Open Applied Calculus

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July 9, 2019

The textbook is adapted from *Modeling, Functions, and Graphs: Algebra for College Students* by Katherine Yoshiwara, *Business Calculus* by Shana Calaway, and *Applied Calculus* by Shana Calaway, Dale Hoffman, and David Lippman. The latter two texts were adapted from *Precalculus: An Investigation of Functions* by David Lippman and Melonie Rasmussen and *Contemporary Calculus*, by Dale Hoffman.

Acknowledgements. I gratefully acknowledge support for the initial development of this book with a grant from the Pennsylvania State System of Higher Education (PASSHE) Faculty Professional Development Committee.

First release, August 2019

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Chapter 0

What is Calculus?

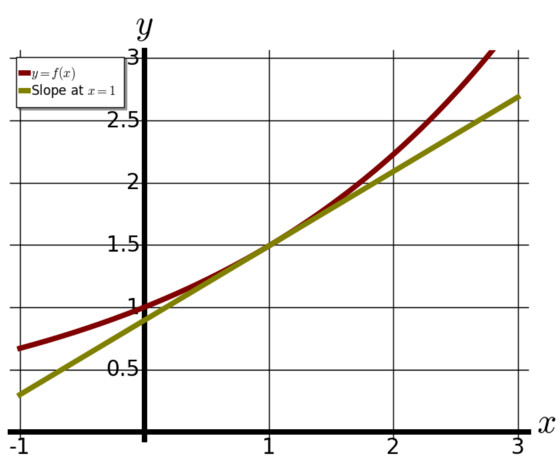
A good speaker will often begin a talk with an outline of what will be discussed in the talk. In the same way, it is incumbent upon the author of a textbook to give the reader the big picture of the subject of the textbook. That is the purpose of this brief chapter.

Calculus is the study of change. Over time, the population of a city may grow. The price of a stock will fluctuate: sometimes increasing and sometimes decreasing. The cumulative profits of a company may grow or wane. These situations represent changing quantities. We may want to dig deeper and find out how quickly the city is growing or the rate at which a company's stock price or profits are growing. Those are questions that calculus attempts to answer.

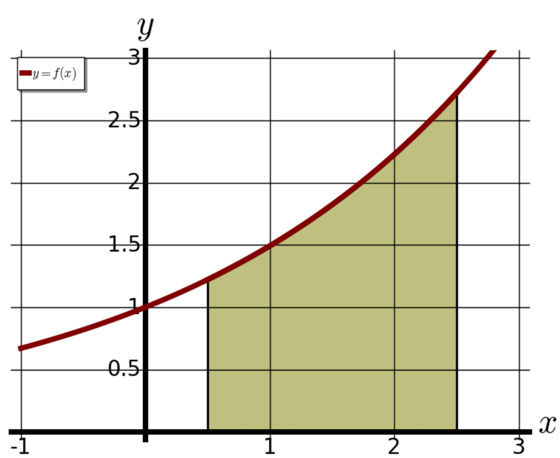
Calculus was first developed in the late 1600s independently by Sir Isaac Newton and Gottfried Wilhelm Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic, and social phenomena and to describe and solve problems in those areas.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields.

0.1 Two Problems



(a) Slope of a Curve at a Point



(b) Area Under a Curve over an Interval

Figure 1: Visualizing the Two Problems of Calculus

Calculus is the study of two seemingly different questions that are actually inverses of each other.

1. What is the **rate of change** of a function?
2. What the **accumulation** of a function?

Geometrically, these questions can be visualized as the **slope of a curve** and the **area under a curve**, respectively, as seen in Figure 1.

In this book, we will unpack these two problems and show how they are useful to business, economics, finance, the social sciences, and the life sciences. We will take a data-driven and a problem-solving approach to this course, working with real and realistic data.

Let's consider a pair of examples to begin.

Example 0.1.1.

Example 0.1.2.

0.2 Problem Solving

Have you ever looked at a math problem and said to yourself, "I don't know where to start!" No matter how you answered that question, this section will offer some tips.

One often thinks of the study of a mathematical subject as mere rote memorization of formulas and step-by-step procedures. On the contrary, the approach that mathematicians take to the subject is much more creative. We see mathematics as a way to apply abstract concepts to solve problems. Often there is just one correct answer. Often, however, the "correct" or "best" answer may be impossible to find, so we are instead interested in a solution that would be considered "good," "plausible," or "reasonable." That is the approach that will drive much of the content of this textbook. In short, one of our objectives is to retrain the way you view mathematics: it's not about knowing or not knowing the answer, it's about figuring out a solution. This textbook is not a book to tell you what to think, but how to think about quantitative problem solving.

In light of this, we introduce a well-known problem-solving process and thoughts on how to implement it in practice. In his famous book *How to Solve It*, mathematician George Pólya described his problem-solving process in four steps, which we have modified to follow the acronym "C.O.P.E."

1. **Comprehend** (Understand) the problem.
    - **What is the problem asking?**
    - It is impossible to solve a problem without this step, but it is often skipped.
    - Read and understand the instructions if there are any.
    - Read the problem over and over until you understand.
    - Look up words and symbols you don't understand.
    - Ask yourself and answer the following questions:
        - Is the problem assuming that I know something that I don't?
        - What knowledge gaps must I fill in order to really understand the problem?
        - Is there enough information given to solve the problem?
        - Do I need to look up information for the problem?
        - Is there any irrelevant information that can be ignored?
        - Do I need to make any assumptions? If so, are these assumptions reasonable?
    - It may help to draw a picture or diagram.
    - It may help to simplify the problem, i.e., work with a simpler "toy" problem.
    - Study examples similar to the problem.
    - It may help to compare the problem to a similar one.

- Be patient with yourself. Sufficient understanding may take time.
- What is a range of plausible solutions?
- Often there is more than one level of comprehension to a problem or concept. A deeper understanding of a problem or concept results in a better or more advanced solution to the problem.
- You may obtain a deeper understanding of the problem by pursuing the next step: "Observe."

2. **Observe** the information and applicable tools and devise a plan.

   - **What information is given in the problem and what tools do you have that apply?**
   - "Look" around. What (mathematical or other) tools, results, theorems, algorithms, etc. apply in this situation, based on the information given?
   - If you studied examples similar to the problem, do the same tools, results, theorems, or algorithms apply in this case? Why or why not?
   - Here are some ideas to work with the information.
     - Define variables to describe components of the problem and solution. Consider units.
     - Express the information we have mathematically.
     - Draw a picture or diagram. Label the picture with variables, numbers, etc.
     - Make a chart or list or plot some of the data or information.
     - Experiment. Play with the problem and the math until something works.
     - Guess and check: Trial and error helps to develop intuition.
     - Be systematic. Use an organized method to check all cases.
     - Look for patterns.
     - Work backwards.
     - Look for counterexamples.
     - Divide and conquer. Split the problem up into pieces and solve the individual pieces.

