

The big formula $\det A = \sum (det P, a_{1,j_1} a_{2,j_2} \dots a_{n,j_n})$ is a direct definition of the determinant.

When $n=4$, -

there are $4! = 24$ terms.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}$$

$$= a_{11} (a_{22} a_{33} - a_{23} a_{32}) + a_{12} (a_{23} a_{31} - a_{21} a_{33}) + a_{13} (a_{21} a_{32} - a_{22} a_{31})$$

$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

"cofactors"

$$C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Choose one entry
from each row and each column

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

Associated with a_{ij} , we cross out row i and column j from A

to get a submatrix M_{ij} of size $n-1$

The cofactor $C_{ij} = (-1)^{i+j} \det M_{ij}$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

Cofactor expansion

Each a_{ij} has its cofactor $C_{ij} = (-1)^{i+j} \det M_{ij}$

$$= a_{11} \det M_{11} - a_{12} \det M_{12} + \dots + (-1)^{1+n} a_{1n} \det M_{1n}$$

$a_{1n} \det M_{1n}$

The determinant of order n
is a combination of n determinants
of order $n-1$.

The cofactor formula along row i is $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$

$\det A = \det A^T$ The cofactor formula along column j is $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$

Example

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} d = d$$

$$C_{12} = (-1)^{1+2} c = -c$$

$$C_{21} = (-1)^{1+2} b = -b$$

$$C_{22} = (-1)^{2+2} a = a$$

$$= a_{11}C_{11} + a_{12}C_{12}$$

$$= 2(-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + (-1)(-1)^{1+2} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$+ (-1)(-1)^{1+2} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 2 \left[2(-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + (-1)(-1)^{1+2} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \right]$$

$$+ (-1)(-1)(-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 2 [2 \times 3 + (-2)] - 3 = 5$$

3.3 Cramer's rule, inverses and volumes

$$\det(AB) = \det A \det B$$

Cramer's rule (formula for the solution to $A\vec{x} = \vec{b}$)

Suppose \vec{x} is a solution to $A\vec{x} = \vec{b}$. $I = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A[\vec{x} \ \vec{e}_2 \ \vec{e}_3] = [\vec{b} \ A\vec{e}_2 \ A\vec{e}_3] = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1$$

Then $\det A x_1 = \det B_1$. If $\det A \neq 0$, $x_1 = \frac{\det B_1}{\det A}$ is the first component of the solution to $A\vec{x} = \vec{b}$.

Similarly, $A \begin{bmatrix} 0 & x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} = B_2$ gives $\det A x_2 = \det B_2$. So $x_2 = \frac{\det B_2}{\det A}$.

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}} = \frac{-4}{-2} = 2 \quad x_2 = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}} = \frac{2}{-2} = -1$$

Example. Solve $3x_1 + 4x_2 = 2$ by Cramer's rule.

$$5x_1 + 6x_2 = 4$$

The solution is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Theorem. If $\det A \neq 0$, $A\vec{x} = \vec{b}$ is solved by determinants as

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \quad \dots, \quad x_n = \frac{\det B_n}{\det A}$$

Each matrix $B_i, i=1, 2, \dots, n$ has the i -th column of A replaced by \vec{b} .

(Cramer's rule)
evaluates $n+1$ determinants.

Example. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and we assume $\det A \neq 0$. We write

$$A^{-1} = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = \begin{bmatrix} \boxed{x_1} & \boxed{y_1} \\ \boxed{x_2} & \boxed{y_2} \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{bmatrix} = I = A A^{-1} = A \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = \begin{bmatrix} \boxed{A\vec{x}} & \boxed{A\vec{y}} \end{bmatrix}$$

By Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\det A} = \frac{d}{\det A}$$

$$x_2 = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\det A} = \frac{-c}{\det A}$$

$$y_1 = \frac{\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}}{\det A} = \frac{-b}{\det A}$$

$$y_2 = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{\det A} = \frac{a}{\det A}$$

We have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Each entry in A^{-1} is a cofactor divided by $\det A$.