Open Applied Calculus

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Chapter 0

What is Calculus?

A good speaker will often begin a talk with an outline of what will be discussed in the talk. In the same way, it is incumbent upon the author of a textbook to give the reader the big picture of the subject of the textbook. That is the purpose of this brief chapter.

Calculus is the study of change. Over time, the population of a city may grow. The price of a stock will fluctuate: sometimes increasing and sometimes decreasing. The cumulative profits of a company may grow or wane. These situations represent changing quantities. We may want to dig deeper and find out how quickly the city is growing or the rate at which a company's stock price or profits are growing. Those are questions that calculus attempts to answer.

Calculus was first developed in the late 1600s independently by Sir Isaac Newton and Gottfried Wilhelm Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic, and social phenomena and to describe and solve problems in those areas.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields.

0.1 Two Problems

(a) Slope of a Curve at a Point   (b) Area Under a Curve over an Interval
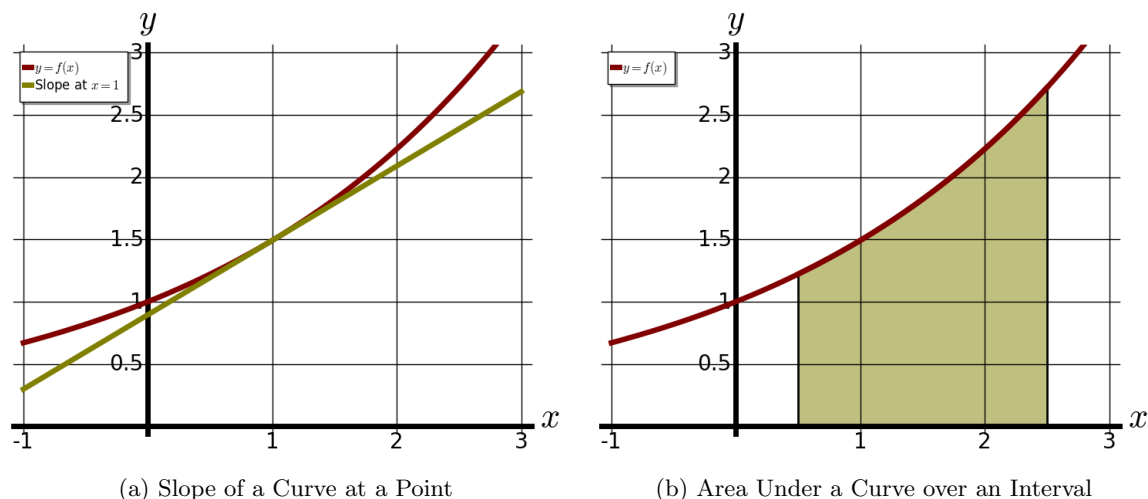
Figure 1: Visualizing the Two Problems of Calculus

Calculus is the study of two seemingly different questions that are actually inverses of each other.

1. What is the **rate of change** of a function?

2. What the **accumulation** of a function?

Geometrically, these questions can be visualized as the **slope of a curve** and the **area under a curve**, respectively, as seen in Figure 1.

In this book, we will unpack these two problems and show how they are useful to business, economics, finance, the social sciences, and the life sciences. We will take a data-driven and a problem-solving approach to this course, working with real and realistic data.

Let's consider a pair of examples to begin.

Example 0.1.1.

Example 0.1.2.

0.2   Problem Solving

Have you ever looked at a math problem and said to yourself, "I don't know where to start!" No matter how you answered that question, this section will offer some tips.

One often thinks of the study of a mathematical subject as mere rote memorization of formulas and step-by-step procedures. On the contrary, the approach that mathematicians take to the subject is much more creative. We see mathematics as a way to apply abstract concepts to solve problems. Often there is just one correct answer. Often, however, the "correct" or "best" answer may be impossible to find, so we are instead interested in a solution that would be considered "good," "plausible," or "reasonable." That is the approach that will drive much of the content of this textbook. In short, one of our objectives is to retrain the way you view mathematics: it's not about knowing or not knowing the answer, it's about figuring out a solution. This textbook is not a book to tell you what to think, but how to think about quantitative problem solving.

In light of this, we introduce a well-known problem-solving process and thoughts on how to implement it in practice. In his famous book *How to Solve It*, mathematician George Pólya described his problem-solving process in four steps, which we have modified to follow the acronym "C.O.P.E."

1. **Comprehend** (Understand) the problem.

   - **What is the problem asking?**
   - It is impossible to solve a problem without this step, but it is often skipped.
   - Read and understand the instructions if there are any.
   - Read the problem over and over until you understand.
   - Look up words and symbols you don't understand.
   - Ask yourself and answer the following questions:
     - Is the problem assuming that I know something that I don't?
     - What knowledge gaps must I fill in order to really understand the problem?
     - Is there enough information given to solve the problem?
     - Do I need to look up information for the problem?
     - Is there any irrelevant information that can be ignored?
     - Do I need to make any assumptions? If so, are these assumptions reasonable?
   - It may help to draw a picture or diagram.
   - It may help to simplify the problem, i.e., work with a simpler "toy" problem.
   - Study examples similar to the problem.
   - It may help to compare the problem to a similar one.

- Be patient with yourself. Sufficient understanding may take time.
- What is a range of plausible solutions?
- Often there is more than one level of comprehension to a problem or concept. A deeper understanding of a problem or concept results in a better or more advanced solution to the problem.
- You may obtain a deeper understanding of the problem by pursuing the next step: "Observe."

2. **Observe** the information and applicable tools and devise a plan.

- **What information is given in the problem and what tools do you have that apply?**
- "Look" around. What (mathematical or other) tools, results, theorems, algorithms, etc. apply in this situation, based on the information given?
- If you studied examples similar to the problem, do the same tools, results, theorems, or algorithms apply in this case? Why or why not?
- Here are some ideas to work with the information.
  - Define variables to describe components of the problem and solution. Consider units.
  - Express the information we have mathematically.
  - Draw a picture or diagram. Label the picture with variables, numbers, etc.
  - Make a chart or list or plot some of the data or information.
  - Experiment. Play with the problem and the math until something works.
  - Guess and check: Trial and error helps to develop intuition.
  - Be systematic. Use an organized method to check all cases.
  - Look for patterns.
  - Work backwards.
  - Look for counterexamples.
  - Divide and conquer. Split the problem up into pieces and solve the individual pieces.

3. **Proceed** with the plan and get a solution.

- **It often helps to work out a solution on scratch paper first.**
- **Make your solution easy to follow for yourself and others.**
- Check your work at each step.
- If you get stuck or the solution doesn't work, go back to the planning stage and try something else. Thomas Edison tested thousands of designs before developing a practical working lightbulb.
- Think about the problem during otherwise unproductive times such as when waiting in line, walking, driving, or falling asleep.
- When stuck, take a break and let your brain work on the problem subconsciously.
- Be patient. It may take a while for the solution to "click."

4. **Evaluate** (in the sense of "Reflect on") your solution and check your work.

- (1) **Does the solution answer the question or solve the problem?**
- (2) **Does the solution make sense?**
- Skipping this step often results in "stupid" mistakes that are easy to catch.
- Are there any obvious errors or contradictions?
- Does the solution fit within a range of plausible solutions?
- Is there another, independent approach to validate the solution or confirm that it is plausible?
- Do the units make sense?
- Check your work. This is why organized solutions are important.

- Proofread any written work, ideally at least a day after you wrote it.

5. **Extend** (Generalize) the solution or problem and reflect.

   - Is there another, possibly easier, way to solve the problem?
   - Can you generalize the problem?
   - Would you do something different to improve the solution?

Chapter 1

Models, Graphs, and Functions

1.1   What is a Function?

1.1.1   Function Concepts

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or **independent variable**, and a corresponding output quantity, or **dependent variable**. From this, we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Definition 1.1.1. A **function** is a rule for a relationship between an **input** (or **independent**) quantity and an **output** (or **dependent**) quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1.1.1. In the height and age example above, is height a function of age? Is age a function of height?
Solution: In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. You cannot have two different heights at any one instant in time.

However, age is not a function of height, since one height input might correspond with more than one output age. Once you have stopped growing, your height essentially remains constant for the rest of your life. ∎

1.1.2   Representing Functions

Functions can be represented in many ways:

1. Words,

2. Tables of values,

3. Graphs, or

4. Formulas.

Example 1.1.1 represented a function in words, but it will be convenient to streamline the discussion of a function. To that end, we introduce notation of functions.

Function Notation.   To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

If we name the function f we could write "height is a function of age" as "h is f of a," or more simply:

$$h = f(a) \ .$$

We could instead name the function h and write $h(a)$, which is read "h of a."

We can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value "a" must be put into the function "h" to get a result.

Remark 1.1.1. Be careful! The parentheses indicate that age is the input into the function. Do not confuse these parentheses with multiplication!

Example 1.1.2. A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ tell us?

Solution: When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function: the year (y). The output value is 300, the number of police officers (N), a value for the output quantity. Remember $N = f(y)$. This tells us that in the year 2005 there were 300 police officers in the town.   ∎

Tables as Functions.   A table lists the input and corresponding output values of a function.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1.1 below represents the age of a child in years and his corresponding height. This represents just some of the data available for the age and height of the child.

(Input:) a, age (years)	4	5	6	7	8	9	10
(Output:) h, height (inches)	40	42	44	47	50	52	54

Table 1.1: Tabulating height as a function of age for a child.

From this, we can create equations such as $h(6) = 44$, meaning that when the child was 6 years old, he was 44 inches tall.

Example 1.1.3. Which of these tables define a function (if any)?

Table A.			Table B.			Table C.	
Input	**Output**		**Input**	**Output**		**Input**	**Output**
2	1		−3	5		1	0
5	3		0	1		5	2
8	6		4	5		5	4

Solution: Tables A and B define functions. In both tables, each input corresponds to exactly one output. Table C does not define a function since the input value of 5 corresponds with two different output values: 2 and 4.   ∎

Graphs as Functions   A function can often be represented as a **graph**, a set of **points** plotted on **coordinate axes**: the **horizontal axis** and the **vertical axis**. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical axis.

Points on the graph are represented by ordered pairs of the form (a, b). Beginning at the **origin** (the point $(0, 0)$), you move a units to the left and b units up to arrive at the point (a, b). If $a < 0$, then movement is to the left and if $b < 0$, then movement is down.

The horizontal and vertical axes are typically called the x-**axis** and the y-**axis**, but these axes can be labeled with any variable name, not just x and y. We say y is a function of x, or $y = f(x)$ when the function is named f. The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

As an example, Figure 1.1 is a plot of the data from Table 1.1.
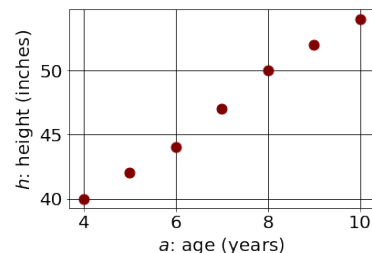
Figure 1.1: A plot of height data.

Example 1.1.4. Which of these graphs defines a function $y = f(x)$?
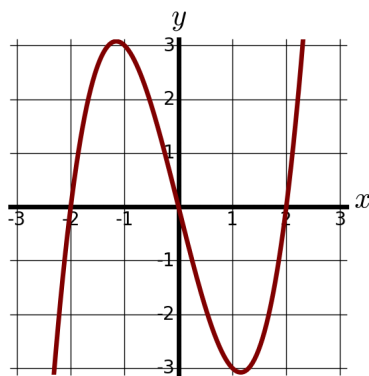
(a) Graph A
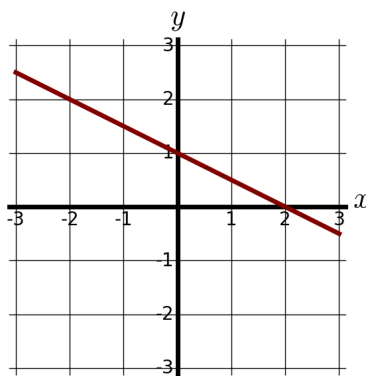
(b) Graph B
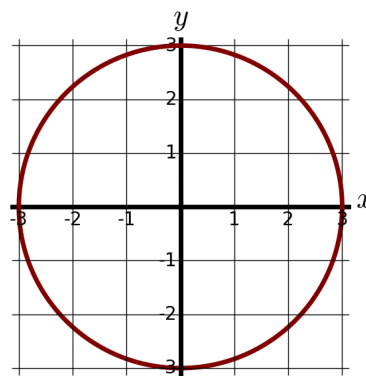
(c) Graph C

Solution: Looking at the graphs above, Graphs A and B define a function $y = f(x)$, since for each input value along the horizontal (x) axis, there is exactly one corresponding output value, determined by the y-value of the graph. Graph C does not define a function $y = f(x)$ since some input values, such as $x = 2$, correspond with more than one output value. ∎

The **vertical line test** is an easy way to determine whether a graph defines a function or not. Imagine drawing vertical lines through the graph. Since a function has exactly one output for every input, if there is a vertical line that would cross the graph more than once, then the graph does not define a function.

