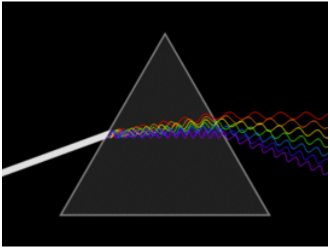
Linear Wave Propagation

Elastic 1D problem

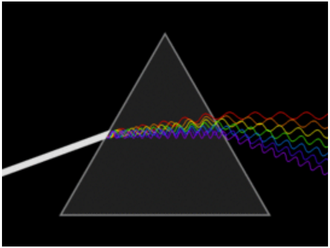
Lúcio de Abreu Corrêa

PROMEC - Federal University of Rio Grande do Sul

Introduction



Introduction



Definition

A wave is the propagation of a disturbance producing in its passage a reversible (or not reversible) variation of the local physical properties of the medium. It moves with a determined speed which depends on the characteristics of the propagation medium.

Introduction

How mechanical waves propagates?

They are created due to local changes of the mechanical properties:

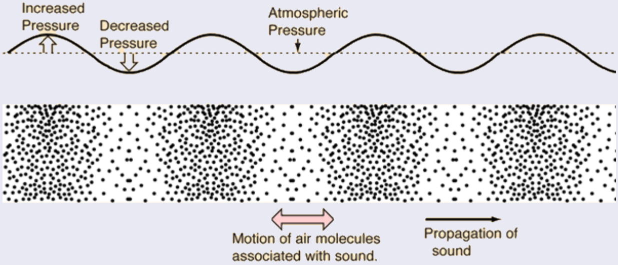
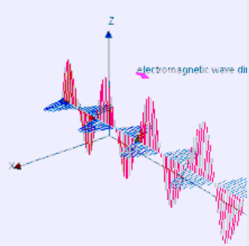


Figure: Pressure variation [from:`http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/tralon.html`]

Introduction

How electromagnetic waves propagates?

They are created due to periodic change of electric or magnetic field:



Introduction

Existence condition

For waves to propagate in a medium, it must be stable! Under the action of an external disturbance, the medium must develop a return mechanism bringing it back to its position of equilibrium. The nature and properties of the wave depend on how this mechanism acts. Thus, for example, for waves, this return mechanism is gravity tending to bring the free surface back to a position of equilibrium. For sound waves, the return mechanism is the tendency of a fluid to even out its pressure.

Introduction

History I

The study of waves can be traced back to antiquity where philosophers, such as Pythagoras (c. 560-480 BC), studied the relation of pitch and length of string in musical instruments. However, it was not until the work of Giovani Benedetti (1530-90), Isaac Beeckman (1588-1637) and Galileo (1564-1642) that the relationship between pitch and frequency was discovered. This started the science of acoustics, a term coined by Joseph Sauveur (1653-1716) who showed that strings can vibrate simultaneously at a fundamental frequency and at integral multiples that he called harmonics. Isaac Newton (1642-1727) was the first to calculate the speed of sound in his Principia. (see page 5 [2])

Introduction

History II

However, he assumed isothermal conditions so his value was too low compared with measured values. This discrepancy was resolved by Laplace (1749-1827) when he included adiabatic heating and cooling effects. The first analytical solution for a vibrating string was given by Brook Taylor (1685-1731). After this, advances were made by Daniel Bernoulli (1700-82), Leonard Euler (1707-83) and Jean d'Alembert (1717-83) who found the first solution to the linear wave equation. Whilst others had shown that a wave can be represented as a sum of simple harmonic oscillations, it was Joseph Fourier (1768-1830) who conjectured that arbitrary functions can be represented by the superposition of an infinite sum of sines and cosines - now known as the Fourier series.

Introduction

History III

However, whilst his conjecture was controversial and not widely accepted at the time, Dirichlet subsequently provided a proof, in 1828, that all functions satisfying Dirichlet's conditions (i.e. non-pathological piecewise continuous) could be represented by a convergent Fourier series. Finally, the subject of classical acoustics was laid down and presented as a coherent whole by John William Strutt (Lord Rayleigh, 1832-1901) in his treatise Theory of Sound.