3. **Proceed** with the plan and get a solution.

   - **It often helps to work out a solution on scratch paper first.**
   - **Make your solution easy to follow for yourself and others.**
   - Check your work at each step.
   - If you get stuck or the solution doesn't work, go back to the planning stage and try something else. Thomas Edison tested thousands of designs before developing a practical working lightbulb.
   - Think about the problem during otherwise unproductive times such as when waiting in line, walking, driving, or falling asleep.
   - When stuck, take a break and let your brain work on the problem subconsciously.
   - Be patient. It may take a while for the solution to "click."

4. **Evaluate** (in the sense of "Reflect on") your solution and check your work.

   - (1) **Does the solution answer the question or solve the problem?**
   - (2) **Does the solution make sense?**
   - Skipping this step often results in "stupid" mistakes that are easy to catch.
   - Are there any obvious errors or contradictions?
   - Does the solution fit within a range of plausible solutions?
   - Is there another, independent approach to validate the solution or confirm that it is plausible?
   - Do the units make sense?
   - Check your work. This is why organized solutions are important.

- Proofread any written work, ideally at least a day after you wrote it.

5. **Extend** (Generalize) the solution or problem and reflect.
   - Is there another, possibly easier, way to solve the problem?
   - Can you generalize the problem?
   - Would you do something different to improve the solution?

Chapter 1

Models, Graphs, and Functions

1.1 What is a Function?

1.1.1 Function Concepts

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or **independent variable**, and a corresponding output quantity, or **dependent variable**. From this, we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Definition 1.1.1. A **function** is a rule for a relationship between an **input** (or **independent**) quantity and an **output** (or **dependent**) quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1.1.1. In the height and age example above, is height a function of age? Is age a function of height?

Solution: In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. You cannot have two different heights at any one instant in time.

However, age is not a function of height, since one height input might correspond with more than one output age. Once you have stopped growing, your height essentially remains constant for the rest of your life. ■

1.1.2 Representing Functions

Functions can be represented in many ways:

1. Words,
2. Tables of values,
3. Graphs, or
4. Formulas.

Example 1.1.1 represented a function in words, but it will be convenient to streamline the discussion of a function. To that end, we introduce notation of functions.

Function Notation. To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

If we name the function f we could write "height is a function of age" as "h is f of a," or more simply:

$$h = f(a) \ .$$

We could instead name the function h and write $h(a)$, which is read "h of a."

We can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value "a" must be put into the function "h" to get a result.

Note 1.1.1. Be careful! The parentheses indicate that age is the input into the function. Do not confuse these parentheses with multiplication!

Example 1.1.2. A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ tell us?

Solution: When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function: the year (y). The output value is 300, the number of police officers (N), a value for the output quantity. Remember $N = f(y)$. This tells us that in the year 2005 there were 300 police officers in the town. ■

Tables as Functions. A table lists the input and corresponding output values of a function.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1.1 below represents the age of a child in years and his corresponding height. This represents just some of the data available for the age and height of the child.

(Input:) a, age (years)	4	5	6	7	8	9	10
(Output:) h, height (inches)	40	42	44	47	50	52	54

Table 1.1: Tabulating height as a function of age for a child.

From this, we can create equations such as $h(6) = 44$, meaning that when the child was 6 years old, he was 44 inches tall.

Example 1.1.3. Which of these tables define a function (if any)?

Table A.

Input	**Output**
2	1
5	3
8	6

Table B.

Input	**Output**
−3	5
0	1
4	5

Table C.

Input	**Output**
1	0
5	2
5	4

Solution: Tables A and B define functions. In both tables, each input corresponds to exactly one output. Table C does not define a function since the input value of 5 corresponds with two different output values: 2 and 4. ■

Graphs as Functions A function can often be represented as a **graph**, a set of **points** plotted on **coordinate axes**: the **horizontal axis** and the **vertical axis**. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical axis.

Points on the graph are represented by ordered pairs of the form (a, b). Beginning at the **origin** (the point $(0, 0)$), you move a units to the left and b units up to arrive at the point (a, b). If $a < 0$, then movement is to the left and if $b < 0$, then movement is down.

The horizontal and vertical axes are typically called the **x-axis** and the **y-axis**, but these axes can be labeled with any variable name, not just x and y. We say y is a function of x, or $y = f(x)$ when the function is named f. The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

As an example, Figure 1.1 is a plot of the data from Table 1.1.

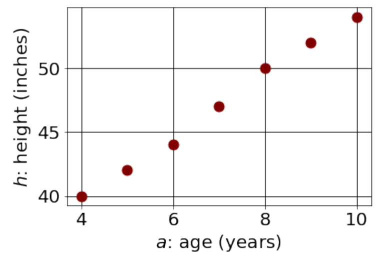
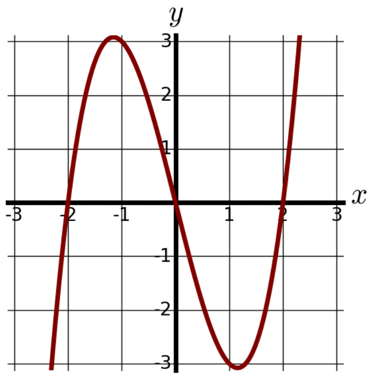
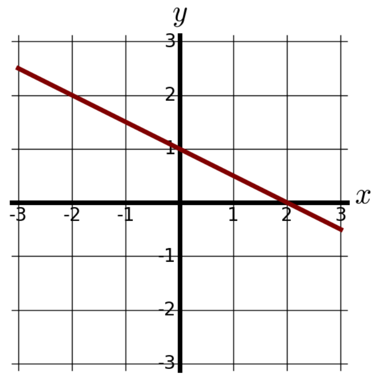
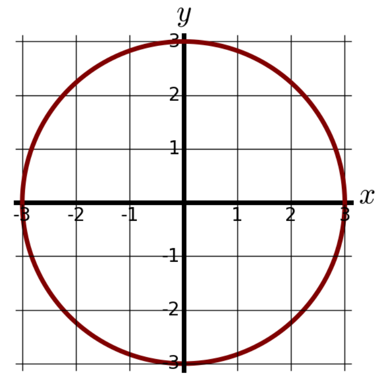


Figure 1.1: A plot of height data.

