The Euclidean Steiner Cable-Trench Problem

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Abstract. The Cable Trench Problem (CTP) is a spanning tree problem on a weighted graph that combines the Minimum Spanning Tree and Shortest Path Spanning Tree Problems. As its name suggests, the CTP is motivated by the problem of minimizing the cost to connect buildings on a campus to a hub housing the central server so that each building is connected directly to the server via a dedicated underground cable. The vertices of a graph represent buildings, edges represent the routes for digging trenches and laying cables between two buildings, and edge weights represent distance. In this paper, we define a new and natural generalization of the CTP, the Euclidean Steiner CTP, which considers the case that the vertices correspond to points in Euclidean space and allows for Steiner vertices to be added to the graph. We will describe a heuristic for approximating optimal solutions and will give preliminary results on the extent to which CTP solutions can be improved with this model.

Keywords: discrete optimization, Steiner points, Cable-Trench Problem, Euclidean Steiner Tree Problem

1 Introduction

The Cable-Trench Problem (CTP) was first described in [11] and establishes a continuum between the Minimum Spanning Tree and Shortest Path Tree Problems on a weighted graph with specified root. The name "Cable-Trench" comes from the problem of minimizing the cost to connect buildings on a campus to a central server so that each building is connected directly to the server via a dedicated underground cable. The problem is modeled as a weighted graph in which the buildings are represented by vertices, the server is represented by the root node, and the edges represent the possible routes for digging trenches and laying cables between two buildings. Weights on the edges generally represent distance.

If we were to connect a number of nodes to a central hub with dedicated cables, it is natural to ask if we could find a lower cost solution if we were not restricted to digging trenches directly between building. Would we benefit by considering auxiliary nodes that would serve as strategic "branching-off" points? This paper addresses this question. To this end, we define a new generalization of the CTP: the Euclidean Steiner CTP (ESCTP). In this case, each vertex of the graph must be a point in Euclidean space so that the auxiliary **Steiner** vertices may be added to the graph in a straightforward way. In practice, this means that an edge of the solution tree may connect any two points in space. We will further describe a solution approach for finding nearly optimal solutions of relatively small instances of the ESCTP using an idea inspired by the pattern search heuristic [3].

In Section 2, we describe the CTP itself and some applications. We also describe another graph-theoretic problem: the Euclidean Steiner Tree Problem (ESTP). We define the ESCTP in Section 3 and show that it is a generalization of both the CTP and the ESTP. In Section 4, we describe a solution approach to the ESCTP and give some computational results in Section 5. Finally, we close with conclusions and areas of future work in Section 6.

2 Background

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, \ldots, v_n\}$, root vertex v_1, edge set E, and $w_{ij} \geq 0$ the weight of the edge $(v_i, v_j) \in E$. Let γ and τ denote the per-unit cable and trench costs, respectively. The CTP is the problem of finding a spanning tree $T = (V, E_T)$ which minimizes

$$\gamma \sum_{v_k \in V} \sum_{(v_i,v_j) \in \mathcal{P}(v_1,v_k)} w_{ij} + \tau \sum_{(v_i,v_j) \in E_T} w_{ij} \ , \tag{1}$$

where $\mathcal{P}(v_1, v_k) \subseteq E_T$ is the path in T from the root node v_1 to the vertex v_k. If $\gamma > 0$ and $\tau = 0$, then a solution to the CTP is any shortest path spanning tree of G with root vertex v_1. In contrast, if $\tau > 0$ and $\gamma = 0$, then a solution to the CTP is any minimum spanning tree of G.