Figure 1.3 illustrates how Graph C from Example 1.1.4 fails the vertical line test. The vertical line $x = 2$ (in gold) intersects Graph C (in maroon) at two points; there are two outputs for for the input $x = 2$.
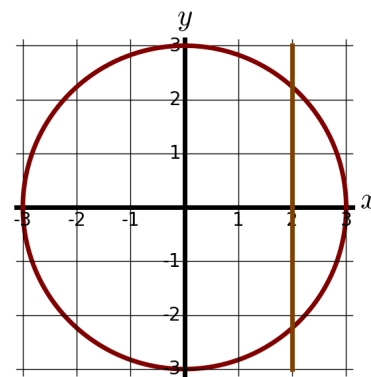
Figure 1.3: A graph failing the vertical line test.

Formulas as Functions When possible, it is very convenient to define relationships between quantities using a formula. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 1.1.5. Express the relationship $2N + 6p = 12$ as a function $p = f(n)$ if possible.

Solution: To express the relationship in this form, we need to be able to write the relationship where p is a function of N, which means writing it as $p =$ something involving N, or p in terms of N. We proceed using algebra.

$$2N + 6p = 12 \qquad\qquad \text{Subtract } 2N \text{ from both sides.}$$
$$6p = 12 - 2N \qquad\qquad \text{Divide both sides by 6 and simplify.}$$
$$p = \frac{12 - 2N}{6}$$
$$p = \frac{12}{6} - \frac{2N}{6}$$
$$p = 2 - \frac{1}{3}N$$

We can now express p as a function of N:

$$p = f(N) = 2 - \frac{1}{3}N \ \ .$$

∎

It is important to note that not every relationship can be expressed as a function with a formula. Consider the examples of the boy's height as a function of age in Table 1.1 or a company's stock price as a function of time. Neither situation allows one to define the relationship using a formula precisely, yet we still may want to analyze aspects of these relationships using tools of calculus. The rest of the chapter will introduce us to some tools to help us with that.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed.

1.1.3   Evaluating Functions.

The fundamental use of a function is to **evaluate** the function: "plugging in" some number into the function, or more precisely, to determine the corresponding output for a given input. In other words, we substitute or replace the input variable of the function with the input value. Evaluating will always produce one result, since each input of a function corresponds to exactly one output. We can evaluate a function from a table, graph, or formula.

Another related use is to determine the input or inputs of a function, given an output of the function. This is called **solving** an equation and could produce more than one solution, since different inputs can produce the same output.

Remark 1.1.2. The concepts of evaluating and solving often get confused. When we use the word *solve*, we will be *solving a problem* or *solving an equation for an unknown quantity or variable*. It does not make sense to solve a function, since a function is merely a mathematical expression and not an equation.

Example 1.1.6. Let $Q = g(n)$ and use the table shown.

$Q = g(n)$ as a table.				
n	1	2	3	4	5
Q	8	6	7	6	8

(a) Evaluate $g(3)$.

   Solution: Evaluating $g(3)$, read "g of 3," means that we need to determine the output value, Q, of the function g given the input value of $n = 3$. Looking at the table, we see the output corresponding to $n = 3$ is $Q = 7$, allowing us to conclude $g(3) = 7$.   ∎

(b) Solve $g(n) = 6$ for n.

Solution: Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we evaluate $g(n)$ at 2, our output is $Q = 6$.

When we evaluate $g(n)$ at 4, our output is also $Q = 6$. ∎

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 1.1.7. Consider the graph of a function $f(x)$ to the right.

(a) Evaluate $f(2)$.

Solution: To evaluate $f(2)$, we find the input of $x = 2$ on the horizontal (x) axis. Moving up to the graph gives the point $(2, 1)$, giving an output of $y = 1$. Therefore, $f(2) = 1$. ∎

(b) Solve $f(x) = 4$ for x.

Solution: To solve $f(x) = 4$, we find the value 4 on the vertical (y) axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: $(-1, 4)$ and $(3, 4)$. These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$.

This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. ∎
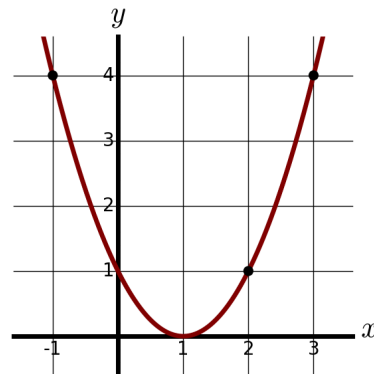
Example 1.1.8. Let $k(t) = t^3 + 2$.

(a) Evaluate $k(2)$.

Solution: To evaluate $k(2)$, we replace t with 2 in the expression $t^3 + 2$, then simplify.

$$\begin{aligned} k(2) &= 2^3 + 2 \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

So $k(2) = 10$. ∎

(b) Solve $k(t) = 1$ for t.

Solution: To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output.

$k(t) = 1$	Substitute the original formula $k(t) = t^3 + 2$.
$t^3 + 2 = 1$	Subtract 2 from each side.
$t^3 = -1$	Evaluate the cube root of each side.
$t = \sqrt[3]{-1} = -1$	

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

To check our work, we want to know if $k(t) = 1$ is a true statement when $t = -1$.

$$\begin{aligned} k(-1) &= (-1)^3 + 2 \\ &= -1 + 2 \\ &= 1 \ , \end{aligned}$$

which was the desired result. ∎

1.1.4   Cost, Revenue, and Profit

Suppose that your club wants to raise funds by selling T-shirts. The screen printing shop that will make the shirts will charge your club $50 to cover overhead costs and $5 per shirt for the shirts themselves. You decide to charge $15 per shirt. Some questions naturally arise. How many shirts need to be sold to **break even**? How much **profit** can be expected?

We can describe situations such as this with functions. The **total cost** to produce these shirts combines **fixed costs** and **variable costs**. The fixed costs are also called **overhead costs** and do not depend on the number of T-shirts made, while the variable costs are the per item cost. If n is the number of T-shirts the screen printing shop will make, $F(n)$ is the fixed costs and $V(n)$ is the variable costs, then the **cost function** to produce n T-shirts is

$$\begin{aligned}
\text{total cost} &= \text{ fixed costs } + \text{ variable costs} \\
C(n) &= F(n) + V(n) \\
&= \$50 + (\$5 \text{ per shirt})(n \text{ shirts}) \\
&= 50 + 5n \text{ dollars}
\end{aligned}$$

If you sell each shirt for $15 and you sell n shirts, then your **revenue** from selling n shirts will be $15n$ dollars. This is your **revenue function**: $R(n) = 15n$ dollars.

Finally, the **profit** that your club will earn from selling n shirts is the revenue minus the total costs. If $P(n)$ is the profit from selling n items, then the **profit function** is

$$P(n) = R(n) - C(n) \ .$$

In this example, we have

$$\begin{aligned}
P(n) &= R(n) - C(n) \\
&= 15n - (50 + 5n) \text{ dollars} \\
&= 15n - 50 - 5n \text{ dollars} \\
&= 10n - 50 \text{ dollars}
\end{aligned}$$

Example 1.1.9. Consider the T-shirt scenario above.

(a) What is the profit from selling 30 shirts?

   Solution: $P(30) = 10 \cdot 30 - 50 = 300 - 50 = \$250$.

   If you sell 30 T-shirts, then you profit $250.   ∎

(b) What is the profit from selling 3 shirts?

   Solution: $P(3) = 10 \cdot 3 - 50 = 30 - 50 = \$(-20)$.

   If you only sell 3 T-shirts, then you lose $20; your profit is negative.   ∎

(c) What is the profit from selling 0 shirts?

   Solution: $P(0) = 10 \cdot 0 - 50 = 0 - 50 = \$(-50)$.

   If you don't sell any T-shirts, then you lose $50; your profit is negative.   ∎

The **break-even point** in this context is the minimum number of T-shirts that must be sold in order to have a profit of at least $0. How do we find this? If our profit is $0, then we turn that sentence into a mathematical equation and solve for the number of shirts. The profit from selling n shirts is $P(n)$, "is" is "=", and 0 is 0.

$$\begin{aligned}
P(n) &= 0 \\
10n - 50 &= 0 \\
10n &= 50 \\
n &= \frac{50}{10} = 5
\end{aligned}$$

So you need to sell at least five T-shirts in order to break even.

Remark 1.1.3. Note the distinction between $P(0)$ and $P(n) = 0$. $P(0)$ is the profit from selling 0 items, while $P(n) = 0$ is an equation whose solution tells you the number of items that must be sold in order to have no profit.

Definition 1.1.2. In summary, the **total cost** to produce n items, $C(n)$, is the combination of **fixed costs** ($F(n)$) and **variable costs** ($V(n)$). The fixed costs are also called **overhead costs** and is constant regardless of the number of items made, while the variable costs depend on the number of items being made. So

$$C(n) = F(n) + V(n) \ .$$

The **revenue** function, $R(n)$ gives the amount of money brought in from selling n items. The difference between revenue and total costs is **profit** ($P(n)$) from selling n items:

$$P(n) = R(n) - C(n) \ .$$

The **break-even point** is the fewest number of items, n, that must be sold in order for $P(n) \geq 0$, in other words, the fewest number of items to guarantee that you won't have negative profit and be losing money.

1.1.5 Domain and Range.

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of the models we create. In our most recent example, it wouldn't make sense to sell -4 T-shirts or 4.4 T-shirts. We also wouldn't expect to sell a billion T-shirts for a university club fund raiser. When using a function to describe a real-world scenario, we need to set common sense boundaries for the input and output. Here is another example.

Table 1.2 shows a relationship between the circumference, c, and height, h, of a tree as it grows. In this table, we would consider height as a function of circumference: $h = h(c)$.

Circumference: c (feet)	1.7	2.5	5.5	8.2	13.7
Height: h (feet)	24.5	31.0	45.2	54.6	92.1

Table 1.2: Height of a tree as a function of its circumference.

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the **domain** and **range** of the function.

Definition 1.1.3. The **domain** of a function is the set of possible input values to the function.

The **range** of a function is the set of possible output values of the function.

Example 1.1.10. Using Table 1.2 above, determine a reasonable domain and range of the function h.
Solution: We can combine the data provided with additional research and our own reason to determine an appropriate domain and range of the function $h = h(c)$. For the domain, it doesn't make sense for the circumference (input) to be negative, so $c \geq 0$. For a maximum circumference, we could make an educated guess at a reasonable value, or look up that the maximum recorded circumference is about 119 feet[1]. With this information, we would say a reasonable domain is $0 \leq c \leq 119$ feet.

Similarly for the range, if we only consider the tree when the sprout has broken through the ground, it doesn't make sense to have negative heights. The maximum recorded height of a tree could be looked up to be 379 feet, so a reasonable range is $0 \leq h \leq 379$ feet. ∎

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[1]`http://en.wikipedia.org/wiki/Tree`, retrieved July 19, 2010

Interval Notation.   A convenient alternative to the notation using inequalities is **interval notation**, in which intervals of values are referred to by the starting and ending values. Parentheses "()" are used for "strictly less than," and square brackets "[]" are used for "less than or equal to." Since infinity, ∞, is not a number, neither $-\infty$ nor ∞ are included in the domain and range of a function, so we always use curved parentheses with $\pm\infty$. Table 1.3 shows how inequalities correspond to interval notation for an arbitrary variable x.

Inequality	Interval Notation
$5 \leq x \leq 10$	$[5, 10]$
$5 < x \leq 10$	$(5, 10]$
$5 \leq x < 10$	$[5, 10)$
$5 < x < 10$	$(5, 10)$
$x < 10$	$(-\infty, 10)$
$5 \leq x$	$[5, \infty)$
All real numbers	$(-\infty, \infty)$

Table 1.3: Interval Notation

To combine two intervals together, we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.
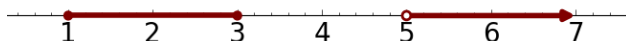

Figure 1.4: A union of intervals.

Example 1.1.11. Describe the intervals of values shown in Figure 1.4 using inequalities and using interval notation.

Solution: To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

As an inequality it is: $1 \leq x \leq 3$ or $x > 5$.

In interval notation: $[1, 3] \cup (5, \infty)$.   ■

Example 1.1.12. Find the domain of each function.

(a) $f(x) = 2\sqrt{x + 4}$

   Solution: Since we cannot evaluate the square root of a negative number, we need the inside of the square root to be non-negative. $x + 4 \geq 0$ when $x \geq -4$. (Subtract 4 from both sides of the inequality.) Therefore, the domain of $f(x)$ is $[-4, \infty)$.   ■

(b) $g(x) = \frac{3}{6 - 3x}$

   Solution: We cannot divide by zero, so we need the denominator to be non-zero. Solving $6 - 3x = 0$ for x, we have $x = 2$, so we must exclude 2 from the domain. Therefore, the domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$.   ■

1.1.6   Exercises

1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

   (a) The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.

   (b) Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

   (a) A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.

   (b) Explain the meaning of the statement $g(100) = 1$.

3. Let $n(t)$ be the number of subscribers to a YouTube channel t years after 2005. Explain the meaning of each statement.

(a) $n(5) = 300$

(b) $n(10) = 4000$

4. Let $p(t)$ be the stock price, in dollars, of Valvoline (VVV) t years after its Initial Public Offering (IPO) on September 23, 2016. Explain the meaning of each statement.

