Chapter 3

Descriptive Statistics: Numerical Methods

In this chapter we study numerical methods for describing the important aspects of a set of measurements. If the measurements are values of a quantitative variable, we often describe (1) what a typical measurement might be and (2) how the measurements vary, or differ, from each other. For example, in the car mileage case we might estimate (1) a typical EPA gas mileage for the new midsize model and (2) how the EPA mileages vary from car to car. Or, in the marketing research case, we might estimate (1) a typical bottle design rating and (2) how the bottle design ratings vary from consumer to consumer.

Taken together, the graphical displays of Chapter 2 and the numerical methods of this chapter give us a basic understanding of the important aspects of a set of measurements. We will illustrate this by continuing to analyze the car mileages, payment times, bottle design ratings, and cell phone usages introduced in Chapters 1 and 2.

Chapter Outline

3.1 Describing Central Tendency

- In addition to describing the shape of a distribution, want to describe the data set's central tendency
- A measure of central tendency represents the center or middle of the data
- May or may not be a typical value

Parameters and Statistics

- *A **population parameter*** is a number calculated using the population measurements that describes some aspect of the population
- A ***sample statistic*** is a number calculated using the sample measurements that describes some aspect of the sample

Point Estimates and Sample Statistics

A *point estimate* is a one-number estimate of the value of a population parameter

A *sample statistic* is a number calculated using sample measurements that describes some aspect of the sample

- Use sample statistics as point estimates of the population parameters

The sample mean, denoted $\bar{x}$, is a sample statistic and is the average of the sample measurements

- The sample mean is a point estimate of the population mean

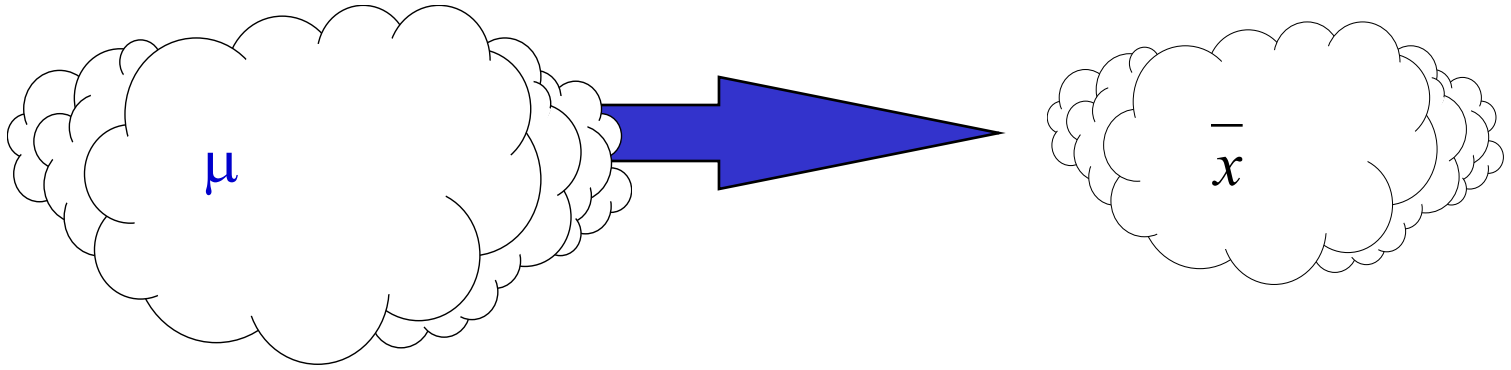
Measures of Central Tendency

Mean, μ	The average or expected value
Median, M_d	The value of the middle point of the ordered measurements
Mode, M_o	The most frequent value

The Mean

Population $X_1, X_2, \ldots, X_N$

Sample $x_1, x_2, \ldots, x_n$



Population Mean

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}$$

Sample Mean

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Example: Car Mileage Case

Example 3.1: Sample mean for first five car mileages from Table 3.1

30.8, 31.7, 30.1, 31.6, 32.1

This point estimate says we estimate that the mean mileage that would be obtained by all of the new midsize cars that will or could potentially be produced this year is 31.56 mpg. Unless we are extremely lucky, however, there will be **sampling error.** That is, the point estimate $\bar{x} = 31.56$ mpg, which is the average of the sample of fifty randomly selected mileages, will probably not exactly equal the population mean μ, which is the average mileage that would be obtained by all cars. Therefore, although $\bar{x} = 31.56$ provides some evidence that μ is at least 31 and thus that the automaker should get the tax credit, it does not provide definitive evidence. In later chapters, we discuss how to assess the *reliability* of the sample mean and how to use a measure of reliability to decide whether sample information provides definitive evidence.

$$\bar{x} = \frac{\sum_{i=1}^{5} x_i}{5} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$$

$$\bar{x} = \frac{30.8 + 31.7 + 30.1 + 31.6 + 32.1}{5} = \frac{156.3}{5} = 31.26$$

The Median

- The median M_d is a value such that 50% of all measurements, after having been arranged in numerical order, lie above (or below) it
  1. If the number of measurements is odd, the median is the middlemost measurement in the ordering
  2. If the number of measurements is even, the median is the average of the two middlemost measurements in the ordering

Example: Median

- Example 3.1 Car Mileage Case: First five observations from Table 3.1:

  30.8, 31.7, 30.1, 31.6, 32.1

- In order: 30.1, 30.8, 31.6, 31.7, 32.1
- There is an odd so median is one in middle, or 31.6

- Six exercise classes example

  15, 30, 30, 34, 41, 60

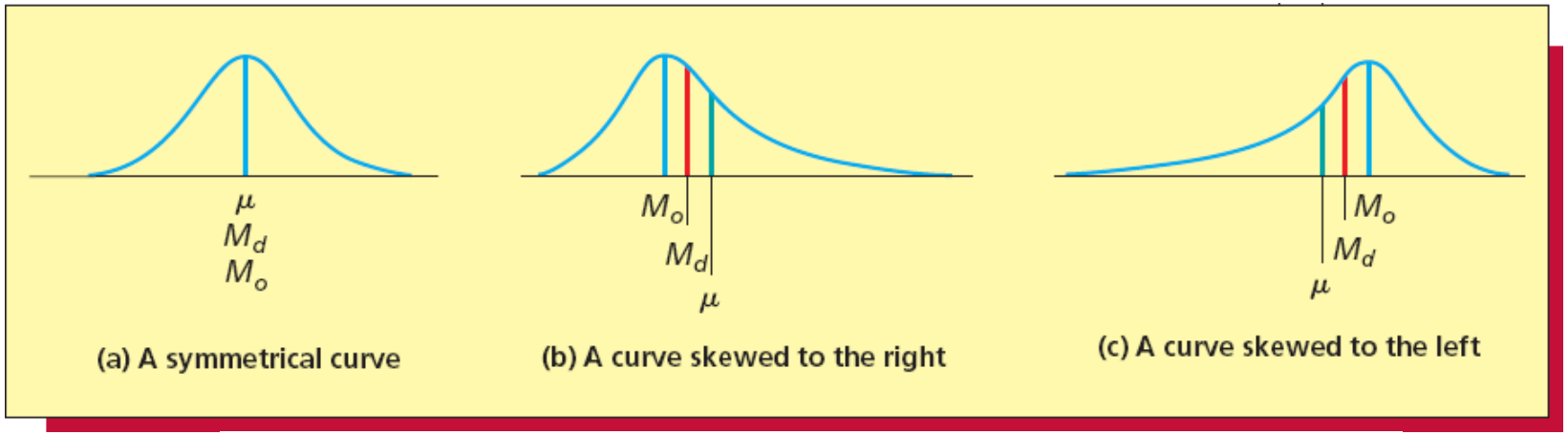
- Median is the average of the two in the middle or (30+34)/2=32

The Mode

- The mode M_o of a population or sample of measurements is the measurement that occurs most frequently
- Modes are the values that are observed “most typically”
- Sometimes higher frequencies at two or more values
  - If there are two modes, the data is *bimodal*
  - If more than two modes, the data is *multimodal*
- When data are in classes, the class with the highest frequency is the modal class

Relationships Among Mean, Median and Mode

Figure 3.3



of the population. Relative frequency curves can have many shapes. Three common shapes are illustrated in Figure 3.3. Part (a) of this figure depicts a population described by a symmetrical relative frequency curve. For such a population, the mean (μ), median (M_d), and mode (M_o) are all equal. Note that in this case all three of these quantities are located under the highest point of the curve. It follows that when the frequency distribution of a sample of measurements is approximately symmetrical, then the sample mean, median, and mode will be nearly the same. For instance, consider the sample of 50 mileages in Table 3.1. Because the histogram of these mileages in Figure 3.2 is approximately symmetrical, the mean—31.56—and the median—31.55—of the mileages are approximately equal to each other.

Figure 3.3(b) depicts a population that is skewed to the right. Here the population mean is larger than the population median, and the population median is larger than the population mode (the mode is located under the highest point of the relative frequency curve). In this case the population mean *averages in* the large values in the upper tail of the distribution. Thus the population mean is more affected by these large values than is the population median. To understand this, we consider the following example.

Examples

EXAMPLE 3.2 Household Incomes

An economist wishes to study the distribution of household incomes in a Midwestern city. To do this, the economist randomly selects a sample of $n = 12$ households from the city and determines last year's income for each household.[1] The resulting sample of 12 household incomes—arranged in increasing order—is as follows (the incomes are expressed in dollars):

7,524	11,070	18,211	26,817	36,551	41,286
49,312	57,283	72,814	90,416	135,540	190,250

EXAMPLE 3.3 The Marketing Research Case: Rating A Bottle Design

BI

The Excel output in Figure 3.4 tells us that the mean and the median of the sample of 60 bottle design ratings are 30.35 and 31, respectively. Because the histogram of the bottle design ratings in Figure 3.5 is not highly skewed to the left, the sample mean is not much less than the sample median. Therefore, using the mean as our measure of central tendency, we estimate that the mean rating of the new bottle design that would be given by all consumers is 30.35. This is considerably higher than the minimum standard of 25 for a successful bottle design.

