$$Jf(x,y,z) = \begin{pmatrix} 2xy+e^x & x^2 & 1 \end{pmatrix} \implies Jf(0,1,-1) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$$

Observamos que $det(\frac{\partial f}{\partial x}) \neq o$ y que f es de clase C^∞, por lo tanto:

$$\begin{aligned} \exists! g\colon A \subset \mathbb{R}^2 &\to \mathbb{R} \\ (y,z) &\mapsto g(y,z) = x \end{aligned} \quad \text{tal que} \quad f(g(y,z),y,z) = 0 \quad \forall (y,z) \in \mathbb{R}^2$$

Calculamos ahora $P_2(g,(1,-1))$ que es:

$$P_2(g,a)(\phi) = g(a) + Jg(a)(\phi - a) + \frac{1}{2}(\phi - a)^t Hg(a)(\phi - a) \quad a = (1,-1)$$

Calculamos ahora $Jg(a)$:

$$Jg(a) = -\left(\frac{\partial f}{\partial x}(p)\right)^{-1} \begin{pmatrix} \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix} = -(1)^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

Calculamos ahora $Hg(a)$. Para ello derivamos implicitamente, sabemos que $f(g(y,z),y,z) = 0$ y por lo tanto (Llamando $g(y,z) = x$):

$$0 = \frac{\partial f(x,y,z)}{\partial y} = x^2 + 2xy\frac{\partial x}{\partial y} + e^x \frac{\partial x}{\partial y} \iff \frac{\partial x}{\partial y}(2xy + e^x) + x^2 = 0$$

Calculamos ahora $\frac{\partial^2 x}{\partial y \partial y}$, repitiendo el mismo proceso:

$$0 = 2x\frac{\partial g}{\partial y} + \frac{\partial^2 x}{\partial y \partial y}(2xy + e^x) + \frac{\partial x}{\partial y}\left(2x\frac{\partial x}{\partial y} + 2x + e^x \frac{\partial x}{\partial y}\right)$$

De donde se obtiene (sustituyendo):

$$\frac{\partial^2 x}{\partial y \partial y}(1,-1) = 0$$

Ahora derivando parcialmente respecto a z en la ecuacion anterior tenemos:

$$0 = 2x\frac{\partial^2 x}{\partial z \partial y} + \frac{\partial^2 x}{\partial z \partial y}(2xy + e^x) + \frac{\partial x}{\partial y}\left(2y\frac{\partial x}{\partial z} + e^x \frac{\partial x}{\partial z}\right) \implies \frac{\partial^2 x}{\partial z \partial y}(1,-1) = 0$$

Repetimos ahora el proceso para obtener $\frac{\partial^2 x}{\partial z \partial z}$:

$$0 = 2xy\frac{\partial x}{\partial z} + e^x \frac{\partial x}{\partial z} + 1 \iff 0 = \frac{\partial x}{\partial z}(2xy + e^x) + 1$$

$$\frac{\partial^2 x}{\partial z \partial z}(2xy + e^x) + \frac{\partial x}{\partial z}\left(2y\frac{\partial x}{\partial z} + e^x \frac{\partial x}{\partial z}\right) = 0 \implies \frac{\partial^2 x}{\partial z \partial z}(1,-1) = -3$$

Y $Hg(a) = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$. Para concluir:

$$P_2(g,a)(y,z) = 0 + \begin{pmatrix} 0 & -1 \end{pmatrix}\left(\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) + \frac{1}{2}\left(\begin{pmatrix} y & z \end{pmatrix} - \begin{pmatrix} 1 & -1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}\left(\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$$P_2(g,a)(y,z) = -3z^2 - 4z + 1$$

Primero calculamos $\frac{\partial F}{\partial x}$ y $\frac{\partial F}{\partial y}$:

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial}{\partial x} h\left(\frac{y}{x}\right) = -h'\left(\frac{y}{x}\right)\frac{y}{x^2}$$

