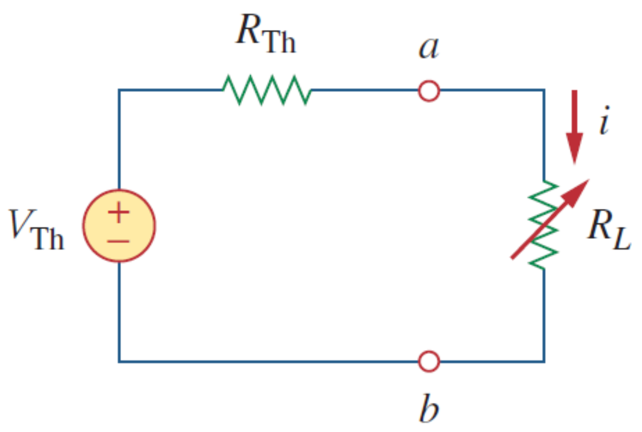
$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

indicating that power is conserved.

8.3 Maximum Average Power Transfer

- Previously we considered how to maximize the power delivered to a resistive load.

- It is shown that maximum power is transferred when the load resistance is equal to the Thevenin's resistance of the supply circuit.



- This theorem can be extended to AC circuits.

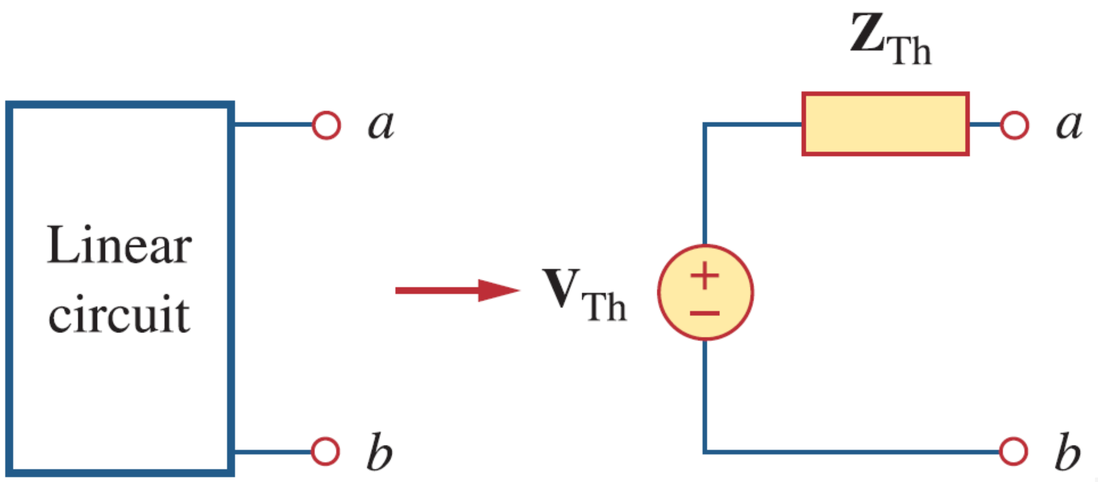
Thevenin’s Theorem

- Thevenin’s and Norton’s theorems can be applied to AC circuits in the same way as to dc circuits.
- A linear circuit is replaced by a voltage source in series with an impedance, where

V_{th} is the open-circuit voltage at the terminals;

Z_{th} is the equivalent impedance at the terminals when the independent sources are turned off.

- The only difference is the fact that we need to calculate complex numbers.



Maximum Average Power

- In rectangular form, the Thevenin impedance Z_{Th} and load impedance Z_L are:

$$Z_{Th} = R_{Th} + jX_{Th}$$
$$Z_L = R_L + jX_L$$

- The current through the load is:

$$I = \frac{V_{Th}}{Z_{Th} + Z_L} = \frac{V_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)}$$

- The average power delivered to the load is:

$$P = \frac{1}{2}|I|^2 R_L = \frac{|V_{Th}|^2 R_L/2}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

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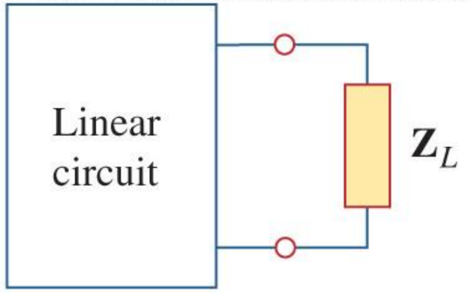
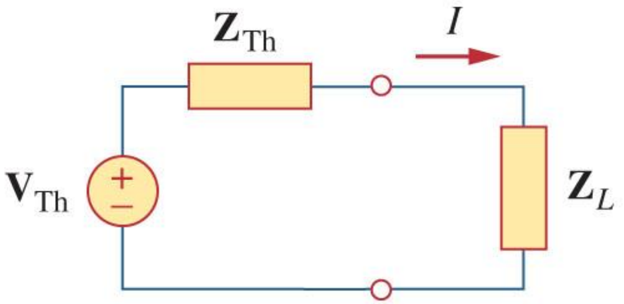


(a)



(b)

Maximum Average Power

- The average power delivered to the load is:

$$P = \frac{1}{2}|I|^2 R_L = \frac{|V_{Th}|^2 R_L/2}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

- The objective is to adjust the load impedance $Z_L = R_L + jX_L$ so that P is maximum.
- We set $\boldsymbol{\partial P / \partial X_L}$=0 and $\boldsymbol{\partial P / \partial R_L}$=0
- For $\boldsymbol{\partial P / \partial X_L}$=0

$$\frac{\partial P}{\partial X_L} = -\frac{|V_{Th}|^2 R_L (X_{Th} + X_L)}{[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} \qquad \boldsymbol{X_L = -X_{Th}}$$

Maximum Average Power

- The objective is to adjust the load impedance $Z_L = R_L + jX_L$ so that P is maximum.

- The average power delivered to the load is:

$$P = \frac{1}{2}|I|^2 R_L = \frac{|V_{Th}|^2 R_L/2}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

- For $\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{R_L}}=0$

$$\frac{\partial P}{\partial R_L} = \frac{|V_{Th}|^2[(R_{Th}+R_L)^2+(X_{Th}+X_L)^2 - 2R_L(R_{Th}+R_L)]}{2[(R_{Th}+R_L)^2+(X_{Th}+X_L)^2]^2}$$

$$\boxed{R_L = \sqrt{R_{Th}^{\ 2} + (X_{Th}+X_L)^2} = R_{Th}}$$

Maximum Average Power Transfer Theorem

- For maximum average power, the load impedance is equal to the complex conjugate of the Thevenin impedance Z_{Th}.

$$Z_L = R_L + jX_L = R_{Th} - jX_{Th} = Z_{Th}^*$$

- The maximum average power is:

$$P = \frac{1}{2}|I|^2 R_L = \frac{|V_{Th}|^2 R_L/2}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2} \Rightarrow P_{\max} = \frac{|V_{Th}|^2}{8R_{Th}}$$

A special case when the load is purely resistive

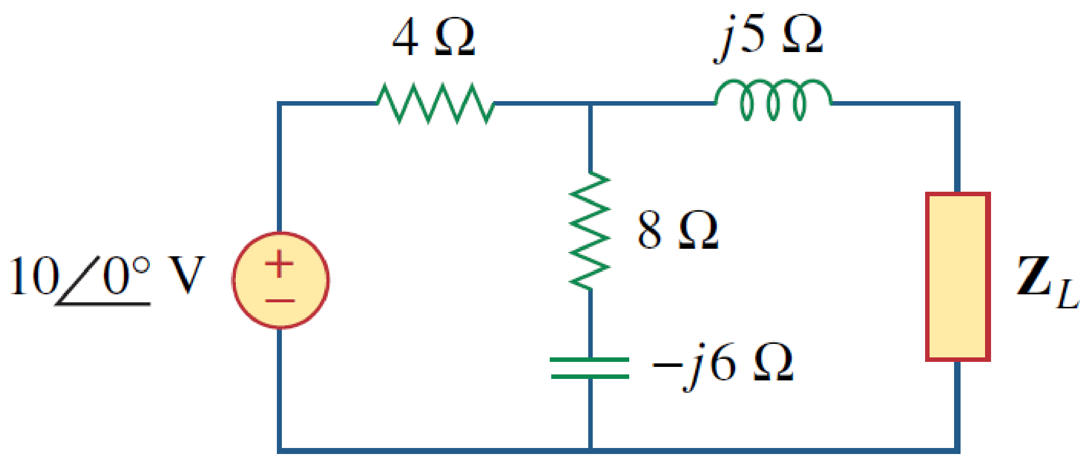
$$Z_{Th} = R_{Th} + jX_{Th} \qquad\qquad Z_L = R_L$$