Introduction

History IV

The study of electromagnetism was again started in antiquity, but very few advances were made until a proper scientific basis was finally initiated by William Gilbert (1544-1603) in his De Magnete. However, it was only late in the 18th century that real progress was achieved when Franz Ulrich Theodor Aepinus (1724-1802), Henry Cavendish (1731-1810), Charles-Augustin de Coulomb (1736-1806) and Alessandro Volta (1745-1827) introduced the concepts of charge, capacity and potential. Additional discoveries by Hans Christian Ørsted (1777-1851), André-Marie Ampère (1775-1836) and Michael Faraday (1791-1867) found the connection between electricity and magnetism and a full unified theory in rigorous mathematical terms was finally set out by James Clerk Maxwell (1831-79) in his Treatise on Electricity and Magnetism.

Introduction

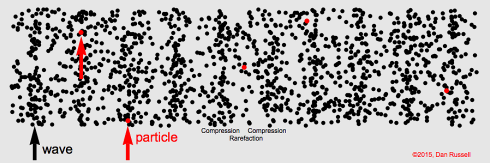
History V

It was in this work that all electromagnetic phenomena and all optical phenomena were first accounted for, including waves. It also included the first theoretical prediction for the speed of light. At the end of the 19th century, when some erroneously considered physics to be very nearly complete, new physical phenomena began to be observed that could not be explained. These demanded a whole new set of theories that ultimately led to the discovery of general relativity and quantum mechanics; which, even now in the 21st century are still yielding exciting new discoveries.

Introduction

Mechanical waves

Mechanical waves propagate through physical matter, the substance of which is deformed. The restorative forces then reverse the deformation.



For example, sound waves propagate through air molecules that collide with their neighbours. When molecules collide, they also bounce against each other. This then prevents the molecules from continuing to move in the direction of the wave;

Introduction

Mechanical waves

- The oscillating wave, which can be periodic, is well illustrated by the ripples caused by the pebble falling into the water.
- The solitary wave or soliton finds a very good example in the tidal bore.
- The shock wave perceived acoustically, for example, when an airplane is flying at supersonic speed.

Introduction

Electromagnetic waves

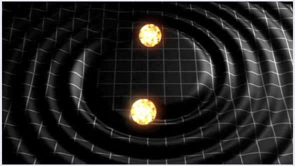
They do not require physical support. Instead, they consist of periodic oscillations of electric and magnetic fields originally generated by charged particles, and therefore can travel through a vacuum (Electromagnetic waves are predicted by the classical laws of electricity and magnetism, known as Maxwell's equations). The equations are:

$$\nabla^2 \mathrm{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2} \quad \nabla^2 \mathrm{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{B}}{\partial t^2}.$$

Introduction

Gravitational waves

Gravitational waves also do not require physical support. These are deformations of the space-time fabric that are propagated.



FYI:Alessandra Buonanno [1]

Introduction

Quantum mechanics

Quantum mechanics has shown that elementary particles can be assimilated to waves, and vice versa, which explains the sometimes undulatory and sometimes corpuscular behaviour of light: the photon can be considered both as a wave and as a particle (see Wave-particle duality); conversely the sound wave (mechanical vibration) can be considered as a corpuscle (see phonon).

$$\lambda = \frac{\hbar}{p} = \frac{\hbar}{mv}\sqrt{1 - \frac{v^2}{c^2}} \tag{1}$$

Linear Waves

Definition I

Linear waves are described by linear equations, i.e. those where in each term of the equation the dependent variable and its derivatives are at most first degree (raised to the first power). Linear waves are modelled by PDEs that are linear in the dependent variable, u, and its first and higher derivatives, if they exist.

$$a_0(x)u + a_1(x)u' + a_2(x)u'' + \cdots + a_n(x)u^{(n)} + b(x) = 0,$$

Linear Waves

Definition II

This means that the superposition principle applies, and linear combinations of simple solutions can be used to form more complex solutions. Thus, all the linear system analysis tools are available to the analyst, with Fourier analysis: expressing general solutions in terms of sums or integrals of well known basic solutions, being one of the most useful.

Establishment of equations I

We are interested in the longitudinal vibrations of the bar. A bar element located between the abscissa x and $x + \Delta x$.

Figure: Longitudinal vibration in elastic beam.

Establishment of equations II

Let's consider:

- Homogeneous beam;
- Density: ρ_0 [kg/m^3]
- Length: L [m]
- Cross section: σ_0 [m^2]
- Mean Tension: T_0 [N]
- Young Modulus: E_0 [N/m^2] [Pa]
- We place ourselves in the hypotheses of small perturbations, which allows us to write linearized equations.

