

## Series Numéricas

Corolario del criterio de Cauchy

$$\sum a_n \text{ convergente} \implies \lim_{n \rightarrow \infty} a_n = 0$$

Serie de Bertrand  $\left( \sum \frac{1}{n^\alpha (\log n)^\beta} \right)$

- $\alpha > 1$  o  $\alpha = 1, \beta > 1 \implies$  convergente
- $\alpha < 1$  o  $\alpha = 1, \beta \leq 1 \implies$  divergente

## Series positivas

Criterio de Condensación ( $a_n$  decreciente,  $a_n \geq 0$ )  $\sum a_n$  convergente  $\iff \sum 2^n a_{2^n}$  convergente

Comparación directa ( $b_n \geq a_n \geq 0 \quad \forall n \geq n_0$ )

- $\sum b_n$  conv.  $\implies \sum a_n$  conv.
- $\sum a_n$  divergente  $\implies \sum b_n$  divergente

Comparación en el límite  $\left( \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \right)$

- $L < +\infty, \sum a_n$  conv.  $\implies \sum b_n$  conv.
- $L > 0, \sum b_n$  div.  $\implies \sum a_n$  div.

Criterio de la raíz y del cociente

$$\left( \lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L \right)$$

- $L < 1 \implies \sum a_n$  convergente
- $L > 1 \implies \sum a_n$  divergente

Criterio de Raabe  $\left( \lim_{n \rightarrow \infty} n \left( 1 - \frac{a_{n+1}}{a_n} \right) = L \right)$

- $L > 1 \implies \sum a_n$  convergente
- $L < 1 \implies \sum a_n$  divergente

Criterio logarítmico  $\left( \lim_{n \rightarrow \infty} \frac{-\log a_n}{\log n} = L \right)$

- $L > 1 \implies \sum a_n$  convergente
- $L < 1 \implies \sum a_n$  divergente

Criterio de Leibnitz  $\left( a_n \text{ dec. } \lim_{n \rightarrow \infty} a_n = 0 \right)$

$$\sum (-1)^{n+1} a_n \text{ convergente}$$

Criterio de la integral ( $a_n = f(n), f$  integ.)

- $\int_M^\infty f$  converge  $\iff \sum a_k$  converge
- $\sum_M^\infty = \sum_M^{N-1} + \int_N^\infty f + \varepsilon_N, \varepsilon_N \in [0, a_N]$

Criterio de Dirichlet ( $\lim_{n \rightarrow \infty} b_n = 0$ ,

$b_n$  dec. Sumas de  $\sum a_n$  acotadas.)  $\sum a_n b_n$  convergente

## Series de Potencias

Teorema de Cauchy-Hadamard

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Radio de convergencia

$$\frac{1}{R} = \lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

## Integrales impropias

Criterio de Cauchy ( $\forall \varepsilon, \exists c_0$ )

$$c_1, c_2 > c_0 \implies \left| \int_{c_1}^{c_2} f \right| < \varepsilon. \implies \text{convergente}$$

Comparación directa ( $g(x) \geq f(x) \geq 0$ )

- $\int_a^\infty g$  converge  $\implies \int_a^\infty f$  converge
- $\int_a^\infty f$  divergente  $\implies \int_a^\infty g$  divergente

Comp. en el límite  $\left( g, f \geq 0, \lim_{x \rightarrow b} \frac{f(x)}{g(x)} = L \right)$

- $L < \infty, \int_a^b g$  conv.  $\implies \int_a^b f$  conv.
- $L > 0, \int_a^b f$  div.  $\implies \int_a^b g$  div.

Criterio de Dirichlet

$$(g \text{ dec.}, \lim_{x \rightarrow b} g(x) = 0, c < b \implies \left| \int_a^c f \right| < M)$$

Entonces  $\int_a^b f(x)g(x)dx$  converge

## Integración múltiple

Conjuntos de medida nula

$(Z \subseteq \mathbb{R}^n \text{ medida nula})$

•  $\text{graph}(f)$  con  $f$  unif. cont.