(a) $p(0) = 23.10$

(b) $p(1) = 23.27$

(c) $p(2) = 21.47$

5. Select all of the following graphs which represent y as a function of x.

Graph A.   Graph B.   Graph C.

Graph D.   Graph E.   Graph F.

6. Select all of the following graphs which represent y as a function of x.

Graph A.   Graph B.   Graph C.

Graph D.   Graph E.   Graph F.
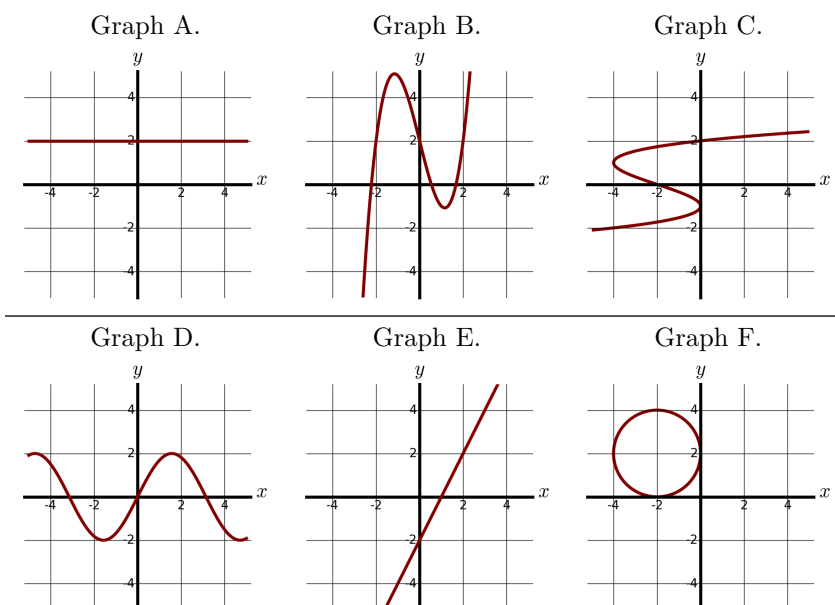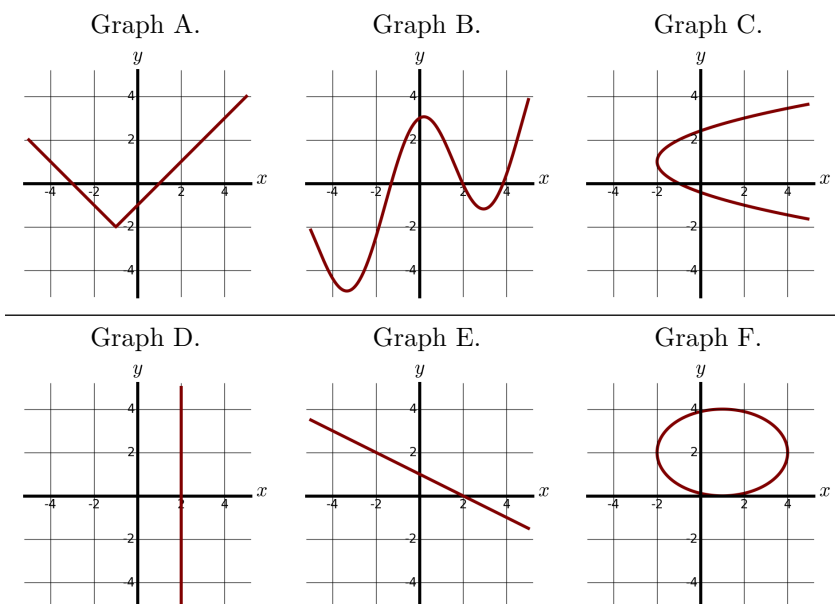
7. Select all of the following tables which represent y as a function of x.

Table A.				Table B.				Table C.		
x	5	10	15	x	5	10	15	x	5	10	10
y	3	8	14	y	3	8	8	y	3	8	14

8. Select all of the following tables which represent y as a function of x.

Table A.				Table B.				Table C.		
x	2	6	13	x	2	6	6	x	2	6	13
y	3	10	10	y	3	10	14	y	3	10	14

9. Select all of the following tables which represent y as a function of x.

Table A.		Table B.		Table C.		Table D.	
x	y	x	y	x	y	x	y
0	-2	-1	-4	0	-5	-1	-4
3	1	2	3	3	1	1	2
4	6	5	4	3	4	4	2
8	9	8	7	9	8	9	7
3	1	12	11	16	13	12	13

10. Select all of the following tables which represent y as a function of x.

Table A.		Table B.		Table C.		Table D.	
x	y	x	y	x	y	x	y
-4	-2	-5	-3	-1	-3	-1	-5
3	2	2	1	1	2	3	1
6	4	2	4	5	4	5	1
9	7	7	9	9	8	8	7
12	16	11	10	1	2	14	12

11. Let $g(x)$ be the function graphed on the right.

    (a) Evaluate $g(2)$.

    (b) Solve $g(x) = 2$ for x.
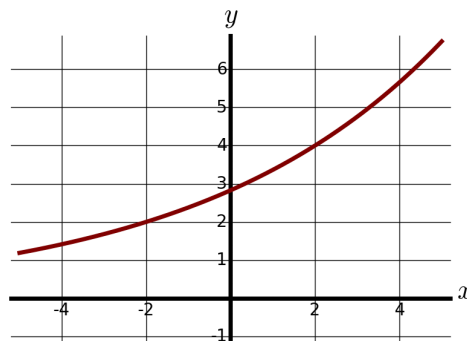
12. Let $f(x)$ be the function graphed on the right.

    (a) Evaluate $f(4)$.

    (b) Solve $f(x) = 4$ for x.

13. Consider the table at right.

    (a) Evaluate $f(3)$.

    (b) Solve $f(x) = 1$ for x.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

14. Consider the table at right.

    (a) Evaluate $g(8)$.

    (b) Solve $g(x) = 7$ for x.

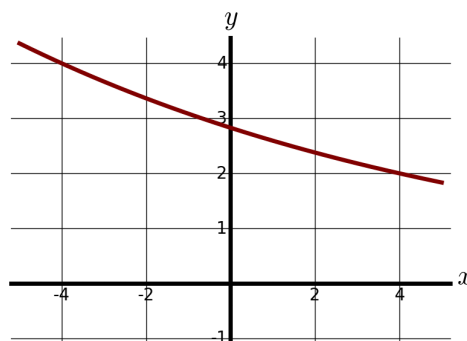| x	0	1	2	3	4	5	6	7	8	9
$g(x)$	62	8	7	38	86	73	70	39	75	34

For Exercises 15-22, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

15. $f(x) = 4 - 2x$

16. $f(x) = 8 - 3x$

17. $f(x) = 8x^2 - 7x + 3$

18. $f(x) = 6x^2 - 7x + 4$

19. $f(x) = 3 + \sqrt{x + 3}$

20. $f(x) = 4 - \sqrt[3]{x - 2}$

21. $f(x) = \frac{x-3}{x+1}$

22. $f(x) = \frac{x-2}{x+2}$

23. Let $f(t) = 3t + 5$.

    (a) Evaluate $f(0)$.
    (b) Solve $f(t) = 0$ for t.

24. Let $g(p) = 6 - 2p$.

    (a) Evaluate $g(0)$.
    (b) Solve $g(p) = 0$ for t.

25. Consider the graph of $f(x)$ on the right.

    (a) Evaluate $f(c)$.

    (b) Solve $f(x) = p$ for x.

    (c) Find the coordinates of points L and K.
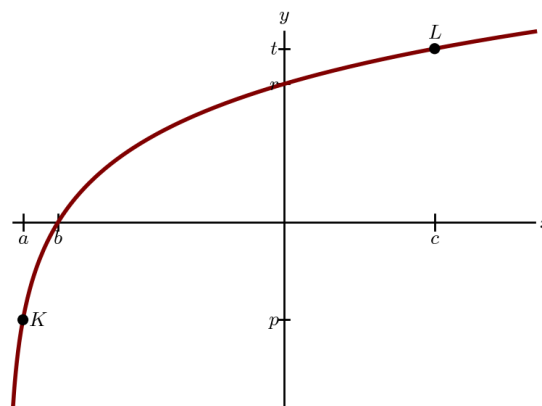
1.2 Models

"All models are wrong; some models are useful." (George E. P. Box)

In the real world, we will, in all likelihood, have to work with data. "Data-driven decisions" is a common phrase heard today. One thing that can be said is that real world data is messy and complex. Countless factors influence every phenomenon. Some factors are designed and some are purely random. Sometimes a data set is incomplete, missing pieces of data. Yet we may have to work with that data set as if it was complete. Sometimes we are tasked with **forecasting**, which is predicting future data points, given the ones that we have.

1.2.1   What is a Mathematical Model?

In many different areas, we need tools to simplify a set of data and work with that simplified version of the data. These simplifications must be based on reasonable assumptions that connect to the larger context of the data. Simplifications serve two purposes. First, it may be impossible to take every variable into account. Second, models must often be communicated to others and a simple model is generally more clear and therefore much easier to communicate. This is the essence of what is called **mathematical modeling**, or simply **modeling**.

Definition 1.2.1. A **model** or a **mathematical model** is a mathematical framework to help describe some phenomenon, specifically how an input or quantity affects or relates to some output. A model has three main components:

1. One or more input variables, with specific descriptions of what these variables represent, including units.

2. One or more functions, with specific descriptions of what the output of the functions represent, including units.

3. A domain and/or range over which the function(s) make sense to use.

Additionally, we may want to consider the following when creating a model.

- Identify underlying assumptions that were used to simplify the situation.

- Perform a sensitivity analysis to determine if the model is relatively unchanged if the data varies slightly.

For example, one could construct a model to speculate how the price of an item affects the demand for that item, and from that, predict revenue from sales of that item.

It is crucial to state that a model is used to simplify reality and does not dictate or reflect past, present, or future reality with absolute precision. Although good models can be useful for forecasting, decision-making, and filling in missing data. That is the essence of the quote at the beginning of the section by the late George Box, a famous British statistician.

The following is an example of a mathematical model, based on an actual data set from "Plant W." We will analyze this scenario in several places in this text and.

Model: Plant W heats an external tank and powers their operations by burning fuel oil. The rate at which they burned this oil in 2016 can be described by the following model. Let m be the month of the year, with 1 referring to January 1, and 12 referring to December 1, so that $1 \leq m < 13$. Let $f(m)$ describe the rate at which the fuel burns, in gallons per month. A model for $f(m)$ is

$$f(m) = 5.76m^3 - 109.98m^2 + 532.58m + 70.17 \text{ gallons per month.}$$

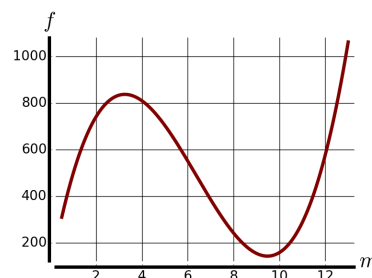Figure 1.5 plots this curve over the given domain.

Figure 1.5: Fuel Oil Usage by Plant W in 2016.

Remark 1.2.1. Note that in this example, we clearly described the input variable, what it represented, a domain that it makes sense over, and gave the units (months). We clearly described the function (the output), what it modeled, and gave the units.

Examining the graph of the function, we can see that the model makes sense, based on the context. If Plant W is heating a large external tank, then in the winter and early spring, we expect more oil to be used to heat it. This tank may also have retained some heat, as fluids tend to do, from the late summer and fall. We expect a significant drop in oil usage through the late spring and summer months and finally a significant increase in oil usage during the fall and early winter.

Is the model precise? Of course not. First, we know that in 2016 (a leap year), there were months with 29, 30, or 31 days, yet we treated each month as equal. Perhaps a better model would have modeled fuel

usage per day. For the sake of clarity, however, we aggregated the days into months; we know immediately that month 6 is June, but do not immediately know which month is day 276, for example.

A second reason that we know that the model is not accurate is that we can't be certain that on January 1, 2016 that Plant W burned oil at a rate of $f(1) = 498.53$ gallons per month. In other words, on January 1, 2016, did Plant W burn $f(1)/31 = 16.08$ gallons of fuel oil? We can be almost certain that they did not, but the actual amount may have been somewhat close to that. Other factors that we didn't consider were the range of temperatures that day and the percentage of the tank that was full. Considering these factors would make the model more accurate, but more complex.

A third item to consider is that in this case, we are dealing with a yearly cycle. We should expect $f(1)$ to be really close to $f(13)$, since both input values would refer to January 1 in 2016 and 2017, respectively. However, we have

$$f(1) = 498.53 \quad \text{and} \quad f(13) = 1061.81 \ ,$$

which is a significant difference. Perhaps a better model would attempt to get these two points either closer to each other or make them exactly equal to each other. Section 1.8 will describe how to work with periodic or seasonal data such as this. ∎

1.2.2 Scatter Plots: Plotting Data in a Spreadsheet

Throughout this book, we will use a spreadsheet to plot and analyze data. In the spirit of using free and open-source resources, our examples will use the well-known office suite *LibreOffice*, specifically its spreadsheet software *Calc*. LibreOffice is compatible with other office suites and the various menus and wizards are very similar to that of other office suites. *LibreOffice* is installed by default in most distributions of the Linux operating system, but the most current version is available for free at `https://www.libreoffice.org/` for Linux and other major operating systems.