Payment Time Case

BI

- Mean=18.108 days
- Median=17.000 days
- Mode=16.000 days
- So:
- Expect the mean payment tim
18.108 days
- A long payment time would b
days and a short payment time would be
< 17 days
- The typical payment time is 16 days

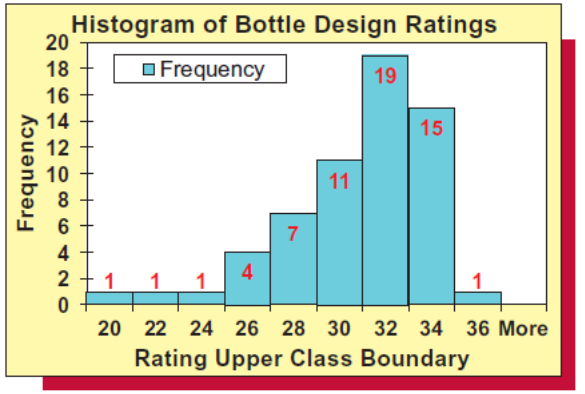
EXAMPLE 3.4 The e-billing Case: Reducing Bill Payment Times C

The MINITAB output in Figure 3.6 gives a histogram of the 65 payment times, and the MINITAB output in Figure 3.7 tells us that the mean and the median of the payment times are 18.108 days and 17 days, respectively. Because the histogram is not highly skewed to the right, the sample mean is not much greater than the sample median. Therefore, using the mean as our measure of central tendency, we estimate that the mean payment time of all bills using the new billing system is 18.108 days. This is substantially less than the typical payment time of 39 days that had been experienced using the old billing system.

FIGURE 3.4 Excel Output of Statistics Describing the 60 Bottle Design Ratings

STATISTICS	
Mean	30.35
Standard Error	0.401146
Median	31
Mode	32
Standard Deviation	3.107263
Sample Variance	9.655085
Kurtosis	1.423397
Skewness	−1.17688
Range	15
Minimum	20
Maximum	35
Sum	1821
Count	60

FIGURE 3.5 Excel Frequency Histogram of the 60 Bottle Design Ratings



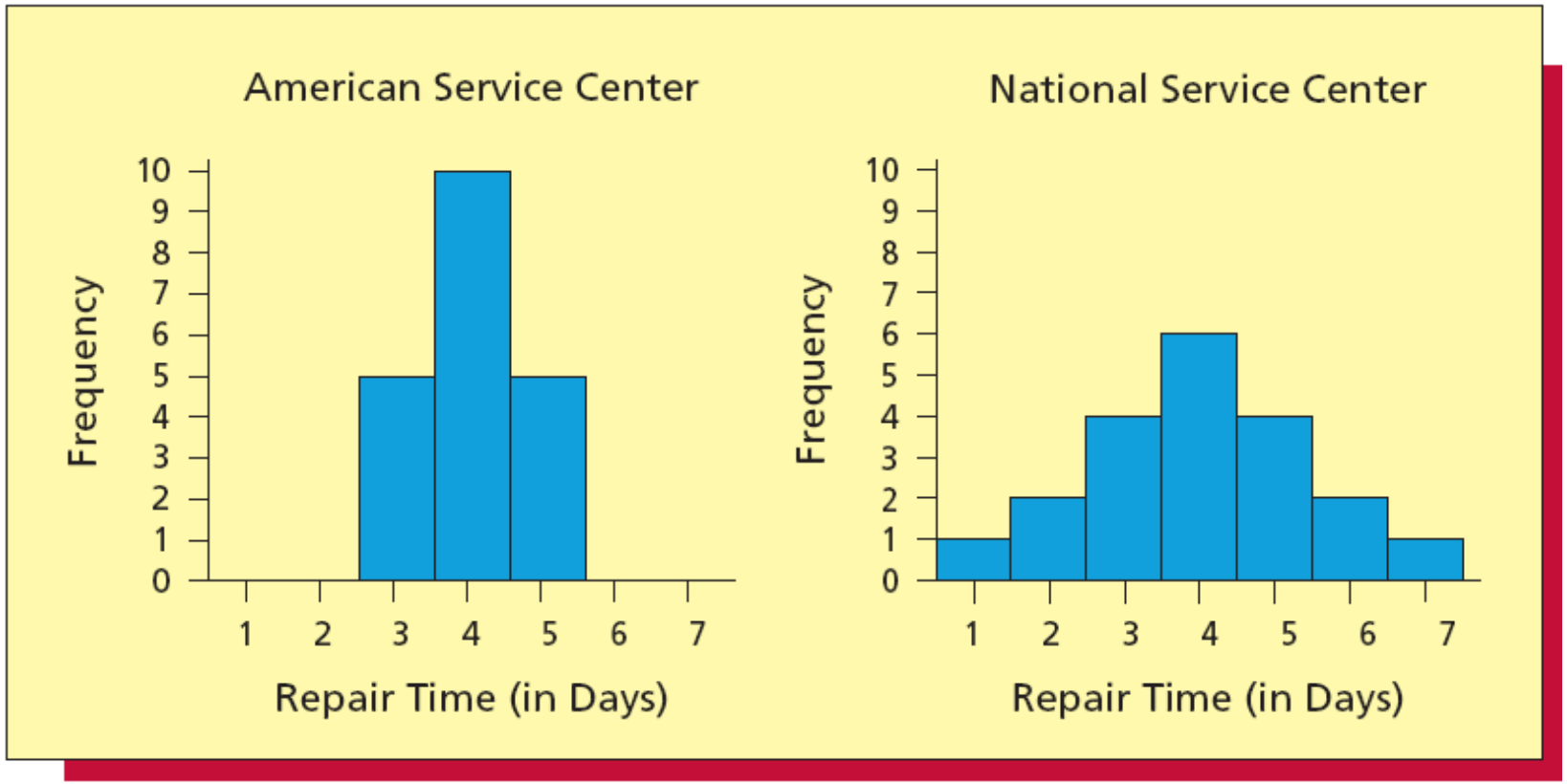
Example 3.5

EXAMPLE 3.5 The Cell Phone Case: Reducing Cellular Phone Costs

Suppose that a cellular management service tells the bank that if its cellular cost per minute for the random sample of 100 bank employees is over 18 cents per minute, the bank will benefit from automated cellular management of its calling plans. Last month's cellular usages for the 100 randomly selected employees are given in Table 1.4 (page 9), and a dot plot of these usages is given in the page margin. If we add together the usages, we find that the 100 employees used a total of 46,625 minutes. Furthermore, the total cellular cost incurred by the 100 employees is found to be \$9,317 (this total includes base costs, overage costs, long distance, and roaming). This works out to an average of \$9,317/46,625 = \$.1998, or 19.98 cents per minute. Because this average cellular cost per minute exceeds 18 cents per minute, the bank will hire the cellular management service to manage its calling plans.

3.2 Measures of Variation

Figure 3.13



Measures of Variation

Range Largest minus the smallest measurement

Variance The average of the squared deviations of all the population measurements from the population mean

Standard Deviation The square root of the variance

The Range

- Largest minus smallest
- Measures the interval spanned by all the data
- For American Service Center, largest is 5 and smallest is 3
  - Range is 5 – 3 = 2 days
- For National Service Center, range is 6

Variance

Population Of Size N

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N} = \frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}$$

Sample Of Size n

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1}$$

Standard Deviation

Population Standard Deviation

$$\sigma = \sqrt{\sigma^2}$$

Sample Standard Deviation

$$s = \sqrt{s^2}$$

Example: The Car Mileage Case

$$s^2 = \frac{\sum_{i=1}^{5}(x_i - \bar{x})^2}{5-1}$$

$$= \frac{(30.8-31.26)^2 + (31.7-31.26)^2 + (30.1-31.26)^2 + (31.6-31.26)^2 + (32.1-31.26)^2}{4}$$

$$= \frac{2.572}{4} = 0.643$$

$$s = \sqrt{s^2} = \sqrt{0.643} = 0.8019$$

Example: The Payment Time Case

- The sample variance
- The sample standard deviation

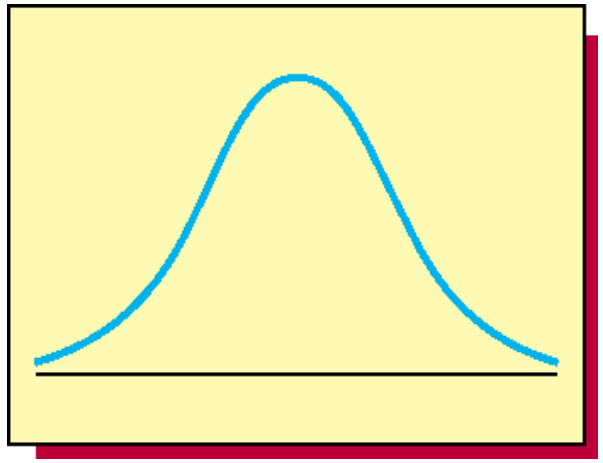
$$\sum_{i=1}^{65} x_i = x_1 + x_2 + \cdots + x_{65} = 22 + 19 + \cdots + 21 = 1{,}177$$

$$\sum_{i=1}^{65} x_i^2 = x_1^2 + x_2^2 + \cdots + x_{65}^2 = (22)^2 + (19)^2 + \cdots + (21)^2 = 22{,}317$$

Therefore $$s^2 = \frac{1}{(65-1)}\left[22{,}317 - \frac{(1{,}177)^2}{65}\right] = \frac{1{,}004.2464}{64} = 15.69135$$

$$s = \sqrt{s^2} = \sqrt{15.69135} = 3.9612$$

The Normal Curve



- Symmetrical and bell-shaped curve for a normally distributed population
- The height of the normal over any point represents the relative proportion of values near that point

The Empirical Rule(经验准则) for Normal Populations

If a population has mean μ and standard deviation σ and is described by a normal curve, then

1. **68.26%** of the population measurements lie within one standard deviation of the mean: **$[\mu-\sigma, \mu+\sigma]$**

2. **95.44%** of the population measurements lie within two standard deviations of the mean: **$[\mu-2\sigma, \mu+2\sigma]$**

3. **99.73%** of the population measurements lie within three standard deviations of the mean: **$[\mu-3\sigma, \mu+3\sigma]$**

The Empirical Rule

- The Empirical Rule holds for normally distributed populations.
- This rule also approximately holds for populations having mound-shaped (single-peaked) distributions that are not very skewed to the right or left.
- For example, recall that the distribution of 65 payment times, it indicates that the empirical rule holds.