Several applications of the CTP and its generalizations have been proposed in recent years. Examples include the application of the CTP to significantly reducing the cost to upgrade and deploy wired and wireless access networks by Nielsen et al. [8]. Marianov et al. generalized the CTP to forests (disjoint unions of trees) to optimize the construction of roads and sawmills for a logging operation and to optimize the construction of canals and wells for irrigation [7]. Schwarze defined the multi-commodity CTP in which a network structure is designed in which multiple cable types (commodities) are laid in the trenches [9]. Zyma et al. applied a generalization of the CTP to connecting an array of radio telescopes [14, 15]. A rather nontrivial application of the CTP is to vascular image analysis by Jiang et al. [4, 5], Vasko et al. [12], and Landquist et al. [6].

Vasko et al. proved that the CTP is NP-hard [11]. However, instances of the CTP with up to 97 vertices have been solved optimally using a multicommodity flow formulation of the CTP [14, 15]. Modifications of Prim's algorithm have been used to find very good solutions for instances with up to 25,000 vertices relatively efficiently [12, 6].

A similar, but older and more studied problem is the **Euclidean Steiner Tree Problem** (ESTP). Given a set of n **terminal** vertices $V \subseteq \mathbb{R}^d$, with $d \geq 2$, the ESTP is the problem of finding a tree, $T = (V_T, E_T)$, such that $V \subseteq V_T$ that minimizes the total length of the edges of T:

$$\sum_{e \in E_T} ||e|| \ ,$$

where $||\cdot||$ is the Euclidean length of edge e. $S = V_T \setminus V$ will denote the set of Steiner points. The tree T is called the **Steiner minimal tree**.

The ESTP is NP-hard [1]. When $d = 2$, however, the GeoSteiner algorithm of Warme et al. has found Steiner minimal trees with 10,000 terminal vertices [13]. Their approach uses the fact that in a planar Steiner minimal tree, angles formed by the edges at each nondegenerate Steiner vertex are 120°. For $d \geq 3$, the problem is far more difficult. Various branch-and-bound techniques have proven to be the most effective, but only find optimal solutions when there are less than 20 terminal vertices [10, 2].

We now pose our question from the introduction again. If the CTP model is used to create a cable network, then why restrict ourselves to digging trenches between buildings? Could we dig trenches between any two points and identify points that would be better “branching off” points in a manner akin to the ESTP? These questions motivate the following section.

3 The Euclidean Steiner CTP

Given a set of n distinct terminal vertices $V = \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{R}^d$, with $d \geq 2$ and root v_1, we wish to determine a set of $n-2$ Steiner points $S = \{v_{n+1}, v_{n+2}, \ldots, v_{2n-2}\} \subseteq \mathbb{R}^d$ and edges E so that the total cable and trench cost of the tree T with vertex set $V_T = V \cup S$ and edge set E is minimized.

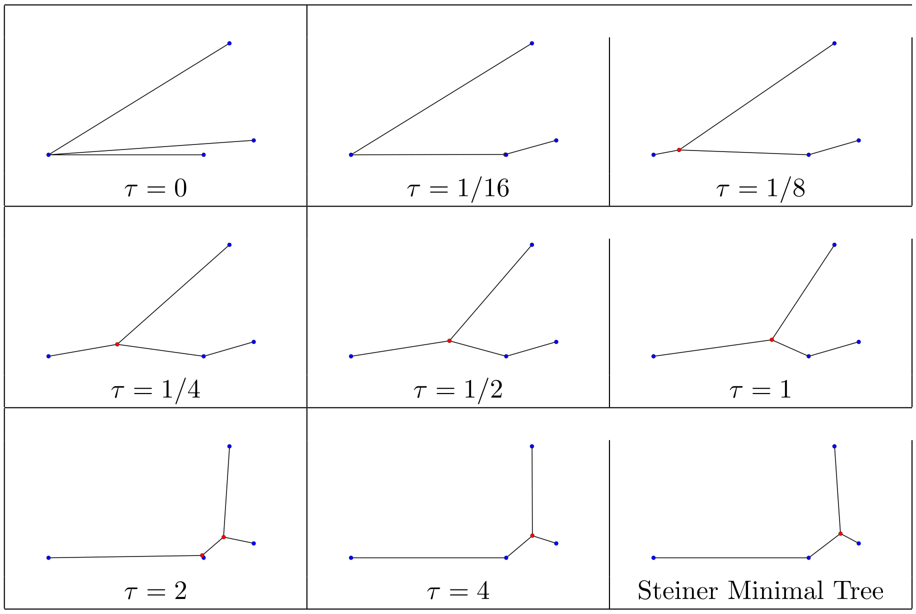
Definition 1. *For fixed cable and trench costs, $\gamma \geq 0$ and $\tau \geq 0$, respectively, the* **ESCTP** *is the problem of finding a spanning tree $T = (V_T, E)$ of a vertex set $V \subseteq V_T$ that minimizes*