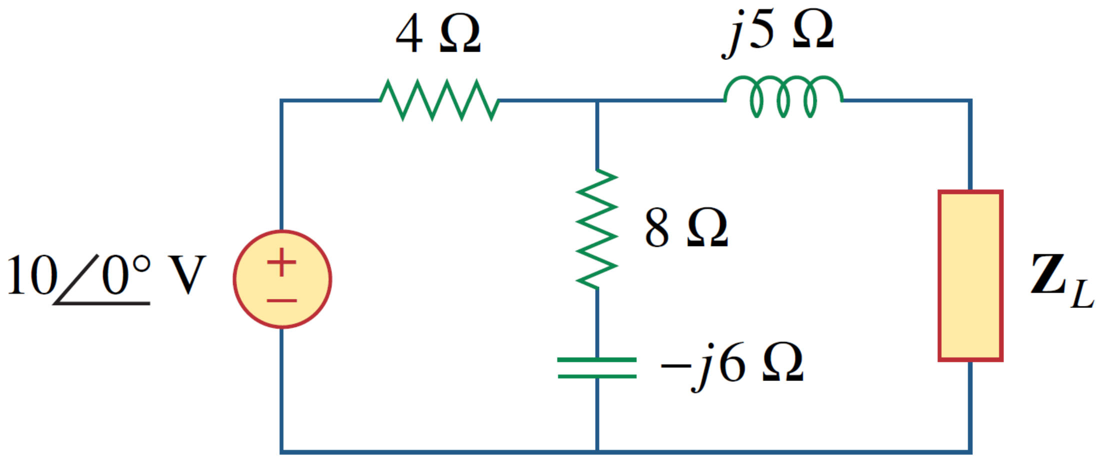
For $\frac{\partial P}{\partial R_L}=0$ $\qquad R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2} = \sqrt{R_{Th}^2 + X_{Th}^2} = |Z_{Th}|$

For maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin's Impedance.

Example

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?





Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{\text{Th}}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{\text{Th}} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

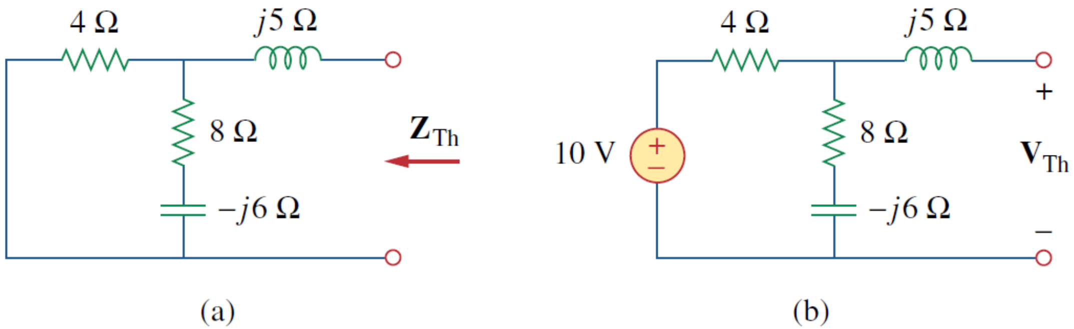


Figure 11.9
Finding the Thevenin equivalent of the circuit in Fig. 11.8.

To find $\mathbf{V}_{\text{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\angle{-10.3^\circ}\ \text{V}$$

The load impedance draws the maximum power from the circuit when

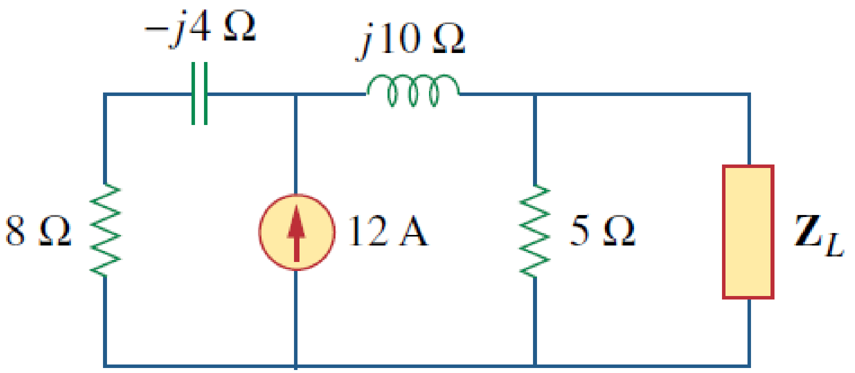
$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

Example

Practice Problem 11.5



For the circuit shown in Fig. 11.10, find the load impedance $\mathbf{Z}_L$ that absorbs the maximum average power. Calculate that maximum average power.

Answer: $3.415 - j0.7317\ \Omega$, 51.47 W.

Example

Example 11.6

In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.

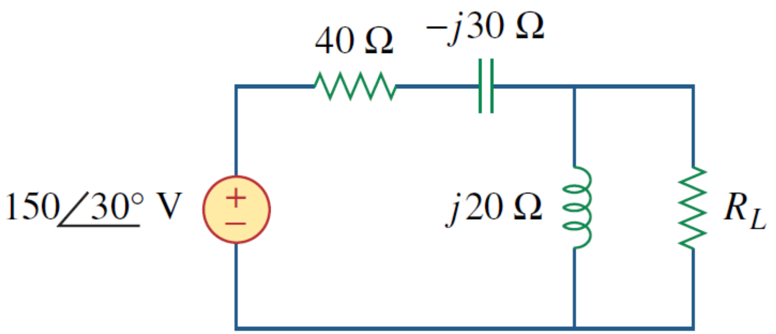


Figure 11.11
For Example 11.6.

Solution:

We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{Th} = (40 - j30) \,\|\, j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{Th} = \frac{j20}{j20 + 40 - j30}(150\underline{/30^\circ}) = 72.76\underline{/134^\circ}\ \text{V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{Th}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + R_L} = \frac{72.76\underline{/134^\circ}}{33.66 + j22.35} = 1.8\underline{/100.42^\circ}\ \text{A}$$

The maximum average power absorbed by R_L is

$$P_{max} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$$

Example

Practice Problem 11.6

In Fig. 11.12, the resistor R_L is adjusted until it absorbs the maximum average power. Calculate R_L and the maximum average power absorbed by it.

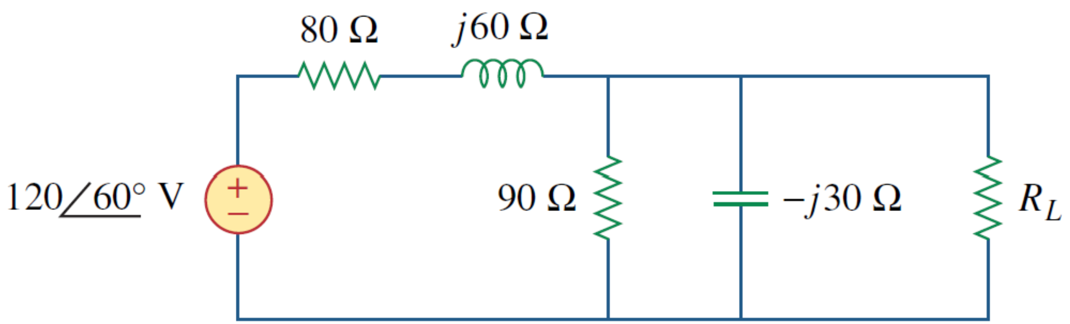


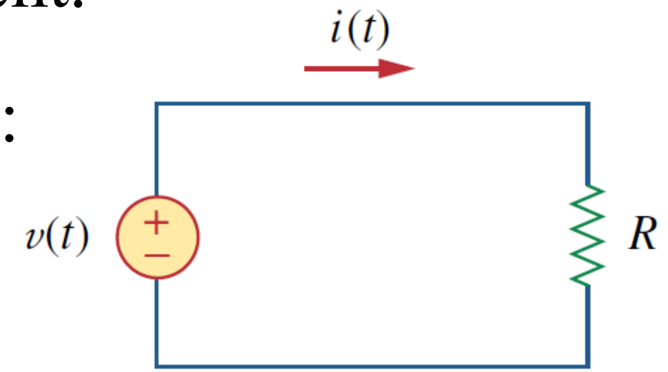
Figure 11.12
For Practice Prob. 11.6.

Answer: 30 Ω, 6.863 W.