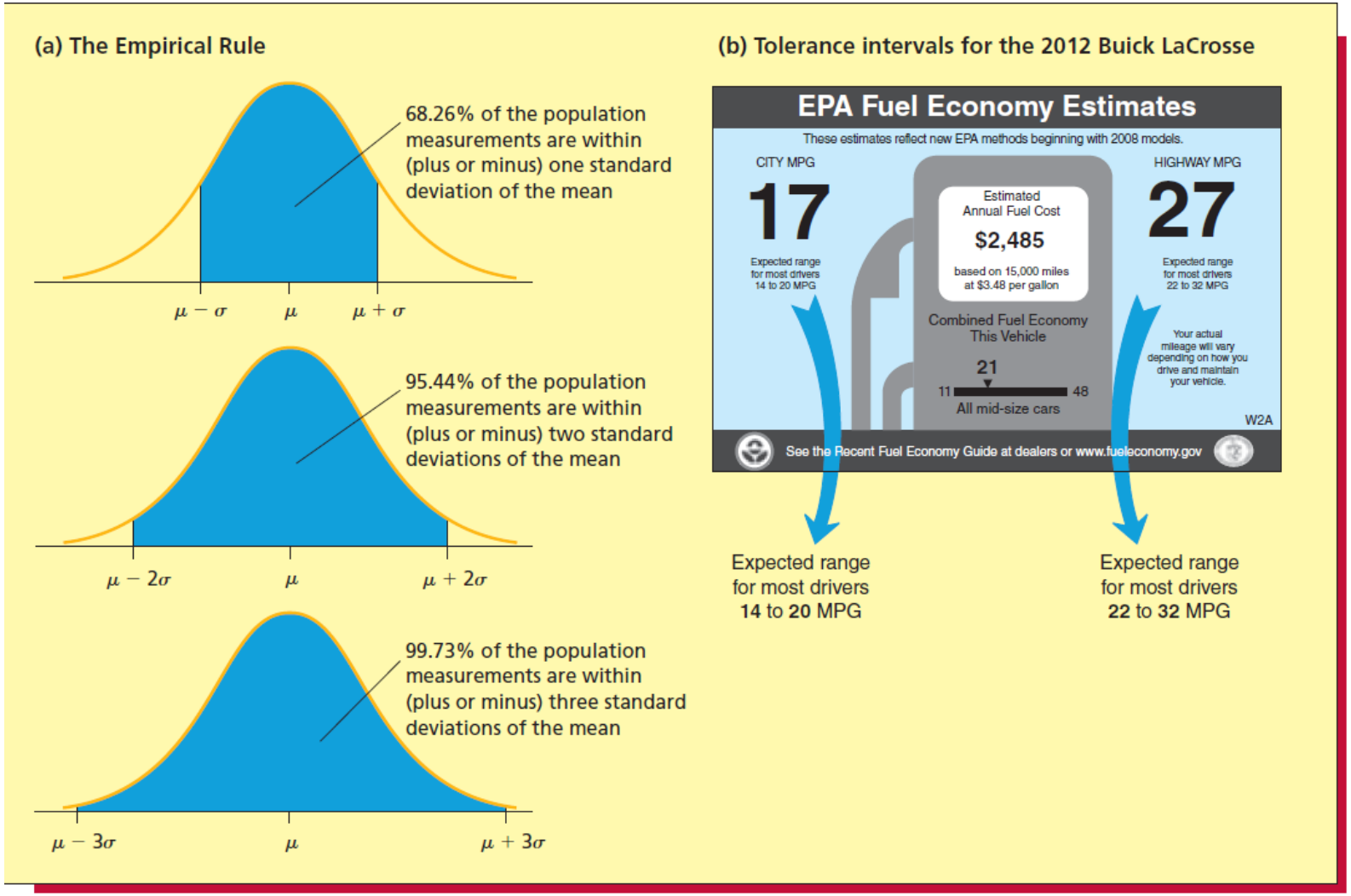
$$\bar{x} = \frac{\sum_{i=1}^{49} x_i}{49} = \frac{1546.1}{49} = 31.5531$$

$$s^2 = \frac{\sum_{i=1}^{49} (x_i - \bar{x})^2}{(49-1)} = \frac{30.66204}{48} = 0.638793$$

$$s = \sqrt{s^2} = \sqrt{0.638793} = 0.7992$$

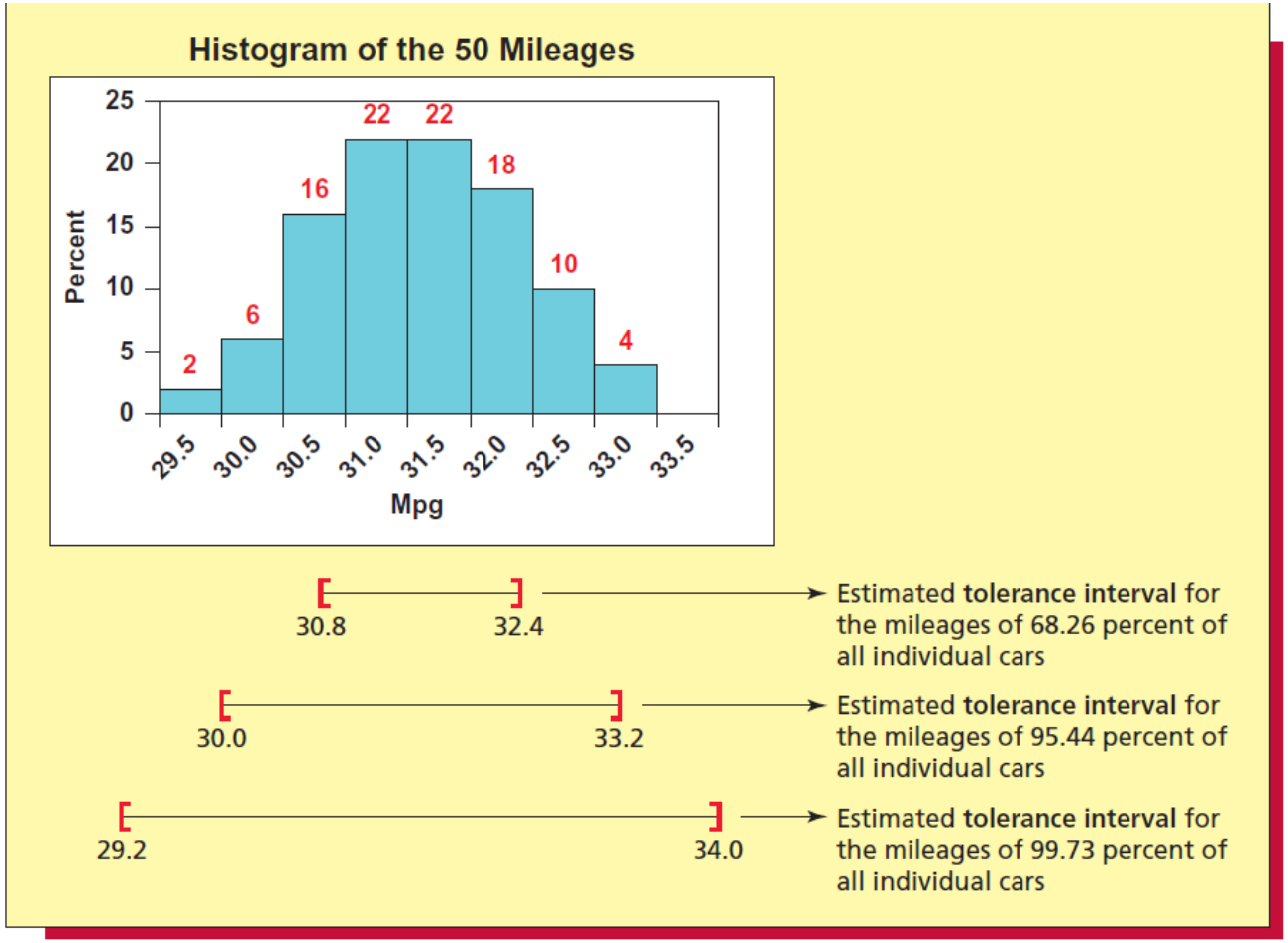
The Empirical Rule for Normal Populations

Figure 3.14



Estimated Tolerance Intervals in Care Mileage Case

Figure 3.15



The Car Mileage Case

- 68.26% of all individual cars will have mileages in the range

$$[\bar{x} \pm s] = [31.6 \pm 0.8] = [30.8, 32.4] \text{ mpg}$$

- 95.44% of all individual cars will have mileages in the range

$$[\bar{x} \pm 2s] = [31.6 \pm 1.6] = [30.0, 33.2] \text{ mpg}$$

- 99.73% of all individual cars will have mileages in the range

$$[\bar{x} \pm 3s] = [31.6 \pm 2.4] = [29.2, 34.0] \text{ mpg}$$

Because the difference between the upper and lower limits of each estimated tolerance interval is fairly small, we might conclude that the variability of the individual car mileages around the estimated mean mileage of 31.6 mpg is fairly small. Furthermore, the interval $[\bar{x} \pm 3s] = [29.2, 34.0]$ implies that almost any individual car that a customer might purchase this year will obtain a mileage between 29.2 mpg and 34.0 mpg.

Tolerance Intervals(容许区间)

An Interval that contains a specified percentage of the individual measurements in a population is called a tolerance interval.

- The one, two, and three standard deviation intervals around μ given in (1), (2) and (3) are tolerance intervals containing, respectively, 68.26 percent, 95.44 percent and 99.73 percent of the measurements in a normally distributed population.