$$\gamma \sum_{v_i \in V} \sum_{e \in \mathcal{P}(v_1, v_i)} ||e|| + \tau \sum_{e \in E} ||e|| \ ,$$

where $\mathcal{P}(v_1, v_i)$ is the path in T from v_1 to the terminal vertex v_i.

We note that there are certain degenerate cases in which multiple Steiner vertices are the same point or in which one or more Steiner points share the same location as a terminal vertex. In such cases we would consider two common points as being incident, but the edge connecting them would have length 0. The example given in Table 3 illustrates this possibility. We give the optimal trench layouts for a “campus” of four buildings for a range of values of τ, with $\gamma = 1$. The buildings are the blue points, with the hub (the root node) on the left. The Steiner vertices are in red. The points are obtained from embedding Example 1 of [11] into $\mathbb{R}^2$ so as to preserve edge weights. Specifically, the points are $(0, 0), (12, 0), (14, 7.74597)$, and $(15.8727, 1.0012)$.

Table 1. Optimal solutions for a 4-vertex ESCTP for various τ values



In the case of the shortest path tree, i.e., when $\tau = 0$, the two Steiner vertices share the same location as the root node. When $\tau = 1/16$, only one Steiner point is in common with the root node. When $\tau = 1/8$, the left-most Steiner point is no longer degenerate, but the right-most point shares a location with another terminal vertex. When $\tau = 2$, both Steiner points are nondegenerate.

On one extreme of the ESCTP, when $\tau = 0$ and $\gamma > 0$, the optimal solution minimizes the total cable cost, so the edge set of the optimal solution is composed solely of those going directly from the root to each vertex. On the other extreme, when $\tau > 0$ and $\gamma = 0$, the optimal solution is a Steiner minimal tree. The ESCTP is a generalization of both the CTP and the ESTP, both of which are NP-hard, so we have the following result.

Theorem 1. *The ESCTP is NP-hard.*

3.1 Mixed Integer Nonlinear Programming Formulation

The ESCTP can be formulated as a mixed integer nonlinear program (MINLP) by generalizing the linear programming formulation of the CTP from [11] and the MINLP formulation of the ESTP in [2] as follows. For every $(i, j) \in E$, we assume that $1 \leq i < j \leq 2n - 2$. Let $E = E_V \cup E_S$, where $E_V = \{(i, j) \in E | 1 \leq i \leq n < j\}$ is the set of edges incident to a Steiner vertex and a terminal vertex and $E_S = \{(j, k) \in E | n < j < k\}$ is the set of edges incident to two

Steiner vertices. We will assume that the Steiner vertices are labeled so that if $(j,k) \in E_S$, then $v_j \in \mathcal{P}(v_1, v_k)$. It follows that $(1, n+1) \in E_V$ and $(1, j) \notin E_V$ for all $j > n+1$. Let $t_{ij} = 0$ if $(i,j) \notin E$ and $t_{ij} = 1$ if $(i,j) \in E$. Finally, let $c_{ij} \in \mathbb{N}_0$ be the number of cables in the trench represented by $(i,j) \in E$.