8.4 Effective or RMS Value

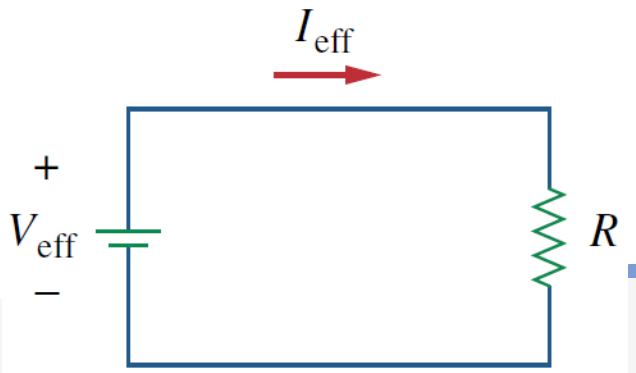
- When a time-varying source is delivering power to a resistive load, we often want to know the effectiveness of the source in delivering power.
- The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.
- For a periodic current, the average power absorbed is:

$$P = \frac{1}{T}\int_0^T i^2 R dt = \frac{R}{T}\int_0^T i^2 dt$$



- For a DC current, the average power absorbed is:

$$P = I_{eff}^2 R$$



Average Power

$$P = \frac{1}{T}\int_{0}^{T} i^2 R dt = \frac{R}{T}\int_{0}^{T} i^2 dt \quad = I_{eff}^2 R$$

- The effective DC current yields:

$$I_{eff} = \sqrt{\frac{1}{T}\int_{0}^{T} i^2 dt}$$

- The effective voltage is found in a similar manner:

$$V_{eff} = \sqrt{\frac{1}{T}\int_{0}^{T} v^2 dt}$$

The effective value is the (square) root of the mean (or average) of the square of the periodic signal.

RMS (root mean square)

- The effective value of a periodic signal is known as the root-mean-square (RMS) value.

- For any periodic function x(t) in general, regardless of its shape, the rms value is given by

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2 dt}$$

RMS

- However, for the sinusoidal waveform $i(t) = I_m \cos(\omega t)$, the RMS value is related to the amplitude as follows:

$$I_{rms} = \sqrt{\frac{1}{T}\int_o^T \mathrm{I}_m^2 \cos^2 \omega t dt} = \frac{\mathrm{I}_m}{\sqrt{2}}$$

- For $v(t) = V_m \cos(\omega t)$,
- $$V_{rms} = \frac{V_m}{\sqrt{2}}$$
- The average power can also be written in terms of RMS values:

$$P = \frac{1}{2}\mathrm{Re}[VI^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}V_m^2/R = I_{rms}^2 R = \frac{V_{rms}^2}{R}$$

RMS

- The power industries specify phasor magnitudes in terms of their rms values rather than peak values.
- In China, the 220V available at every household is the rms value of the voltage from the power company.
- Analog voltmeters and ammeters are designed to read directly the rms value of voltage and current.

Example

Example 11.7

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

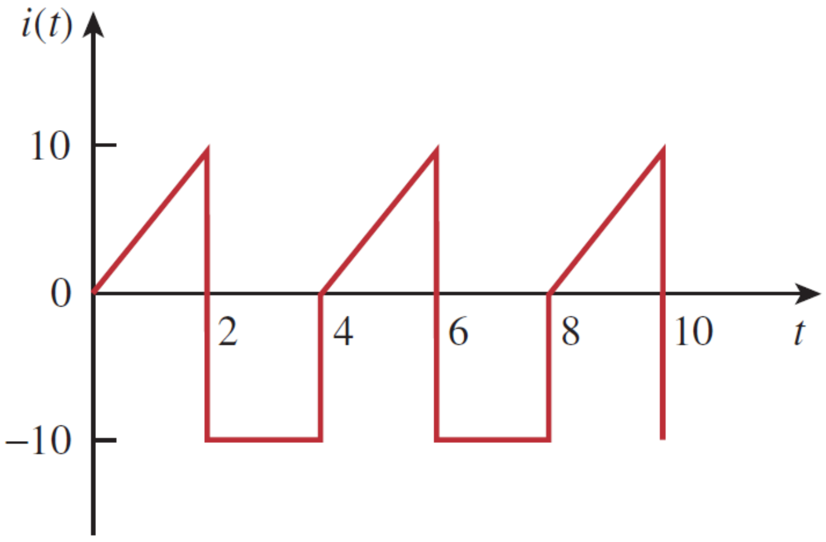


Figure 11.14
For Example 11.7.

Solution:

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\Big|_0^2 + 100t\Big|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165\text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3\text{ W}$$

Example

Example 11.8

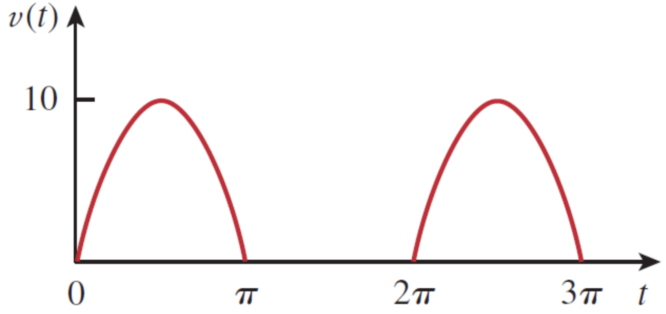


Figure 11.16
For Example 11.8.

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^\pi (10\sin t)^2\,dt + \int_\pi^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^\pi \frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Bigg|_0^\pi$$

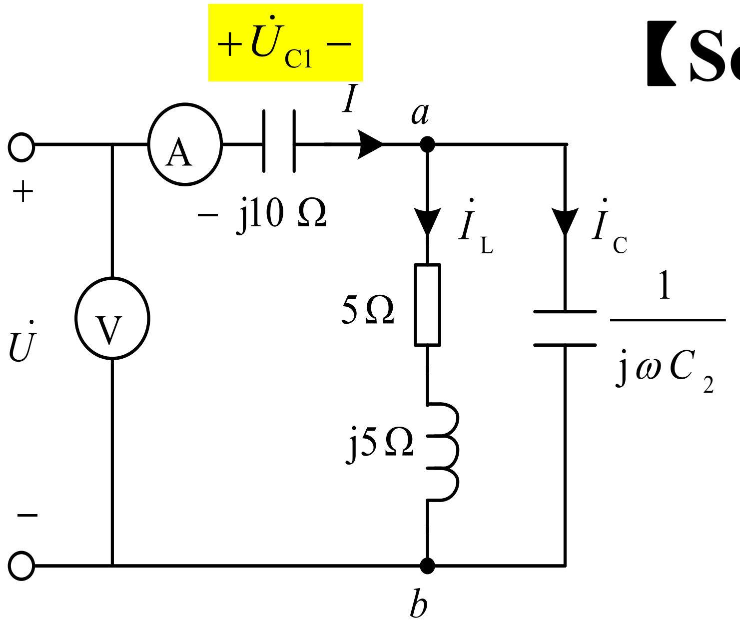
$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

Example Given the magnitude of $\dot{I}_c$ $\dot{U}_{ab}$ is: $I_C = 10\text{A}$, $U_{ab} = 100\text{V}$ 。

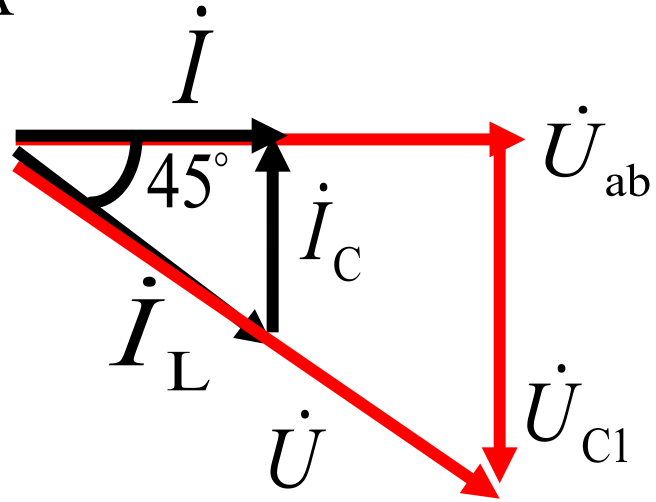
Find the readings of the voltmeter V and the ammeter A.



【Solution】 $\dot{U}_{ab} = 100\angle 0^\circ\ \text{V}$

$$\dot{I}_L = \frac{\dot{U}_{ab}}{5+\text{j}5} = \frac{100\angle 0^\circ}{5+\text{j}5} = 10\sqrt{2}\angle -45^\circ\ \text{A}$$

$$\dot{I}_C = 10\angle 90^\circ = \text{j}10\text{A}$$



$$\dot{I} = \dot{I}_C + \dot{I}_L = 10\angle 90^\circ + 10\sqrt{2}\angle -45^\circ = 10\angle 0^\circ\ \text{A}$$

$$\dot{U}_{C1} = \dot{I}(-\text{j}10) = 10\times(-\text{j}10) = -\text{j}100\text{V}$$

$$\dot{U} = \dot{U}_{C1} + \dot{U}_{ab} = -\text{j}100\text{V} + 100\text{V} = 141.1\angle -45^\circ\ \text{V}$$

The reading on the voltmeter is 100 V， and the readings of the ammeter A is 7.07A。

8.5 Apparent Power

Suppose:

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$\boldsymbol{V} = V_m \angle \theta_v \qquad \boldsymbol{I} = I_m \angle \theta_i$$

The average power:

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$= \boldsymbol{V_{rms} I_{rms}} \cos(\theta_v - \theta_i)$$

$$= \boldsymbol{S} \cos(\theta_v - \theta_i)$$

The average power is a product of two terms.