Establishment of equations III

Let's consider:

- A point on the bar is identified by its abscissa $x \in [0, L]$.
- We denote by $u(x, t)$ is the longitudinal displacement at time t of the point M located at x when the bar is at rest and $T(x, t)$ the tension of the bar at x at time t.
- $T(x, t)$ denotes the force exerted at time t by the portion $[x, L]$ of the bar on the portion $[0, x]$.

Establishment of equations IV

Its length at rest is $l = \Delta x$. At instant t, we observe an elongation δl such that:

$$\delta l = u(x + \Delta x, t) - u(x, t) \approx \frac{\partial u(x, t)}{\partial x} \Delta x. \tag{2}$$

However, according to Hooke's law, for an element of length l subjected to an average tension T_0, an elongation of length δl requires an increase in tension δT such that:

$$\delta T = E_0 \sigma_0 \frac{\delta l}{l}. \tag{3}$$

It can be write as:

$$T(x, t) - T_0 = E_0 \sigma_0 \frac{\partial u(x, t)}{x} \tag{4}$$

Establishment of equations V

Moreover, according to the Fundamental Principle of Dynamics, applied to the bar element $[x, x + \Delta x]$:

$$\rho_0 \sigma_0 \Delta x \frac{\partial^2 u(x,t)}{\partial t^2} = T(x + \Delta x, t) - T(x, t) \tag{5}$$

The first approximation can be write as:

$$\rho_0 \sigma_0 \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial T(x,t)}{\partial x} \tag{6}$$

Establishment of equations VI

$$\rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} = E_0 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{7}$$

Relation between celerity and mechanical properties

$$c = \sqrt{\frac{E_0}{\rho_0}} \quad \left[\frac{m}{s}\right] \tag{8}$$

Establishment of equations VII

1D wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{9}$$

Remarks I

Some numbers

Material	ρ_0	E_0	c
Steel	7900	210e9	5155
Aluminium	2700	70e9	5091

Table: Material property table.

Question 1

Explain the difference between the longitudinal velocity in the material and the wave velocity predict by Eq. (8).

Remarks II

Equivalence for transverse vibrations in strings

The transverse vibrations of a stretched rope (that is, subjected to a tension $T_0 \geq 0$) can be treated in the same way. We then obtain the following equation for the transverse displacement w of the rope:

$$\frac{\partial^2 w(x,t)}{\partial t^2} - c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \tag{10}$$

with $c = \sqrt{\frac{T_O}{\tau_0}}$, where τ_0 represents the linear density of the string. For this reason, the one-dimensional wave equation is often referred to in literature as the vibrating string equation.

Remarks III

Equivalence for acoustic wave equation

The pressure vibrations can be treated in the same way. We then obtain the following equation for the pressure (p)

$$\frac{\partial^2 p(x,t)}{\partial t^2} - c^2 \frac{\partial^2 p(x,t)}{\partial x^2} = 0 \tag{11}$$

with $c = \sqrt{\frac{\partial p}{\partial \rho_0}}$.

Remarks IV

Equivalence for acoustic wave equation

- The isothermal gas case

$$p = \frac{\rho_o R T_0}{MW|} \therefore c = \sqrt{\frac{RT_0}{MW|}} \tag{12}$$

- The isentropic gas case

$$\frac{p}{\rho^\gamma} = K \Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma K \rho^{\gamma-1} = \frac{T_O}{MW|} \therefore c = \sqrt{\frac{\gamma R T_0}{MW|}} \tag{13}$$

Remarks V

Equivalence for acoustic wave equation

- The isothermal liquid case

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{\beta}{\rho_0} \therefore c = \sqrt{\frac{\beta}{\rho_0}} \tag{14}$$

Remarks VI

Add some numbers to the equations...

For atmospheric air at standard conditions we have $p = 10132\text{Pa}$, $T_0 = 293.15\text{K}$, $R = 8.3145\text{J/mol/K}$, $\gamma = 1.4$ and $MW = 0.028965\text{kg/mol}$, which gives:

- isothermal:$c = 290m/\text{s}$,
- isentropic:$c = 343m/\text{s}$.

For liquid distilled water at 20^oC we have $\beta = 2.18\text{GPa}$ and $\rho = 1,000\text{kg/m}^3$, which gives liquid:

- $c = 1476\text{m/s}$

Remarks VII

About the hyperbolicity of the wave equation

The solutions of hyperbolic equations are "wave-like". If a disturbance is made in the initial data of a hyperbolic differential equation, then not every point of space feels the disturbance at once. Relative to a fixed time coordinate, disturbances have a finite propagation speed. They travel along the characteristics of the equation. This feature qualitatively distinguishes hyperbolic equations from elliptic partial differential equations and parabolic partial differential equations. A perturbation of the initial (or boundary) data of an elliptic or parabolic equation is felt at once by essentially all points in the domain.

