Properties of determinants   Let A denote an n by n matrix

(1) $\det I = 1$

(2) If two rows (or columns) of A are interchanged, the determinant of the resulting matrix is $-\det A$

(3) If a row (or column) of A is multiplied by l, $l(\det A)$

$$\det(lA) = l^n(\det A)$$

(4) If two rows (or columns) of A are identical, $\det A = 0$ (By rule 9, a matrix with two equal rows ~~is not~~ invertible, $\det A$

(5) If a multiple of one row (or column) of A is added to a different row (column), $\det A$

(6) If A has a row of zeros, $\det A = 0$

(7) If A is triangular, $\det A$ = products of entries on the main diagonal.

(8) $\det(AB) = (\det A)(\det B)$ $\det(ABC) = (\det A)(\det B)(\det C)$. $\det(A^k) = (\det A)^k$.

Every rule for the rows can be apply to the columns

(9) A is invertible if and only if $\det A \neq 0$. (A is singular if and only if $\det A = 0$). (10) $\det(A^T) = \det A$.

not invertible

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ When $\frac{a}{c} = \frac{b}{d}$, the two rows are parallel, and $\det A = 0$.

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

A is invertible if and only if $\underbrace{ad - bc}_{\det A} \neq 0$. $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det(A^T) = ad - cb = \det A$.

Proof of rule 9: Suppose $A \xrightarrow{\text{elimination}}$ upper triangular U without row exchange

if A is invertible, then U has the n pivots along its diagonal. $\det A \overset{\text{rule 5}}{=} \det U \overset{\text{rule 7}}{=}$ product of the pivots $\neq 0$.

if A is singular, then U has a row of zeros $\det A = \det U \overset{\text{rule 6}}{=} 0$. #

Proof of rule 10: We start with $PA = LU$. Firstly, by rule 8, $(\det P)(\det A) = (\det L)(\det U)$.

Then $A^T P^T = U^T L^T$ gives $(\det A^T)(\det P^T) = (\det U^T)(\det L^T)$

Both the lower triangular L and the upper triangular L^T have 1's on the diagonal. By rule 7, $\det L = \det L^T = 1$.

Similarly, $\det U = \det U^T =$ product of the pivots.

Permutation matrices have $PP^T = I$. By rule 8, $(\det P)(\det P^T) = 1$. Then both $\det P$ and $\det P^T$ equal 1 or -1.

So P, L, U have the same determinants as P^T, L^T, U^T. It leaves $\det A = \det A^T$

Example. By row and column operations, find the determinant of $A = \begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix}$

$$A \xrightarrow{r_1 - r_3} \begin{bmatrix} -a & 0 & a \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix} \xrightarrow{r_2 - r_3} \begin{bmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{bmatrix}$$

$$\xrightarrow{c_3 + c_1} \begin{bmatrix} -a & 0 & 0 \\ 0 & -a & a \\ 1 & 1 & 2-a \end{bmatrix} \xrightarrow{c_3 + c_2} \begin{bmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 1 & 1 & 3-a \end{bmatrix}$$

By rules 5 and 7, $\det A = (-a)(-a)(3-a) = a^2(3-a)$.

A is NOT invertible if $a = 0$ or $a = 3$.

When $a = 0$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. By rule 4, $\det A = 0$. When $a = 3$, $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[r_1 + r_3]{r_1 + r_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

3.2 Permutations and cofactors

The pivot formula. When elimination leads to $A = LU$, the pivots $d_1, d_2, \dots, d_n$ are on the diagonal of U.

$$\det A = (\det L)(\det U) = (1)(d_1 d_2 \cdots d_n)$$

With a possibility of row exchanges, $PA = LU$. It follows $\det A = \pm(d_1 d_2 \cdots d_n)$

Example. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = PA$.

$\det P = (-1)^{1}$ So $\det A = (-1)\, 4 \times 2 \times 1 = -8$.

The odd number of row exchanges gives $\det P = -1$, while the even number gives $\det P = 1$

Example. $A \xrightarrow{r_2 + \frac{1}{2} r_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{r_3 + \frac{2}{3} r_2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{r_4 + \frac{3}{4} r_3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$

So $\det A = \cancel{2} \times \frac{3}{\cancel{2}} \times \frac{4}{3} \times \frac{5}{4} = 5$

Find the determinant of

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

The big formula

$$\det A \overset{\text{linearity in row 1}}{=} \begin{vmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$[2 \ -1 \ 0 \ 0] = [2 \ 0 \ 0 \ 0] + [0 \ -1 \ 0 \ 0]$$

$$\overset{\text{elimination}}{=} \begin{vmatrix} 2 & 0 & 0 & 0 \\ \boxed{0 \ \ 2 \ \ -1 \ \ 0} \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ \boxed{-1 \ \ 0 \ \ -1 \ \ 0} \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$\det = 0$

$$\overset{\text{elimination}}{=} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \boxed{0 \ \ 0 \ \ 2 \ \ -1} \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \boxed{0 \ \ 0 \ \ 2 \ \ -1} \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$\overset{\text{elimination}}{=} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 2 & & & \\ & 2 & & \\ & & -1 & \\ & & & -1 \end{vmatrix} \qquad = (-1)\begin{vmatrix} 2 & & & \\ & -1 & & \\ & & -1 & \\ & & & 2 \end{vmatrix} \qquad = (-1)\begin{vmatrix} -1 & & & \\ & -1 & & \\ & & 2 & \\ & & & 2 \end{vmatrix} \qquad = (-1)^2\begin{vmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{vmatrix}$$

$$= 2^4 + (-1)\,2^2(-1)^2 + (-1)\,2\times(-1)^2\times 2 + (-1)(-1)^2\times 2^2 + (-1)^2(-1)^4 = 16-4-4-4+1 = 5.$$