The product of the rms values $\boldsymbol{V_{rms} I_{rms}}$ is known as the **apparent power S**.

The apparent power is measured in **VA**, to distinguish it from the average or real power (**W**).

The factor $\boldsymbol{\cos(\theta_v - \theta_i)}$ is called the **power factor (pf)**.

Power Factor

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- The power factor is the ratio of the average power dissipated in the load to the apparent power.
- The power factor is the cosine of the phase difference between voltage and current.
- The angle $\boldsymbol{\theta_v - \theta_i}$ is called the power factor angle.
- This is equal to the angle of the load impedance, if ***V*** is the voltage across the load and ***I*** is the current through it.

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle\left(\theta_v - \theta_i\right)$$

Power Factor

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- The power factor can range from zero to unity.
- For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf=1, the average power is equal to the apparent power.
- For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$ and pf=0, the average power is zero.
- In between, pf is said to be leading or lagging.
- Leading power factor means that current leads voltage, implying a capacitive load.
- Lagging power factor means that current lags voltage, implying an inductive load.

Example

Example 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\underline{/-20^\circ}}{4\underline{/10^\circ}} = 30\underline{/-30^\circ} = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = \cancel{-j}15 = \cancel{-}\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Example

Example 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\underline{/-13.24^\circ}\ \Omega$$

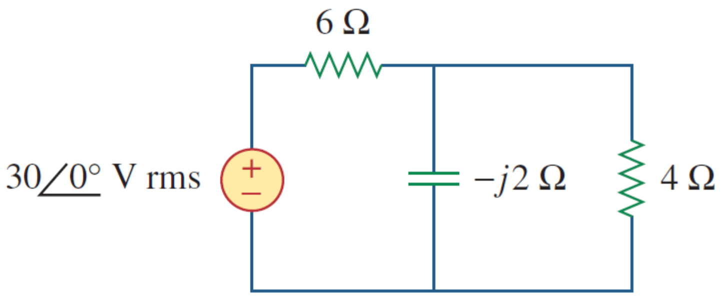


Figure 11.18
For Example 11.10.

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\underline{/0^\circ}}{7\underline{/-13.24^\circ}} = 4.286\underline{/13.24^\circ}\ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\, \text{pf} = (30)(4.286)0.9734 = 125\ \text{W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2 (6.8) = 125\ \text{W}$$

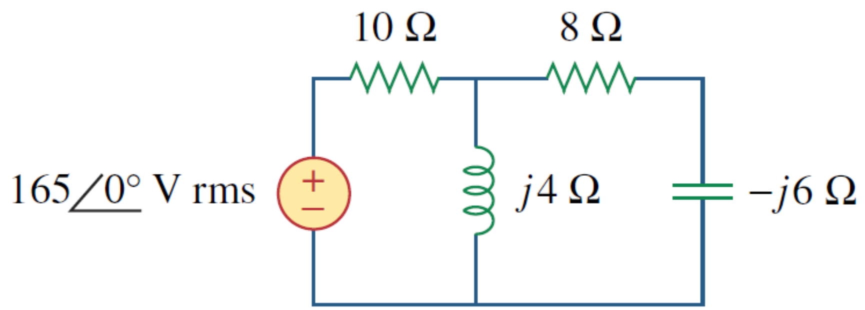
where R is the resistive part of $\mathbf{Z}$.

Example

Practice Problem 11.10

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

Answer: 0.936 lagging, 2.008 kW.

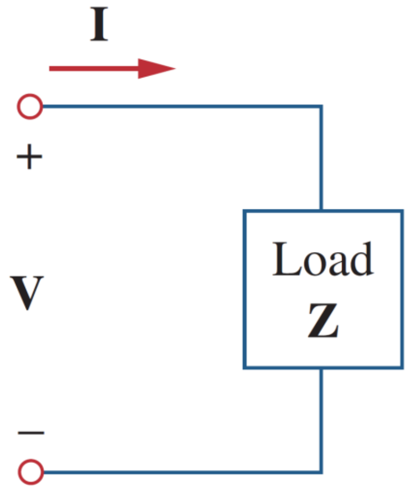


8.6 Complex Power

- Complex power is important in power analysis because it contains all the relevant power information in a given load.

$$\boldsymbol{V} = V_m \angle \theta_v$$

$$\boldsymbol{I} = I_m \angle \theta_i$$



- The complex power $\boldsymbol{S}$, is the product of the voltage phasor and the complex conjugate of the current phasor:

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^* = \boldsymbol{V}_{rms}\,\boldsymbol{I}_{rms}^{\ *}$$

- In terms of the RMS values

$$\boldsymbol{V}_{rms} = \frac{\boldsymbol{V}}{\sqrt{2}} = V_{rms}\angle\theta_v \qquad \boldsymbol{I}_{rms} = \frac{\boldsymbol{I}}{\sqrt{2}} = I_{rms}\angle\theta_i$$

$$S = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + jV_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

- The magnitude of the complex power is the apparent power, the angle of the complex power is the power factor angle.

Complex Power

- The complex power may be expressed in terms of load impedance $\boldsymbol{Z}$

$$Z = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{\boldsymbol{V}_{rms}}{\boldsymbol{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle\left(\theta_v - \theta_i\right)$$

$$S = \boldsymbol{V}_{rms}\boldsymbol{I}_{rms}^{*} = I_{rms}^{2}\boldsymbol{Z} = \frac{V_{rms}^{2}}{\boldsymbol{Z}^{*}} \qquad \boldsymbol{Z} = R + jX$$

$$S = I_{rms}^{2}\left(R + jX\right) = P + jQ$$

- Where:

$$P = \mathrm{Re}(S) = I_{rms}^{2}R$$

$$Q = \mathrm{Im}(S) = I_{rms}^{2}X$$

- P is the average or real power and it depends on the load resistance R
- Q depends on the load's reactance X and is called reactive power

Real and Reactive Power

$$S = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + jV_{rms} I_{rms} \sin(\theta_v - \theta_i) = P + jQ$$

- The real power P is the average power delivered to a load (**W**).
- The reactive power Q is a measure of the energy exchange between the source and the reactive load (**volt-ampere reactive, VAR**).
- The reactive power is being transferred back and forth between the load and the source, it represents a lossless interchange.
- Q=0 for resistive loads (unity pf)
- Q<0 for capacitive loads (leading pf)
- Q>0 for inductive loads (lagging pf)

Summarizing Power

Introducing the complex power enables us to obtain the real and reactive power directly from voltage and current phasors:

Complex Power= $\boldsymbol{S} = P + jQ = \boldsymbol{V}_{rms}(\boldsymbol{I}_{rms})^*$

$= V_{rms} I_{rms} \angle(\theta_v - \theta_i)$

Apparent Power= $S = |\boldsymbol{S}| = V_{rms} I_{rms} = \sqrt{P^2 + Q^2}$

Real Power= $P = \mathrm{Re}(\boldsymbol{S}) = S\cos(\theta_v - \theta_i)$

Reactive Power= $Q = \mathrm{Im}(\boldsymbol{S}) = S\sin(\theta_v - \theta_i)$

Power Factor= $\frac{P}{S} = \cos(\theta_v - \theta_i)$

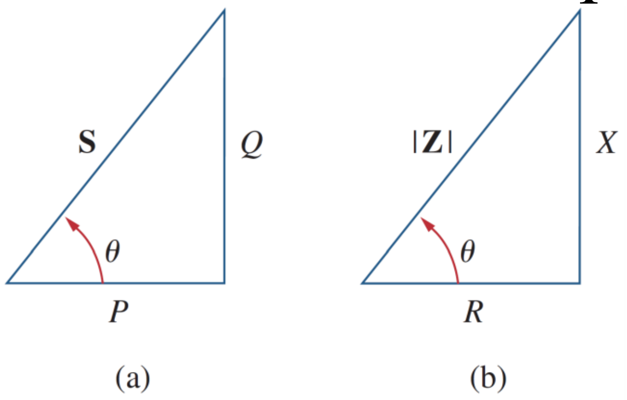


Figure 11.21
(a) Power triangle, (b) impedance triangle.

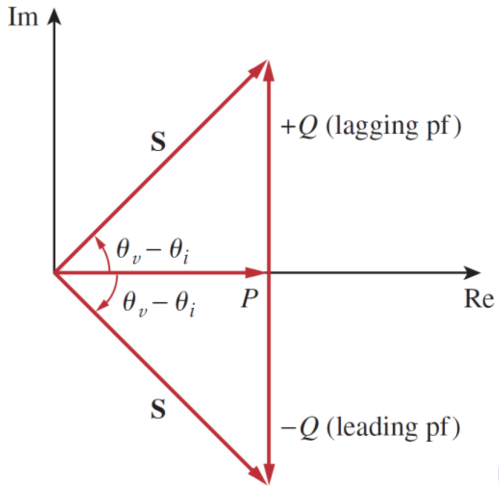


Figure 11.22
Power triangle.