Minimize:

$$\gamma \left(\sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} ||(i,j)|| \right) c_{ij} + \tau \left(\sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} ||(i,j)|| \right) t_{ij} \tag{2}$$

subject to:

$$t_{1,n+1} = 1 \tag{3}$$

$$c_{1,n+1} = n-1 \tag{4}$$

$$\sum_{j \in S} t_{ij} = 1 \qquad \text{for all } i \in V \setminus \{1\} \tag{5}$$

$$\sum_{i<j \in V} t_{ij} = 0 \qquad \text{for all } i \in V \tag{6}$$

$$\sum_{j \in S} c_{ij} = 1 \qquad \text{for all } i \in V \setminus \{1\} \tag{7}$$

$$\sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n-2} t_{ij} = 2n-1 \tag{8}$$

$$\sum_{i \in V} t_{ij} + \sum_{k<j, k \in S} t_{kj} + \sum_{k>j} t_{jk} = 3 \qquad \text{for all } j \in S \tag{9}$$

$$\sum_{n<k<j} t_{kj} = 1 \qquad \text{for all } j \in S \setminus \{n+1\} \tag{10}$$

$$\sum_{i \in V} c_{ij} + \sum_{k>j} c_{jk} - \sum_{k<j} c_{kj} = 0 \qquad \text{for all } j \in S \setminus \{n+1\} \tag{11}$$

$$\sum_{1<i \in S} c_{i,n+1} + \sum_{k \in S} c_{n+1,k} = n+1 \tag{12}$$

$$t_{ij} - c_{ij} = 0 \qquad \text{for all } 1 < i \leq n < j. \tag{13}$$

$$(n-1)t_{jk} - c_{jk} \geq 0 \qquad \text{for all } n < j < k. \tag{14}$$

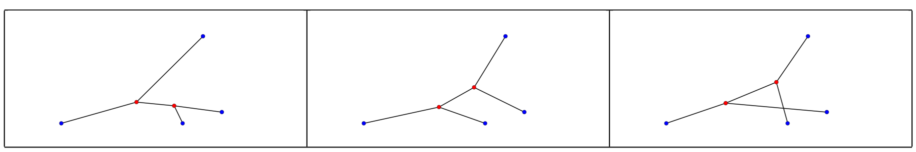
Constraint 3 explicitly connects the root to Steiner vertex v_{n+1} and Constraint 4 fills that trench with all $n-1$ cables. Constraint 5 connects each non-root terminal vertex to precisely one Steiner point. Combined with Constraint 6, we guarantee that each terminal vertex has degree 1. Constraint 7 then states that each building is connected with exactly one cable and specifically from the Steiner point it is incident to. Constraint 8 guarantees that exactly $2n-1$ trenches are dug. Constraint 9 says that each Steiner point is incident to exactly three other vertices, including degenerate Steiner points. Constraint 10 prevents cycles among the Steiner points. Constraints 11 and 12 guarantee that no cable terminates at a Steiner point. Finally, Constraints 13 and 14 ensure that cables are not laid unless a trench is dug.

We note that this MINLP could be improved by incorporating bounds on the edge lengths as in [2] and utilizing a multi-commodity flow formulation as in [7, 14, 15]. We focus instead on a graph-theoretic approach to solving the ESCTP. One item to note that is not explicitly stated in the MINLP formulation is how the Steiner nodes are connected to each other and to the terminal vertices. We elaborate on this in the next section.

3.2 Tree Topologies

The **topology** of a tree specifies how the vertices are connected, but does not necessarily give the locations of the Steiner points. To illustrate this concept, we notice that in Table 3, there was a change in the topology of the solution tree between $\tau = 1$ and $\tau = 2$; the Steiner points changed which terminals they were incident to. For Steiner trees with four terminal vertices, there are three topologies, as demonstrated in Table 3.2.