- The *three-sigma* interval $[\mu \pm 3\sigma]$ to be a tolerance interval that contains *almost all* of the measurements in a normally distributed population.

Skewness and the Empirical Rule

- The Empirical Rule holds for a normally distributed population
- It approximately holds for populations having mound-shaped, single-peaked distributions
  - As long as they are not ***very*** skewed to the right or left
- In some situations, skewness can make it tricky to know whether to use the Empirical Rule

Chebyshev's Theorem

- Let μ and σ be a population's mean and standard deviation, then for any value k> 1
- At least $100(1 - 1/k^2)$% of the population measurements lie in the interval [μ-kσ, μ+kσ]
- Holds for any distribution
- Only useful for non-mound-shaped distribution population that is not very skewed

Although Chebyshev's Theorem technically applies to any population, it is only of practical use when analyzing a **non-mound-shaped** (for example, a double-peaked) **population that is not *very* skewed to the right or left.** Why is this? First, **we would not use Chebyshev's Theorem to describe a mound-shaped population that is not very skewed because we can use the Empirical Rule** to do this. In fact, the Empirical Rule is better for such a population because it gives us a shorter interval that will contain a given percentage of measurements. For example, if the Empirical Rule can be used to describe a population, the interval $[\mu \pm 3\sigma]$ will contain

It is also not appropriate to use Chebyshev's Theorem—or any other result making use of the population standard deviation σ—to describe a population that is very skewed. This is because, if a population is very skewed, the measurements in the long tail to the left or right will inflate σ. This implies that tolerance intervals calculated using σ will be too long to be useful. In this case, it is best to measure variation by using **percentiles,** which are discussed in the next section.

z Scores

- For any x in a population or sample, the associated z score is

$$z = \frac{x - \text{mean}}{\text{standard deviation}}$$

- The z score is the number of standard deviations that x is from the mean
  - A positive z score is for x above the mean
  - A negative z score is for x below the mean
  - The mean has a z score of zero

The z-score, which is also called the *standardized value,* is the number of standard deviations that x is from the mean. A positive z-score says that x is above (greater than) the mean, while a negative z-score says that x is below (less than) the mean. For instance, a z-score equal to 2.3 says that x is 2.3 standard deviations above the mean. Similarly, a z-score equal to -1.68 says that x is 1.68 standard deviations below the mean. A z-score equal to zero says that x equals the mean.

A z-score indicates the relative location of a value within a population or sample. For example, below we calculate the z-scores for each of the profit margins for five competing companies in a particular industry. For these five companies, the mean profit margin is 10% and the standard deviation is 3.406%.

Company	Profit margin, x	x − mean	z-score
1	8%	$8 - 10 = -2$	$-2/3.406 = -.59$
2	10	$10 - 10 = 0$	$0/3.406 = 0$
3	15	$15 - 10 = 5$	$5/3.406 = 1.47$
4	12	$12 - 10 = 2$	$2/3.406 = .59$
5	5	$5 - 10 = -5$	$-5/3.406 = -1.47$

Values in two different populations or samples having the same z-score are the same number of standard deviations from their respective means and, therefore, have the same relative locations. For example, suppose that the mean score on the midterm exam for students in Section A of a statistics course is 65 and the standard deviation of the scores is 10. Meanwhile, the mean score on the same exam for students in Section B is 80 and the standard deviation is 5. A student in Section A who scores an 85 and a student in Section B who scores a 90 have the same relative locations within their respective sections because their z-scores, $(85 - 65)/10 = 2$ and $(90 - 80)/5 = 2$, are equal.

Coefficient of Variation

- Measures the size of the standard deviation relative to the size of the mean

$$\text{coefficient of variation} = \frac{\text{standard deviation}}{\text{mean}} \times 100$$

- Used to:
  - Compare the variability of values about the mean
  - Compare variability of populations or samples with different means and standard deviations
  - Measure risk

The coefficient of variation compares populations or samples having different means and different standard deviations. For example, suppose that the mean yearly return for a particular stock fund, which we call Stock Fund 1, is 10.39 percent with a standard deviation of 16.18 percent, while the mean yearly return for another stock fund, which we call Stock Fund 2, is 7.7 percent with a standard deviation of 13.82 percent. It follows that the coefficient of variation for Stock Fund 1 is $(16.18/10.39) \times 100 = 155.73$, and that the coefficient of variation for Stock Fund 2 is $(13.82/7.7) \times 100 = 179.48$. This tells us that, for Stock Fund 1, the standard deviation is 155.73 percent of the value of its mean yearly return. For Stock Fund 2, the standard deviation is 179.48 percent of the value of its mean yearly return.

In the context of situations like the stock fund comparison, the coefficient of variation is often used as a measure of *risk* because it measures the variation of the returns (the standard deviation) relative to the size of the mean return. For instance, although Stock Fund 2 has a smaller standard deviation than does Stock Fund 1 (13.82 percent compared to 16.18 percent), Stock Fund 2 has a higher coefficient of variation than does Stock Fund 1 (179.48 versus 155.73). This says that, *relative to the mean return,* the variation in returns for Stock Fund 2 is higher. That is, we would conclude that investing in Stock Fund 2 is riskier than investing in Stock Fund 1.

3.3 Percentiles, Quartiles, and Box-and-Whiskers Displays

- For a set of measurements arranged in increasing order, the p^{th} percentile is a value such that p percent of the measurements fall at or below the value and (100-p) percent of the measurements fall at or above the value
- The first quartile $\mathbf{Q_1}$ is the 25^{th} percentile
- The second quartile (or median) is the 50^{th} percentile
- The third quartile $\mathbf{Q_3}$ is the 75^{th} percentile
- The interquartile range IQR is $Q_3 - Q_1$

Steps Calculating Percentiles

1. Arrange the measurements in increasing order
2. Calculate the index i=(p/100)n where p is the percentile to find
3. Calculating the percentile

a) If i is not an integer, round up and the next integer greater than i denotes the p^{th} percentile

b) If i is an integer, the p^{th} percentile is the average of the measurements in the i and i+1 positions

Example (p=10th Percentile)

7,524	11,070	18,211	26,817	36,551	41,286
49,312	57,283	72,814	90,416	135,540	190,250

- i=(10/100)12=1.2
- Not an integer so round up to 2
- 10th percentile is in the second position so 11,070
- Q_1 = 22,514
- Q_2 = M_d = 45,299
- Q_3 = 81,615

Note that the second quartile is simply another name for the median. Furthermore, the procedure we have described here that is used to find the 50th percentile (second quartile) will always give the same result as the previously described procedure (see Section 3.1) for finding the median. To illustrate how the quartiles divide a set of measurements into four parts, consider the following display of the sampled incomes, which shows the first quartile (the 25th percentile), $Q_1 = 22{,}514$, the median (the 50th percentile), $M_d = 45{,}299$, and the third quartile (the 75th percentile), $Q_3 = 81{,}615$:

7,524	11,070	18,211	26,817	36,551	41,286	
		$Q_1 = 22{,}514$			$M_d = 45{,}299$	
49,312	57,283	72,814	90,416	135,540	190,250	
		$Q_3 = 81{,}615$				

Using the quartiles, we estimate that for the household incomes in the Midwestern city: (1) 25 percent of the incomes are less than or equal to \$22,514, (2) 25 percent of the incomes are between \$22,514 and \$45,299, (3) 25 percent of the incomes are between \$45,299 and \$81,615, and (4) 25 percent of the incomes are greater than or equal to \$81,615. In addition, to assess some of the lowest and highest incomes, the 10th percentile estimates that 10 percent of the incomes are less than or equal to \$11,070, and the 90th percentile estimates that 10 percent of the incomes are greater than or equal to \$135,540.

In general, unless percentiles correspond to very high or very low percentages, they are resistant (like the median) to extreme values. For example, the 75th percentile of the household incomes would remain \$81,615 even if the largest income (\$190,250) were, instead, \$7,000,000. On the other hand, the standard deviation in this situation would increase. In general, if a population is highly skewed to the right or left, the standard deviation is so large that using it to describe variation does not provide much useful information. For example, the standard deviation of the 12 household incomes is inflated by the large incomes \$135,540 and \$190,250 and can be calculated to be \$54,567. Because the mean of the 12 incomes is \$61,423, Chebyshev's Theorem says that we estimate that at least 75 percent of all household incomes in the city are in the interval $[\bar{x} \pm 2s] = [61{,}423 \pm 2(54{,}567)] = [-47{,}711, 170{,}557]$; that is, are \$170,557 or less. This is much less informative than using the 75th percentile, which estimates that 75 percent of all household incomes are less than or equal to \$81,615. In general, if a population is highly skewed to the right or left, it can be best to describe the variation of the population by using various percentiles. This is what we did when we estimated the variation of the household incomes in the city by using the 10th, 25th, 50th, 75th, and 90th percentiles of the 12 sampled incomes and when we depicted this variation by using the five-number summary. Using other percentiles can also be informative. For example, the Bureau of the Census sometimes assesses the variation of all household incomes in the United States by using the 20th, 40th, 60th, and 80th percentiles of these incomes.