Example

The voltage across a load is $v(t) = 60 \cos(\omega t - 10°)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5 \cos(\omega t + 50°)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\angle{-10°}, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\angle{+50°}$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^{*} = \left(\frac{60}{\sqrt{2}}\angle{-10°}\right)\left(\frac{1.5}{\sqrt{2}}\angle{-50°}\right) = 45\angle{-60°} \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle{-60°} = 45[\cos(-60°) + j\sin(-60°)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60°) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle{-10°}}{1.5\angle{+50°}} = 40\angle{-60°}\ \Omega$$

which is a capacitive impedance.

Example

Practice Problem 11.11

For a load, $\mathbf{V}_{rms} = 110\angle 85^\circ$ V, $\mathbf{I}_{rms} = 0.4\angle 15^\circ$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\angle 70^\circ$ VA, 44 VA, (b) 15.05 W, 41.35 VAR, (c) 0.342 lagging, $94.06 + j258.4\ \Omega$.

Example

A load **Z** draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf $= \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\underline{/0^\circ}} = 85.6 + j51.7 \text{ A} = 100\underline{/31.13^\circ} \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\underline{/-31.13^\circ}$ and the peak current is

$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{120\underline{/0^\circ}}{100\underline{/-31.13^\circ}} = 1.2\underline{/31.13^\circ}\ \Omega$$

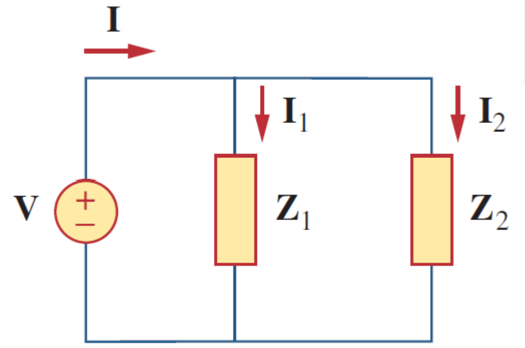
which is an inductive impedance.

8.7 Conservation of AC Power

- The principle of conservation of power applies to ac circuits as well as to dc circuits.
- Whether the loads are connected in series or in parallel, the total complex power **supplied** by the source equals the total complex power **absorbed** by the loads.

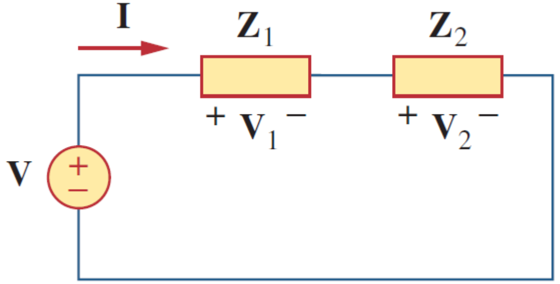
$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \cdots + \boldsymbol{S}_N$$

- It is also true of real power and reactive power, but not true of apparent power.
- The complex, real, and reactive powers from sources in a network equal the respective sums of the complex, real, and reactive powers of the individual loads.



$$I = I_1 + I_2$$

$$\boldsymbol{S} = \boldsymbol{V}\boldsymbol{I}^* = \boldsymbol{V}\left(\boldsymbol{I}_1^* + \boldsymbol{I}_2^*\right) = \boldsymbol{V}\boldsymbol{I}_1^* + \boldsymbol{V}\boldsymbol{I}_2^* = \boldsymbol{S}_1 + \boldsymbol{S}_2$$



$$V = V_1 + V_2$$

$$\boldsymbol{S} = \boldsymbol{V}\boldsymbol{I}^* = \left(\boldsymbol{V}_1 + \boldsymbol{V}_2\right)\boldsymbol{I}^* = \boldsymbol{V}_1\boldsymbol{I}^* + \boldsymbol{V}_2\boldsymbol{I}^* = \boldsymbol{S}_1 + \boldsymbol{S}_2$$

Example 11.13

Example

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

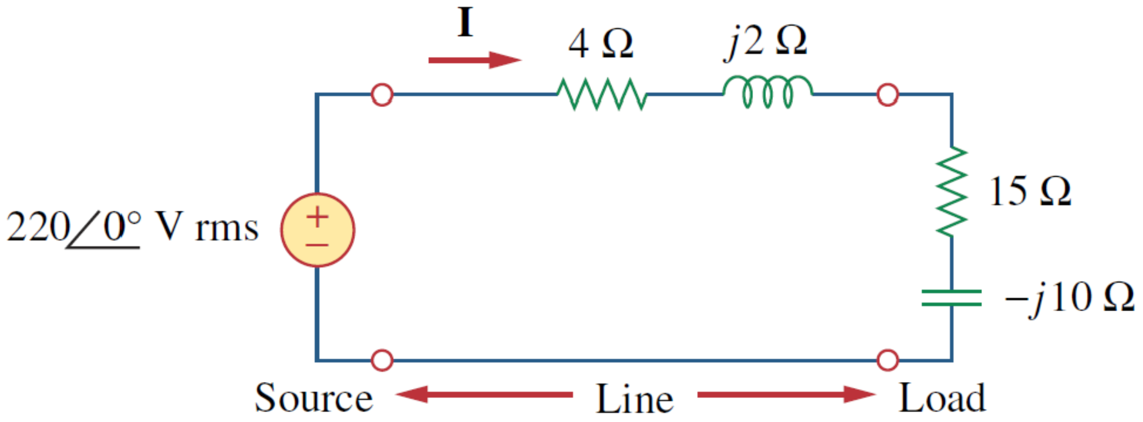
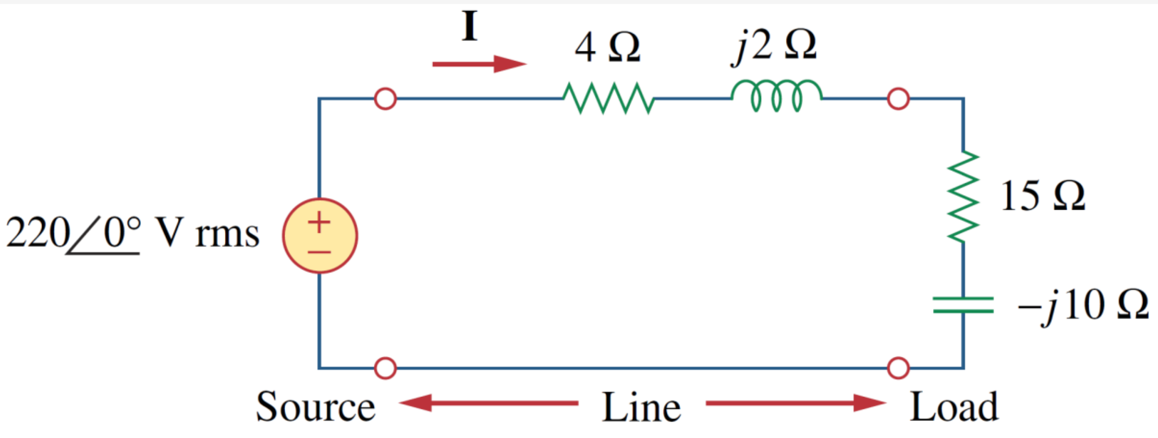


Figure 11.24
For Example 11.13.



Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle -22.83^\circ\ \Omega$$

The current through the circuit is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle 0^\circ}{20.62\angle -22.83^\circ} = 10.67\angle 22.83^\circ \text{ A rms}$$

(a) For the source, the complex power is

$$\begin{aligned}\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* &= (220\angle 0^\circ)(10.67\angle -22.83^\circ)\\ &= 2347.4\angle -22.83^\circ = (2163.5 - j910.8) \text{ VA}\end{aligned}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\begin{aligned}\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} &= (4.472\angle 26.57^\circ)(10.67\angle 22.83^\circ)\\ &= 47.72\angle 49.4^\circ \text{ V rms}\end{aligned}$$

The complex power absorbed by the line is

$$\begin{aligned}\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* &= (47.72\angle 49.4^\circ)(10.67\angle -22.83^\circ)\\ &= 509.2\angle 26.57^\circ = 455.4 + j227.7 \text{ VA}\end{aligned}$$

or

$$\mathbf{S}_{\text{line}} = |\mathbf{I}|^2\mathbf{Z}_{\text{line}} = (10.67)^2(4 + j2) = 455.4 + j227.7 \text{ VA}$$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).