In this section, we will see how to create two different kinds of **scatter plots** in *Calc*. A scatter plot is simply a representation of data in which each element of a data set can be represented as a point on a set of coordinate axes. A scatter plot helps to visualize the relationship between variables in a data set.

The first kind of scatter plot is ideal for **time series**, a data set that is in chronological order and there is at most one data point for every unit of time. Examples include economic data, such as a country's gross domestic product (GDP) over time; financial data, such as the price of a stock or commodity; biological data, such as the population of a species in a region over time; business data, such as a company's sales over time; and so on. Such plots help to identify trends and the strength of that trend.

The following spreadsheet shows the oil usage data of Plant W in 2016.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Month**	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
2	**Oil (Gal.)**	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0		638.0

To make a scatter plot of this data:

- Highlight the data (including the labels).

- Select `Insert` from the `LibreOffice Calc` menu.

- Select `Chart ...` to bring up the **Chart Wizard**.

- Select `Line` and the **Points Only** plot.
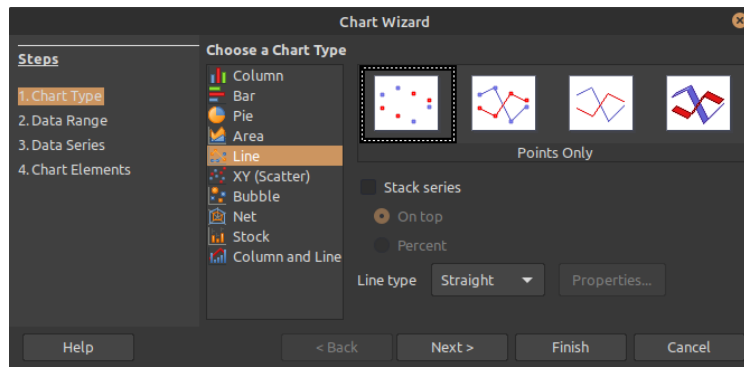
- Click on `Next >`. If you've already highlighted the data, then the wizard should have the correct range.

- Select `Data series in rows` and check `First column as label` since the first column are the axis labels of the data.
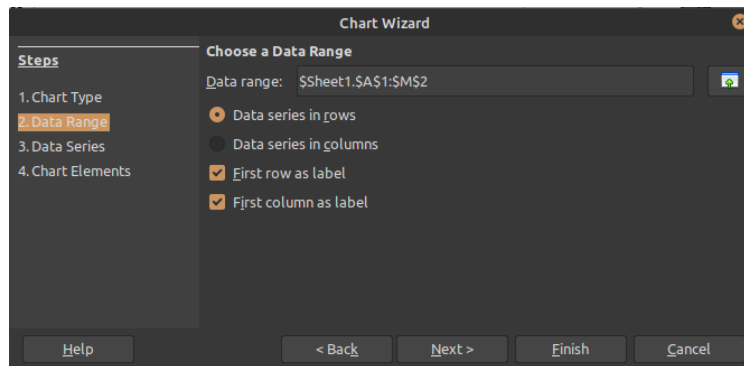
- Select `4. Chart Elements` on the left panel of the wizard.

- Put in appropriate titles and axis labels for the plot.

- Other options in this part of the wizard will make the chart more readable.
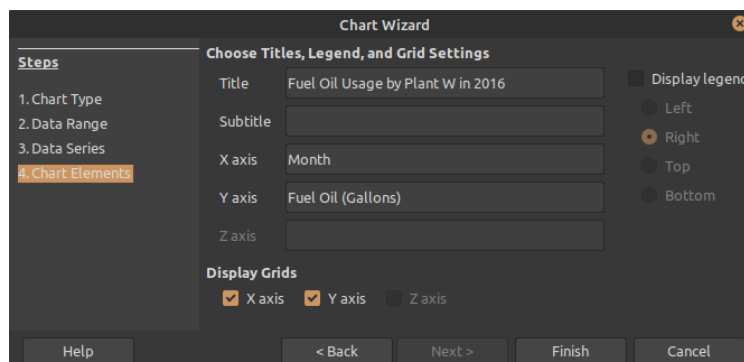
- Select `Finish`.

- Once the wizard is complete, we can make other changes to the chart.

- Double-clicking on the plotted points allows you to change the color and shape of the plotted points.

- Double-click on the y-axis and select the `Scale` tab. Changing the `Major interval` to 200 makes the axis a little more readable.

Fuel Oil Usage by Plant W in 2016

[Figure: scatter plot titled "Fuel Oil Usage by Plant W in 2016" with x-axis "Month" (Jan. through Dec.) and y-axis "Fuel Oil (Gallons)" from 0 to 1000]

The second type of scatter plot is more general and seeks to understand the relationship between two variables in a particular phenomenon.

Suppose that you are a new realtor in King County, Washington. You want to understand the housing market in the county in order to advise potential home buyers and sellers on reasonable prices for homes. You acquire a data set giving information on 21,613 home sales in King County, WA between May 2, 2014 and May 27, 2015. The following spreadsheet is a very small portion of this data set. To get started, you want to understand the relationship between living space area and the sale price of a home, though there are certainly more variables to consider. It makes more sense to think of the sale price as a function of the living space area, so the x-axis will be living space area in square feet and the sale price (in U.S. dollars) will be along the y-axis.

	A	B
1	Price ($)	Living Area (ft^2)
2	313000	1340
3	2384000	3650
4	342000	1930
...	...	...
21613	445500	1390
21614	1310000	3750

Table 1.6: King County Home Sales from May 2, 2014 to May 27, 2015

To make a scatter plot of this data:

- Highlight the data (including the labels).

- Select `Insert` from the `LibreOffice Calc` menu.

- Select `Chart ...` to bring up the **Chart Wizard**.
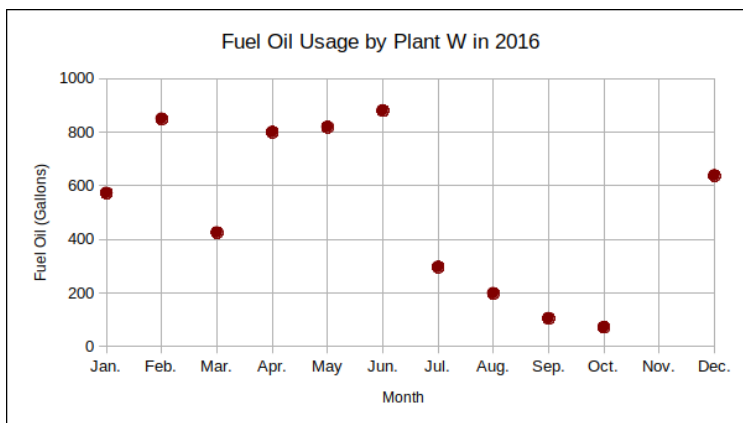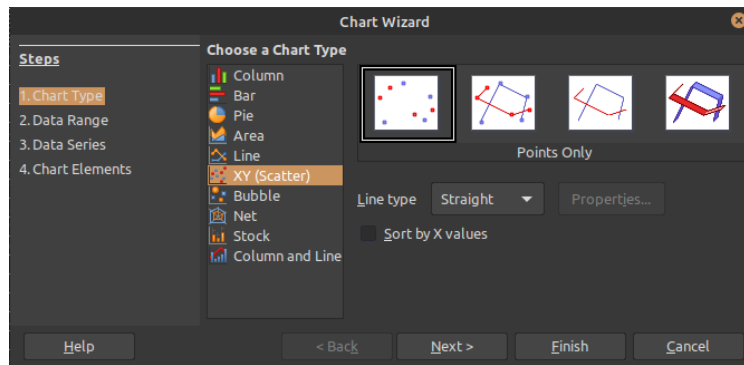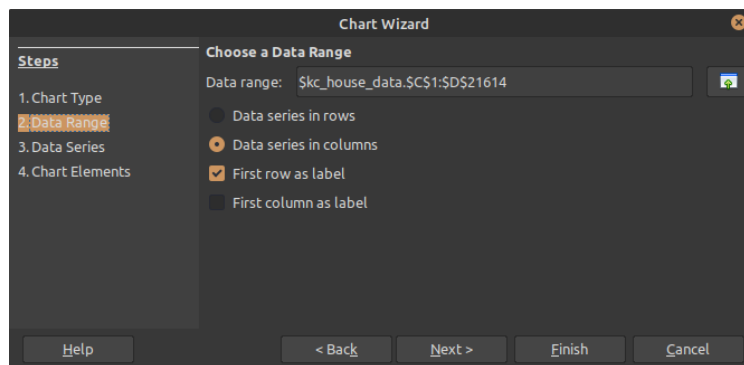
- Select `XY (Scatter)` and the **Points Only** plot.

- Click on `Next >`. If you've already highlighted the data, then the wizard should have the correct range.

- Select `Data series in columns` and check `First row as label` since the first row are the axis labels of the data.

- Click on `Next >`.

- Notice that the x-axis data is in Column B and the y-axis data is in Column A. In this `Data Series` window, manually adjust this.

- In the `Data ranges:` area, click on `X-Values` and `Y-Values` and change the `Range for X-Values` and `Range for Y-Values`, respectively.
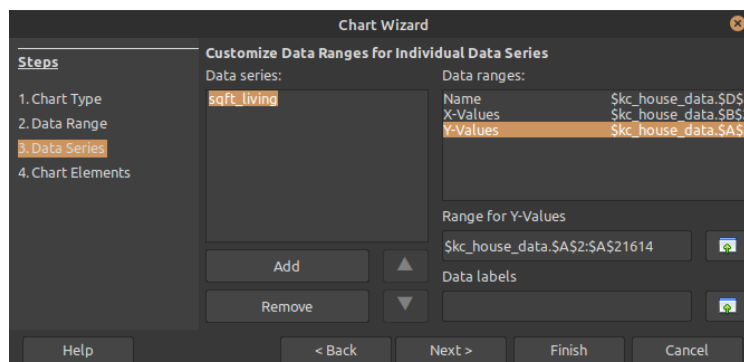
- Click on `Next >` to go to `4. Chart Elements`.

- Put in appropriate titles and axis labels for the plot.

- Other options in this part of the wizard will make the chart more readable.

- Select `Finish`.

- We will adjust the chart as we did above to make it more readable.

King County Home Sales from May 2, 2014 to May 27, 2015
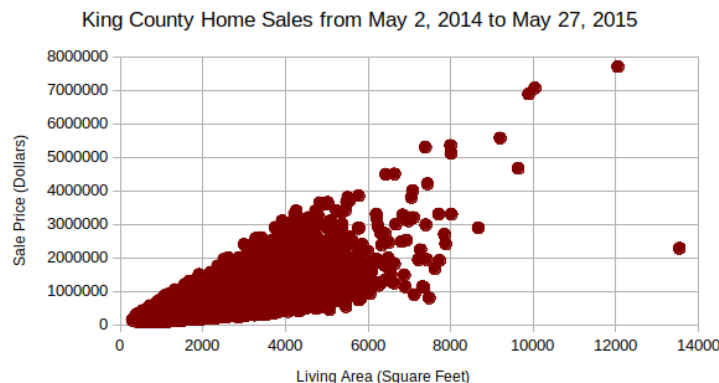
We immediately notice a number of things. First, with such a large data set, it is apparent that data is messy. There appears to be a predictable trend but for any living area, there is a very wide price range. Most people looking to buy or sell a house will also not be interested in the data of the very expensive houses, so the data on the large and expensive houses could be ignored. Also, King County covers a large area: Seattle, many of its suburbs, and areas within the Cascade Range. Houses in the city and upscale areas of the suburbs will sell for much more than a similarly sized house in the mountains. For this reason, data and models need to be considered in context.

Now suppose that you are helping a family with three children find a house in Issaquah. The father is a software engineer and the mother homeschools the children and based on their budget, they would consider a house no more than $400,000. Would you show them this chart? Of course not. We will sort the data and consider homes in Issaquah (ZIP codes 98027, 98029, and 98075) that have sold for $400,000 or less. However, to attempt to establish a more accurate trend, we will plot data on house prices up to $600,000.

- Highlight the spreadsheet by clicking on the empty box next to the `A` column designation.

- Click on `Data` in the menu.

- Select `Sort...`

- On the `Options` tab, select `Range contains column labels`.

- Go back to `Sort Criteria`. Under `Sort Key 1`, select the ZIP code and `Ascending`.

- Under `Sort Key 2`, select the price and `ascending`.

Gathering this data into a separate spreadsheet, we now have 464 entries. The family wants at least three bedrooms and at least two bathrooms. This narrows our data set to 345 entries, which is more readable.

Issaquah Home Prices with 3+ Bedrooms and 2+ Bathrooms
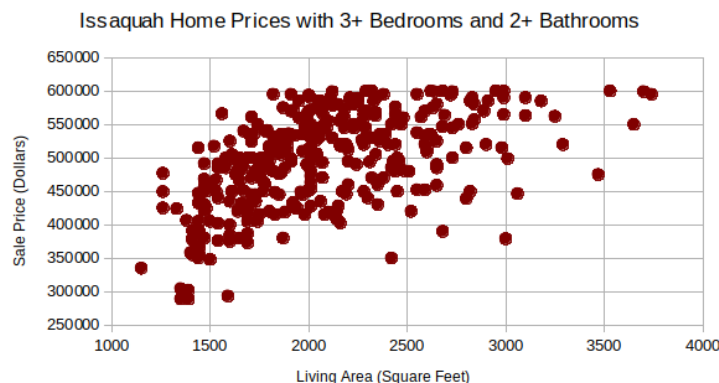
Given this level of context, you, as the real estate agent, and the family have a more accurate picture of the relationship between living area and home price. This would better equip you to make a reasonable offer on a house that comes up for sale.