Example 1.1.4. Which of these graphs defines a function $y = f(x)$?


Graph A.


Graph B.


Graph C.

Solution: Looking at the graphs above, Graphs A and B define a function $y = f(x)$, since for each input value along the horizontal (x) axis, there is exactly one corresponding output value, determined by the y-value of the graph. Graph C does not define a function $y = f(x)$ since some input values, such as $x = 2$, correspond with more than one output value. ■

The **vertical line test** is an easy way to determine whether a graph defines a function or not. Imagine drawing vertical lines through the graph. Since a function has exactly one output for every input, if there is a vertical line that would cross the graph more than once, then the graph does not define a function.

Figure 1.2 illustrates how Graph C from Example 1.1.4 fails the vertical line test. The vertical line $x = 2$ (in gold) intersects Graph C (in maroon) at two points; there are two outputs for for the input $x = 2$.

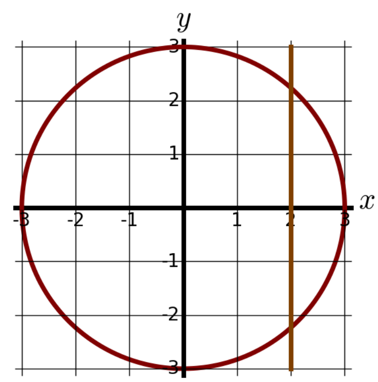


Figure 1.2: A graph failing the vertical line test.

Formulas as Functions When possible, it is very convenient to define relationships between quantities using a formula. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 1.1.5. Express the relationship $2N + 6p = 12$ as a function $p = f(n)$ if possible.

Solution: To express the relationship in this form, we need to be able to write the relationship where p is a function of N, which means writing it as $p =$ something involving N, or p in terms of N. We proceed using algebra.

$2N + 6p = 12$	Subtract $2N$ from both sides.
$6p = 12 - 2N$	Divide both sides by 6 and simplify.
$p = \dfrac{12 - 2N}{6}$	
$p = \dfrac{12}{6} - \dfrac{2N}{6}$	
$p = 2 - \dfrac{1}{3}N$	

We can now express p as a function of N:

$$p = f(N) = 2 - \frac{1}{3}N \ .$$

■

It is important to note that not every relationship can be expressed as a function with a formula. Consider the examples of the boy's height as a function of age in Table 1.1 or a company's stock price as a function of time. Neither situation allows one to define the relationship using a formula precisely, yet we still may want to analyze aspects of these relationships using tools of calculus. The rest of the chapter will introduce us to some tools to help us with that.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed.

1.1.3 Evaluating Functions.

The fundamental use of a function is to **evaluate** the function: "plugging in" some number into the function, or more precisely, to determine the corresponding output for a given input. In other words, we substitute or replace the input variable of the function with the input value. Evaluating will always produce one result, since each input of a function corresponds to exactly one output. We can evaluate a function from a table, graph, or formula.

Another related use is to determine the input or inputs of a function, given an output of the function. This is called **solving** an equation and could produce more than one solution, since different inputs can produce the same output.

Note 1.1.2. The concepts of evaluating and solving often get confused. When we use the word *solve*, we will be *solving a problem* or *solving an equation for an unknown quantity or variable.* It does not make sense to solve a function, since a function is merely a mathematical expression and not an equation.

Example 1.1.6. Let $Q = g(n)$ and use the table shown.

$Q = g(n)$ as a table.

n	1	2	3	4	5
Q	8	6	7	6	8

(a) Evaluate $g(3)$.

Solution: Evaluating $g(3)$, read "g of 3," means that we need to determine the output value, Q, of the function g given the input value of $n = 3$. Looking at the table, we see the output corresponding to $n = 3$ is $Q = 7$, allowing us to conclude $g(3) = 7$. ■

(b) Solve $g(n) = 6$ for n.

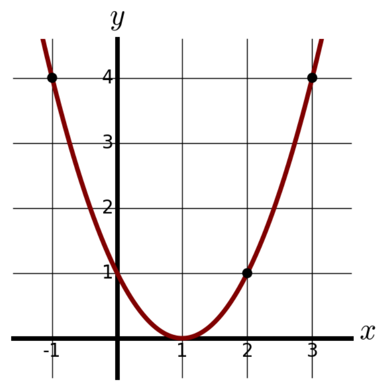
Solution: Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we evaluate $g(n)$ at 2, our output is $Q = 6$.

When we evaluate $g(n)$ at 4, our output is also $Q = 6$. ■

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 1.1.7. Consider the graph of a function $f(x)$ to the right.



(a) Evaluate $f(2)$.

Solution: To evaluate $f(2)$, we find the input of $x = 2$ on the horizontal (x) axis. Moving up to the graph gives the point $(2, 1)$, giving an output of $y = 1$. Therefore, $f(2) = 1$. ■

(b) Solve $f(x) = 4$ for x.

Solution: To solve $f(x) = 4$, we find the value 4 on the vertical (y) axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: $(-1, 4)$ and $(3, 4)$. These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$.

This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. ■

Example 1.1.8. Let $k(t) = t^3 + 2$.

(a) Evaluate $k(2)$.

Solution: To evaluate $k(2)$, we replace t with 2 in the expression $t^3 + 2$, then simplify.

$$\begin{aligned} k(2) &= 2^3 + 2 \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

So $k(2) = 10$. ■

(b) Solve $k(t) = 1$ for t.

Solution: To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output.

$$\begin{aligned} k(t) &= 1 && \text{Substitute the original formula } k(t) = t^3 + 2. \\ t^3 + 2 &= 1 && \text{Subtract 2 from each side.} \\ t^3 &= -1 && \text{Evaluate the cube root of each side.} \\ t &= \sqrt[3]{-1} = -1 \end{aligned}$$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

To check our work, we want to know if $k(t) = 1$ is a true statement when $t = -1$.

$$\begin{aligned} k(-1) &= (-1)^3 + 2 \\ &= -1 + 2 \\ &= 1 \ , \end{aligned}$$

which was the desired result. ■

1.1.4 Cost, Revenue, and Profit

Suppose that your club wants to raise funds by selling T-shirts. The screen printing shop that will make the shirts will charge your club \$50 to cover overhead costs and \$5 per shirt for the shirts themselves. You decide to charge \$15 per shirt. Some questions naturally arise. How many shirts need to be sold to **break even**? How much **profit** can be expected?