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial}{\partial y} h\left(\frac{y}{x}\right) = \frac{h'\left(\frac{y}{x}\right)}{x}$$

Y evidentemente la ecuacion se cumple:

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = -xh'\left(\frac{y}{x}\right)\frac{y}{x^2} + y\frac{h'\left(\frac{y}{x}\right)}{x} = 0$$

Ahora, definimos la function:

$$\begin{aligned} G\colon A \subset \mathbb{R}^2 &\to \mathbb{R}^2 \\ (u,v) &\mapsto (x,y) = (u, uv) \end{aligned}$$

Y tenemos que:

$$D(F\circ G)(x) = DF(G(x)) \circ DG(x)$$

$$DF(G(x)) = \begin{pmatrix} \frac{\partial F}{\partial x}(u,uv) & \frac{\partial F}{\partial y}(u,uv) \end{pmatrix}$$

$$DG(x) = \begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix}$$

Y por lo tanto:

$$D(F\circ G)(u,v) = \begin{pmatrix} \frac{\partial F}{\partial x}(u,uv) + \frac{\partial F}{\partial y}(u,uv)v & \frac{\partial F}{\partial y}(u,uv)u \end{pmatrix}$$

Observamos ahora que, por hipotesis:

$$u\frac{\partial (F\circ G)}{\partial u}(u,uv) = 0 \implies \frac{\partial (F\circ G)}{\partial u}(u,uv) = 0 \quad \forall u,v \in A$$

Y que por lo tanto $(F\circ G)_{|A}$ se puede expresar como:

$$(F\circ G)_{|A}(u,v) = h(v) \implies (F\circ G\circ G^{-1})_{|A}(x,y) = h\left(G_2^{-1}(x,y)\right)$$

Pero $G^{-1}(x,y) = \left(x, \frac{y}{x}\right)$ y por lo tanto:

$$F_{|A} = h\left(\frac{y}{x}\right)$$

c) $m = f(p)$ donde p es el minimo calculado en el apartado b

$$f\left(\frac{x}{\|x\|}\right) = \frac{f(x)}{\|x\|^2} \implies f(x) = \|x\|^2 f\left(\frac{x}{\|x\|}\right) \geq \|x\|^2 f(p) = \|x\|^2 m$$

d) inmediato a partir de e
e)

$$0 = \lim_{h\to 0} \frac{f(h) - f(0) - Df(0)(h)}{\|h\|} \iff \lim_{h\to 0} \frac{f(h) - Df(0)(h)}{\|h\|} = 0$$

Pero

$$\lim_{h\to 0} \frac{f(h)}{\|h\|} = \lim_{h\to 0} f\left(\frac{h}{\sqrt{\|h\|}}\right) = \lim_{h\to 0} f\left(\frac{h}{\|h\|}\frac{\|h\|}{\sqrt{\|h\|}}\right) =$$

$$= \lim_{h\to 0} f\left(\sqrt{\|h\|}\frac{h}{\|h\|}\right) = f\left(\lim_{h\to 0} \sqrt{\|h\|}\frac{h}{\|h\|}\right) = f(0) = 0$$

Por lo tanto, tomando $Df(0) = 0$, tenemos que:

$$0 = \lim_{h\to 0} \frac{f(h) - f(0) - Df(0)(h)}{\|h\|}$$

Y f es diferenciable en el origen

$$Jf(x,y) = \begin{pmatrix} -y^2e^{y-x^2}2x & 2ye^{y-x^2} + y^2e^{y-x^2} \end{pmatrix}$$

De la primera componente observamos que vale 0 $\iff \begin{cases} x = 0 \\ y = 0 \end{cases}$