Fourier Transform

Fourier Transform

The Fourier transformation $\mathcal{F}$ is an operation which transforms an integrable function on $\mathbb{R}$ into another function, describing the frequency spectrum of the latter. If f is an integrable function on $\mathbb{R}$, its Fourier transform is the function $\mathcal{F}(f) = \hat{f}$ given by the formula:

$$\mathcal{F}(f) : \xi \mapsto \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} f(x)\, \mathrm{e}^{-\mathrm{i}\xi x}\, \mathrm{d}x. \tag{15}$$

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} \hat{f}(\xi)\, \mathrm{e}^{+\mathrm{i}\xi x}\, \mathrm{d}\xi \tag{16}$$

Cauchy Problem I

Suppose we have an infinite bar and that we try to solve the following problem, called Cauchy's problem, whose unknown is the displacement $u(x, t)$:

Cauchy Problem

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 & \forall x \in \mathbb{R}, \forall t > 0, \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R} \\ \frac{\partial u(x,0)}{\partial t} = u_1(x) & \forall x \in \mathbb{R} \end{cases} \tag{17}$$

Cauchy Problem II

Considering $\hat{u}(\xi, t)$ the Fourier Transform of $u(x, t)$:

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x, t) \mathrm{e}^{-\mathrm{i}\xi x} \mathrm{d}x. \tag{18}$$

Cauchy Problem

$$\begin{cases} \frac{\partial^2 \hat{u}(\xi,t)}{\partial t^2} - c^2 \xi^2 \hat{u} = 0 & \forall \xi \in \mathbb{R}, \forall t > 0, \\ \hat{u}(\xi, 0) = \hat{u}_0(\xi) & \forall \xi \in \mathbb{R} \\ \frac{\partial \hat{u}(\xi,0)}{\partial t} = \hat{u}_1(\xi) & \forall \xi \in \mathbb{R} \end{cases} \tag{19}$$

Cauchy Problem III

For a fix ξ the Eq (19) is a ODE in time. The solution is the follow:

$$\hat{u}(\xi, t) = \hat{f}(\xi)\mathrm{e}^{-i\xi ct} + \hat{g}(\xi)\mathrm{e}^{+i\xi ct} \tag{20}$$

Finally:

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\hat{f}(\xi)\mathrm{e}^{-i\xi ct} + \hat{g}(\xi)\mathrm{e}^{+i\xi ct} \right) \mathrm{e}^{i\xi x} \mathrm{d}\xi. \tag{21}$$

Cauchy Problem IV

Assuming f and g as inverse Fourier Transform of $\hat{f}$ and $\hat{g}$:

$$u(x,t) = f(x-ct) + g(x+ct) \tag{22}$$

Remark

In fact, it can be shown that any solution of the wave equation is written in this form. This means that the solution is the sum of two waves propagating at speed c, one to the right and the other to the left.
We can notice that the Fourier transformation in space allowed us to represent the solution $u(x,t)$ of the transient problem as a superposition of harmonic waves in time of the form:

$$\mathrm{e}^{i\xi(x \pm ct)} \tag{23}$$

Cauchy Problem V

Let us now try to express the functions f and g using the initial data u_0 and u_1. By identification, we find:

$$\begin{cases} u_0(x) = f(x) + g(x), \\ u_1(x) = -cf'(x) + cg'(x), \end{cases} \tag{24}$$

where

$$\begin{cases} \frac{\partial u_0(x)}{\partial x} = f'(x) + g'(x), \\ \frac{1}{c} u_1(x) = -f'(x) + g'(x), \end{cases} \tag{25}$$

Solving the 2×2 system we have

$$\begin{cases} f'(x) = \frac{1}{2}\left(\frac{\partial u_0(x)}{\partial x} - \frac{1}{c} u_1(x)\right), \\ g'(x) = \frac{1}{2}\left(\frac{\partial u_0(x)}{\partial x} + \frac{1}{c} u_1(x)\right), \end{cases} \tag{26}$$

Cauchy Problem VI

Using $u_0 = f(x) + g(x)$

$$\begin{cases} f(x) = \frac{1}{2}\left(u_0(x) - \frac{1}{c}\int_0^x u_1(s)\mathrm{d}s\right) + a, \\ g(x) = \frac{1}{2}\left(u_0(x) + \frac{1}{c}\int_0^x u_1(s)\mathrm{d}s\right) - a, \end{cases} \tag{27}$$

where a is an arbitrary constant.

