Chapter 2 Solving linear equations

2.1 Vectors and linear equations

Example 1 $x - 2y = 1$

$3x + 2y = 11$

$A\vec{x} = \vec{b}$: $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

The row way : $A\vec{x} = \begin{bmatrix} (1,-2)\cdot(x,y) \\ (3,2)\cdot(x,y) \end{bmatrix} = \begin{bmatrix} x-2y \\ 3x+2y \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$: the coefficient matrix of the system.

$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

The column way

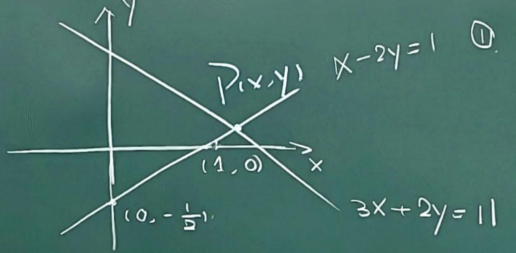
$A\vec{x} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} x-2y \\ 3x+2y \end{bmatrix}$

matrix form $A\vec{x} = \vec{b}$; A is of size n by n.

when $n = 4$, then 4 equations with 4 unknowns gives 4 hyperplanes.

$A\vec{x} = \vec{b}$

row picture



The two lines produced by the two equations meet at a Point P(x, y)

In other words, this P lies on both lines, and it solves both equation

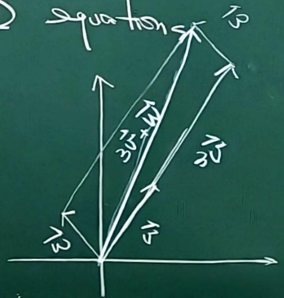
$3 \times ①$: $3x - 6y = 3$

$② - 3 \times ①$: $3x - 3x + 2y - (-6y) = 11 - 3$: $8y = 8$. gives $y = 1$

Then $x - 2 \times 1 = 1$ gives $x = 3$. So $P = (3, 1)$

With the two intersecting lines, the row picture show that: the 2 lines meet at a single point, which is the solution of the 2 equations

Column picture $A = [\, \vec{v} \;\, \vec{w} \,]$, where $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$



The problem is to find the suitable coefficients x and y such that $x\vec{v} + y\vec{w}$ produces $\vec{b}$

With the choice $x = 3$, $y = 1$. $3\vec{v} + \vec{w} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \vec{b}$

In the column picture, we combine the columns of A on the left side to produce $\vec{b}$ on the right side.

Example 2 $x + 2y + 3z = 6$

$2x + 5y + 2z = 4$

$6x - 3y + z = 2$

We look for numbers x, y, z that solve all equations at once.

Before solving it, we visualize the problem in both the row and column ways.

In the row picture, each equation produce a plane in the xyz space. The three planes meet at $P = P(0, 0, 2)$.

The first plane and the second plane intersect in a line L, (The usual result of 2 equations with 3 unknowns is a line of solutions)

The third plane cut this line ℓ at a single point P

So the 3 plane meet at this single point P. It solves all 3 equations.

Exception $x + 2y + 3z = 6$

$x + 2y + 3z = 0$

In the column picture. $A\vec{x} = x\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. $A = [\vec{u}\ \vec{v}, \vec{w}]$

Since $\vec{b} = 2\vec{w}$.

the coefficients we need are

$x=0,\ y=0,\ z=2$

Since all the combinations of $\vec{u}, \vec{v}, \vec{w}$ fill the whole xyz space

for any $\vec{b}$, we can find a typical combination of the columns that equals $\vec{b}$

Example 3 Show the system

$$x + 3y + 5z = 4 \quad ①$$
$$x + 2y - 3z = 5 \quad ②$$
$$2x + 5y + 2z = 8 \quad ③$$

has no solution.

Vector equation: $A\vec{x} = \vec{b}$

$$\begin{bmatrix} x+3y+5z \\ x+2y-3z \\ 2x+5y+2z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

Proof: Method I

① + ②: $2x + 5y + 2z = 9$

a linear combination of the 3 equations with coefficients $1, 1, -1$

$\boxed{1\times① + 1\times② + (-1)\times③}$: $0 = 1$. It is impossible. So this system has NO solution.

In other words, we deduce from $A\vec{x} = \vec{b}$ that $0 = 1$. It makes a contradiction.

Method II: We take the dot product of $A\vec{x} = \vec{b}$ on both sides with $\vec{n} = (1, 1, -1)$.

$$\vec{n}\cdot\vec{b} = (1, 1, -1)\cdot(4, 5, 8) = 1\times4 + 1\times5 + (-1)\times8 = 1.$$

$$\vec{n}\cdot A\vec{x} = 1\times(x+3y+5z) + 1\times(x+2y-3z) + (-1)\times(2x+5y+2z) = 0.$$

In the row picture, the 3 equations determine 3 planes. Although no two planes are parallel, the 3 planes don't meet at any point.

From the column perspective, no combination of the 3 columns of the coefficient matrix A can produce $\vec{b}$.

equations

quations

$\Big] = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

matrix form $A\vec{x} = \vec{b}$; A is of size n by n.

when $n=4$, then 4 equations with 4 unknowns gives 4 hyperplanes.

$$\begin{array}{cccc} & \vec{u} & \vec{v} & \vec{w} \\ A = & \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & -3 \\ 2 & 5 & 2 \end{bmatrix} \end{array}$$

let $x = 19,\ y = -8,\ z = 1$ go into $\vec{x}$.

The typical combination $A\vec{x} = 19\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + (-8)\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + (1)\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So $\vec{w} = -19\vec{u} + 8\vec{v}$,

all the linear combinations of $\vec{u}$, $\vec{v}$ and $\vec{w}$ remain on the plane of $\vec{u}$ and $\vec{v}$

But $\vec{b}$ is not on the plane of $\vec{u}$ and $\vec{v}$.

(how to prove it?)

There is No solution to

$$c\vec{u} + d\vec{v} = \vec{b}$$