

Chapter 3 Determinants

3.1 The properties of determinants

The determinant of a square matrix A is a single number. It is written as $\det A$ or $|A|$.

The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Three basic properties for the determinant. (can be apply to any n by n formula)

(1) The n by n identity matrix I has $\det I = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1 \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \times 1 - 0 \times 0 = 1.$

(2), (sign reversal) The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = - (ad - bc) = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\det(A + B) \neq \det A + \det B$$

$$\det(2I) = 2^n$$

The determinant of any permutation matrix $\det P = 1$ or -1 .

$$\det P_{21} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \stackrel{\text{rule 2}}{=} -1. \quad \text{In general } \det I_s = 1 \xrightarrow{\text{exchange two rows}} \det P_1 = -1 \xrightarrow{\text{exchange two more rows}} \det P_2 = 1$$

(3) linearity: The determinant is linear with respect to every row separately.

$$13-1) \begin{vmatrix} l\alpha & l\beta \\ c & d \end{vmatrix} = l\alpha \cdot d - l\beta \cdot c = l(\alpha d - \beta c) = l \begin{vmatrix} \alpha & \beta \\ c & d \end{vmatrix}, \text{ for any number } l.$$

$$13-2) \begin{vmatrix} \alpha + \omega & \beta + f \\ \underline{c} & d \end{vmatrix} = (\alpha + \omega)d - \underline{c}(\beta + f) = (\alpha d - \beta c) + (\omega d - fc) = \begin{vmatrix} \alpha & \beta \\ \underline{c} & d \end{vmatrix} + \begin{vmatrix} \omega & f \\ \underline{c} & d \end{vmatrix}$$

Note, this rule only applies when other rows stay fixed. It is a linear combination in one row.

All those three rules completely determine $\det A$ for any square matrix.

Example 1 For $A = \begin{bmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\det A = \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$ rule 4 $= 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4 \left(\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \right)$

Example 2 $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4 \times 1 = 4$. $S=1^2=1$ $S=(2)^2=4$.

(4) If two rows of A are equal, then $\det A = 0$. e.g. $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = a \times b - b \times a = 0$.

Proof. When we exchange the two equal rows, by rule 2, the determinant is supposed to change sign.

So $\det A = -\det A$ forces $\det A = 0$.

(5) Subtracting a multiple of one row from another row leaves $\det A$ unchanged.

$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} \stackrel{\text{rule 3}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\stackrel{\text{rule 3}}{=} -l \begin{vmatrix} a & b \\ a & b \end{vmatrix} \stackrel{\text{rule 4}}{=} -l \times 0 = 0.$$

(6) A matrix A with a row of zeros has $\det A = 0$. e.g. $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$ Now we have two equal rows.

Proof: We can add the other row to this zero row, by rule 5, the determinant is unchanged. By rule 4, $\det A = 0$.

In the elimination, $A \xrightarrow{\text{row operations}} U$. By rule 5, $\det A = \det U$.

7. If A is triangular, upper triangular, lower triangular, or even diagonal, then $\det A = a_{11}a_{22}\dots a_{nn}$.

$$\text{eg. } \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad.$$

Proof: Suppose all diagonal entries are nonzero. $A \xrightarrow{\text{elimination}} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & a_{nn} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \xrightarrow{\text{det}}$

$$\text{By rule 5, } \det A = \frac{a_{11}}{a_{22}} \cdots a_{nn} \xrightarrow{\text{rule 3}} = a_{11}a_{22}\dots a_{nn} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \xrightarrow{\text{rule 1}} = a_{11}a_{22}\dots a_{nn} \times 1$$

If a diagonal entry $a_{ii} = 0$. $A = \begin{bmatrix} a_{11} & * & \cdots & * \\ a_{21} & 0 & \cdots & * \\ \vdots & \vdots & \ddots & * \\ a_{n1} & 0 & \cdots & 0 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 - \frac{c}{a}R_1} \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix} \quad \text{if } a \neq 0$$

$$\det A \stackrel{\text{rule 5}}{=} \det U \stackrel{\text{rule 7}}{=} a(d - \frac{c}{a}b) \xrightarrow{PA} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{a \neq 0} \begin{bmatrix} c & d \\ 0 & b - \frac{a}{c}d \end{bmatrix}$$

With no row exchange, the product of the pivots is the determinant.

$$\text{By rule 6, } \det A = 0 = a_{11} \cdots a_{nn}$$

$$\begin{aligned} \det PA &\stackrel{\text{rule 5}}{=} (c(b - \frac{a}{c}d)) \\ &= cb - ad = -(ad - bc) \\ &= -\det A = (\det P)(\det A) \end{aligned}$$

18. The determinant of AB is $\det A$ times $\det B$. $|\det AB| = |\det A||\det B|$

Factorization $PA = LU$

$$\det PA = \det(LU) \stackrel{\substack{1 \\ \text{if}}}{=} (\det L)(\det U) = \det U$$

$\det P \cdot \det A = \det(PA) \stackrel{\substack{1 \\ \text{if}}}{=} \det(LU)$

$\sim = \pm 1$

So $\det A = \pm \det U$

When A is invertible, the determinant of A^{-1} is $\frac{1}{\det A}$.

Proof: $I = AA^{-1}$, then $1 = \det I = \det(AA^{-1}) \stackrel{\text{rule 8}}{=} (\det A)(\det A^{-1})$.