The data is still quite messy. It's called a scatter plot for a reason! Though we have eliminated much of the data by drilling down and focusing on certain values for some variables, there are more variables to consider. Throughout the book, we will explore various ways to model this data with a curve.

1.2.3   Curve Fitting, Interpolation, and Extrapolation

A common and simple way to create a mathematical model from a scatter plot of a data set is to determine a function that is a reasonable fit to the curve, given the scenario. Figure 1.6 gives an example of a scatter plot of data from Table 1.7 that will be used in Example 1.2.1. **Curve fitting** is a technique in which one creates a continuous function to smooth out discrete data. The curve will generally not "connect the dots" of a scatter plot of the data, but will give the general behavior of the data. The most common and well-known means to fit a curve to data is by creating a **regression curve** of the data. The mathematical details of how to make these curves is outside the scope of this text. It is sufficient to understand that regression curves are **curves of best fit** or **best fit curves**.
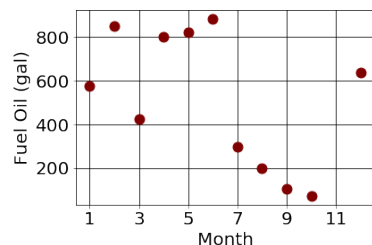
With an appropriate curve fit to data, we can **interpolate** and **extrapolate** the data.

Figure 1.6: Fuel Oil Usage by Plant W in 2016.

Definition 1.2.2. Using the graph of a function to estimate values between known data points (i.e., within the domain) is called **interpolation**. Making predictions beyond the domain of known data is called **extrapolation**. When a model no longer applies after some point, and extrapolation is unreasonable it is sometimes called **model breakdown**.

Interpolation.   In cases in which there is missing data, we can use interpolation techniques to make educated guesses for the actual data. This is just one example in which interpolation is used. The following example uses regression curves and another simpler technique.

Example 1.2.1. Plant W, mentioned earlier, powers their operations by burning fuel oil. Table 1.7 below shows how much oil they burned in 2016, but they are missing data from November. (Figure 1.6 gave a scatter plot of this data.) What are some reasonable values for the amount of fuel oil that they burned in November 2016?

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Oil (gal)	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0	??	638.0

Table 1.7: Fuel Oil Usage at Plant W in 2016

Solution: We will show two ways to estimate the missing data point. The first is straightforward. The other will use a model developed from a regression curve, applying concepts and techniques that we will learn more about in Section 1.4.

A simple way to interpolate a missing data point is to find the average (or mean) of the data point before and the data point after the missing point. With this approach, we have an estimate for the November 2016 fuel oil consumption of

$$\frac{72.0 + 638.0}{2} \text{ gallons} = \frac{710.0}{2} \text{ gallons} = 355 \text{ gallons}.$$

The second method creates a model by finding a curve the best fits the data in Table 1.7. The volume of fuel oil burned by Plant W in month m of 2016 is

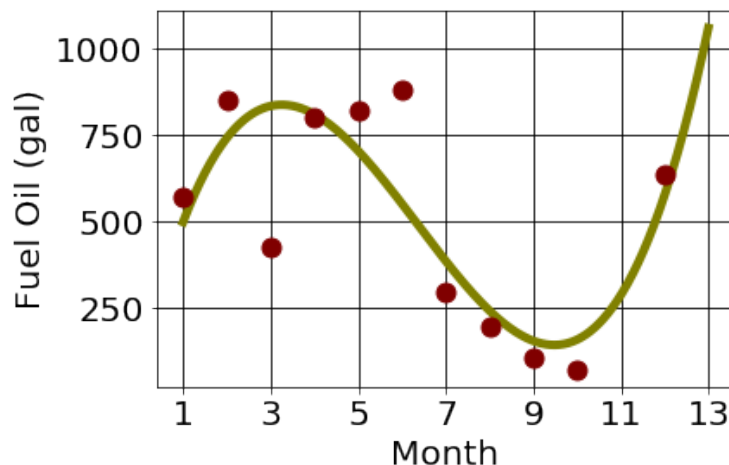$$f(m) = 5.76m^3 - 109.98m^2 + 532.58m + 70.17 \text{ gallons}.$$

Figure 1.7: Fuel Oil Usage by Plant W in 2016.

This model is plotted with the data in Figure 1.7, showing that the model is reasonable.

From this model, we can estimate that Plant W burned

$$f(11) = 5.76 \cdot 11^3 - 109.98 \cdot 11^2 + 532.58 \cdot 11 + 70.17 \text{ gallons} = 287.53 \text{ gallons}$$

in November (month $m = 11$). ∎

Extrapolation. In contrast to interpolation, extrapolation is more difficult because it involves predicting data points beyond the domain of the data using the trends that currently exist. Extra care must be used to create and justify assumptions used to develop the model that is used to extrapolate.

Let's discuss the following table and plot of winning Men's and Women's 100 meter dash times in the Olympics. As of this writing, the next summer Olympics will be in 2020, so we can extrapolate from the given data to predict the winning times in the next Olympics. When we plot the data, however, many more questions will naturally arise and the answers to those questions will vary depending on the model that we use to describe the data.

Year	1928	1932	1936	1948	1952	1956	1960	1964	1968	1972	1976
Time (M, s)	10.8	10.3	10.3	10.3	10.4	10.5	10.2	10.0	9.95	10.14	10.06
Time (W, s)	12.2	11.9	11.5	11.9	11.5	11.5	11.0	11.4	11.0	11.07	11.08
Year	1980	1984	1988	1992	1996	2000	2004	2008	2012	2016	2020
Time (M, s)	10.25	9.99	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81	????
Time (W, s)	11.06	10.97	10.54	10.82	10.94	11.12	10.93	10.78	10.75	10.71	????

Table 1.8: Winning Men's and Women's 100 Meter Dash Times in the Olympics

Figure 1.8 gives a scatter plot of the data. It's clear that there has been a downward trend in the gold medal times over the past century, but it's not a smooth trend. The data is choppy. The simplest curve-fitting model to smooth out the data is to use a best fit line. In the next section, we will learn how to find best fit lines, but for now, we will describe the model and discuss the model.

Model: Let y be the year and $1928 \leq y \leq 2016$. Then the winning men's and women's 100-meter dash times in the Olympics in year y can be described by $m(y)$ and $w(y)$, respectively.

$$m(y) = -0.009498y + 28.841 \text{ seconds} \tag{1.1}$$
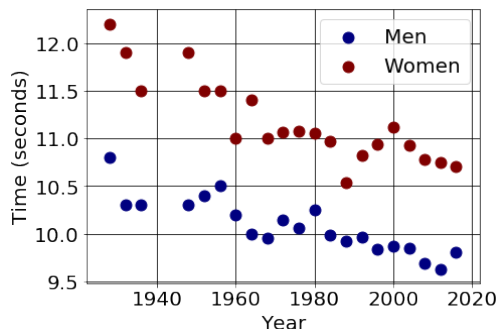$$w(y) = -0.014185y + 39.188 \text{ seconds}$$

Figure 1.8: Winning Olympic 100-Meter Dash Times

Note that we have all the necessary components of a model: (1) a description of the input variable (y), with the units (years), (2) a description of the functions (m and w) and their units (seconds), and (3) a domain over which the models make sense. A plot of the models with the scatter plot of the data shows that the models make sense.

Example 1.2.2. Given the data in Table 1.8 and the models in (1.1), what are reasonable predictions for the winning times in the 100-meter dash at the 2020 Olympics?
Solution: Since y is the year of the Olympics, then we predict the winning time for the men's 100-meter dash to be

$$m(2020) = -0.009498 \cdot 2020 + 28.841 \text{ seconds} = 9.66 \text{ seconds}$$

and for the women's 100-dash:

$$w(2020) = -0.014185 \cdot 2020 + 39.188 \text{ seconds} = 10.53 \text{ seconds}.$$

These times aren't completely unreasonable, given the data, but taken in context, one might be a bit skeptical of these predictions. For the men's 100-meter dash, the time of 9.66 would be only 0.03 slower than the Olympic record, held by Usain Bolt of Jamaica. Bolt ran the three fastest Olympic 100-meter dashes at the 2008, 2012, and 2016 Olympics, but has since retired from sprinting. The forecast women's time would beat Florence Griffith-Joyner's world record, set in 1988, by 0.01 second.   ■

To create a better model, we could include data from other races, not just the Olympics. We could also consider curves other than lines. To see why extrapolation has it's limitations with the linear model, consider the predicted winning times for the Olympics in the year 3000. For the men:

Figure 1.9: Winning Olympic 100-Meter Dash Times with Models

$$m(3000) = -0.009498 \cdot 3000 + 28.841 \text{ seconds} = 0.347 \text{ seconds}$$

and for the women's 100-meter dash:

$$w(3000) = -0.014185 \cdot 3000 + 39.188 \text{ seconds} = -3.37 \text{ seconds}.$$

These times are clearly absurd. The men's time would require a runner to run faster than a race car and for the women, no one can run a race in negative time. Therefore, the further out we attempt to extrapolate, the less plausible and more uncertain the results are.

The plot also gives us a question to consider. Notice that the women's times are decreasing more rapidly than the men's times. Will a woman ever run the 100 meter dash faster than a man in a single Olympics? The model predicts that this could happen, but the winning times would again be absurd.
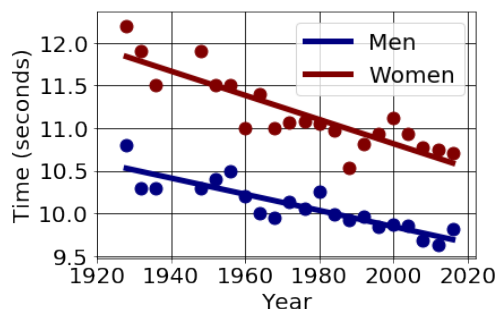
1.3   Linear Functions

1.3.1   Linear Function Basics

In Section 1.1.4, we had the example of a screen printing shop that charges $50 to cover the overhead costs of setting the screen to make T-shirts and $5 per shirt for the shirts themselves. Using descriptive variables, we chose n for the number of T-shirts and C for the cost in dollars as a function of the number of shirts: $C(n)$.

We know that $C(0) = 50$, since the overhead, or fixed, costs are charged regardless of the number of T-shirts made. Since $5 is added for each T-shirt ordered, then $C(1) = \$50 + \$5 = \$55$, $C(2) = \$55 + \$5 = \$60$, and so on.

If n T-shirts are ordered, then

$$C(n) = \$50 + \left(\frac{\$5}{\text{T-shirt}}\right)(n \text{ T-shirts}) = 50 + 5n \text{ dollars } .$$

Notice how the units (or dimensions) make sense in this equation. Each term has units of dollars, after cancelling the unit of "T-shirt" in the second term.

Notice this equation consisted of two quantities. The first is the fixed $50 charge which does not change based on the value of the input. When the no T-shirts are ordered, the cost is $50, giving the point $(0, 50)$ on the graph of the function. This is the **vertical** or y-intercept. The second is the $5 per T-shirt, which is a **rate of change**. In $C(n)$, this rate of change is multiplied by the input value. It is important here to note that in this equation, the rate of change is **constant**; over any interval, the rate of change is the same.

Looking at $C(n)$ as a table, we can also see the cost changes by $5 for each additional T-shirt.

n	0	1	2	3
$C(n)$	50	55	60	65

(Even though we cannot order a fractional number of T-shirts, we still graph $y = C(n)$ on all $n \geq 0$.)

The graph is increasing in a straight line from left to right because for each T-shirt, the cost goes up by $5; this rate remains consistent.

In this example, you have seen the T-shirt cost modeled in words, an equation, a table, and as a graph. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all four representations for a problem and so being able to work with all four forms is very important.

The function $C(n)$ is an example of a **linear function**. This name comes from the fact that a graph of a linear function is a line.

Definition 1.3.1. A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$f(x) = mx + b \ ,$$

where:

- b is the initial or starting value of the function (when the input, $x = 0$), and

- m is the constant rate of change of the function.

This form of the line is called the **slope-intercept form**.

Many people like to write linear functions in the form $y = mx + b$ because it corresponds to the way we tend to speak: "The output starts at b and increases at a rate of m."

Slope and Increasing/Decreasing   The constant rate of change of a linear function, m, is also called the **slope** of the function. The slope determines if the function is an increasing function or a decreasing function.

- $f(x) = mx + b$ is an **increasing** function if $m > 0$.

- $f(x) = mx + b$ is a **decreasing** function if $m < 0$.

If $m = 0$, then the rate of change of $f(x) = mx + b$ is zero, and $f(x) = 0 \cdot x + b = b$, so its graph is just the horizontal line passing through the point $(0, b)$, neither increasing nor decreasing.