Table 2. The three topologies for $n = 4$ terminal vertices



In the last example, edges cross, but this can only be resolved by moving one of the Steiner nodes outside of the convex hull of the terminals. Therefore, this third topology will not yield an optimal solution. This restricts the number of topologies that we would realistically consider for a given example.

For n terminals, there are $\frac{(2n-4)!}{2^{n-2}(n-2)!} = \Omega(n!)$ possible tree topologies. Therefore, considering all possible topologies and finding the optimal location of each Steiner node for each topology is computationally infeasible for graphs with even 20 terminal vertices; there are roughly 8 billion topologies when $n = 15$ and about $2.2 \cdot 10^{20}$ topologies when $n = 20$, for example. This is another proof that the ESCTP is NP-hard.

We may consider natural restrictions on the topologies, as we observed in Table 3.2. One approach is to consider various triangulations of the terminal nodes. Given n terminal nodes, a triangulation will make at least $n-2$ triangles. (If the terminals give rise to a convex n-gon, then there will be exactly $n-2$ triangles. Otherwise there will be more.) Therefore, we can make a feasible topology by initializing the Steiner nodes at the center of distinct triangles. In this case, we allow an edge between Steiner nodes if their triangles share a common side and allow an edge between a Steiner node and a terminal if the terminal is a vertex of the triangle at hand. For a given triangulation, there are multiple feasible topologies, and the number of triangulations of a convex n-gon is the $(n-2)$-nd Catalan number: $\frac{(2n-4)!}{((n-2)!)(n-1)!} = O\left(\frac{4^n}{n^{3/2}}\right)$. Even with such a restriction, the number of topologies grows rapidly and is computationally infeasible to check each for even 25 terminal vertices. This leads us to explore one possible solution approach.

4 Iterative Midpoint Pattern Search

We describe a heuristic to find solutions to the ESCTP, within some tolerance that is inspired by the pattern search algorithm [3], but modified to work with a topology induced by a triangulation. For small n, we may exhaust all possible topologies, but for larger n, we work with fewer topologies. When considering larger instances of the ESCTP, we may consider those topologies arising from a feasible solution of the corresponding CTP found via the Modified Prim's algorithm in [12]. Given a triangulation of the terminal vertices, our heuristic, the Iterated Midpoint Pattern Search (IMPS), proceeds as follows.

1. Initialize the Steiner points at the centroid of distinct triangles.
2. Connect the Steiner points and terminal vertices as described in Section 3.2.
3. For Steiner vertices $v_j \in \{v_{n+1}, v_{n+2}, \ldots, v_{2n-2}\}$:
4. Find the centroids of the three smaller triangles induced by the Steiner node v_j and consider these as candidate Steiner nodes.
5. Determine which centroid yields the smallest cost and set that as v_j.
6. If v_j remains unchanged:
   (a) If the difference between the highest cost induced by the three other centroids and that of v_j is within the tolerance, then quit.
   (b) Otherwise, consider the triangle formed by the three other centroids.
   (c) Go to Step 3.
7. Otherwise, that is, if v_j is moved to a new location:
   (a) If the difference between the previous and new costs is within the tolerance, then quit.
   (b) Otherwise, consider the triangle it was the centroid of.
   (c) Go to Step 3.

The following figures, Figures 4-4, illustrate one pass through IMPS on a 3-vertex graph. The solid black points represent terminal vertices and the open vertices represent potential locations of the Steiner vertex.

