Review. $A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ & & \vdots & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$

$\det M_{ij}$ denote the submatrix of size $(n-1)$ by $(n-1)$

The (i,j) cofactor $C_{ij} = (-1)^{i+j} \det M_{ij}$

The cofactor expansion of $\det A$ along row 1 of A is

$$\det A = a_{11}C_{11} + \cdots + a_{1j}C_{1j} + \cdots + a_{1n}C_{1n}$$

$\det A$ can be computed by using the cofactor expansion along any row or column of A

Cramer's rule. For an n by n invertible matrix A, the solution to $A\vec{x} = \vec{b}$ is given by

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \quad \ldots, \quad x_n = \frac{\det B_n}{\det A},$$

where each B_i, $i = 1, 2, \ldots, n$, is obtained from A by replacing column i by $\vec{b}$

A square matrix A is invertible $\Longleftrightarrow$ $\det A \neq 0$, $\det(A^{-1}) = \dfrac{1}{\det A}$.

Formula for A^{-1} by using determinants. When $n=2$. if $ad - bc \neq 0$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$

. Each entry of A^{-1} is a cofactor of A divided by $\det A$.

When $n=3$, let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Suppose $\det A \neq 0$. $[A\vec{x}_1 \ A\vec{x}_2 \ A\vec{x}_3] = A A^{-1} = I = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]$

To find column 1 of A^{-1}, which satisfies $A\vec{x}_1 = \vec{e}_1$, we have

$$B_1 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad B_2 = \begin{bmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}, \quad B_3 = \begin{bmatrix} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$$

$$\det B_1 = \overset{1}{b_{11}} C_{11} + \overset{0}{b_{21}} C_{21} + \overset{0}{b_{31}} C_{31}$$

$$= 1(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \underbrace{C_{11}}_{(1,1) \text{ cofactor of } A}$$

$$\det B_2 = 1 \times (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = C_{12}$$

$$\det B_3 = C_{13}$$

Cramer's rule

So $\vec{x}_1 \overset{\text{Cramer's rule}}{=} \begin{bmatrix} \dfrac{\det B_1}{\det A} \\ \dfrac{\det B_2}{\det A} \\ \dfrac{\det B_3}{\det A} \end{bmatrix} = \begin{bmatrix} \dfrac{C_{11}}{\det A} \\ \dfrac{C_{12}}{\det A} \\ \dfrac{C_{13}}{\det A} \end{bmatrix}$ is the first column of A^{-1}, where C_{11}, C_{12}, C_{13} are the corresponding cofactors associated with entries along row 1 of A.

To find column 2 of A^{-1}, it is equivalent to solve $A\vec{x}_2 = \vec{e}_2$.

By Cramer's rule, we have to compute the determinants of

$$B_1 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad B_2 = \begin{bmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}, \quad B_3 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{bmatrix}$$

So $\vec{x}_2 = \left(\frac{\det B_1}{\det A}, \frac{\det B_2}{\det A}, \frac{\det B_3}{\det A}\right) = \left(\frac{C_{21}}{\det A}, \frac{C_{22}}{\det A}, \frac{C_{23}}{\det A}\right)$

When $n=3$, if $\det A \neq 0$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

In general, the (i,j) entry x_{ij} of A^{-1} is the i-th component of the j-th column $\vec{x}_j$ of A^{-1} such that $A\vec{x}_j = \vec{e}_j$

$$x_{ij} = \frac{\det B_i}{\det A}, \quad \text{where } B_i = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix} \leftarrow j\text{-th row} \\ \uparrow \\ i\text{-th column} \end{array}$$

and $\det B_i = (1)\, C_{ji} = C_{ji}$. So $x_{ij} = \frac{C_{ji}}{\det A}$.

Let the cofactors C_{ij} go into the "cofactor matrix" C. If $\det A \neq 0$, $\boxed{A^{-1} = \frac{C^T}{\det A}}$

$$AC^T = (\det A)\, I$$ When $n=3$, it says that

remains true even if A is not invertible

when $\det A = 0$, $AC^T = 0\,I = 0_{n\times n}$ is a zero matrix

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

Proof: The (1, 1) entry of the product AC^T is

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \det A$$, is the cofactor expansion of $\det A$ along row 1,

Similarly, the (2, 2), (3, 3) entries of AC^T are ...

For instance, the (2, 1) entry of AC^T is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} \overset{?}{=} 0$ ✓.

$$\det A_* = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$ $$\det A_* = 0 = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$$

Example. By Cramer's rule, find the inverse of

$$A = \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}$$

Sol. To find $\vec{x}_3$ of A^{-1}, we have to compute

$$\det B_1 = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 1 \times (-1)^{3+1} \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4.$$
$\parallel C_{31}$

$$\det B_2 = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2,$$
$\parallel C_{32}$

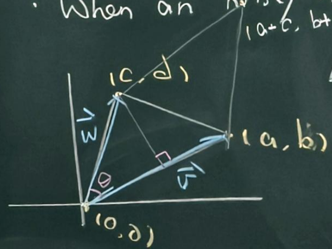
$$\det B_3 = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2.$$
$\parallel C_{33}$

And $\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = 5\times 4 + 9 \times (-2) = 2$

So $$A^{-1} = \frac{1}{2}\begin{bmatrix} * & * & 4 \\ * & * & -2 \\ * & * & 2 \end{bmatrix}$$

Area of triangle

- Expand an n-dimensional box by l, and its volume increases by l^n.
- When an n by n matrix A is multiplied by l. $\det(lA) = l^n \det A$



$$\text{Area} = \frac{1}{2}\,\text{base} \times \text{height}$$

$\det\ \vec{v} = (a, b),\ \vec{w} = (c, d)$

$$2\text{Area} = \|\vec{v}\|\left(\|\vec{w}\|\sin\theta\right) = \|\vec{v}\|\,\|\vec{w}\|\sqrt{1-\cos^2\theta}$$

$$= \|\vec{v}\|\,\|\vec{w}\|\sqrt{1-\frac{(\vec{v}\cdot\vec{w})^2}{\|\vec{v}\|^2\|\vec{w}\|^2}} = \sqrt{\|\vec{v}\|^2\|\vec{w}\|^2-(\vec{v}\cdot\vec{w})^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)-(ac+bd)^2} = \sqrt{a^2d^2+b^2c^2-2acbd}$$

$$= \sqrt{(ad-bc)^2} = |ad-bc| = \text{the absolute value of} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The area of a triangle with corners (a,b), (c,d), (e,f) is $S =$ the absolute value of

$$\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

When $(e, f) = (0, 0)$,

$S =$ the absolute value of

$$\frac{1}{2}(-1)^{1+1}\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$