We can describe situations such as this with functions. The **total cost** to produce these shirts combines **fixed costs** and **variable costs**. The fixed costs are also called **overhead costs** and do not depend on the number of T-shirts made, while the variable costs are the per item cost. If n is the number of T-shirts the screen printing shop will make, $F(n)$ is the fixed costs and $V(n)$ is the variable costs, then the **cost function** to produce n T-shirts is

$$\begin{aligned}\text{total cost} \;&=\; \text{fixed costs} \;+\; \text{variable costs}\\ C(n) &= F(n) + V(n)\\ &= \$50 + (\$5 \text{ per shirt})\,(n \text{ shirts})\\ &= 50 + 5n \text{ dollars}\end{aligned}$$

If you sell each shirt for \$15 and you sell n shirts, then your **revenue** from selling n shirts will be $15n$ dollars. This is your **revenue function**: $R(n) = 15n$ dollars.

Finally, the **profit** that your club will earn from selling n shirts is the revenue minus the total costs. If $P(n)$ is the profit from selling n items, then the **profit function** is

$$P(n) = R(n) - C(n) \;\; .$$

In this example, we have

$$\begin{aligned}P(n) &= R(n) - C(n)\\ &= 15n - (50 + 5n) \text{ dollars}\\ &= 15n - 50 - 5n \text{ dollars}\\ &= 10n - 50 \text{ dollars}\end{aligned}$$

Example 1.1.9. Consider the T-shirt scenario above.

(a) What is the profit from selling 30 shirts?

Solution: $P(30) = 10 \cdot 30 - 50 = 300 - 50 = \$250$.

If you sell 30 T-shirts, then you profit \$250. ■

(b) What is the profit from selling 3 shirts?

Solution: $P(3) = 10 \cdot 3 - 50 = 30 - 50 = \$(-20)$.

If you only sell 3 T-shirts, then you lose \$20; your profit is negative. ■

(c) What is the profit from selling 0 shirts?

Solution: $P(0) = 10 \cdot 0 - 50 = 0 - 50 = \$(-50)$.

If you don't sell any T-shirts, then you lose \$50; your profit is negative. ■

The **break-even point** in this context is the minimum number of T-shirts that must be sold in order to have a profit of at least \$0. How do we find this? If our profit is \$0, then we turn that sentence into a mathematical equation and solve for the number of shirts. The profit from selling n shirts is $P(n)$, "is" is "=", and 0 is 0.

$$\begin{aligned}P(n) &= 0\\ 10n - 50 &= 0\\ 10n &= 50\\ n &= \frac{50}{10} = 5\end{aligned}$$

So you need to sell at least five T-shirts in order to break even.

Note 1.1.3. Note the distinction between $P(0)$ and $P(n) = 0$. $P(0)$ is the profit from selling 0 items, while $P(n) = 0$ is an equation whose solution tells you the number of items that must be sold in order to have no profit.

Definition 1.1.2. In summary, the **total cost** to produce n items, $C(n)$, is the combination of **fixed costs** ($F(n)$) and **variable costs** ($V(n)$). The fixed costs are also called **overhead costs** and is constant regardless of the number of items made, while the variable costs depend on the number of items being made. So

$$C(n) = F(n) + V(n) \ .$$

The **revenue** function, $R(n)$ gives the amount of money brought in from selling n items. The difference between revenue and total costs is **profit** ($P(n)$) from selling n items:

$$P(n) = R(n) - C(n) \ .$$

The **break-even point** is the fewest number of items, n, that must be sold in order for $P(n) \geq 0$, in other words, the fewest number of items to guarantee that you won't have negative profit and be losing money.

1.1.5 Domain and Range.

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of the models we create. In our most recent example, it wouldn't make sense to sell -4 T-shirts or 4.4 T-shirts. We also wouldn't expect to sell a billion T-shirts for a university club fund raiser. When using a function to describe a real-world scenario, we need to set common sense boundaries for the input and output. Here is another example.

Table 1.2 shows a relationship between the circumference, c, and height, h, of a tree as it grows. In this table, we would consider height as a function of circumference: $h = h(c)$.

Circumference: c (feet)	1.7	2.5	5.5	8.2	13.7
Height: h (feet)	24.5	31.0	45.2	54.6	92.1

Table 1.2: Height of a tree as a function of its circumference.

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the **domain** and **range** of the function.

Definition 1.1.3. The **domain** of a function is the set of possible input values to the function.

The **range** of a function is the set of possible output values of the function.

Example 1.1.10. Using Table 1.2 above, determine a reasonable domain and range of the function h.

Solution: We can combine the data provided with additional research and our own reason to determine an appropriate domain and range of the function $h = h(c)$. For the domain, it doesn't make sense for the circumference (input) to be negative, so $c \geq 0$. For a maximum circumference, we could make an educated guess at a reasonable value, or look up that the maximum recorded circumference is about 119 feet[1]. With this information, we would say a reasonable domain is $0 \leq c \leq 119$ feet.

Similarly for the range, if we only consider the tree when the sprout has broken through the ground, it doesn't make sense to have negative heights. The maximum recorded height of a tree could be looked up to be 379 feet, so a reasonable range is $0 \leq h \leq 379$ feet. ■

[1] `http://en.wikipedia.org/wiki/Tree`, retrieved July 19, 2010

Interval Notation. A convenient alternative to the notation using inequalities is **interval notation**, in which intervals of values are referred to by the starting and ending values. Parentheses "()" are used for "strictly less than," and square brackets "[]" are used for "less than or equal to." Since infinity, ∞, is not a number, neither $-\infty$ nor ∞ are included in the domain and range of a function, so we always use curved parentheses with $\pm\infty$. Table 1.3 shows how inequalities correspond to interval notation for an arbitrary variable x.

To combine two intervals together, we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Inequality	Interval Notation
$5 \leq x \leq 10$	$[5, 10]$
$5 < x \leq 10$	$(5, 10]$
$5 \leq x < 10$	$[5, 10)$
$5 < x < 10$	$(5, 10)$
$x < 10$	$(-\infty, 10)$
$5 \leq x$	$[5, \infty)$
All real numbers	$(-\infty, \infty)$

Table 1.3: Interval Notation

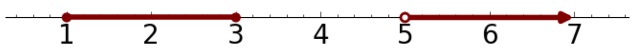


Figure 1.3: A union of intervals.

Example 1.1.11. Describe the intervals of values shown in Figure 1.3 using inequalities and using interval notation.

Solution: To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

As an inequality it is: $1 \leq x \leq 3$ or $x > 5$.

In interval notation: $[1, 3] \cup (5, \infty)$. ■

Example 1.1.12. Find the domain of each function.