La segunda, vale 0 $\iff \begin{cases} y = 0 \\ y = -2 \end{cases}$

De donde obtenemos que los puntos criticos son $(0, -2)$ y $(x, 0) \quad \forall x \in \mathbb{R}$. Calculamos ahora $Hf(x, y)$:

$$Hf(x,y) = \begin{pmatrix} 4y^2x^2e^{y-x^2} - 2y^2e^{y-x^2} & -4xye^{y-x^2} - 2xy^2e^{y-x^2} \\ -4xye^{y-x^2} - 2xy^2e^{y-x^2} & 2e^{y-x^2} + 2ye^{y-x^2} + 2ye^{y-x^2} + y^2e^{y-x^2} \end{pmatrix}$$

$$Hf(0,-2) = \begin{pmatrix} -8e^{-2} & 0 \\ 0 & -2e^{-2} \end{pmatrix} \implies f(0,-2) \text{ es maximo}$$

$$Hf(x,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2e^{-x^2} \end{pmatrix} \quad \text{que es semidefinida positiva}$$

Para caracterizar los puntos $(x, 0)$ observamos que $f(x, y) \geq 0 \quad \forall (x, y) \in \mathbb{R}^2$ y que $f(x, 0) = 0 \quad \forall x \in \mathbb{R}^2$ y por lo tanto son minimos

Los puntos criticos de f en K son $(0, 0)$. Miramos ahora los puntos de la frontera $x = 0$. Para ello definimos la funcion:

$$F(x,y,\lambda) = y^2e^{y-x^2} + \lambda x$$

$$JF(x,y,\lambda) = \begin{pmatrix} -2xy^2e^{y-x^2} + \lambda & 2ye^{y-x^2} + y^2e^{y-x^2} & x \end{pmatrix}$$

$$JF(x,y,\lambda) = 0 \iff \begin{cases} -2xy^2e^{y-x^2} + \lambda = 0 \\ 2ye^{y-x^2} + y^2e^{y-x^2} = 0 \\ x = 0 \end{cases} \implies x = 0, \lambda = 0, y = \begin{cases} -2 \\ 0 \end{cases}$$

Ahora la frontera $y = 1$:

$$F(x,y,\lambda) = y^2e^{y-x^2} + \lambda(y-1) \implies JF(x,y,\lambda) = \begin{pmatrix} -2xy^2e^{y-x^2} & 2ye^{y-x^2} + y^2e^{y-x^2} + \lambda & y-1 \end{pmatrix}$$

$$JF(x,y,\lambda) = 0 \iff x = 0, y = 1, \lambda = -3e$$

Con la frontera $x^2 = y$:

$$F(x,y,\lambda) = y^2e^{y-x^2} + \lambda(x^2-y) \implies JF(x,y,\lambda) = \begin{pmatrix} -2xy^2e^{y-x^2} + 2\lambda x & 2ye^{y-x^2} + y^2e^{y-x^2} - \lambda & x^2 - y \end{pmatrix}$$

$$JF(x,y,\lambda) = 0 \iff x = 0, y = 0, \lambda = 0$$

Ahora con la restriccion $x = 0$ y $y = 1$:

$$F(x, y, \lambda_1, \lambda_2) = y^2 e^{y-x^2} + \lambda_1 x + \lambda_2 (y-1) \implies$$

$$\implies JF(x, y, \lambda_1, \lambda_2) = \begin{pmatrix} -2xy^2e^{y-x^2} + \lambda_1 & 2ye^{y-x^2} + y^2e^{y-x^2} + \lambda_2 & x & y \end{pmatrix}$$

$$JF(x, y, \lambda_1, \lambda_2) = 0 \iff x = 0, y = 0, \lambda_1 = 0, \lambda_2 = 0$$

Ahora con la restriccion $x = 0$ y $x^2 = y$:

$$F(x, y, \lambda_1, \lambda_2) = y^2 e^{y-x^2} + \lambda_1 x + \lambda_2 (x^2 - y) \implies$$

$$\implies JF(x, y, \lambda_1, \lambda_2) = \begin{pmatrix} -2xy^2e^{y-x^2} + \lambda_1 + 2\lambda_2 x & 2ye^{y-x^2} + y^2e^{y-x^2} - \lambda_2 & x & x^2 - y \end{pmatrix}$$

$$JF(x, y, \lambda_1, \lambda_2) = 0 \iff x = 0, y = 0, \lambda_1 = 0, \lambda_2 = 0$$