Remember

$$u(x,t) = f(x-ct) + g(x+ct)$$

D'Alembert solution

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)\mathrm{d}s. \tag{28}$$

Cauchy Problem VII

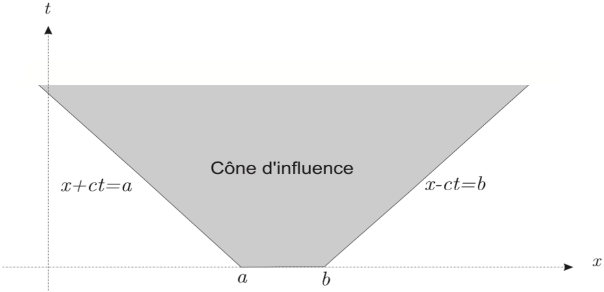


Figure: Cone of influence.

Energy Conservation I

Let us finally consider the general case of a bar, finite or infinite, occupying the interval $[a, b]$ $(-\infty \leq a < b \leq +\infty)$. We then consider the Cauchy problem which consists in finding $u(x, t)$ solution of the vibrating string equation such that

- $u_0(x, 0)$ and $\frac{\partial u(x,0)}{\partial t}$ are two functions given by u_0 and u_1 (they are smooth and have finite support);
- If the beam is finite or semi-finite, u is verified at the boundaries of the beam by the follow boundary conditions:
  - $u = 0$ if the beam is fixed at this boundary (Dirichlet)
  - $\frac{\partial u}{\partial x} = 0$ if the beam is free at this boundary (Neumann)

Energy Conservation II

Energy conservation

$$\frac{d}{dt}E(t) = 0 \tag{29}$$

$$E(t) = E_c(t) + E_p(t)$$

By multiplying Eq. (9) by $\frac{\partial u(x,t)}{\partial t}$ and integrating over the interval $[a, b]$, we get:

$$\int_a^b \left(\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 - c^2 \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} \right) \mathrm{d}x = 0 \tag{30}$$

Energy Conservation III

$$E_c(t) = \frac{1}{2c^2}\int_a^b \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x \quad E_p(t) = \frac{1}{2}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \qquad (31)$$

The quantity $E(t)$ represents the energy of the system at time t. It is the sum of the kinetic energy $E_c(t)$ and the potential energy $E_p(t)$. Thus, we have:

$$E(t) = \frac{1}{2c^2}\int_a^b u_1(x)^2 \mathrm{d}x + \frac{1}{2}\int_a^b \left(\frac{\partial u_0(x)}{\partial x}\right)^2 \mathrm{d}x \qquad (32)$$

This shows in particular that the solution of the Cauchy problem is unique. Indeed, if $u_0 = u_1 = 0$, then $E(t) = 0 \quad \forall t > 0$, hence $u(x,t) = 0 \quad \forall x \in \mathrm{R} \quad \forall t > 0$.

Energy Conservation IV

Question 2

Why the consideration

$$\frac{d}{dt}E(t) = 0 \tag{33}$$

allows us to say that the solution is unique?

Dispersion Relation I

Remember

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

Solving: using Fourier Transform

$$\left(-\omega^2 + c^2 \xi^2\right) \tilde{u} = 0 \qquad (34)$$

Dispersion Relation II

Solutions

- trivial solution

$$\tilde{u} = 0 \tag{35}$$

- non trivial solution

$$c = \frac{\omega}{\xi} = \frac{\omega}{k} \tag{36}$$

Dispersion Relation III

Birds of a feather

- ω is angular frequency $[rad/s]$
- f is frequency $[1/s][Hz]$
- λ is wavelength $[m]$
- k is wave-number $[1/m]$ or $[rad/m]$
- c_0 is velocity(celerity) $[m/s]$
- T is period $[s]$

$$k = \frac{2\pi}{\lambda} \quad v = \lambda f \quad T = \frac{1}{f} \quad \omega = 2\pi f$$