(a) $f(x) = 2\sqrt{x+4}$

Solution: Since we cannot evaluate the square root of a negative number, we need the inside of the square root to be non-negative. $x+4 \geq 0$ when $x \geq -4$. (Subtract 4 from both sides of the inequality.) Therefore, the domain of $f(x)$ is $[-4, \infty)$. ■

(b) $g(x) = \frac{3}{6-3x}$

Solution: We cannot divide by zero, so we need the denominator to be non-zero. Solving $6 - 3x = 0$ for x, we have $x = 2$, so we must exclude 2 from the domain. Therefore, the domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$. ■

1.1.6 Exercises

1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

   (a) The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.

   (b) Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

   (a) A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.

   (b) Explain the meaning of the statement $g(100) = 1$.

3. Let $n(t)$ be the number of subscribers to a YouTube channel t years after 2005. Explain the meaning of each statement.

(a) $n(5) = 300$

(b) $n(10) = 4000$

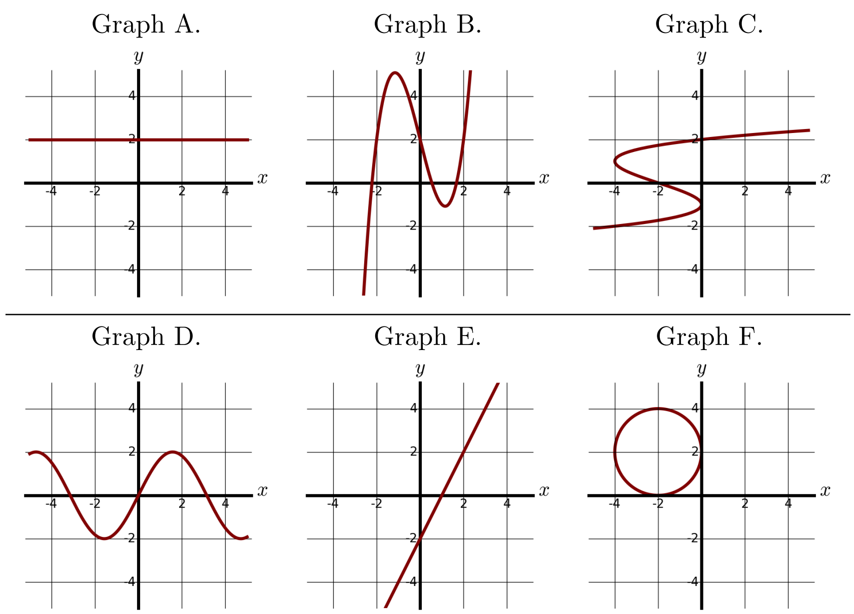
4. Let $p(t)$ be the stock price, in dollars, of Valvoline (VVV) t years after its Initial Public Offering (IPO) on September 23, 2016. Explain the meaning of each statement.

   (a) $p(0) = 23.10$

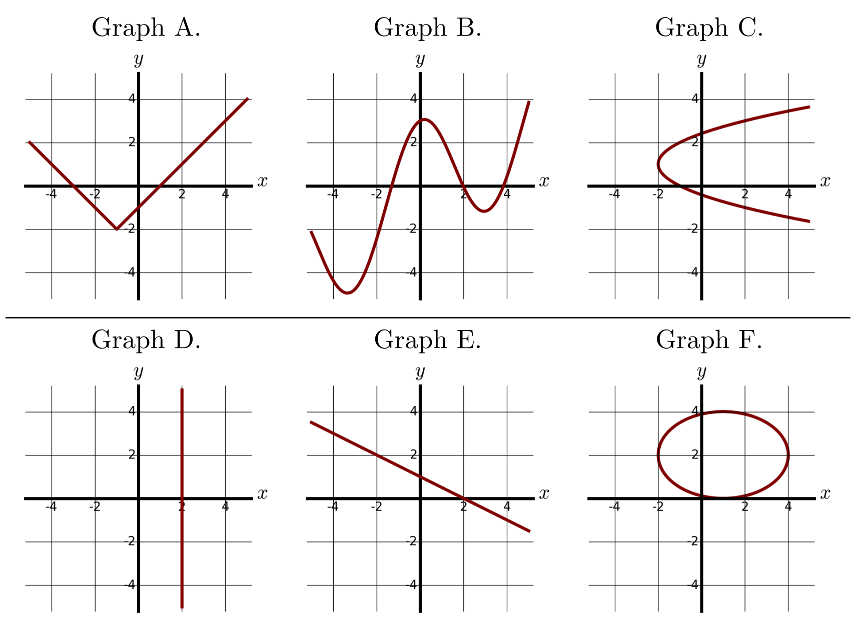
   (b) $p(1) = 23.27$

   (c) $p(2) = 21.47$

5. Select all of the following graphs which represent y as a function of x.



6. Select all of the following graphs which represent y as a function of x.



7. Select all of the following tables which represent y as a function of x.

Table A.

x	5	10	15
y	3	8	14

Table B.

x	5	10	15
y	3	8	8

Table C.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent y as a function of x.

Table A.

x	2	6	13
y	3	10	10

Table B.

x	2	6	6
y	3	10	14

Table C.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent y as a function of x.

Table A.

x	y
0	−2
3	1
4	6
8	9
3	1

Table B.

x	y
−1	−4
2	3
5	4
8	7
12	11

Table C.

x	y
0	−5
3	1
3	4
9	8
16	13

Table D.

x	y
−1	−4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent y as a function of x.

Table A.

x	y
−4	−2
3	2
6	4
9	7
12	16

Table B.

x	y
−5	−3
2	1
2	4
7	9
11	10

Table C.

x	y
−1	−3
1	2
5	4
9	8
1	2

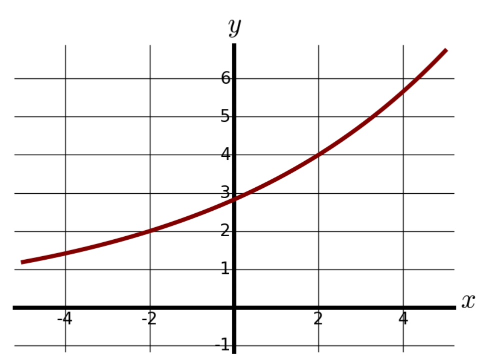
Table D.

x	y
−1	−5
3	1
5	1
8	7
14	12

11. Let $g(x)$ be the function graphed on the right.

(a) Evaluate $g(2)$.