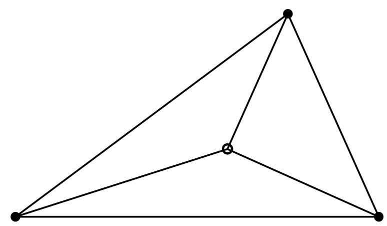


Fig. 1. Initialization of the Steiner point with connections

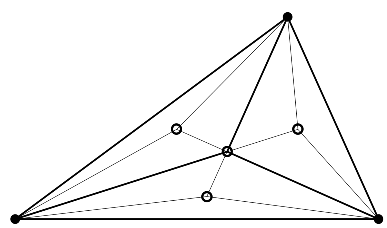


Fig. 2. Placement of the three new candidate Steiner points at the centroids

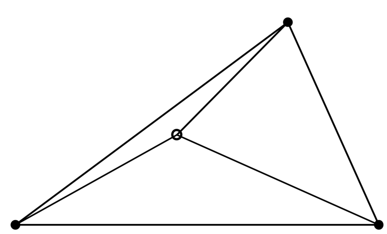


Fig. 3. Location of the best of the four possible points, with new connections

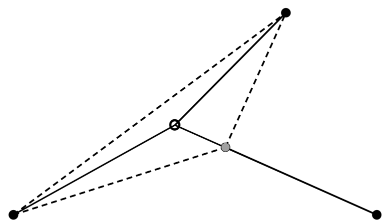


Fig. 4. Outline of the new triangle to consider

5 Results

We tested IMPS on the 4-vertex example based on Example 1 of [11], namely the graph with vertices $(0,0),(12,0),(14,7.74597)$, and $(15.8727,1.0012)$, found in Table 3. We applied IMPS to the two topologies that would yield feasible solutions for a range of values of τ and γ. We fixed $\gamma = 1$ except for the last row when $\gamma = 0$ and $\tau = 1$. The tolerance was set to 0.0001 to guarantee three decimal places of precision on the optimal solution. Opt., CTP, and CTP: K_4 refer to the cost of the optimal solution of the ESCTP, the CTP in Example 1 of [11], and the CTP on the complete graph determined by Euclidean distances of Example 1 of [11], respectively. The last two columns give the percentage improvement over the two CTP models.

Table 3. Comparison of optimal solutions of the ESCTP and CTP on Example 1 of [11]

τ/γ	Opt.	CTP	CTP: K_4	%Impr./CTP	%Impr./K_4
0	43.904	44	43.904	0.22	0
1/16	46.000	46	46	0.00	0.00
1/8	47.966	48	48	0.07	0.07
1/4	51.518	52	52	0.93	0.93
1/2	58.055	60	60	3.24	3.24
1	70.482	72	72	2.11	2.11
2	94.479	96	96	1.58	1.58
4	139.918	143	143	2.16	2.16
8	230.534	235	235	1.90	1.90
∞	22.630	23	23	1.61	1.61

6 Conclusions and Future Work

In this paper, we defined the Euclidean Steiner Cable-Trench Problem (ESCTP) as a common generalization of both the Cable-Trench Problem (CTP) and the Euclidean Steiner Tree Problem as a new graph theoretic problem, along with a formulation as a mixed integer nonlinear program. We described a heuristic, called the Iterated Midpoint Pattern Search (IMPS), which is based on the pattern search algorithm and was able to determine optimal solutions for a 4-vertex example. The ESCTP model yielded improvements of up to 3.24% over the CTP in this example.

There are several questions that naturally arise from this paper. What are the best methods to determine optimal or nearly optimal solutions to instances of the ESCTP? How well does IMPS scale up to larger problems? Would other heuristics, such as the original pattern search, teaching-learning-based optimization, branch-and-bound techniques, or a genetic algorithm yield faster and/or better results? Would extending the MINLP formulation yield improved results? How does the situation in $\mathbb{R}^2$ compare with higher dimensions? What is the biggest problem that we can solve optimally or within certain tolerances? How do we find good lower bounds of optimal solutions? On the theoretical side, what kind of geometric properties do optimal solutions have? Could the answer to this question be leveraged to improve the performance of various heuristics? Is there a connection between optimal CTP solutions and the topology of optimal ESCTP solutions? How much does an ESCTP formulation improve on the corresponding CTP?

As the above questions indicate, this paper opens the door for several avenues of study on a new problem with many natural applications. Answers to these questions will result in improvements to the solutions of applications listed in Section 2.

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