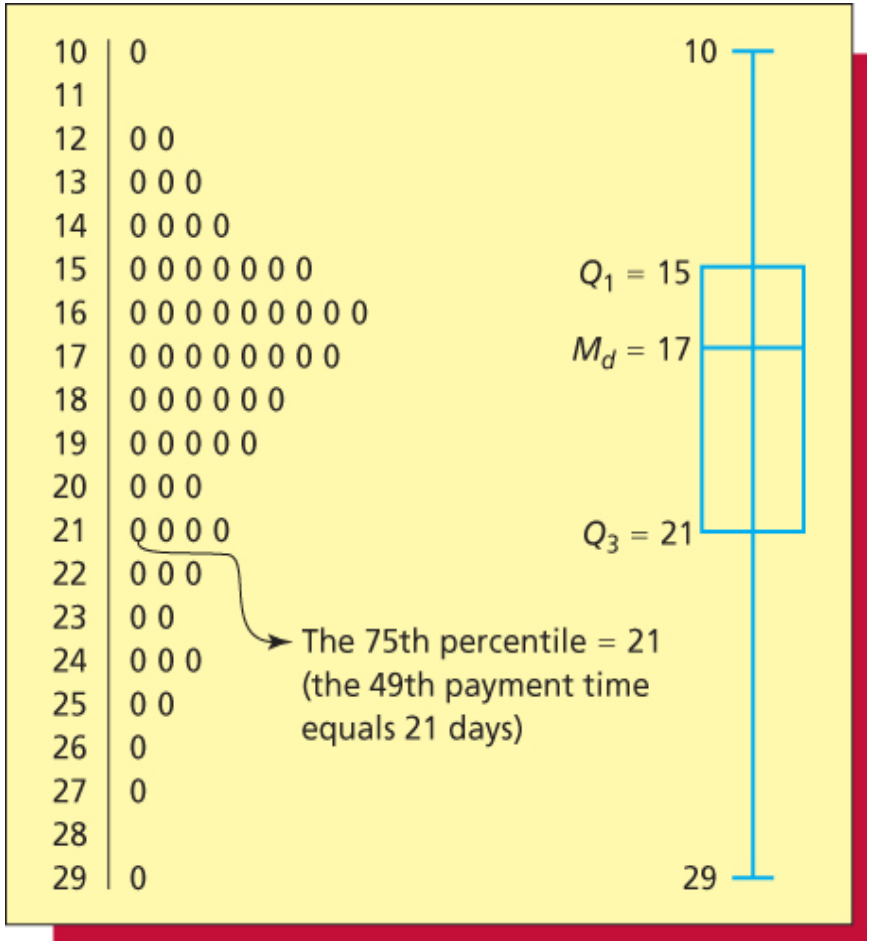
Five Number Summary

1. Smallest measurement
2. First quartile, Q_1
3. Median, M_d
4. Third quartile, Q_3
5. Interquartile range

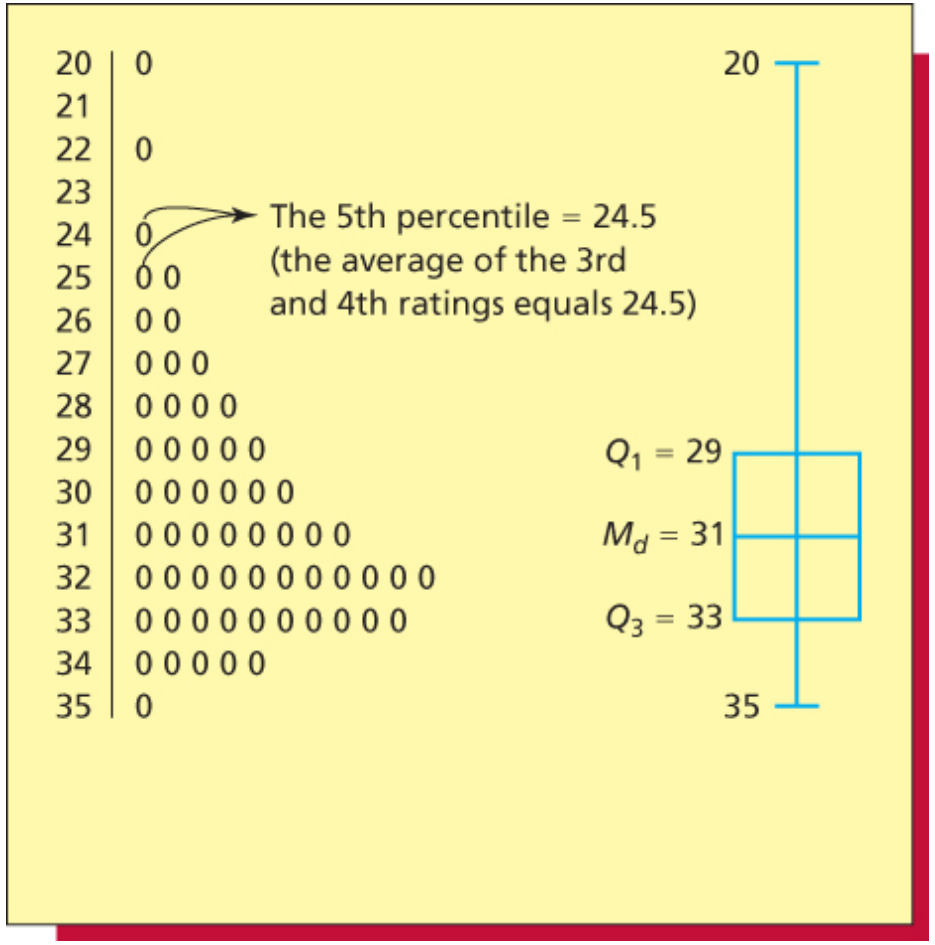
- Displayed visually using a box-and-whiskers plot

Using stem-and-leaf displays to find percentiles.

(a) The 75th percentile of the 65 payment times, and a five-number summary



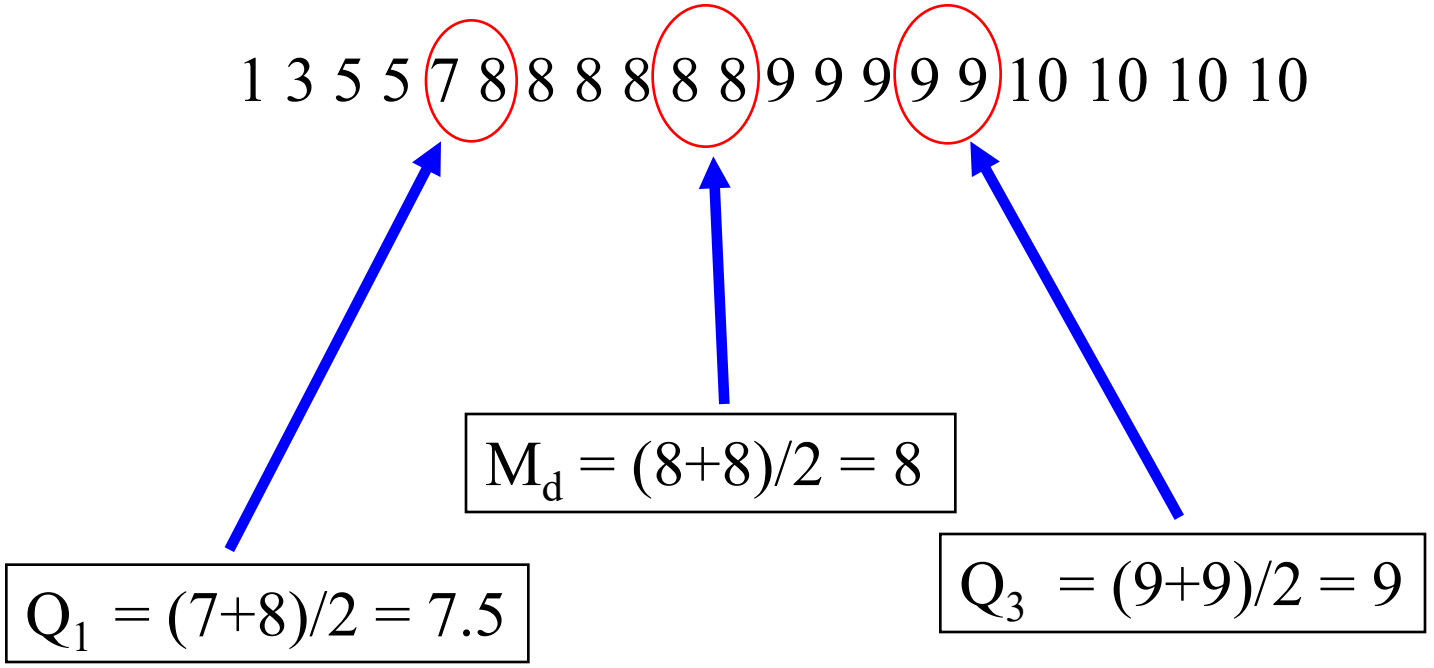
(b) The 5th percentile of the 60 bottle design ratings and a five-number summary



DVD Recorder Satisfaction

20 customer satisfaction ratings:

1 3 5 5 7 8 8 8 8 8 8 9 9 9 9 9 10 10 10 10



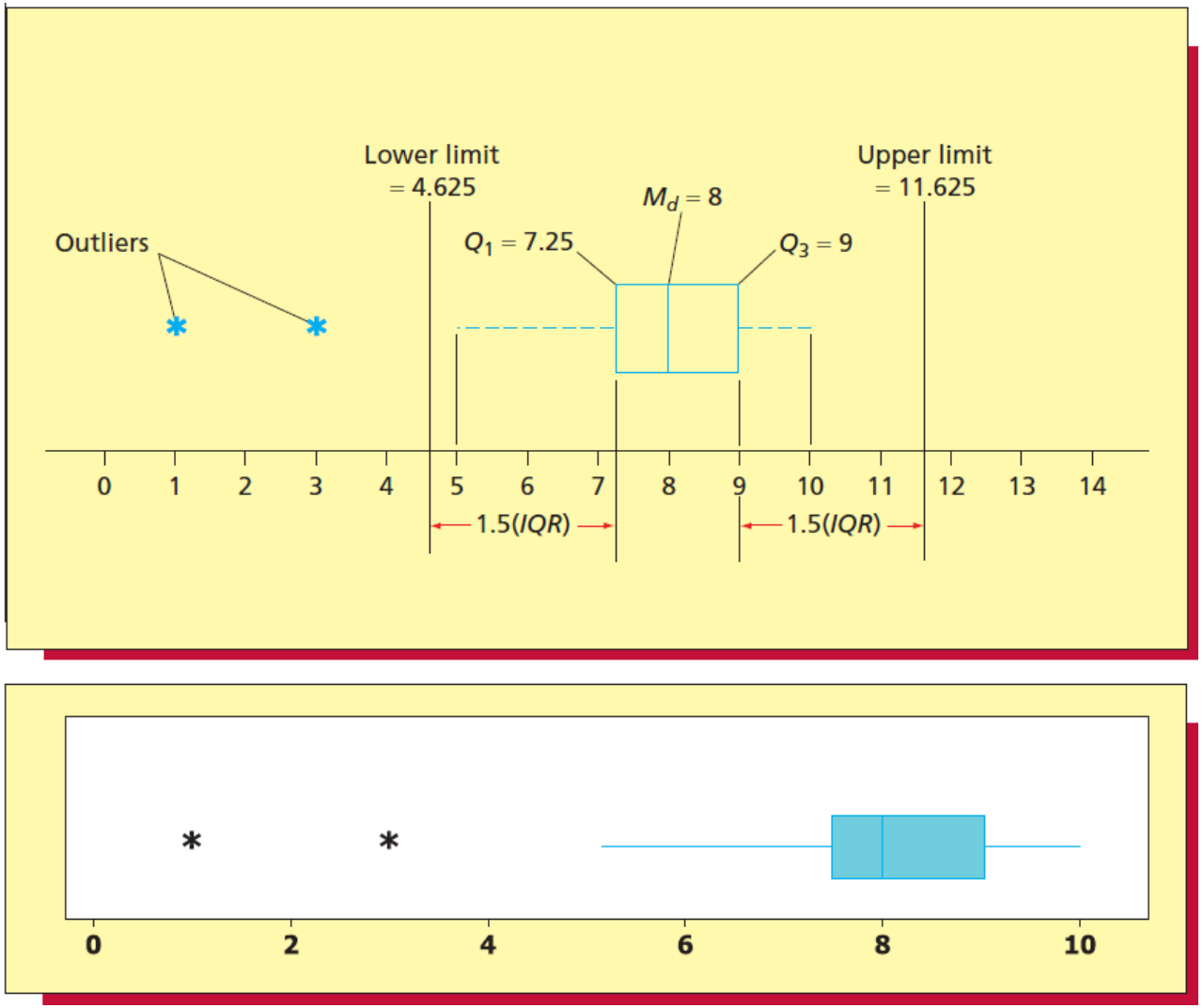
$$\text{IQR} = Q_3 - Q_1 = 9 - 7.5 = 1.5$$

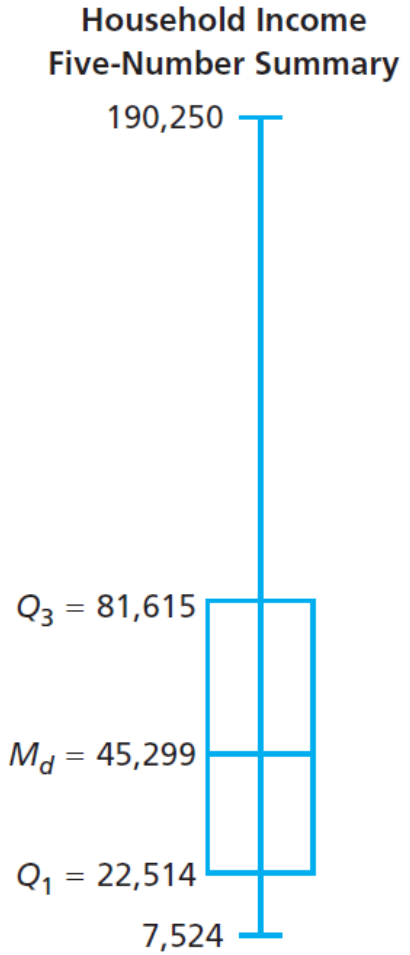
The Box-and-Whiskers Plots

- The box plots the:
  - first quartile, Q_1
  - median, M_d
  - third quartile, Q_3
  - inner fences, located 1.5×IQR away from the quartiles:
    - $= Q_1 - (1.5 \times IQR)$
    - $= Q_3 + (1.5 \times IQR)$
  - outer fences, located 3×IQR away from the quartiles:
    - $= Q_1 - (3 \times IQR)$
    - $= Q_3 + (3 \times IQR)$

- The “whiskers” are dashed lines that plot the range of the data
  - A dashed line drawn from the box below Q_1 down to the smallest measurement
  - Another dashed line drawn from the box above Q_3 up to the largest measurement
- Note: Q_1, M_d, Q_3, the smallest value, and the largest value are sometimes referred to as the five number summary

Box-and-Whiskers Plots





The MINITAB output in Figure 3.16 says that for these ratings $Q_1 = 7.25$, $M_d = 8$, $Q_3 = 9$, and $IQR = Q_3 - Q_1 = 9 - 7.25 = 1.75$. To construct a box-and-whiskers display, we first draw a box that extends from Q_1 to Q_3. As shown in Figure 3.17, for the satisfaction ratings data this box extends from $Q_1 = 7.25$ to $Q_3 = 9$. The box contains the middle 50 percent of the data set. Next a vertical line is drawn through the box at the value of the median M_d. This line divides the data set into two roughly equal parts. We next define what we call the **lower** and **upper limits.** The **lower limit** is located $1.5 \times IQR$ below Q_1 and the **upper limit** is located $1.5 \times IQR$ above Q_3. For the satisfaction ratings data, these limits are

$$Q_1 - 1.5(IQR) = 7.25 - 1.5(1.75) = 4.625 \quad \text{and} \quad Q_3 + 1.5(IQR) = 9 + 1.5(1.75) = 11.625$$

The lower and upper limits help us to draw the plot's **whiskers:** dashed lines extending below Q_1 and above Q_3 (as in Figure 3.17). One whisker is drawn from Q_1 to the smallest measurement between the lower and upper limits. For the satisfaction ratings data, this whisker extends from $Q_1 = 7.25$ down to 5, because 5 is the smallest rating between the lower and upper limits 4.625 and 11.625.

Constructing a Box-and-Whiskers Display (Box Plot)

1 Draw a **box** that extends from the first quartile Q_1 to the third quartile Q_3. Also draw a vertical line through the box located at the median M_d.

2 Determine the values of the **lower** and **upper limits.** The **lower limit** is located $1.5 \times IQR$ below Q_1 and the **upper limit** is located $1.5 \times IQR$ above Q_3. That is, the lower and upper limits are

$$Q_1 - 1.5(IQR) \quad \text{and} \quad Q_3 + 1.5(IQR)$$

3 Draw **whiskers** as dashed lines that extend below Q_1 and above Q_3. Draw one whisker from Q_1 to the *smallest* measurement that is between the lower and upper limits. Draw the other whisker from Q_3 to the *largest* measurement that is between the lower and upper limits.

4 A measurement that is less than the lower limit or greater than the upper limit is an **outlier.** Plot each outlier using the symbol *.

Outliers

- Outliers are measurements that are very different from other measurements
  - They are either much larger or much smaller than most of the other measurements
- Outliers lie beyond the fences of the box-and-whiskers plot
- Outliers are plotted with an “*”

The other whisker is drawn from Q_3 to the largest measurement between the lower and upper limits. For the satisfaction ratings data, this whisker extends from $Q_3 = 9$ up to 10, because 10 is the largest rating between the lower and upper limits 4.625 and 11.625. The lower and upper limits are also used to identify *outliers*. An **outlier** is a measurement that is separated from (that is, different from) most of the other measurements in the data set. A measurement that is less than the lower limit or greater than the upper limit is considered to be an outlier. We indicate the location of an outlier by plotting this measurement with the symbol *. For the satisfaction rating data, the ratings 1 and 3 are outliers because they are less than the lower limit 4.625. Figure 3.18 gives the MINITAB output of a box-and-whiskers plot of the satisfacting ratings.

We now summarize how to construct a box-and-whiskers plot.

When interpreting a box-and-whiskers display, keep several points in mind. First, the box (between Q_1 and Q_3) contains the middle 50 percent of the data. Second, the median (which is inside the box) divides the data into two roughly equal parts. Third, if one of the whiskers is longer than the other, the data set is probably skewed in the direction of the longer whisker. Last, observations designated as outliers should be investigated. Understanding the root causes behind the outlying observations will often provide useful information. For instance, understanding why two of the satisfaction ratings in the box plot of Figure 3.18 are substantially lower than the great majority of the ratings may suggest actions that can improve the DVD recorder manufacturer's product and/or service. Outliers can also be caused by inaccurate measuring, reporting, or plotting of the data. Such possibilities should be investigated, and incorrect data should be adjusted or eliminated.

Graphical five-number summaries and box-and-whiskers displays are perhaps best used to compare different sets of measurements. We demonstrate this use of such displays in the following example.

Example 3.9: S&P 500 Case

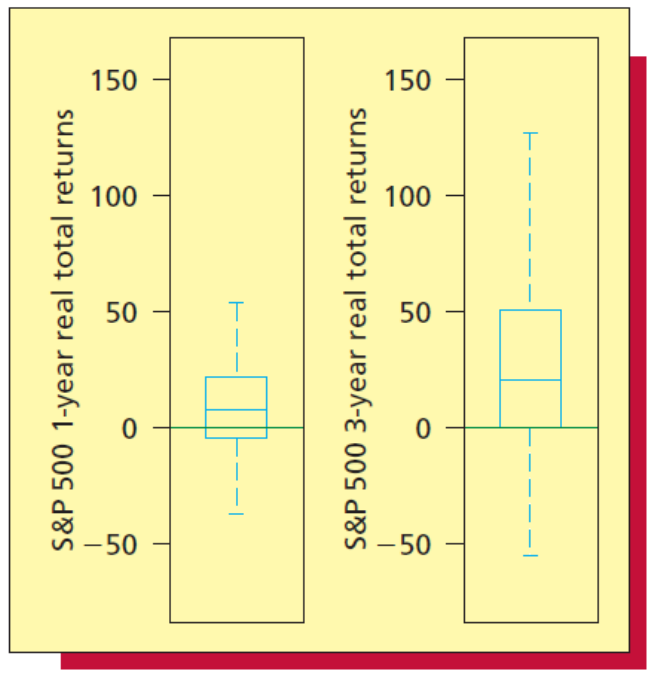
EXAMPLE 3.9 The Standard and Poor's 500 Case