The concepts of slope and rate of change are major component of calculus. It is crucial to understand each facet of these two concpets that we will discuss in this section since these conepts will be generalized from linear functions to a wide array of functions in Chapter 2.

Example 1.3.1. Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year?

Solution: The initial value for this function is 200, since he currently owns 200 songs, $N(0) = 200$ songs. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula:

$$N(m) = 200 \text{ songs} + \left(\frac{15 \text{ songs}}{\text{month}} \right) (m \text{ months}) = 200 + 15m \text{ songs } .$$

With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 200 + 15 \cdot 12 \text{ songs} = 200 + 180 = 380 \text{ songs } .$$

Marcus will have 380 songs in a year.   ∎

Calculating Rate of Change   Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, (x_1, y_1) and (x_2, y_2), we can find a linear function that contains both points. First we calculate the rate of change, m, of the function.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$$

It is also customary to write Δv for a change (or difference) in the variable v. We read this "delta v". With this notation, we can write

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \ .$$

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \ .$$

Once we have computed m, we can use either of the given points to find b using algebra.

$$y_1 = mx_1 + b$$
$$b = y_1 - mx_1$$

Remark 1.3.1. It is a waste of your time to make a special effort to memorize this following formula. This is a concept to understand.

Example 1.3.2. The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

Solution: The rate of change will relate the change in population to the change in time. The population increased by $27800 - 23400 = 4400$ people over the four-year time interval. To find the rate of change, the number of people per year the population changed by:

$$m = \frac{27800 - 23400 \text{ people}}{2006 - 2002 \text{ years}} = \frac{4400 \text{ people}}{4 \text{ years}} = 1100 \text{ people per year} \ .$$

■

Notice that we knew the population was increasing, so we expected $m > 0$. This is a quick way to check if the solution is reasonable.

Example 1.3.3. The pressure, P, in pounds per square inch (psi) on a diver depends upon his depth below the water surface, d, in feet, following the equation

$$P(d) = 14.696 + 0.434d \ .$$

Interpret the components of this function.

Solution: The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{psi}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 psi for each foot his depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 psi. ■

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

Example 1.3.4. Write an equation for the linear function graphed below.
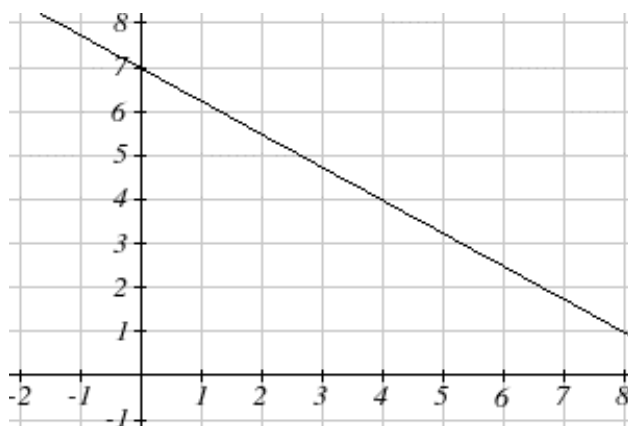
Solution: Looking at the graph, we might notice that it passes through the points $(0, 7)$ and $(4, 4)$. From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4 - 7}{4 - 0} = \frac{-3}{4} \ .$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x \ .$$

■

Example 1.3.5. If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.
Solution: The rate of change (or slope) of the function is $m = \frac{1-(-2)}{8-3} = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will

have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3:

$$f(3) = b + \frac{3}{5} \cdot 3 \ .$$

Since we know that $f(3) = -2$, we can substitute on the left side

$$-2 = b + \frac{3}{5} \cdot 3 \ .$$

This leaves us with an equation we can solve for the initial value

$$b = -2 - \frac{9}{5} = -\frac{19}{5} \ .$$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = -\frac{19}{5} + \frac{3}{5}x \ .$$

$\blacksquare$

Example 1.3.6. Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so his weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.
Solution: The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$m = \frac{920 - 760 \text{ dollars}}{5 - 3 \text{ policies}} = \frac{160}{2} \text{ dollars per policy} = \$80 \text{ per policy} \ .$$

Keeping track of units, as we did above, can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value $I(n) = b + 80n$ then when n $= 3$, $I(3) = 760$, giving $760 = b + 80 \cdot 3 = b + 240$ this allows us to solve for b $b = 760 - 240 = 520$

This form allows us to see the starting value for the function: 520. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week. $\blacksquare$

1.3.2   Graphs of Linear Functions

When we are working with a new function, it is useful to know as much as we can about the function: its graph, where the function is zero, and any other special behaviors of the function. We will begin this exploration of linear functions with a look at graphs.

When graphing a linear function, there are two basic ways to graph it.

1. Plot at least two points and draw a line through the points.

2. Use the initial value (output when $x = 0$) and the rate of change (slope).

Example 1.3.7. Graph $f(x) = 5 - \frac{2}{3}x$ by plotting points.
Solution: In general, we evaluate the function at two or more inputs to find at least two points on the graph. Usually it is best to pick input values that will "work nicely" in the equation. In this equation, multiples of 3 will work nicely due to the in the equation, and of course using $x = 0$ to get the vertical intercept. Evaluating $f(x)$ at $x = 0, 3$, and 6:

$f(0) = 5 - \frac{2}{3} \cdot 0 = 5$
$f(3) = 5 - \frac{2}{3} \cdot 3 = 3 \qquad f(6) = 5 - \frac{2}{3} \cdot 6 = 1$

These evaluations tell us that the points $(0, 5)$, $(3, 3)$, and $(6, 1)$ lie on the graph of the line. Plotting these points and drawing a line through them gives us the graph. $\blacksquare$

When using the initial value and rate of change to graph, we need to consider the graphical interpretation of these values. Remember the initial value of the function is the output when the input is 0, so in the equation $f(x) = b + mx$, the graph includes the point $(0, b)$. On the graph, this is the vertical intercept – the point where the graph crosses the vertical axis.

For the rate of change, it is helpful to recall that we calculated this value as

$$m = \frac{\text{change of output}}{\text{change of input}}$$

From a graph of a line, this tells us that if we divide the vertical difference, or rise, of the function outputs by the horizontal difference, or run, of the inputs, we will obtain the rate of change, also called slope of the line.

$$m = \frac{\text{change of output}}{\text{change of input}} = \frac{\text{rise}}{\text{run}}$$

Notice that this ratio is the same regardless of which two points we use.

Graphical Interpretation of a Linear Equation. Graphically, in the equation, $f(x) = b + mx$,

- b is the **vertical intercept** of the graph and tells us we can start our graph at $(0, b)$

- m is the **slope of the line** and tells us how far to rise and run to get to the next point.

Once we have at least two points, we can extend the graph of the line to the left and right.

Example 1.3.8. Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope.

Solution: The vertical intercept of the function is $(0, 5)$, giving us a point on the graph of the line. The slope is $\frac{-2}{3}$. This tells us that for every 3 units the graph "runs" in the horizontal, the vertical "rise" decreases by 2 units. In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value $(0, 5)$, the slope tells us that if we move to the right 3 units, we will move down 2 units, moving us to the point $(3, 3)$. We can continue this again to find a third point at $(6, 1)$. Finally, extend the line to the left and right, containing these points. ■
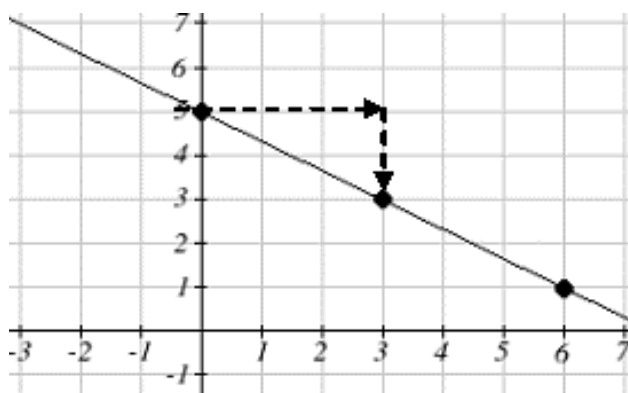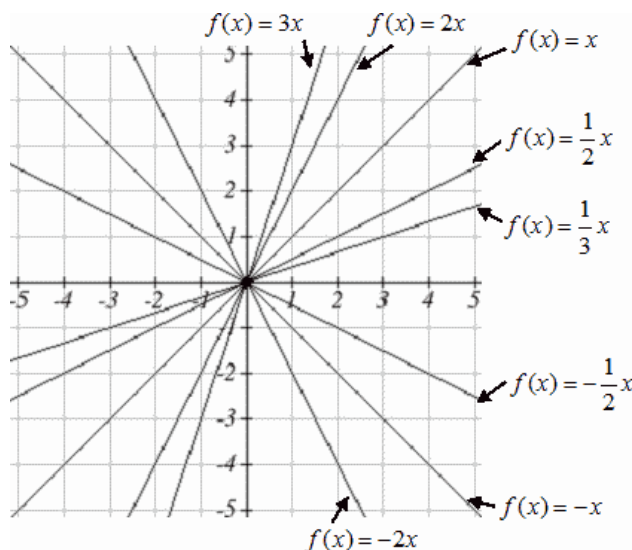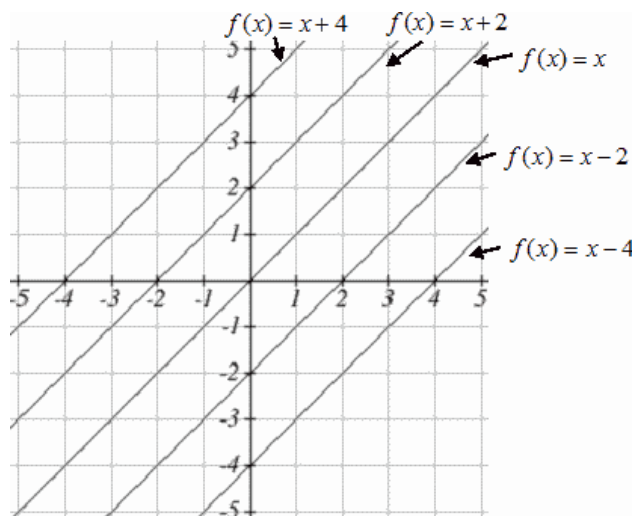

Figure 1.10

Another option for graphing is to use transformations of the identity function $f(x) = x$. In the equation $f(x) = mx$, the m acts as the vertical stretch of the identity function. When $m < 0$, there is also a vertical reflection of the graph. Looking at some examples will also help show the effect of slope on the shape of the graph.

In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Here are some examples.

Figure 1.11: $f(x) = mx$ for several values of m.

Figure 1.12: $f(x) = mx$ for several values of b.

Example 1.3.9. Match each function with one of the lines in the graph below.
Solution: Only one graph has a vertical intercept of -3, so we can immediately match that graph with $g(x)$. For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with $h(x)$. Of the other two, the steeper line would have a larger slope, so we can match that graph with $f(x)$, and the flatter line with $j(x)$.

■

   In addition to understanding the basic behavior of a linear function (increasing or decreasing, recognizing the slope and vertical intercept), it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

Finding Horizontal Intercepts

Definition 1.3.2. The **horizontal intercept** of the function is where the graph crosses the horizontal axis. If a function has a horizontal intercept, you can always find it by solving $f(x) = 0$ for x.
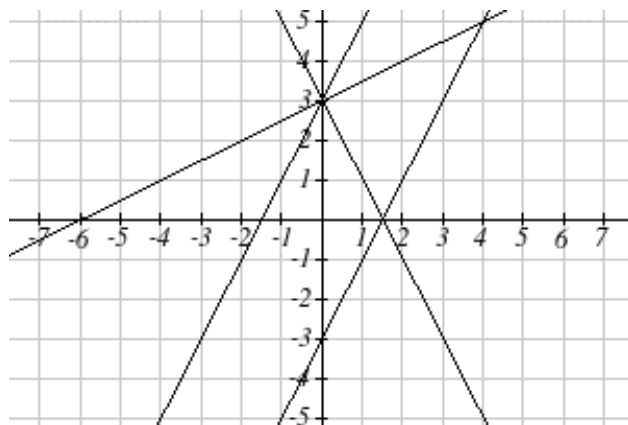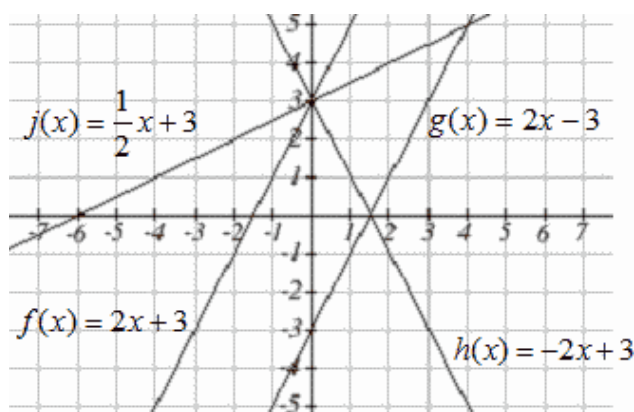
Figure 1.13

Figure 1.14

Example 1.3.10. Find the horizontal intercept of $f(x) = -3 + \frac{1}{2}x$

Solution: Setting the function equal to zero to find what input will put us on the horizontal axis,

$$0 = -3 + \frac{1}{2}x$$
$$3 = \frac{1}{2}x$$
$$x = 6 \ .$$

The graph crosses the horizontal axis at the point $(6, 0)$. ∎

Intersections of Lines The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

Economics tells us that in a free market, the price for an item is related to the quantity that producers will supply and the quantity that consumers will demand. Increases in prices will decrease demand, while supply tends to increase with prices. Sometimes supply and demand are modeled with linear functions.

