Given an m by n matrix $A = [\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_n]$,

the column space of A, denoted by $C(A)$, is a subspace of $\mathbb{R}^m$, $\underset{\in \mathbb{R}^m}{\vec{0}} \in C(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$

the nullspace of A, $\underset{\in \mathbb{R}^n}{\vec{0}} \in N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$ $\leftarrow$ solved from $U\vec{x} = \vec{0}$

Every solution to $A\vec{x} = \vec{0}$ is a linear combination of certain special solutions

Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{(k)}$ be the special solutions to $A\vec{x} = \vec{0}$. $N(A) = \text{Span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{n-r}\}$.

" number of free variables

Counting theorem $r + (n-r) = n$

number of pivots $\leq \min\{m, n\} \leq m$

A short wide matrix always has nonzero vectors in its nullspace. When $n > m \geq r$, there must be at least $n-m$ free variables. Then $A\vec{x} = \vec{0}$ has at least $n-m$ special solutions.

Theorem. Suppose $A\vec{x} = \vec{0}$ has more unknowns than equations, then $A\vec{x} = \vec{0}$ has nonzero solutions.

Reduced row echelon form

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -2 & 0 & -4 \end{bmatrix} = U.$$ Continue the elimination with two more steps.

(1) produce zeros above the pivots $U \xrightarrow{r_1 + r_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -4 \end{bmatrix}$

(2) produce ones in the pivots $\xrightarrow{\frac{r_2}{-2}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = R = \mathrm{rref}(A)$ It is completely determined by A.

free columns

- The reduced row echelon form of A, denoted by $\boxed{R = \mathrm{rref}(A)}$, has all pivots equal to 1, with zeros below and above.

The pivot columns of R contains I. easier to solve

$$x_1 + 2x_3 = 0$$
$$x_2 + 2x_4 = 0$$

that $\vec{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{s}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

- $N(A) = N(U) = N(R)$ We see from $R\vec{x} = \vec{0}$, i.e.,

We see from $R = \mathrm{rref}(A)$ that col 3 $= 2$(col 1) $= 2$(col 1) $+ 0$(col 2). The same holds for A.

col 4 $= 2$(col 2) $= 0$(col 1) $+ 2$(col 2) $\therefore$ Every free column is a linear combination of the pivot columns

Let $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{r_2 - 3r_1} U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -2 & 0 & -4 \end{bmatrix}.$$

$$N(A) = \operatorname{span}\{\boldsymbol{s}_1, \boldsymbol{s}_2\} = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The complete solution to $A\vec{x}=\vec{0}$ is a linear combination of the $n-r$ special solutions.

The 2 key steps are: (1) reducing A to its reduced row echelon form R.

Each free column leads to a special solution

(2) finding $n-r$ special solutions to $R\vec{x}=\vec{0}$, or $A\vec{x}=\vec{0}$.

If j-th column of R is free, there is a special solution with $x_j = 1$, and other free variables being zeros

Rank of A.

Definition. The rank of A is the number of pivots, and it is denoted by r.

Example. $A=\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \xrightarrow[r_3-r_1]{r_2-r_1} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{r_3-2r_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \xrightarrow{r_1-r_2} \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 2 & 3 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} = R$

The rank of A is 2.