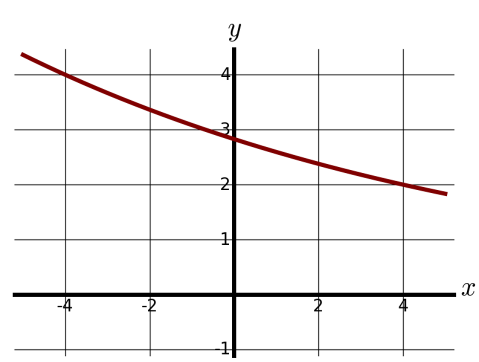
(b) Solve $g(x) = 2$ for x.



12. Let $f(x)$ be the function graphed on the right.

(a) Evaluate $f(4)$.

(b) Solve $f(x) = 4$ for x.



13. Consider the table at right.

    (a) Evaluate $f(3)$.

    (b) Solve $f(x) = 1$ for x.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

14. Consider the table at right.

    (a) Evaluate $g(8)$.

    (b) Solve $g(x) = 7$ for x.

x	0	1	2	3	4	5	6	7	8	9
$g(x)$	62	8	7	38	86	73	70	39	75	34

For Exercises 15-22, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

15. $f(x) = 4 - 2x$

16. $f(x) = 8 - 3x$

17. $f(x) = 8x^2 - 7x + 3$

18. $f(x) = 6x^2 - 7x + 4$

19. $f(x) = 3 + \sqrt{x+3}$

20. $f(x) = 4 - \sqrt[3]{x-2}$

21. $f(x) = \frac{x-3}{x+1}$

22. $f(x) = \frac{x-2}{x+2}$

23. Let $f(t) = 3t + 5$.

    (a) Evaluate $f(0)$.

    (b) Solve $f(t) = 0$ for t.

24. Let $g(p) = 6 - 2p$.

    (a) Evaluate $g(0)$.

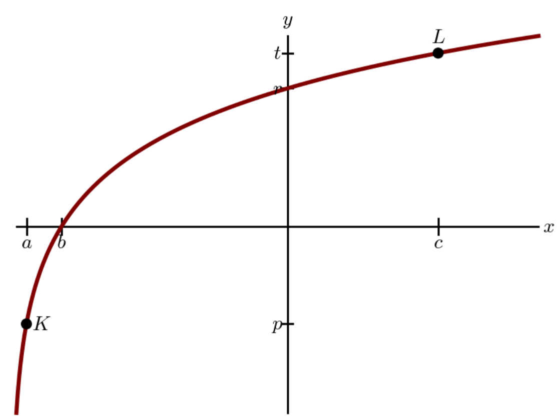
    (b) Solve $g(p) = 0$ for t.

25. Consider the graph of $f(x)$ on the right.

    (a) Evaluate $f(c)$.

    (b) Solve $f(x) = p$ for x.

    (c) Find the coordinates of points L and K.



1.2 Models

"All models are wrong; some models are useful." (George E. P. Box)

In the real world, we will, in all likelihood, have to work with data. "Data-driven decisions" is a common phrase heard today. One thing that can be said is that real world data is messy and complex. Countless factors influence every phenomenon. Some factors are designed and some are purely random. Sometimes a data set is incomplete, missing pieces of data. Yet we may have to work with that data set as if it was complete. Sometimes we are tasked with **forecasting**, which is predicting future data points, given the ones that we have.

1.2.1 What is a Mathematical Model?

In many different areas, we need tools to simplify a set of data and work with that simplified version of the data. These simplifications must be based on reasonable assumptions that connect to the larger context of the data. Simplifications serve two purposes. First, it may be impossible to take every variable into account. Second, models must often be communicated to others and a simple model is generally more clear and therefore much easier to communicate. This is the essence of what is called **mathematical modeling**, or simply **modeling**.

Definition 1.2.1. A **model** or a **mathematical model** is a mathematical framework to help describe some phenomenon, specifically how an input or quantity affects or relates to some output. A model has three main components:

1. One or more input variables, with specific descriptions of what these variables represent, including units.

2. One or more functions, with specific descriptions of what the output of the functions represent, including units.

3. A domain and/or range over which the function(s) make sense to use.

Additionally, we may want to consider the following when creating a model.

- Identify underlying assumptions that were used to simplify the situation.

- Perform a sensitivity analysis to determine if the model is relatively unchanged if the data varies slightly.

For example, one could construct a model to speculate how the price of an item affects the demand for that item, and from that, predict revenue from sales of that item.

It is crucial to state that a model is used to simplify reality and does not dictate or reflect past, present, or future reality with absolute precision. Although good models can be useful for forecasting, decision-making, and filling in missing data. That is the essence of the quote at the beginning of the section by the late George Box, a famous British statistician.

The following is an example of a mathematical model, based on an actual data set from "Plant W." We will analyze this scenario in several places in this text and.

Model: Plant W heats an external tank and powers their operations by burning fuel oil. The rate at which they burned this oil in 2016 can be described by the following model. Let m be the month of the year, with 1 referring to January 1, and 12 referring to December 1, so that $1 \leq m < 13$. Let $f(m)$ describe the rate at which the fuel burns, in gallons per month. A model for $f(m)$ is

$$f(m) = 5.76m^3 - 109.98m^2 + 532.58m + 70.17 \text{ gallons per month.}$$

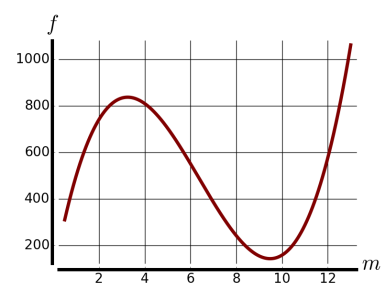


Figure 1.4: Fuel Oil Usage by Plant W in 2016.

Figure 1.4 plots this curve over the given domain.

Commentary: Note that in this example, we clearly described the input variable, what it represented, a domain that it makes sense over, and gave the units (months). We clearly described the function (the output), what it modeled, and gave the units.

Examining the graph of the function, we can see that the model makes sense, based on the context. If Plant W is heating a large external tank, then in the winter and early spring, we expect more oil to be used to heat it. This tank may also have retained some heat, as fluids tend to do, from the late summer and fall. We expect a significant drop in oil usage through the late spring and summer months and finally a significant increase in oil usage during the fall and early winter.

Is the model precise? Of course not. First, we know that in 2016 (a leap year), there were months with 29, 30, or 31 days, yet we treated each month as equal. Perhaps a better model would have modeled fuel usage per day. For the sake of clarity, however, we aggregated the days into months; we know immediately that month 6 is June, but do not immediately know which month is day 276, for example.