Example 1.3.11. The supply, in thousands of items, for custom phone cases can be modeled by the equation $s(p) = 0.5 + 1.2p$ while the demand can be modeled by $d(p) = 8.70.7p$, where p is in dollars. Find the equilibrium price and quantity, the intersection of the supply and demand curves.

Solution: Setting $s(p) = d(p)$, we find

$$0.5 + 1.2p = 8.70.7p$$
$$1.9p = 8.2$$
$$p = \frac{8.2}{1.9} \approx 4.32$$

We can find the output value of the intersection point by evaluating either function at this input:

$$s(4.32) = 0.5 + 1.2(4.32) \approx 5.68 \ .$$

These lines intersect at the point $(4.32, 5.68)$. Therefore, the equilibrium price is \$4.32 and the equilibrium quantity is 5,680 items. Looking at the graph, this result seems reasonable.   ∎
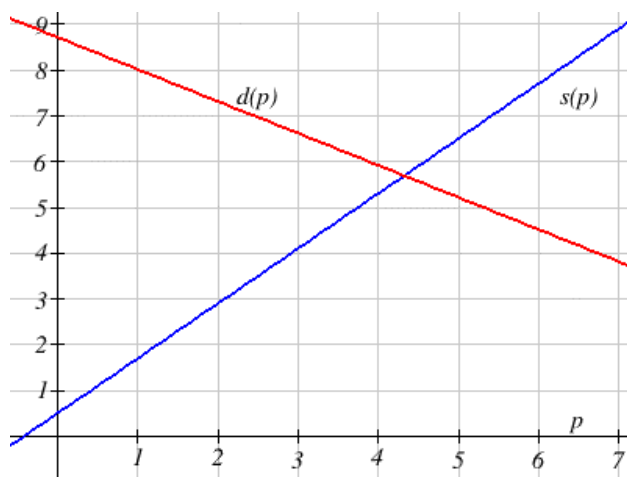

Figure 1.15

1.3.3   Modeling with Linear Functions

Here are a number of examples of modeling with linear functions.

Example 1.3.12. Emily saved up \$3500 for her summer visit to Seattle. She anticipates spending \$400 each week on rent, food, and fun. Find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

Solution: In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.

　　Output: M, money remaining, in dollars

　　Input: t, time, in weeks

Reading the problem, we identify two important values. The first, \$3500, is the initial value for M. The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

　　To answer the first question, looking for the horizontal intercept, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation: $M(t) = 3500 - 400t$.

To find the horizontal intercept, we set the output to zero, and solve for the input, t:

$$0 = 3500 - 400t$$
$$400t = 3500$$
$$t = \frac{3500}{400} = 8.75$$

The horizontal intercept is 8.75 weeks. Since this represents the input value where the output will be 0, interpreting this, we could say: "Emily will have no money left after 8.75 weeks."

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than 0. It is also likely that this model is not valid after the horizontal intercept (unless Emily's going to start using a credit card and go into debt).

The domain represents the set of input values and so the reasonable domain for this function is $0 \leq t \leq 8.75$.

However, in a real world scenario, the rental might be weekly or nightly. She may not be able to stay a partial week and so all options should be considered. Emily could stay in Seattle for 0 to 8 full weeks (and a couple of days), but would have to go into debt to stay 9 full weeks, so restricted to whole weeks, a reasonable domain without going in to debt would be $0 \leq t \leq 8$, or $0 \leq t \leq 9$ if she went into debt to finish out the last week.

The range represents the set of output values and she starts with \$3500 and ends with \$0 after 8.75 weeks so the corresponding range is $0 \leq M(t) \leq 3500$. If we limit the rental to whole weeks, however, the range would change. If she left after 8 weeks because she didn't have enough to stay for a full 9 weeks, she would have $M(8) = 3500 - 400 \cdot 8 = \$300$ left after 8 weeks, giving a range of $300 \leq M(t) \leq 3500$. If she wanted to stay the full 9 weeks she would be \$100 in debt giving a range of $-100 \leq M(t) \leq 3500$. ■

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

Example 1.3.13. Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of \$20, then 59 cents per mile. The second, Budget, charges an up-front fee of \$16, then 63 cents per mile.[2] When will U-Haul be the better choice for Jamal?

Solution: The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input: m, miles driven

Outputs:

$Y(m)$: cost, in dollars, for renting from U-Haul

$B(m)$: cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: \$0.59 per mile for U-Haul, and \$0.63 per mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where $Y(m)$ is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$
$$B(m) = 16 + 0.63m$$

---

[2]Rates retrieved Aug 2, 2010 from `http://www.budgettruck.com` and `http://www.uhaul.com/`.

These graphs are sketched to the right, with $Y(m)$ drawn dashed.

[INSERT GRAPH] To find the intersection, we set the equations equal to each other and solve for m.

$$Y(m) = B(m)$$
$$20 + 0.59m = 16 + 0.63m$$
$$4 = 0.04m$$
$$m = \frac{4}{0.04} = 100$$

This tells us that the cost from the two companies will be the same if you drive 100 miles. Either by looking at the graph, or noting that $Y(m)$ is growing at a slower rate, we can conclude that U-Haul would be the cheaper option if you drive more than 100 miles. ∎

Example 1.3.14. A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. If this trend continues,

a) Predict the population in 2013.

b) When will the population reach 15,000?

Solution: The two changing quantities are the population and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2004.

Input: t, years since 2004

Output: $P(t)$, the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2009 would correspond to $t = 5$, giving us the point $(5, 8100)$.

a) To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15000, we would need to solve for the input that would provide an output of 15000. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{8100 - 6200}{5 - 0} = \frac{1900}{5} = 380 \text{ people per year }.$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$P(t) = 6200 + 380t \ .$$

To predict the population in 2013, we evaluate our function at $t = 9$:

$$P(9) = 6200 + 380 \cdot 9 = 9620 \ .$$

If the trend continues, our model predicts a population of 9,620 in 2013.

b) To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$P(t) = 15000$$
$$6200 + 380t = 15000$$
$$380t = 8800$$
$$t = \frac{8800}{380} \approx 23.158$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

∎

1.3.4   Fitting Linear Models to Data

In the real world, things rarely follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use a linear model to make predictions about the data, inside and outside of the data range.

Example 1.3.15. The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit[3]. Plot this data, and determine whether the data appears to be linearly related.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temp. (°F)	80.5	70.5	57	66	68	72	52	73.5	53

Table 1.9: Cricket Chirps in 15 Seconds Versus Temperature.

Solution: Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so. We will plot this data in a spreadsheet. First, we put the data in a spreadsheet.

	A	B	C	D	E	F	G	H	I	J
1	**Chirps**	44	35	20.4	33	31	35	18.5	37	26
2	**Temp. (°F)**	80.5	70.5	57	66	68	72	52	73.5	53

Table 1.10

First we make a scatter plot. See Section 1.2.2 for the details on how we create a scatterplot in *LibreOffice Calc*.

Remark 1.3.2. The default scatter plot is on the left and a revised plot is on the right. Notice that the default plot includes a lot of unnecessary "white space," with a range of 0°F to 90°F. A more reasonable range is 50°F to 85°F. Within *Calc*, clicking on the *y*-axis will allow us to adjust the chart to make it more presentable. We can likewise trim the domain slightly.
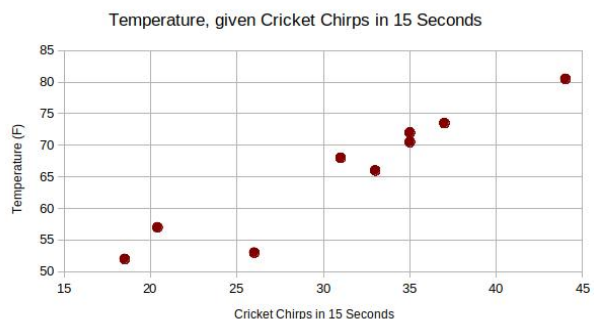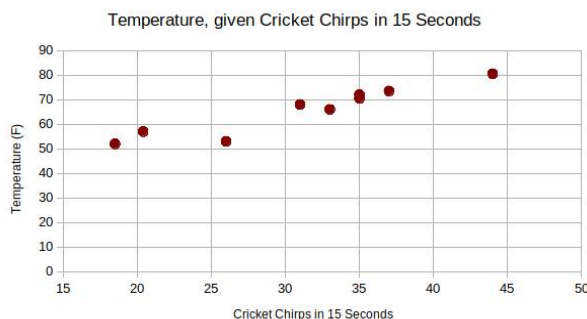
The simplest way to find an equation to approximate this data is to try to "eyeball" a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

    Questions to Consider:

---

[3]Selected data from `http://classic.globe.gov/fsl/scientistsblog/2007/10/`. Retrieved Aug 3, 2010

- What descriptive variables would you choose to represent temperature and chirps?

- Which variable is the independent variable and which is the dependent variable?

- Based on this data and the graph, what is a reasonable domain and range?

Example 1.3.16. Using the table of values from the previous example, find a linear function that fits the data by "eyeballing" a line that seems to fit.

Solution: On a graph, we could try sketching in a line. The figure below has a rough line that approximates the trend of the data.
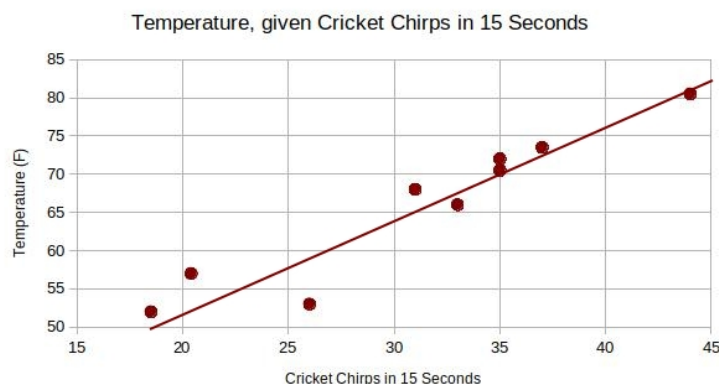
To find an equation of the line, we pick two points on the line. The best places to pick points are he intersection of grid lines, such as $(35, 70)$. Others would be points near extreme points of the line on a grid line, such as $(20, 52)$ and $(45, 82)$; 52 and 82 are estimates. Based on these latter two points, the line has a slope of $m = \dfrac{82 - 52}{45 - 20} = \dfrac{30}{25} = 1.2$. Now we find the vertical intercept at 30. Since the line has an equation of the form $y = 1.2x + b$ and one point has $x = 35$ and $y = 70$, we have

$$70 = 1.2 \cdot 35 + b$$
$$= 42 + b$$
$$b = 70 - 42 = 28 \quad .$$

This gives us the model

$$T(c) = 28 + 1.2c \ ,$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit.   ■

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. Recall the notions of interpolation and extrapolation.

Example 1.3.17.   a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

Solution:

a) With our cricket data, the number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so this would be a case of interpolation. Using our model:

$$T(28) = 28 + 1.2 \cdot 30 = 64°\text{F} \ .$$

Based on the data we have, this value seems reasonable.

b) The temperature values varied from 52°F to 80.5°F. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$40 = 28 + 1.2c$$
$$12 = 1.2c$$
$$c = \frac{12}{1.2} \approx 10$$

Our model predicts the crickets would chirp 10 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, crickets generally stop chirping altogether below around 50°F. Therefore, our prediction is likely unreasonable.

∎

1.3.5 Fitting Lines with a Spreadsheet.

In this section, we will use *LibreOffice Calc* to compute the **line of best fit** or **regression line** for various data sets.

Example 1.3.18. Find the least-squares regression line using the cricket chirp data from Table 1.10.
Solution: Double-click on the chart made from the data in Table 1.10.

- Click on a data point so that the points turn green.

- Select the menu item `Trend Line...`. This will bring up a regression curve wizard.

- Click on the `Type` tab.

- Make the selections given in Figure 1.16.

The new scatter plot in Figure 1.17 now has the linear regression line. By default, the regression line has far too much precision than we need for this context. Two decimal place of precision is enough. We can make that change by clicking on the equation for the line of best fit and going to the `Numbers` tab. Therefore, the best-fit line is

$$T(c) = 1.14c + 30.28 \text{ degrees Fahrenheit.}$$

This line is very similar to the equation we "eyeballed," but it fits the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$T(30) = 1.14 \cdot 30 + 30.28 = 64.48°F \ .$$

∎

Notice that the plot also includes another number: $R^2 = 0.90$. Most calculators and computer software will also provide you with this number, called the **coefficient of determination**, or the related **correlation coefficient**. These numbers measure how well a function models a set of data.

Definition 1.3.3 (Correlation Coefficient)**.** The **correlation coefficient** is a value, $-1 \leq r \leq 1$ that measures the strength of the relationship between two variables, x and y. $|r|$ measures the proportion of the output variable, y, can be explained by the input variable, x.

- $r > 0$ suggests a positive (increasing) relationship between x and y.