Dispersion Relation IV

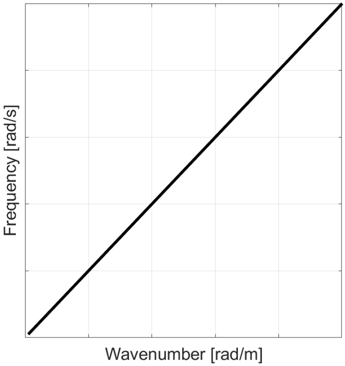


Figure: Dispersion curve

Forced motion of an infinite string I

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} = p(x,t) = \delta(x-\zeta)\delta(t-\tau), \tag{37}$$

where $\delta(x)$ is the Dirac Function.
This represents a unit load occurring at time $t = \tau$ at a location $x = \zeta$ on the string. The practice of replacing general loads with an impulse load and determining the system response to this load is widely used in applied mathematics. The resulting system response is usually designated as the Green's function (G) of the system. For the present one-dimensional problem, this would be written as

$$G = G(x, t/\zeta, \tau) \tag{38}$$

Forced motion of an infinite string II

The problem is thus one of considering

$$\frac{\partial^2 G}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 G}{\partial t^2} = p(x,t) \tag{39}$$

A double-transform approach will be taken. We first take the Fourier transform on the space variable, giving

$$-\xi^2 \hat{G} - \frac{1}{c_0^2}\frac{\partial^2 \hat{G}}{\partial t^2} = \frac{e^{i\xi\zeta}}{\sqrt{2\pi}}\delta(t-\tau) \tag{40}$$

We assume the system is initially at rest, so that $u(x,0) = \dot{u}(x,0) = 0$. Taking the Laplace transform of Eq. (40) gives:

$$-\xi^2 \bar{G} - \frac{s^2}{c_0^2}\bar{G} = \frac{e^{i\xi\zeta}}{\sqrt{2\pi}}e^{s\tau} \tag{41}$$

Forced motion of an infinite string III

Finally,

$$\bar{G} = \frac{-c_0^2}{\sqrt{2\pi}} \frac{e^{i\xi\zeta} e^{s\tau}}{s^2 + c_0^2 \xi^2} \tag{42}$$

In inverting the above, we first consider the Laplace inversion.

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + c_0^2 \xi^2} \right\} = \frac{1}{c_o \xi} \sin c_o \xi t \tag{43}$$

and that in general

$$\mathcal{L}^{-1} \left\{ e^{s\tau} F_L(s) \right\} = F(t - \tau) \quad \text{and} \quad F(t) = 0, \quad t < 0 \tag{44}$$

Forced motion of an infinite string IV

The Laplace inverted results is then

$$\hat{G} = \frac{-c_0^2}{\sqrt{2\pi}} \frac{e^{i\xi\zeta}}{c_0\xi} \sin c_o\xi(t-\tau)H\langle t-\tau\rangle \tag{45}$$

where $H\langle t\rangle$ is the Heaviside function step function. Next we should inverse the Fourier Transform

$$G = \frac{-c_0^2}{2\pi} H\langle t-\tau\rangle \int_{-\infty}^{+\infty} \frac{\sin c_o\xi(t-\tau)}{c_o\xi} e^{i\xi(x-\zeta)} d\xi \tag{46}$$

Now it is trivial to show that:

$$G = \frac{c_0}{2} H\langle t'/c_0\rangle \left(H\langle x'+t'\rangle - H\langle x'-t'\rangle\right) \tag{47}$$

Forced motion of an infinite string V

where $t' = c_0(t-\tau)$, $x' = x - \zeta$

$$\mathcal{F}^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{\sin a\gamma}{\gamma}\right\} = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases} \tag{48}$$

$$\begin{aligned} G(x, t/\zeta, \tau) = -\frac{c_0}{2} H\langle t-\tau\rangle \{ H\langle x - \zeta + c_0(t-\tau)\rangle \\ - H\langle x - \zeta - c_0(t-\tau)\rangle \} \end{aligned} \tag{49}$$

The solution for the response of the string to a general loading $p(x,t)$ is obtained from the Green's function result by the following double integral:

$$u(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} G(x, t/\zeta, \tau) p(\zeta, t) d\zeta \tag{50}$$

References I

Francis Bernardeau, Christophe Grojean, and Jean Dalibard. *Particle physics and cosmology: the fabric of spacetime: lecture notes of the Les Houches Summer School 2006*. Elsevier, 2007.

Karl F Graff. *Wave motion in elastic solids*. Courier Corporation, 2012.