C

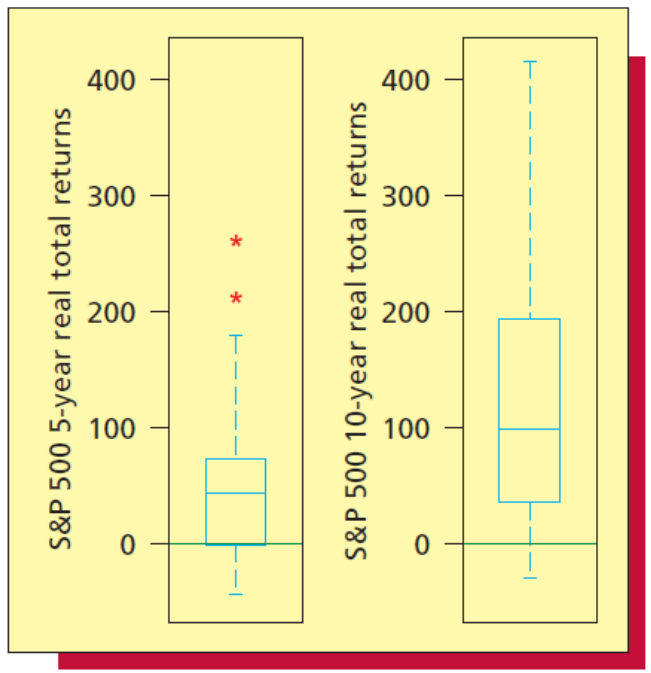
Figure 3.19 shows box plots of the percentage returns of stocks on the Standard and Poor's 500 (S&P 500) for different time horizons of investment. Figure 3.19(a) compares a 1-year time horizon with a 3-year time horizon. We see that there is a 25 percent chance of a negative return (loss) for the 3-year horizon and a 25 percent chance of earning more than 50 percent on the principal during the three years. Figures 3.19(b) and (c) compare a 5-year time horizon with a 10-year time horizon and a 10-year time horizon with a 20-year time horizon. We see that there is still a positive chance of a loss for the 10-year horizon, but the median return for the 10-year horizon almost doubles the principal (a 100 percent return, which is about 8 percent per year compounded). With a 20-year horizon, there is virtually no chance of a loss, and there were two positive outlying returns of over 1000 percent (about 13 percent per year compounded).

FIGURE 3.19 Box Plots of the Percentage Returns of Stocks on the S&P 500 for Different Time Horizons of Investment

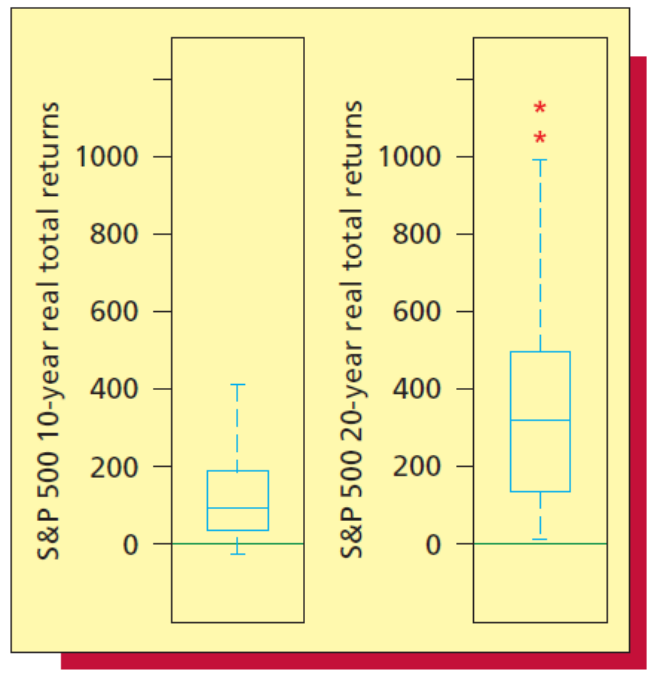
(a) 1-year versus 3-year



(b) 5-year versus 10-year



(c) 10-year versus 20-year



Data from Global Financial Data
Source: http//junkcharts.typepad.com/junk_charts/boxplot/.

3.4 Covariance, Correlation, and the Least Squares Line

(Optional)

- Sample covariance

$$s_{xy} = \frac{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{n-1}$$

- A positive covariance indicates a positive linear relationship between x and y
  - As x increases, y increases
- A negative covariance indicates a negative linear relationship between x and y
  - As x increases, y decreases

FIGURE 3.22 The Sales Volume Data, and a Scatter Plot

(a) The sales volume data SalesPlot

Sales Region	Advertising Expenditure, x	Sales Volume, y
1	5	89
2	6	87
3	7	98
4	8	110
5	9	103
6	10	114
7	11	116
8	12	110
9	13	126
10	14	130

(b) A scatter plot of sales volume versus advertising expenditure

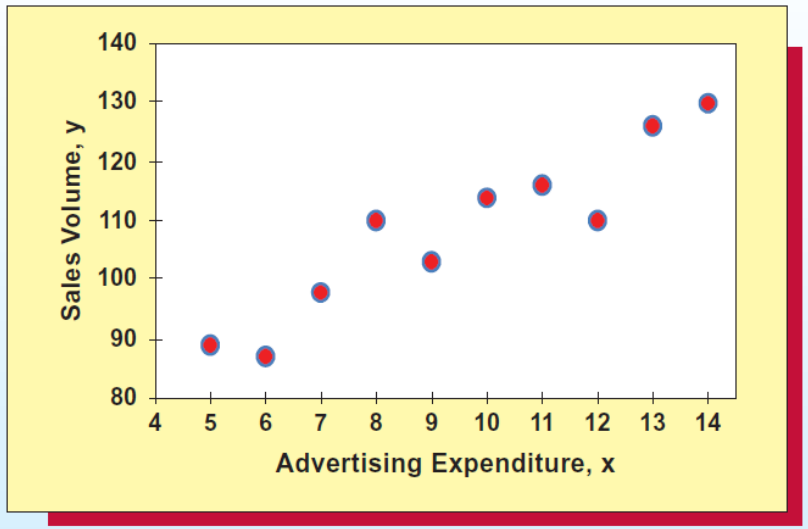
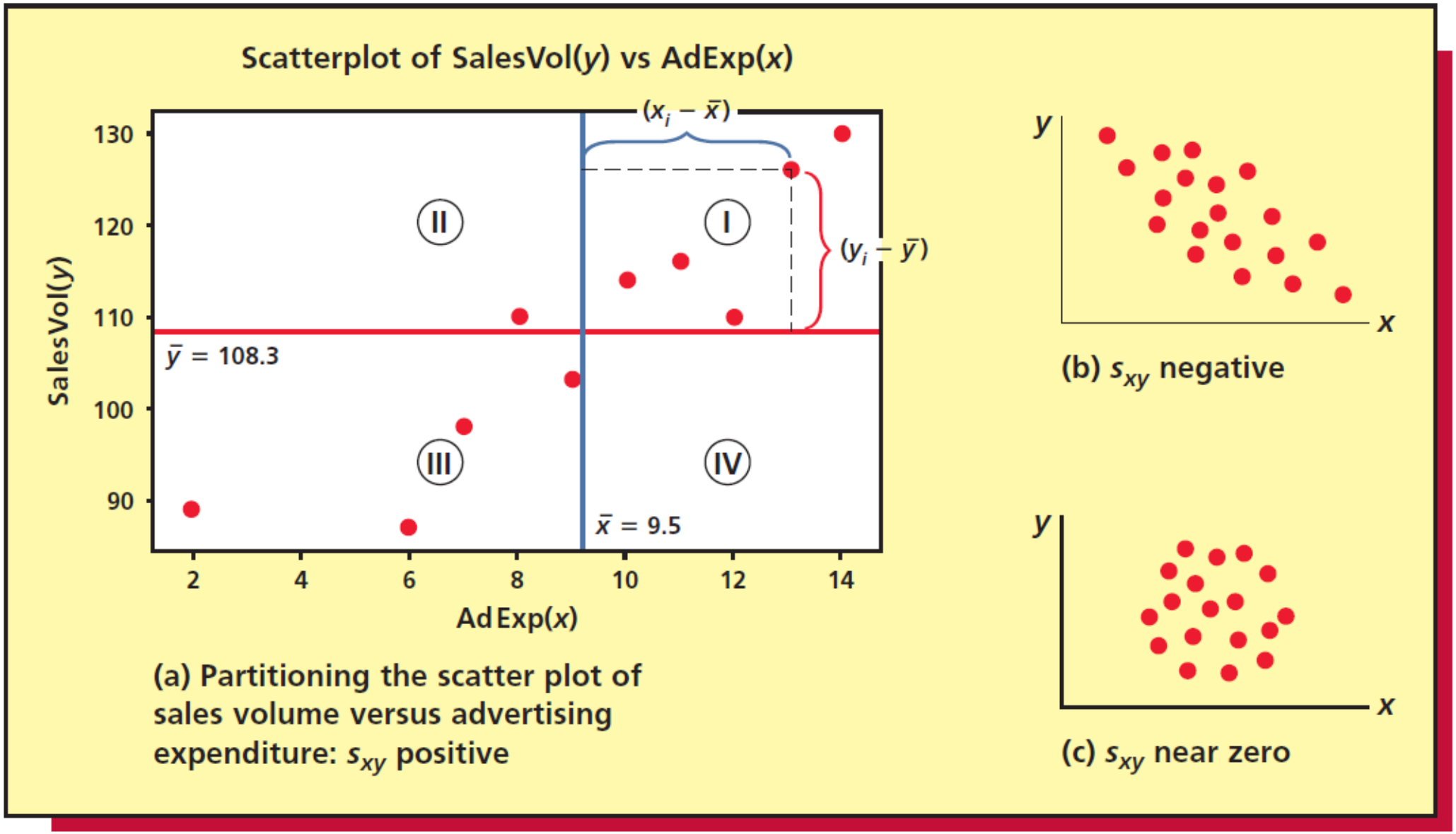