rank r = number of pivots = nonzero rows in R

$N(A) = ?$ There are $n-r=2$ special solutions. We solve from

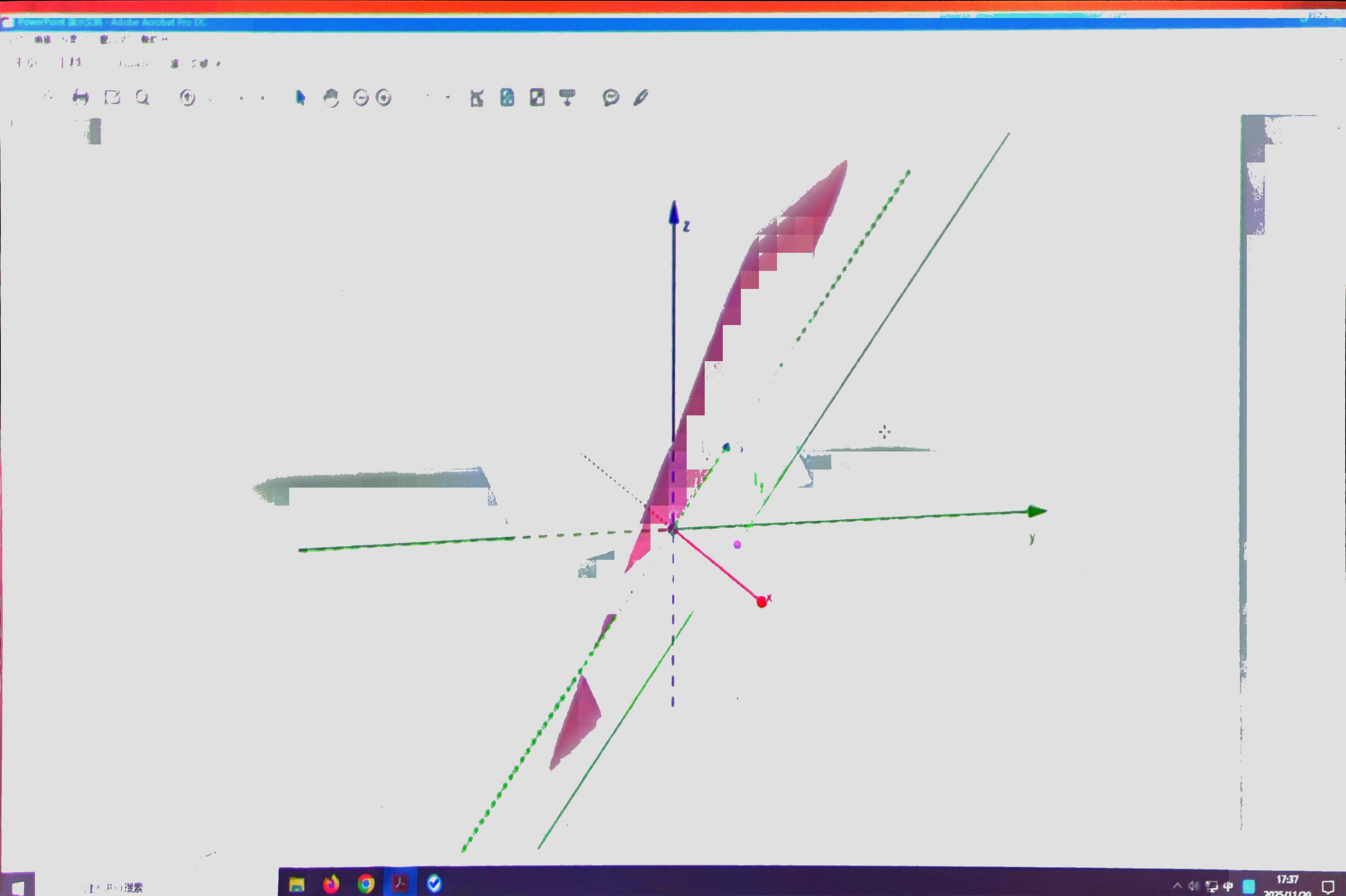
$x_1 \quad + 2x_3 + 3x_4 = 0$

$x_2 \quad + x_4 = 0$

that $\vec{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{s}_2 = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

span$\{\vec{s}_1, \vec{s}_2\}$

The 3-rd and 4-th column of R, as well as those of A, are free


z
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Col 3 = $2 \cdot$ Col 1 $+ 0$ (Col 2) With signs reversed, $-2, 0$ show up in $\vec{s}_1$

Col 4 = $3 \cdot$ Col 1 $+ 1$ (Col 2) $-3, -1$ $\vec{s}_2$

Example. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix}$, where c is some constant

1) Find $R = \text{rref}(A)$, and the rank of A

2) Find the special solutions to $A\vec{x} = \vec{0}$, and $N(A)$.

$$A \xrightarrow[r_3 - 4r_1]{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & c-4 \end{bmatrix}$$

Case I When $c \neq 4$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & c-4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{r_2}{c-4}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

The rank of A is 2.

There is $\underset{}{1}^{=3-2}$ special solution, by letting the free variable $x_2 = 1$.

We solve from $R\vec{x} = \vec{0}$ that $\vec{s} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. $N(A) = \text{span}\{\vec{s}\}$

Case II When $c = 4$, $R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$r = 1$,

there are 2 special solutions $\vec{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{s}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$N(A) = \text{span}\{\vec{s}_1, \vec{s}_2\}$.

Example

$x + y + z = 3$

$x + 2y - z = 4$

$A\vec{x} = \vec{b}$

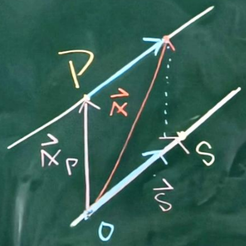
$x + y + z = 0$

$x + 2y - z = 0$

$A\vec{x} = \vec{0}$

l_1 : solutions to $A\vec{x} = \vec{b}$

l_2 = solutions to $A\vec{y} = \vec{0}$



We can choose one point on l_1, $\vec{x}_p$ and call it a particular solution to $A\vec{x} = \vec{b}$

Q: How to find the complete solution to $A\vec{x} = \vec{b}$?

$\vec{x} = \vec{x}_p + c\vec{s}$

The solutions to $A\vec{x} = \vec{0}$ are shifted to a parallel line l_2 through $(0, 0, 0)$