(c) For the load, the voltage is

$$\begin{aligned}\mathbf{V}_L = (15 - j10)\mathbf{I} &= (18.03\angle -33.7^\circ)(10.67\angle 22.83^\circ)\\ &= 192.38\angle -10.87^\circ \text{ V rms}\end{aligned}$$

The complex power absorbed by the load is

$$\begin{aligned}\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* &= (192.38\angle -10.87^\circ)(10.67\angle -22.83^\circ)\\ &= 2053\angle -33.7^\circ = (1708 - j1139) \text{ VA}\end{aligned}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

Example

Practice Problem 11.13

In the circuit in Fig. 11.25, the 60-Ω resistor absorbs an average power of 240 W. Find **V** and the complex power of each branch of the circuit. What is the overall complex power of the circuit? (Assume the current through the 60-Ω resistor has no phase shift.)

Answer: $240.7\angle 21.45^\circ$ V (rms); the 20-Ω resistor: 656 VA; the $(30 - j10)$ Ω impedance: $480 - j160$ VA; the $(60 + j20)$ Ω impedance: $240 + j80$ VA; overall: $1376 - j80$ VA.

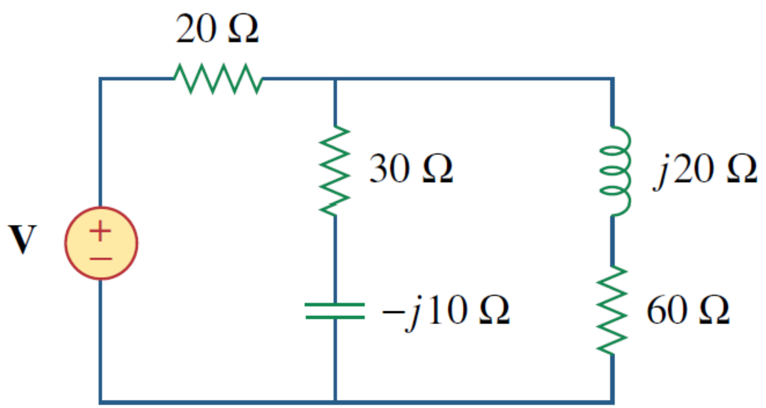
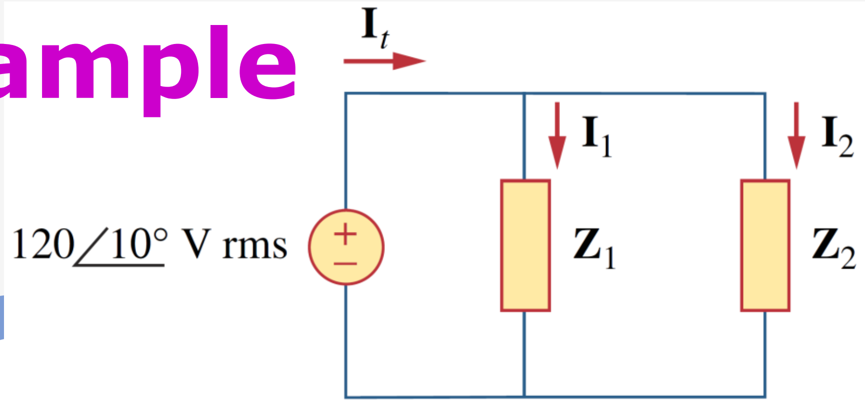


Figure 11.25
For Practice Prob. 11.13.

Example



In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\angle -30^\circ\ \Omega$ and $\mathbf{Z}_2 = 40\angle 45^\circ\ \Omega$. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf, supplied by the source and seen by the source.

Solution:

The current through $\mathbf{Z}_1$ is

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{120\angle 10^\circ}{60\angle -30^\circ} = 2\angle 40^\circ \text{ A rms}$$

while the current through $\mathbf{Z}_2$ is

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{120\angle 10^\circ}{40\angle 45^\circ} = 3\angle -35^\circ \text{ A rms}$$

The complex powers absorbed by the impedances are

$$\mathbf{S}_1 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_1^*} = \frac{(120)^2}{60\angle 30^\circ} = 240\angle -30^\circ = 207.85 - j120 \text{ VA}$$

$$\mathbf{S}_2 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_2^*} = \frac{(120)^2}{40\angle -45^\circ} = 360\angle 45^\circ = 254.6 + j254.6 \text{ VA}$$

The total complex power is

$$\mathbf{S}_t = \mathbf{S}_1 + \mathbf{S}_2 = 462.4 + j134.6 \text{ VA}$$

(a) The total apparent power is

$$|\mathbf{S}_t| = \sqrt{462.4^2 + 134.6^2} = 481.6 \text{ VA}.$$

(b) The total real power is

$$P_t = \text{Re}(\mathbf{S}_t) = 462.4 \text{ W or } P_t = P_1 + P_2.$$

(c) The total reactive power is

$$Q_t = \text{Im}(\mathbf{S}_t) = 134.6 \text{ VAR or } Q_t = Q_1 + Q_2.$$

(d) The pf $= P_t/|\mathbf{S}_t| = 462.4/481.6 = 0.96$ (lagging).

We may cross check the result by finding the complex power $\mathbf{S}_s$ supplied by the source.

$$\begin{aligned}\mathbf{I}_t = \mathbf{I}_1 + \mathbf{I}_2 &= (1.532 + j1.286) + (2.457 - j1.721)\\ &= 4 - j0.435 = 4.024\angle -6.21^\circ \text{ A rms}\\ \mathbf{S}_s = \mathbf{V}\mathbf{I}_t^* &= (120\angle 10^\circ)(4.024\angle 6.21^\circ)\\ &= 482.88\angle 16.21^\circ = 463 + j135 \text{ VA}\end{aligned}$$

which is the same as before.

Example

Practice Problem 11.14

Two loads connected in parallel are respectively 2 kW at a pf of 0.75 leading and 4 kW at a pf of 0.95 lagging. Calculate the pf of the two loads. Find the complex power supplied by the source.

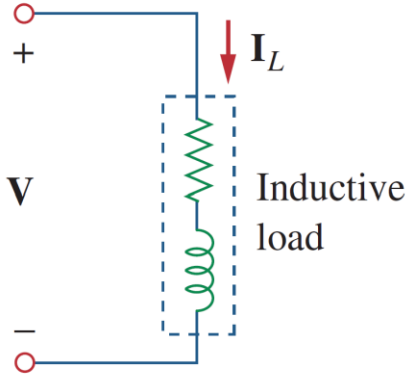
Answer: 0.9972 (leading), $6 - j0.4495$ kVA.

8.8 Power Factor Correction

- Most domestic and industrial loads, such as washing machines, air conditioners, and induction motors are inductive.
- They have a low, lagging power factor.
- The load cannot be changed, but the power factor can be increased without altering the voltage or current to the original load.
- This is referred to as power factor correction.

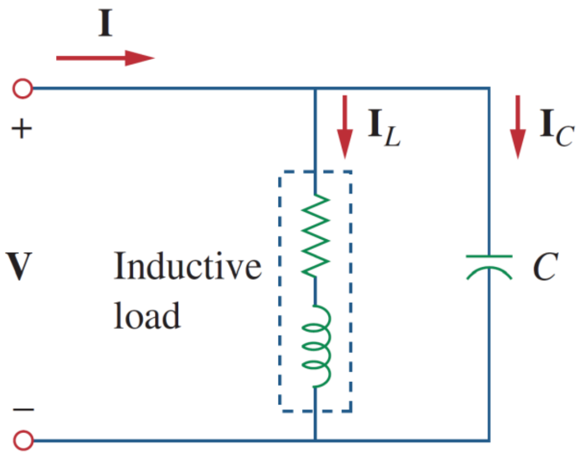
Adding a Capacitor

- An inductive load is modeled as a series combination of an inductor and a resistor.
- To mitigate the inductive aspect of the load, a capacitor is added in parallel with the load.
- From the phasor diagram, it shows that the power factor has been improved after adding the capacitor.