•  $f(Z)$  con  $f$  lipschitziana  
( $d(f(x), f(y)) \leq d(x, y)$ )

•  $f(Z)$  con  $f$  de clase  $\mathcal{C}^1$

•  $M$  subvariedad regular de  $\dim M < n$

Teorema de Lebesgue:  $f$  integrable en  $A$  sii  $\text{disc}(f) \cap A$  tiene medida nula

Conjuntos admisibles ( $A, A'$  admisibles)

- $A \cap A', A \cup A', A \setminus A'$  son admisibles
- rectángulos acotados y bolas

Medida de Jordan ( $C \subseteq \mathbb{R}^n$  admisible)

$$\text{vol}(C) = \int_C 1$$

Propiedades de la integral ( $f, g$  integrables)

- $f + g$  integrable
- $fg$  integrable
- $f \leq g \implies \int_E f \leq \int_E g$
- $m \leq f \leq M \implies m \text{vol}(E) \leq \int_E f \leq M \text{vol}(E)$
- $\text{vol}(E) = 0 \implies \int_E f = 0$
- $E$  conexo,  $f$  continua  
 $\implies \int_E f = f(x_0) \text{vol}(E)$
- $h$  continua  $\implies h \circ f$  integrable
- $|\int_E f| \leq \int_E |f|$
- $\int_{A \cup B} f = \int_A f + \int_B f - \int_{A \cap B} f$

Teorema de Fubini ( $f$  continua)

$$\int_{A \times B} f(x, y) dx dy = \int_A dx \left( \int_B dy f(x, y) \right)$$

Región elemental ( $\psi, \phi$  cont.  $D$  elemental)

$$E = \{ (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid_{\phi(x) \leq y \leq \psi(x)}^{x \in D} \}$$

TCV ( $V \in \mathbb{R}^n$  abierto,  $\varphi: V \mapsto \mathbb{R}^n$  inyectiva, de clase  $\mathcal{C}^1$ ,  $\det D\varphi \neq 0$ ),  $f: U = \varphi(V) \mapsto \mathbb{R}$  integrable). Entonces  $\int_U = \int_V (f \circ \varphi) |\det D\varphi|$

- $\int x^n dx = \frac{1}{n+1} x^{n+1}$
- $\int \frac{1}{x} dx = \log(|x|)$
- $\int e^x = e^x$
- $\int a^x dx = \frac{a^x}{\log(a)}$
- $\int \sin(x) dx = -\cos(x)$
- $\int \cos(x) dx = \sin(x)$
- $\int \tan(x) dx = -\log(|\cos(x)|)$
- $\int \arcsin\left(\frac{x}{a}\right) dx = x \arcsin\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} \quad a > 0$
- $\int \arccos\left(\frac{x}{a}\right) dx = x \arccos\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} \quad a > 0$
- $\int \arctan\left(\frac{x}{a}\right) dx = x \arctan\left(\frac{x}{a}\right) - \frac{a}{2} \log(a^2 + x^2) \quad a > 0$
- $\int \sin^2(mx) dx = \frac{1}{2m} (mx - \sin(mx) \cos(mx))$
- $\int \cos^2(mx) dx = \frac{1}{2m} (mx + \sin(mx) \cos(mx))$
- $\int \sec^2(x) dx = \tan(x)$
- $\int \csc^2(x) dx = -\cot(x)$
- $\int \sin^n(x) dx = -\frac{\sin^{n-1}(x) \cos(x)}{n} + \frac{n-1}{n} \int \sin^{n-2}(x) dx$
- $\int \cos^n(x) dx = \frac{\cos^{n-1}(x) \sin(x)}{n} + \frac{n-1}{n} \int \cos^{n-2}(x) dx$
- $\int \tan^n(x) dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x) dx$
- $\int \sinh(x) dx = \cosh(x)$
- $\int \cosh(x) dx = \sinh(x)$
- $\int \tanh(x) dx = \log(|\cosh(x)|)$
- $\int \sinh^2(x) dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x$
- $\int \cosh^2(x) dx = \frac{1}{4} \cosh(2x) + \frac{1}{2} x$
- $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \log(x + \sqrt{a^2 + x^2})$