Ahora con $y = 1$ y $x^2 = y$:

$$F(x, y, \lambda_1, \lambda_2) = y^2 e^{y-x^2} + \lambda_1 (y-1) + \lambda_2 (x^2 - y) \implies$$

$$\implies JF(x, y, \lambda_1, \lambda_2) = \begin{pmatrix} -2xy^2e^{y-x^2} + 2\lambda_2 x & 2ye^{y-x^2} + y^2e^{y-x^2} + \lambda_1 - \lambda_2 & y - 1 & x^2 - y \end{pmatrix}$$

$$JF(x, y, \lambda_1, \lambda_2) = 0 \iff x = \pm 1, y = 1, \lambda_1 = 1 - 3e, \lambda_2 = 1$$

Por ultimo, tomamos $x = 0$, $y = 1$, $x^2 = y$:

$$F(x, y, \lambda_1, \lambda_2, \lambda_3) = y^2 e^{y-x^2} + \lambda_1 x + \lambda_2 (y-1) + \lambda_3 (x^2 - y) \implies$$

$$\implies JF(x, y, \lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} -2xy^2e^{y-x^2} + \lambda_1 + 2\lambda_2 x & 2ye^{y-x^2} + y^2e^{y-x^2} + \lambda_2 - \lambda_3 & x & y - 1 & x^2 - y \end{pmatrix}$$

$$JF(x, y, \lambda_1, \lambda_2, \lambda_3) \neq 0 \quad \forall (x, y, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^5$$

Los candidatos a maximo y aminimo son:

$$\begin{cases} (0,0) \\ (0,1) \\ (1,1) \end{cases} \implies \begin{cases} f(0,0) = 0 \\ f(0,1) = e \\ f(1,1) = 1 \end{cases} \implies \begin{cases} \text{maximo: } (0,1) \\ \text{minimo: } (0,0) \end{cases}$$

La funcion de Lagrange es $F(x,y,z,\lambda) = x + y + z + \lambda(x^2 + y^2 - 2z^2 - 6)$

$$JF(x,y,z,\lambda) = \begin{pmatrix} 1 + 2x\lambda & 1 + 2y\lambda & 1 - 4z\lambda & x^2 + y^2 - 2z^2 - 6 \end{pmatrix}$$

$$JF(x,y,z,\lambda) = 0 \iff \begin{cases} (x,y,z,\lambda) = \left(2, 2, -1, \frac{1}{4}\right) \\ (x,y,z,\lambda) = \left(-2, -2, 1, \frac{-1}{4}\right) \end{cases}$$

Primero calculamos $T_{(2,2,-1)}M$ y $T_{(-2,-2,1)}M$, que en este caso es:

$T_{(2,2,-1)}M = Nuc(DF(2,2,-1)) = Nuc \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \left\{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\right\}$

$T_{(-2,-2,-1)}M = Nuc(DF(-2,-2,1)) = Nuc \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \left\{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\right\}$

Y $HF(x,y,z,\lambda)$:

$$HF(x,y,z,\lambda) = \begin{pmatrix} 2\lambda & 0 & 0 & 2x \\ 0 & 2\lambda & 0 & 2y \\ 0 & 0 & -4\lambda & -4z \\ 2x & 2y & -4z & 0 \end{pmatrix}$$

f es de clase C^∞ y $det\left(\frac{\partial f}{\partial z}\right) \neq 0$ y por lo tanto existe $h(x,y)$ de clase C^∞ y (llamando $z = h(x,y)$):

$$1 + \frac{\partial z}{\partial x} = 0 \iff \frac{\partial z}{\partial x} = -1$$

$$1 + \frac{\partial z}{\partial y} = 0 \iff \frac{\partial z}{\partial y} = -1$$