A second reason that we know that the model is not accurate is that we can't be certain that on January 1, 2016 that Plant W burned oil at a rate of $f(1) = 498.53$ gallons per month. In other words, on January 1, 2016, did Plant W burn $f(1)/31 = 16.08$ gallons of fuel oil? We can be almost certain that they did not, but the actual amount may have been somewhat close to that. Other factors that we didn't consider were the range of temperatures that day and the percentage of the tank that was full. Considering these factors would make the model more accurate, but more complex.

A third item to consider is that in this case, we are dealing with a yearly cycle. We should expect $f(1)$ to be really close to $f(13)$, since both input values would refer to January 1 in 2016 and 2017, respectively. However, we have

$$f(1) = 498.53 \quad \text{and} \quad f(13) = 1061.81 \ ,$$

which is a significant difference. Perhaps a better model would attempt to get these two points either closer to each other or make them exactly equal to each other. Section 1.7 will describe how to work with periodic or seasonal data such as this. ∎

1.2.2 Curve Fitting, Interpolation, and Extrapolation

A common way to create a mathematical model is to create a **scatter plot** of a data set and determine a graph that is a reasonable fit to the curve, given the scenario. Figure 1.5 gives an example of a scatter plot of data from Table 1.6 that will be used in Example 1.2.1. A scatter plot is simply a representation of data in which each element of a data set can be represented as a point on a set of coordinate axes.

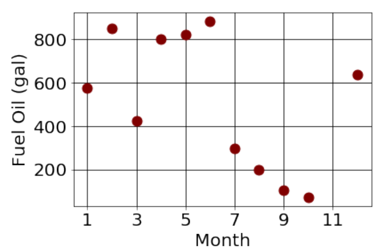


Figure 1.5: Fuel Oil Usage by Plant W in 2016.

Curve fitting is a technique in which one creates a continuous function to smooth out discrete data. The curve will generally not "connect the dots" of a scatter plot of the data, but will give the general behavior of the data. The most common and well-known means to fit a curve to data is by creating a **regression curve** of the data. The mathematical details of how to make these curves is outside the scope of this text. It is sufficient to understand that regression curves are **curves of best fit** or **best fit curves**.

With an appropriate curve fit to data, we can **interpolate** and **extrapolate** the data.

Definition 1.2.2. Using the graph of a function to estimate values between known data points (i.e., within the domain) is called **interpolation**. Making predictions beyond the domain of known data is called **extrapolation**.

Interpolation. In cases in which there is missing data, we can use interpolation techniques to make educated guesses for the actual data. This is just one example in which interpolation is used. The following example uses regression curves and another simpler technique.

Example 1.2.1. Plant W, mentioned earlier, powers their operations by burning fuel oil. Table 1.6 below shows how much oil they burned in 2016, but they are missing data from November. (Figure 1.5 gave a scatter plot of this data.) What are some reasonable values for the amount of fuel oil that they burned in November 2016?

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Oil (gal)	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0	??	638.0

Table 1.6: Fuel Oil Usage at Plant W in 2016

Solution: We will show two ways to estimate the missing data point. The first is straightforward. The other will use a model developed from a regression curve, applying concepts and techniques that we will learn more about in Section 1.3.

A simple way to interpolate a missing data point is to find the average (or mean) of the data point before and the data point after the missing point. With this approach, we have an estimate for the November 2016 fuel oil consumption of

$$\frac{72.0 + 638.0}{2} \text{ gallons} = \frac{710.0}{2} \text{ gallons} = 355 \text{ gallons.}$$

The second method creates a model by finding a curve the best fits the data in Table 1.6. The volume of fuel oil burned by Plant W in month m of 2016 is

$$f(m) = 5.76m^3 - 109.98m^2 + 532.58m + 70.17 \text{ gallons.}$$

This model is plotted with the data in Figure 1.6, showing that the model is reasonable.

From this model, we can estimate that Plant W burned

$$f(11) = 5.76 \cdot 11^3 - 109.98 \cdot 11^2 + 532.58 \cdot 11 + 70.17 \text{ gallons} = 287.53 \text{ gallons}$$

in November (month $m = 11$).

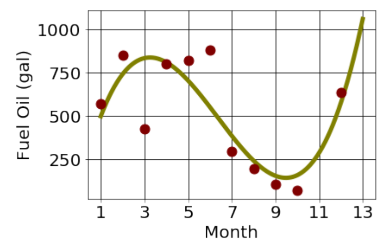


Figure 1.6: Model of Fuel Oil Usage by Plant W in 2016.

Extrapolation. In contrast to interpolation, extrapolation is more difficult because it involves predicting data points beyond the domain of the data using the trends that currently exist. Extra care must be used to create and justify assumptions used to develop the model that is used to extrapolate.

Let's discuss the following table and plot of winning Men's and Women's 100 meter dash times in the Olympics. As of this writing, the next summer Olympics will be in 2020, so we can extrapolate from the given data to predict the winning times in the next Olympics. When we plot the data, however, many more questions will naturally arise and the answers to those questions will vary depending on the model that we use to describe the data.

Year	1928	1932	1936	1948	1952	1956	1960	1964	1968	1972	1976
Time (M, s)	10.8	10.3	10.3	10.3	10.4	10.5	10.2	10.0	9.95	10.14	10.06
Time (W, s)	12.2	11.9	11.5	11.9	11.5	11.5	11.0	11.4	11.0	11.07	11.08
Year	1980	1984	1988	1992	1996	2000	2004	2008	2012	2016	2020
Time (M, s)	10.25	9.99	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81	????
Time (W, s)	11.06	10.97	10.54	10.82	10.94	11.12	10.93	10.78	10.75	10.71	????

Table 1.7: Winning Men's and Women's 100 Meter Dash Times in the Olympics

Figure 1.7 gives a scatter plot of the data. It's clear that there has been a downward trend in the gold medal times over the past century, but it's not a smooth trend. The data is choppy. The simplest curve-fitting model to smooth out the data is to use a best fit line. In the next section, we will learn how to find best fit lines, but for now, we will describe the model and discuss the model.