- $\int \frac{1}{a^2 + x^2} dx = \frac{1}{2} \arctan \frac{x}{a}$
- $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$
- $\int (a^2 - x^2)^{\frac{3}{2}} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}$
- $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a}$
- $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{x+a}{x-a} \right|$
- $\int \frac{1}{(a^2 - x^2)^{\frac{3}{2}}} dx = \frac{x}{a^2 \sqrt{a^2 - x^2}}$
- $\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \log |x \pm \sqrt{x^2 \pm a^2}|$
- $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}|$
- $\int \frac{1}{x(a+bx)} dx = \frac{1}{a} \log \left| \frac{x}{a+bx} \right|$
- $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{\frac{3}{2}}}{15b^2}$
- $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx$
- $\int \frac{x}{\sqrt{a+bx}} dx = \frac{2(bx-2a)\sqrt{a+bx}}{3b^2}$
- $\int \frac{1}{x\sqrt{a+bx}} dx = \frac{1}{\sqrt{a}} \log \left| \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} \right|$
- $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \log \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|$
- $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{\frac{3}{2}}$
- $\int x^2 \sqrt{a^2 - x^2} dx = \frac{a}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}$
- $\int \frac{1}{x\sqrt{a^2 - x^2}} dx = -\frac{1}{a} \log \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|$
- $\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2}$
- $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$
- $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \log \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|$
- $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - \arcsin \frac{x}{a}$

- $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{\frac{3}{2}}$
- $\int \frac{1}{x\sqrt{x^2 + a^2}} dx = \frac{1}{a} \log \left| \frac{x}{a + \sqrt{x^2 + a^2}} \right|$
- $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \arccos \frac{a}{|x|}$
- $\int \frac{1}{x^2 \sqrt{x^2 \pm a^2}} dx = \pm \frac{\sqrt{x^2 \pm a^2}}{a^2 x}$
- $\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2}$
- $\int \frac{1}{ax^2 + bx + c} dx = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \log \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| & (b^2 > 4ac) \\ \frac{2}{\sqrt{b^2 - 4ac}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} & (b^2 < 4ac) \end{cases}$
- $\int \frac{x}{ax^2 + bx + c} dx = \frac{1}{2a} \log |ax^2 + bx + c| - \frac{b}{2a} \int \frac{1}{ax^2 + bx + c} dx$
- $\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \begin{cases} \frac{1}{\sqrt{a}} \log |2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}| & (a > 0) \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}} & (a < 0) \end{cases}$
- $\int \sqrt{ax^2 + bx + c} dx = \frac{2ax+b}{4a} \sqrt{ax^2 + bx + c} - \frac{4ac-b^2}{8a} \int \frac{1}{\sqrt{ax^2 + bx + c}} dx$
- $\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{1}{\sqrt{ax^2 + bx + c}} dx$
- $\int x^3 \sqrt{x^2 + a^2} dx = \left(\frac{1}{5}x^2 - \frac{2}{15}a^2\right) \sqrt{(a^2 + x^2)^3}$
- $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \frac{\pm \sqrt{(x^2 \pm a^2)^3}}{3a^2 x^3}$
- $\int \sin(ax) \sin(bx) dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$
- $\int \sin(ax) \cos(bx) dx = \frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$
- $\int \cos(ax) \cos(bx) dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$
- $\int x^n \log(ax) dx = x^{n+1} \left( \frac{\log(ax)}{n+1} - \frac{1}{(n+1)^2} \right)$
- $\int e^{ax} \sin bxdx = \frac{e^{ax}(b \sin(bx) - b \cos(bx))}{a^2 + b^2}$
- $\int e^{ax} \cos bxdx = \frac{e^{ax}(b \sin(bx) + b \cos(bx))}{a^2 + b^2}$