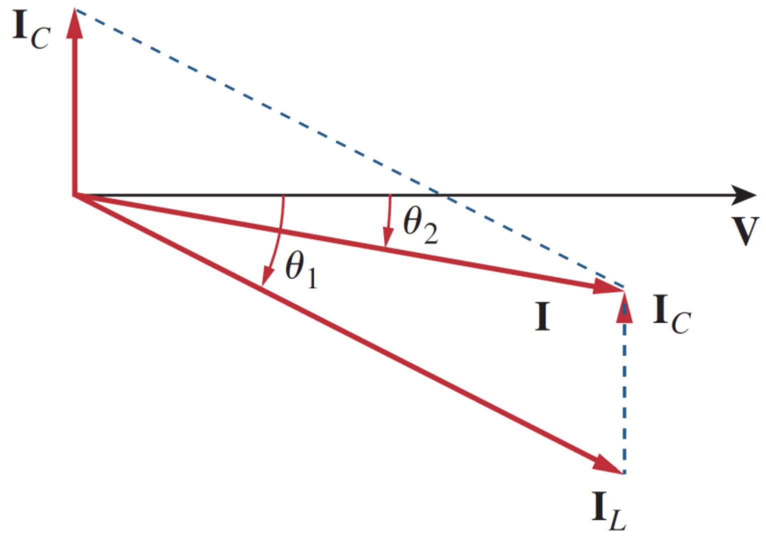
(a)

Original inductive load



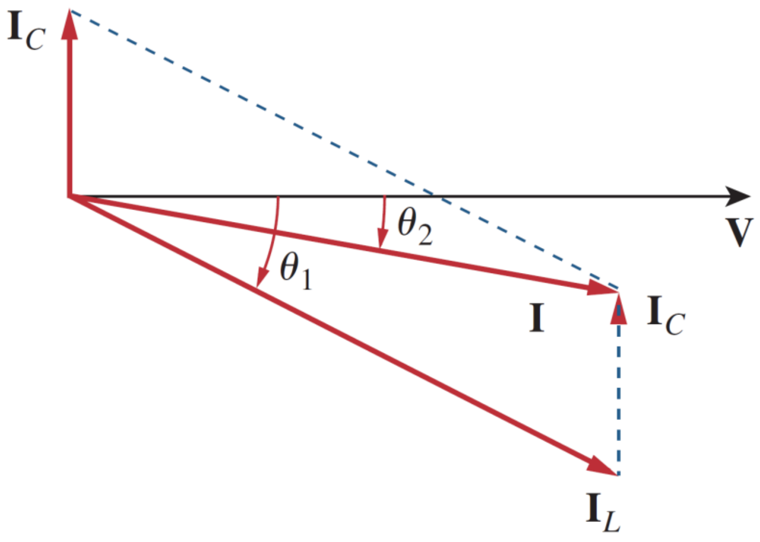
(b)

Inductive load with improved power factor



Adding a Capacitor

- The phase angle between the supplied voltage and current is reduced from θ_1 to θ_2, thereby increasing the power factor.
- With the same supplied voltage, the current drawn is less by adding the capacitor (from I_L to I).
- Power companies charge more for larger currents because they lead to larger power losses.
- Overall, the power factor correction benefits the power company and the consumer.
- By choosing a suitable size for the capacitor, the power factor can be made to be unity.



Adding a Capacitor

- Note that the real power P dissipated in the load is not affected by the shunt capacitor.

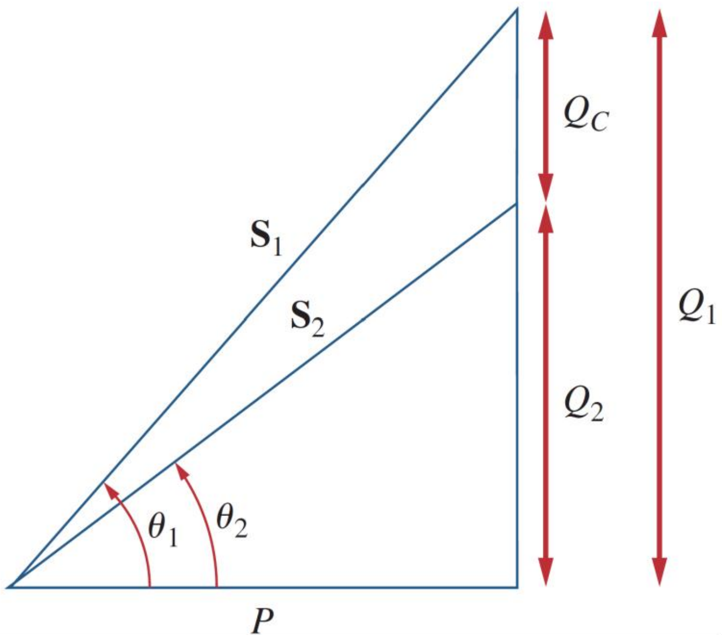
- If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power, the reactive power changes from Q_1 to Q_2, where $Q_1 = P \tan\theta_1$ and $Q_2 = P \tan\theta_2$.

- $|Q_c| = Q_1 - Q_2$

- The capacitor needed in order to shift the power factor angle is:

$$C = \frac{|Q_c|}{\omega V_{rms}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{rms}^2}$$

- Although it is not common, if a load is capacitive, an inductor should be connected across the load for power factor correction.



Example

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution:

If the pf = 0.8, then

$$\cos\theta_1 = 0.8 \qquad \Rightarrow \qquad \theta_1 = 36.87^\circ$$

where θ_1 is the phase difference between voltage and current. We obtain the apparent power from the real power and the pf as

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000 \text{ VA}$$

The reactive power is

$$Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95,

$$\cos\theta_2 = 0.95 \qquad \Rightarrow \qquad \theta_2 = 18.19^\circ$$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA}$$

The new reactive power is

$$Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

and

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

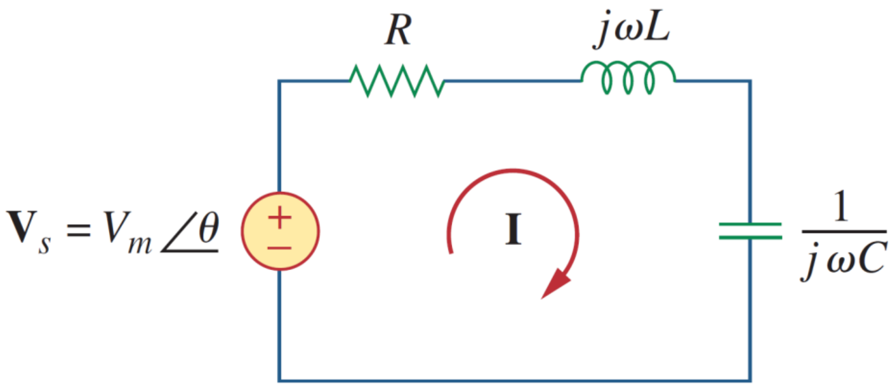
Note: Capacitors are normally purchased for voltages they expect to see. In this case, the maximum voltage this capacitor will see is about 170 V peak. We would suggest purchasing a capacitor with a voltage rating equal to, say, 200 V.

8.9 Series resonant circuit

- The most prominent feature of the frequency response of a circuit may be the sharp peak (resonant peak) in the amplitude characteristic.
- Resonance occurs in any system that has a complex conjugate pair of poles.
- It is the cause of oscillations of stored energy from one form to another.
- It requires at least one capacitor and one inductor.
- Resonance is a condition in an RLC circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.
- Resonant circuits (series or parallel) are useful for constructing filters, are used in many applications such as selecting the desired stations in radio and TV receivers.

Series Resonance

- A series resonant circuit consists of an inductor and capacitor in series.
- Resonance occurs when the imaginary part of Z is zero.
- The value of $\boldsymbol{\omega}$ that satisfies this condition is called the *resonant frequency* $\boldsymbol{\omega_0}$



$$Z = R + \boldsymbol{j}X = R + \boldsymbol{j}\left(\omega L - \frac{1}{\omega C}\right)$$

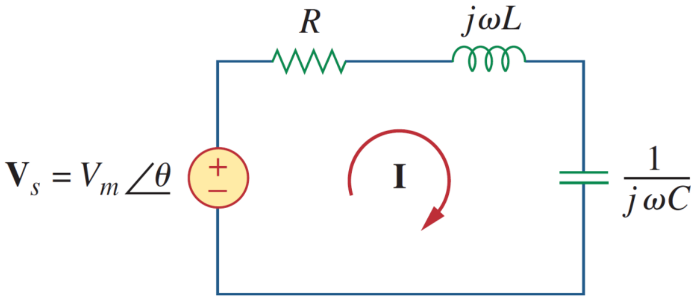
$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \text{ rad/s}$$

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

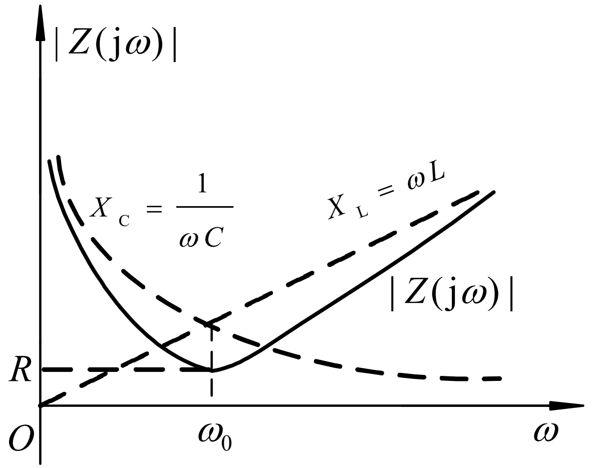
Series Resonance

At resonance:

- The impedance is purely resistive, $Z=R$.
- The voltage $\boldsymbol{V_s}$ and the current $\boldsymbol{I}$ are in phase, so that the power factor is unity.
- The magnitude of the transfer function $Z(\omega)$ is minimum.
- The inductor and capacitor voltages can be much more than the source voltage. *Q* is the ***quality factor***.



$$|Z(j\omega)| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$



$$|V_C| = \frac{V_m}{R}\frac{1}{\omega_o C} = QV_m \qquad |V_L| = \frac{V_m}{R}\omega_o L = QV_m \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

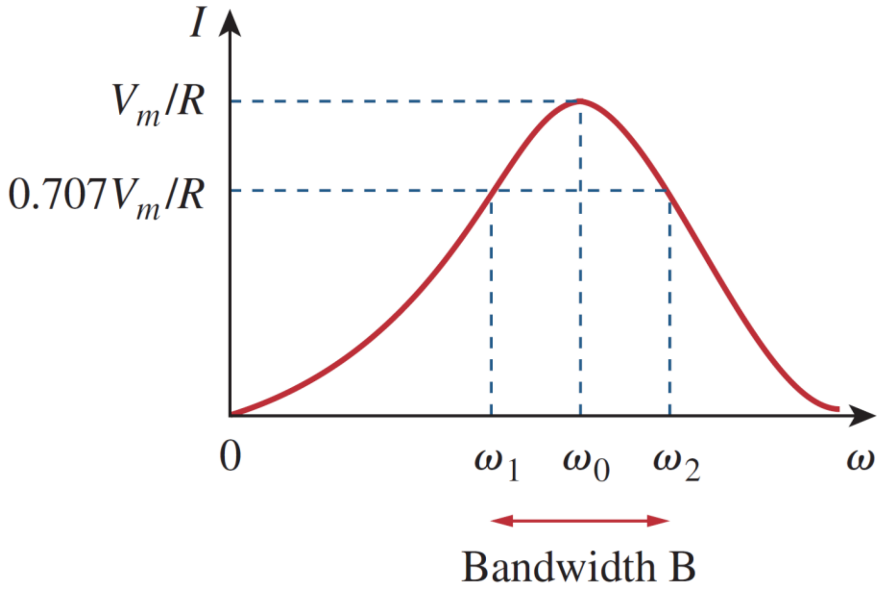
Quality Factor

$$|I(j\omega)| = \frac{V_m}{|Z(j\omega)|} = \frac{V_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

- The highest power dissipated occurs at resonance, when $I = V_m/R$, so $P = \frac{1}{2} I^2 R = \frac{1}{2}\frac{V_m^{\ 2}}{R}$
- At certain frequencies $\omega=\omega_1$, ω_2, the dissipated power is half the maximum value.

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}\frac{\left(V_m/\sqrt{2}\right)^2}{R} = \frac{V_m^{\ 2}}{4R}$$

- ω_1, ω_2 are called the half-power frequencies.
- The width of the response curve depends on the **bandwidth *B***, which is the difference between the two half-power frequencies.



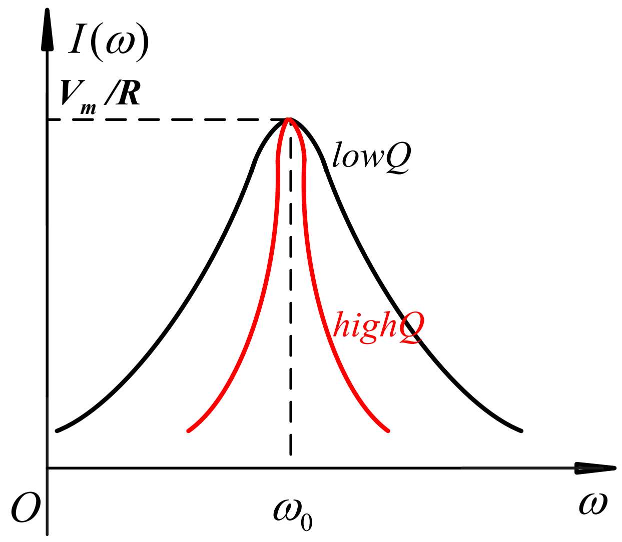
$$B = \omega_2 - \omega_1$$

Quality Factor

- The "sharpness" of the resonance is measured by the quality factor Q.
- The higher the value of $\boldsymbol{Q}$, the more selective the circuit is, but the smaller the bandwidth $\boldsymbol{B}$.
- The ***selectively*** of an RLC circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies.

$$|I(j\omega)| = \frac{V_m}{|Z(j\omega)|} = \frac{V_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$$= \frac{V_m}{R\sqrt{1 + Q^2(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})^2}}$$



Resonant circuit is widely used in radio and TV receivers to select a desired station from many stations available. The tuning circuit includes an inductor L, a resistor R and variable capacitor C, which is adjustable to match the resonant frequency of the circuit to the frequency of the desired radio station. The voltage across C is to be amplified by the subsequent circuits. If $R = 2\Omega$, $L = 300\mu H$。

A signal of f_s= 600kHz and v_{rms}=1mV is to be received. Find:

（1） What size of capacitance is required? （2） Resonant current;

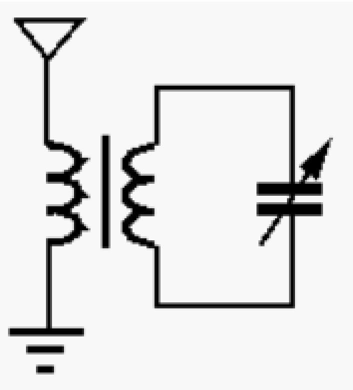
（3） The voltage across the capacitor; （4） The quality factor Q.

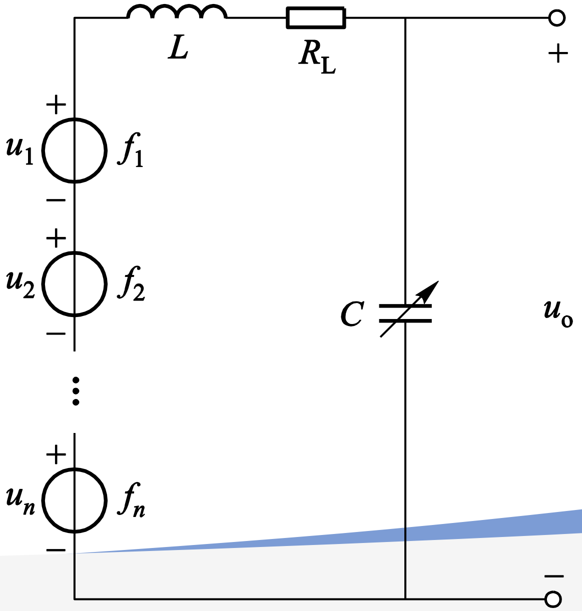
【Solution】

（1） $$f_0 = \frac{1}{2\pi\sqrt{LC}} = f = 600 \times 10^3\,\text{Hz}$$

$$C = \frac{1}{(2\pi f_0)^2 L} = \frac{1}{\left(2\pi \times 600 \times 10^3\right)^2 \times 300 \times 10^{-6}}$$

$$= 235\text{pF}$$





（2）Resonant current

$$I_0 = \frac{U}{R} = \frac{1\times10^{-3}}{2} = 0.5\times10^{-3}\,\mathrm{A} = 0.5\mathrm{mA}$$

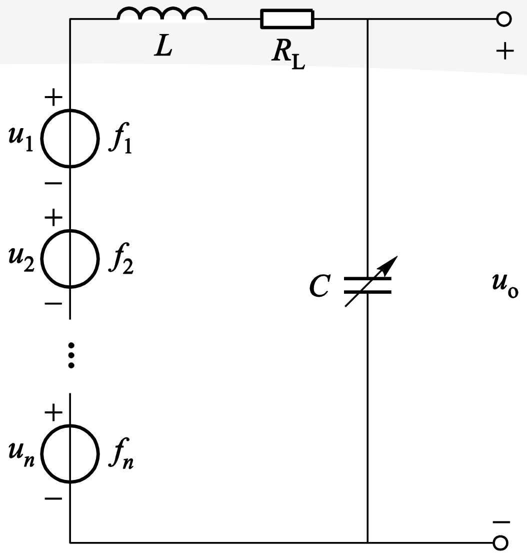
（3） The voltage across the capacitor

$$U_\mathrm{C} = I_0 X_\mathrm{C} = I_0\omega_0 L = I_0 2\pi f_0 L$$

$$= 0.5\times10^{-3}\times6.28\times600\times10^{3}\times300\times10^{-6}$$

$$= 565\mathrm{mV}$$

（4） The quality factor $Q = \frac{U_\mathrm{C}}{U} = \frac{565\mathrm{mV}}{1\mathrm{mV}} = 565$



$L = 300\mu\mathrm{H}$

$f_0 = 600\mathrm{kHz}$

$U = 1\mathrm{mV}$

It can be seen that the input signal is amplified 565 times when the resonance occurs.