Model: Let y be the year and $1928 \leq y \leq 2016$. Then the winning men's and women's 100-meter dash times in the Olympics in year y can be described by $m(y)$ and $w(y)$, respectively.

$$\begin{aligned} m(y) &= -0.009498y + 28.841 \text{ seconds} \\ w(y) &= -0.014185y + 39.188 \text{ seconds} \end{aligned} \tag{1.1}$$

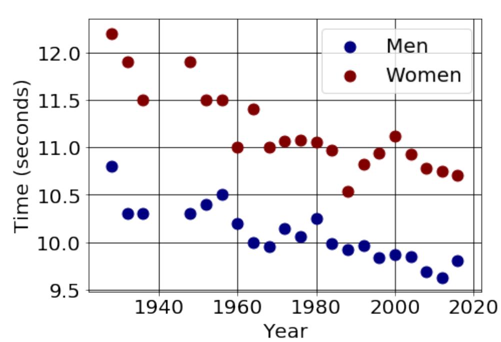


Figure 1.7: Winning Olympic 100-Meter Dash Times

Note that we have all the necessary components of a model: (1) a description of the input variable (y), with the units (years), (2) a description of the functions (m and w) and their units (seconds), and (3) a domain over which the models make sense. A plot of the models with the scatter plot of the data shows that the models make sense.

Example 1.2.2. Given the data in Table 1.7 and the models in (1.1), what are reasonable predictions for the winning times in the 100-meter dash at the 2020 Olympics?
Solution: Since y is the year of the Olympics, then we predict the winning time for the men's 100-meter dash to be

$$m(2020) = -0.009498 \cdot 2020 + 28.841 \text{ seconds} = 9.66 \text{ seconds}$$

and for the women's 100-dash:

$$w(2020) = -0.014185 \cdot 2020 + 39.188 \text{ seconds} = 10.53 \text{ seconds}.$$

These times aren't completely unreasonable, given the data, but taken in context, one might be a bit skeptical of these predictions. For the men's 100-meter dash, the time of 9.66 would be only 0.03 slower than the Olympic record, held by Usain Bolt of Jamaica. Bolt ran the three fastest Olympic 100-meter dashes at the 2008, 2012, and 2016 Olympics, but has since retired from sprinting. The forecast women's time would beat Florence Griffith-Joyner's world record, set in 1988, by 0.01 second. ■

To create a better model, we could include data from other races, not just the Olympics. We could also consider curves other than lines. To see why extrapolation has it's limitations with the linear model, consider the predicted winning times for the Olympics in the year 3000. For the men:

$$m(3000) = -0.009498 \cdot 3000 + 28.841 \text{ seconds} = 0.347 \text{ seconds}$$

and for the women's 100-meter dash:

$$w(3000) = -0.014185 \cdot 3000 + 39.188 \text{ seconds} = -3.37 \text{ seconds}.$$

These times are clearly absurd. The men's time would require a runner to run faster than a race car and for the women, no one can run a race in negative time. Therefore, the further out we attempt to extrapolate, the less plausible and more uncertain the results are.

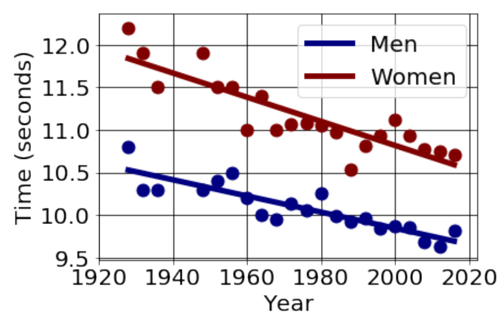


Figure 1.8: Winning Olympic 100-Meter Dash Times with Models

The plot also gives us a question to consider. Notice that the women's times are decreasing more rapidly than the men's times. Will a woman ever run the 100 meter dash faster than a man in a single Olympics? The model predicts that this could happen, but the winning times would again be absurd.

1.3 Polynomial Functions

1.4 Exponential Functions

1.5 Logistic Functions

1.6 Logarithmic Functions

1.7 Trigonometric Functions

Most applied calculus texts skip **trigonometric** functions, but they are extremely powerful for modeling periodic or seasonal data. We will keep the treatment of these models very basic and intuitive, but independent from the rest of the text; it is optional. For the scientist, trigonometric functions are indispensable. However, for our purposes, we will focus our attention exclusively on two related functions: the **sine** and **cosine** functions.

1.7.1 Definitions

A **sine wave** is of the form

$$f(x) = v + a \cdot \sin\left(\frac{2\pi(x-h)}{p}\right) \quad . \tag{1.2}$$

where v is the **vertical shift** (mean of the data), a is the **amplitude** of the wave (roughly the difference between the maximum and the mean of the data), h is the **horizontal shift** (where the data is at the mean and begins to rise to the maximum), and p is the **period**.

1.7.2 Modeling with Sine Waves

Consider the following table that we saw in Sections 1.2 and 1.3 on the fuel oil usage of Plant W in 2016. One task that we were unable to complete before was to determine a model that would reflect the seasonality of the data. In other words, since the data represents a calendar year, we want a model that will oscillate and finish at the same output value as the output value of the beginning point. A **trigonometric model** serves that very purpose.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Oil (gal)	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0	??	638.0

Table 1.8: Fuel Oil Usage at Plant W in 2016

Given our data, we don't know the vertical shift, the amplitude, or the horizontal shift, but we do know the period. Since the input is in months and there are 12 months in a year, we choose $p = 12$ from 1.2. We then fit a sine wave to the data and get the following model.

If m is the mth month of the year, then Plant W burned oil at a rate of

$$f(m) = 492.82 - 334.33 \sin\left(\frac{2\pi(m + 29.49)}{12}\right) \text{ gallons per month}$$

in 2016.

Graphing $y = f(m)$ with the scatter plot of Table 1.8 gives us the following plot to verify that the model makes sense.

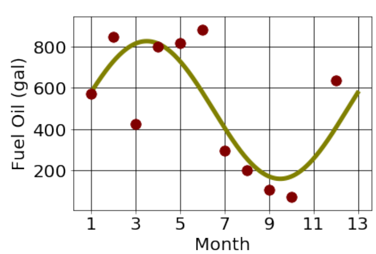


Figure 1.9: Fuel Oil Usage by Plant W in 2016 Fitted by a Sine Wave.

1.8 Combining Functions

Chapter 2

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2.2 Tangent Lines and Instantaneous Rates of Change

2.3 Basic Applications of Slope

2.4 The Derivative

2.5 Limits of Functions

2.6 Finding Derivatives Algebraically

2.7 The Power Rule

2.8 The Product and Quotient Rules

2.9 The Chain Rule

2.10 Higher Order Derivatives

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3.3 Concavity and Inflection Points

3.4 Maximum and Minimum Function Values

3.5 Curve Sketching

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4.2 Approximating Area

4.3 Integrals

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4.5 Integral Formulas

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