TABLE 3.5 The Calculation of the Numerator of s_{xy}

	x_i	y_i	$(x_i - 9.5)$	$(y_i - 108.3)$	$(x_i - 9.5)(y_i - 108.3)$
	5	89	−4.5	−19.3	86.85
	6	87	−3.5	−21.3	74.55
	7	98	−2.5	−10.3	25.75
	8	110	−1.5	1.7	−2.55
	9	103	−0.5	−5.3	2.65
	10	114	0.5	5.7	2.85
	11	116	1.5	7.7	11.55
	12	110	2.5	1.7	4.25
	13	126	3.5	17.7	61.95
	14	130	4.5	21.7	97.65
Totals	95	1083	0	0	365.50

FIGURE 3.23 Interpretation of the Sample Covariance



(a) Partitioning the scatter plot of sales volume versus advertising expenditure: s_{xy} positive

From the previous discussion, it might seem that a large positive value for the covariance indicates that x and y have a strong positive linear relationship and a very negative value for the covariance indicates that x and y have a strong negative linear relationship. However, one problem with using the covariance as a measure of the strength of the linear relationship between x and y is that the value of the covariance depends on the units in which x and y are measured. A measure of the strength of the linear relationship between x and y that does not depend on the units in which x and y are measured is the **correlation coefficient.**

Correlation Coefficient

- Magnitude of covariance does not indicate the strength of the relationship
- Correlation coefficient (r) is a measure of the strength of the relationship that does not depend on the magnitude of the data

$$r = \frac{S_{xy}}{S_x S_y}$$

Correlation Coefficient Continued

- Always between ±1
  - Near -1 shows strong negative correlation
  - Near 0 shows no correlation
  - Near +1 shows strong positive correlation
- Sample correlation coefficient is the point estimate for the population correlation coefficient ρ

It can be shown that the sample correlation coefficient r is always between -1 and 1. A value of r near 0 implies little linear relationship between x and y. A value of r close to 1 says that x and y have a strong tendency to move together in a straight-line fashion with a positive slope and, therefore, that x and y are highly related and **positively correlated.** A value of r close to -1 says that x and y have a strong tendency to move together in a straight-line fashion with a negative slope and, therefore, that x and y are highly related and **negatively correlated.** Note that if $r = 1$, the (x, y) points fall exactly on a positively sloped straight line, and, if $r = -1$, the (x, y) points fall exactly on a negatively sloped straight line. For example, since $r = .93757$ in the sales volume example, we conclude that advertising expenditure (x) and sales volume (y) have a strong tendency to move together in a straight-line fashion with a positive slope. That is, x and y have a strong positive linear relationship.

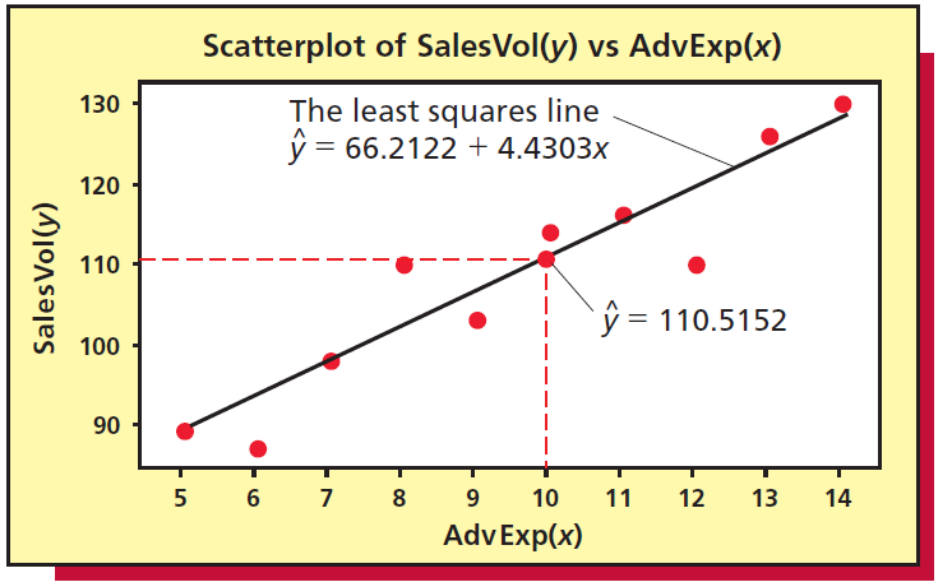
Least Squares Line

$$b_1 = \frac{s_{xy}}{s_x^2}$$

$$b_0 = \bar{y} - b_1 \bar{x}$$

- b_0 is the y-intercept
- b_1 is the slope

FIGURE 3.24 The Least Squares Line for the Sales Volume Data



3.5 Weighted Means and Grouped Data

(Optional)

- Sometimes, some measurements are more important than others
- Assign numerical “weights” to the data
- Weights measure relative importance of the value

$$\frac{\sum w_i x_i}{\sum w_i}$$

In order to illustrate the need for a weighted mean and the required calculations, suppose that an investor obtained the following percentage returns on different amounts invested in four stock funds:

Stock Fund	Amount Invested	Percentage Return
1	$50,000	9.2%
2	$10,000	12.8%
3	$10,000	−3.3%
4	$30,000	6.1%

If we wish to compute a mean percentage return for the total of $100,000 invested, we should use a weighted mean. This is because each of the four percentage returns applies to a different amount invested. For example, the return 9.2 percent applies to $50,000 invested and thus should count more heavily than the return 6.1 percent, which applies to $30,000 invested.

The percentage return measurements are $x_1 = 9.2$ percent, $x_2 = 12.8$ percent, $x_3 = -3.3$ percent, and $x_4 = 6.1$ percent, and the weights applied to these measurements are $w_1 = \$50{,}000$, $w_2 = \$10{,}000$, $w_3 = \$10{,}000$, and $w_4 = \$30{,}000$. That is, we are weighting the percentage returns by the amounts invested. The weighted mean is computed as follows:

$$\mu = \frac{50{,}000(9.2) + 10{,}000(12.8) + 10{,}000(-3.3) + 30{,}000(6.1)}{50{,}000 + 10{,}000 + 10{,}000 + 30{,}000}$$

$$= \frac{738{,}000}{100{,}000} = 7.38\%$$

In this case the unweighted mean of the four percentage returns is 6.2 percent. Therefore, the unweighted mean understates the percentage return for the total of $100,000 invested.

Descriptive Statistics for Grouped Data

- Data already categorized into a frequency distribution or a histogram is called grouped data
- Can calculate the mean and variance even when the raw data is not available
- Calculations are slightly different for data from a sample and data from a population

Descriptive Statistics for Grouped Data

(Continued)

Sample

$$\bar{x} = \frac{\sum f_i M_i}{\sum f_i} = \frac{\sum f_i M_i}{n}$$

$$s^2 = \frac{\sum f_i \left(M_i - \bar{x}\right)^2}{n-1}$$

Population

$$\mu = \frac{\sum f_i M_i}{\sum f_i} = \frac{\sum f_i M_i}{N}$$

$$\sigma^2 = \frac{\sum f_i \left(M_i - \bar{x}\right)^2}{N}$$

Sample Mean and Sample Variance of the Satisfaction Rates

Calculating the Sample Mean Satisfaction Rating

Satisfaction Rating	Frequency (f_i)	Class Midpoint (M_i)	f_iM_i
36–38	4	37	4(37) = 148
39–41	15	40	15(40) = 600
42–44	25	43	25(43) = 1,075
45–47	19	46	19(46) = 874
48–50	2	49	2(49) = 98
	$n = 65$		2,795

$$\bar{x} = \frac{\sum f_i M_i}{n} = \frac{2{,}795}{65} = 43$$

Calculating the Sample Variance of the Satisfaction Ratings

Satisfaction Rating	Frequency f_i	Class Midpoint M_i	Deviation $(M_i - \bar{x})$	Squared Deviation $(M_i - \bar{x})^2$	$f_i(M_i - \bar{x})^2$
36–38	4	37	37 − 43 = −6	36	4(36) = 144
39–41	15	40	40 − 43 = −3	9	15(9) = 135
42–44	25	43	43 − 43 = 0	0	25(0) = 0
45–47	19	46	46 − 43 = 3	9	19(9) = 171
48–50	2	49	49 − 43 = 6	36	2(36) = 72
	65				$\sum f_i(M_i - \bar{x})^2 = 522$

$$s^2 = \text{sample variance} = \frac{\sum f_i(M_i - \bar{x})^2}{n - 1} = \frac{522}{65 - 1} = 8.15625$$

Table 3.8 and Table 3.8

3.6 The Geometric Mean (Optional)

- For rates of return of an investment, use the geometric mean
- Suppose the rates of return are R_1, R_2, ⋯, R_n for periods 1, 2, ⋯, n
- The mean of all these returns is the calculated as the geometric mean:

$$R_g = \sqrt[n]{(1+R_1)\times(1+R_2)\times\cdots\times(1+R_n)}-1$$

Summary

Chapter Summary

We began this chapter by presenting and comparing several measures of **central tendency.** We defined the **population mean** and we saw how to estimate the population mean by using a **sample mean.** We also defined the **median** and **mode,** and we compared the mean, median, and mode for symmetrical distributions and for distributions that are skewed to the right or left. We then studied measures of **variation** (or *spread*). We defined the **range, variance,** and **standard deviation,** and we saw how to estimate a population variance and standard deviation by using a sample. We learned that a good way to interpret the standard deviation when a population is (approximately) normally distributed is to use the **Empirical Rule,** and we studied **Chebyshev's Theorem,** which gives us intervals containing reasonably large fractions of the population units no matter what the population's shape might be. We also saw that, when a data set is highly skewed, it is best to use **percentiles** and **quartiles** to measure variation, and we learned how to construct a **box-and-whiskers plot** by using the quartiles.

After learning how to measure and depict central tendency and variability, we presented several optional topics. First, we discussed several numerical measures of the relationship between two variables. These included the **covariance,** the **correlation coefficient,** and the **least squares line.** We then introduced the concept of a **weighted mean** and also explained how to compute descriptive statistics for **grouped data.** Finally, we showed how to calculate the **geometric mean** and demonstrated its interpretation.

Thank you!