- $r < 0$ suggests a negative (decreasing) relationship between x and y.

- The closer r is to 0, the more scattered or **noncorrelated** the data.

- The closer r is to 1 or -1, the stronger the relationship the data is.

The **coefficient of determination** is the value $0 \leq r^2 \leq 1$, which measures the proportion of the variable y that is determined, or predicted, by x.
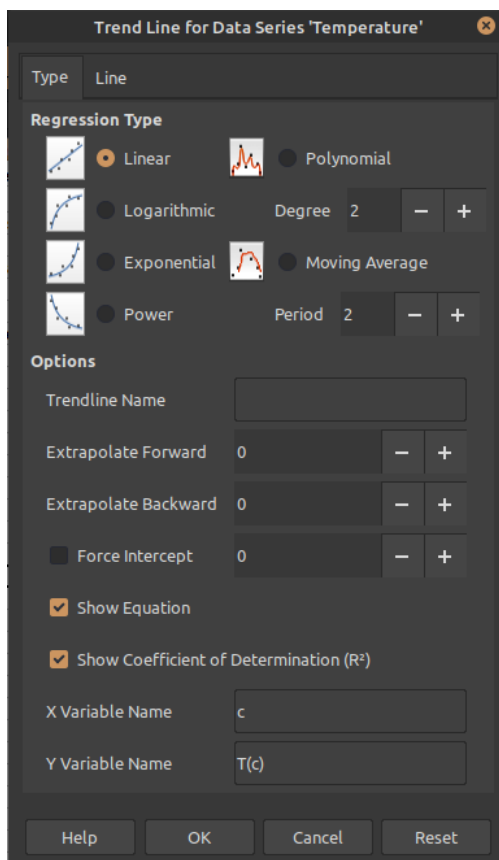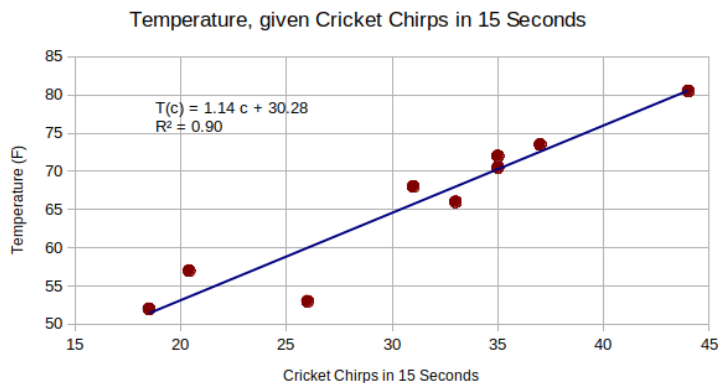
Figure 1.16

Figure 1.17

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of r and the graph of the data, here are some large data sets with their correlation coefficients: [Provide graph]

Example 1.3.19. The coefficient of determination of the cricket chirp data is $r^2 = 0.90$. Since the linear regression line has a positive slope, the correlation coefficient is $r = \sqrt{0.9} = 0.95$. This is a very strong relationship between a cricket's chirp rate and the temperature.   ∎

We can compute the correlation coefficient directly in *Calc*.

Example 1.3.20. Compute the correlation coefficient of the following data, repeated from Table 1.10.
Solution: In a new box, type the following command and hit `Enter`.

	A	B	C	D	E	F	G	H	I	J
1	**Chirps**	44	35	20.4	33	31	35	18.5	37	26
2	**Temp. (°F)**	80.5	70.5	57	66	68	72	52	73.5	53

```
=CORREL(B1:J1, B2:J2)
```

We find $r = 0.9509$.   ∎

Example 1.3.21. Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below.[4] Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Gasoline consumption (billion gallons)	113	116	118	119	123	125	128	126	131	133	136

Table 1.11: Gasoline consumption in the United States from 1994 to 2004.

Solution: To make things simpler, a new input variable is introduced, t, representing years since 1994. We make the following spreadsheet.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Year**	0	1	2	3	4	5	6	7	8	9	10
2	**Gas Consumption**	113	116	118	119	123	125	128	126	131	133	136

The following is a scatterplot of the data.
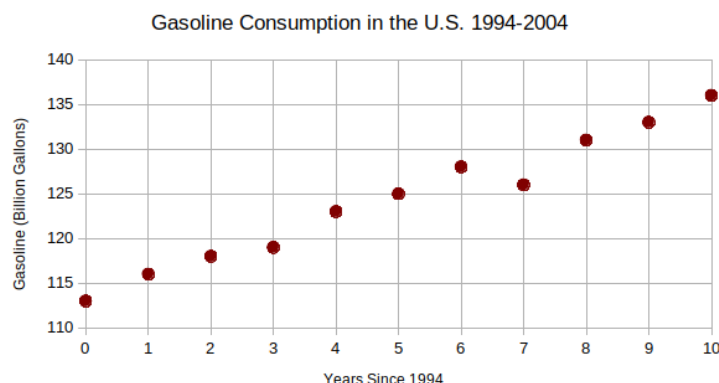
The data appears linear. Let's compute the correlation coefficient.
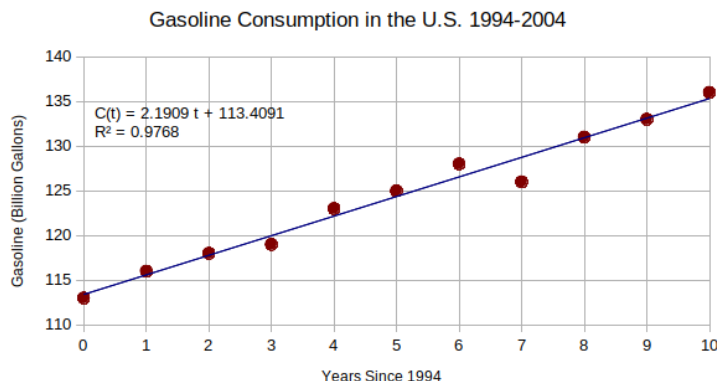
```
=CORREL(B1:L1, B2:L2)
```

We have $r = 0.9883$, suggesting a very strong increasing linear trend.

Let t be the number of years since 1994 and $C(t)$ be the volume of gasoline consumed in the United States, in billions of gallons. Then the least-squares regression equation is:

$$C(t) = 113.41 + 2.19t \ .$$

Here is the scatter plot with the best fit line.

---
[4]http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

Gasoline Consumption in the U.S. 1994-2004

$C(t) = 2.1909\,t + 113.4091$
$R^2 = 0.9768$

Gasoline (Billion Gallons)

Years Since 1994

Using this to predict gasoline consumption in 2008 ($t = 14$), we have

$$C(14) = 113.41 + 2.19 \cdot 14 = 144.07 \text{ billion gallons.}$$

■

1.3.6   Exercises

1. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation,for the population t years after 2003.

2. A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation,for the population t years after 2005.

3. Sonya is currently 10 miles from home, and is walking away from home at 3 miles per hour. Write an equation for her distance from home t hours from now.

4. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

5. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

6. At noon, a barista notices she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

1.4   Polynomial Functions

1.4.1   Quadratic Functions

Quadratics are transformations of the function $f(x) = x^2$. Quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 1.4.1. A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose three sides, and will put the fourth side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

Solution: In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the fourth side or backyard fence. Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L: $W = 80 - 2L$.
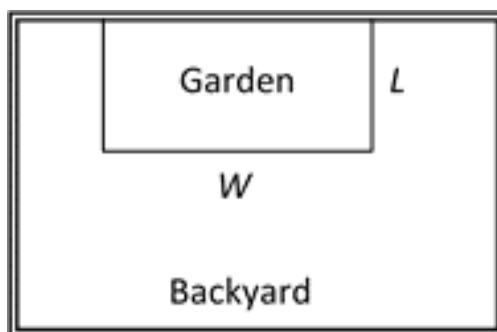
Figure 1.18

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so $A = LW = L(80 - 2L)$, so

$$A(L) = 80L - 2L^2 \ \ .$$

This formula represents the area of the fence in terms of the variable length L. ∎

Definition 1.4.1 (Forms of Quadratic Functions)**.** The **standard form** of a **quadratic function** is $f(x) = ax^2 + bx + c$.

The **transformation form** of a quadratic function is $f(x) = a(x - h)^2 + k$.

The **vertex** of the quadratic function is located at the point (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

Example 1.4.2. Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$.
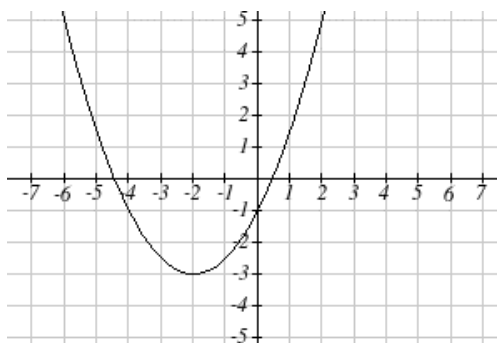

Figure 1.19

Solution: We can see the graph is the basic quadratic shifted to the left 2 and down 3, putting the vertex at the point $(-2, -3)$, giving a formula in the form $g(x) = a(x + 2)^2 - 3$. By plugging in a point that falls on the grid, such as $(0, -1)$, we can solve for the stretch factor:

$$-1 = a(0 + 2)^2 - 3$$
$$2 = 4a$$
$$a = \frac{1}{2}$$

The equation for this formula is

$$g(x) = \frac{1}{2}(x + 2)^2 - 3 \ \ .$$

∎

Short run Behavior: Intercepts As with any function, we can find the **vertical intercepts** of a quadratic by evaluating the function at an input of 0, and we can find the **horizontal intercepts** by solving for when the output will be 0. Notice that depending upon the location of the graph, we might have 0, one, or two horizontal intercepts.

(a) No horizontal intercepts   (b) One horizontal intercept   (c) Two horizontal intercepts
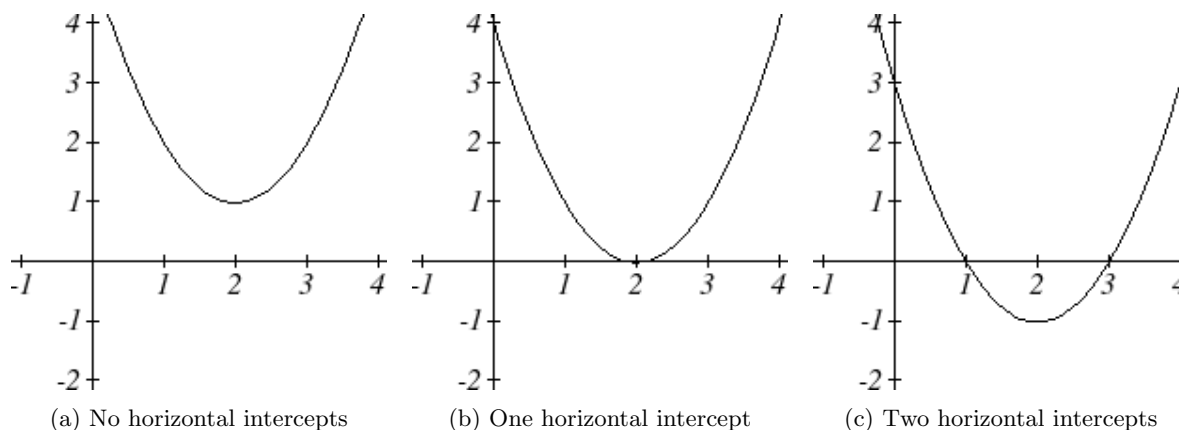
Notice that in the standard form of a quadratic, the constant term c reveals the vertical intercept of the graph, since $f(0) = a(0)2 + b(0) + c = c$.

Example 1.4.3. Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.
Solution: We can find the vertical intercept by evaluating the function at an input of 0:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2 \ .$$

So the vertical intercept is at the point $(0, -2)$.
For the horizontal intercepts, we find the 0es (or roots) of $f(x)$, solving $f(x) = 0$ for x:

$$0 = 3x^2 + 5x - 2 \ .$$

In this case, the quadratic can be factored easily, providing the simplest method for solution:

$$0 = (3x - 1)(x + 2) \ ,$$

so either

$$0 = 3x - 1$$
$$x = \frac{1}{3}$$

or

$$0 = x + 2$$
$$x = -2$$

So the Horizontal intercepts are at the points $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.   ∎

When a quadratic is not factorable or is hard to factor, we can turn to the quadratic formula.

Theorem 1.4.1 (Quadratic Formula). For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1.4.4. A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation

$$H(t) = -16t^2 + 80t + 40 \ .$$

When does the ball hit the ground?

Solution: To find when the ball hits the ground, we need to determine when the height is 0, i.e., when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. ∎

1.4.2 Polynomial Functions

Definition 1.4.2 (Terminology of Polynomial Functions)**.** A **polynomial** is a function that can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Each of the a_i constants are called **coefficients** and can be positive, negative, or 0, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any $a_i x_i$. Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots a_2 x^2 + a_1 x + a_0$$

Example 1.4.5. Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 3 + 2x^2 - 4x^3$.

Solution: The degree is 3, the highest power of x. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4. ∎

Short Run Behavior: Intercepts As with any function, the **vertical intercept** can be found by evaluating the function at an input of 0. Since this is evaluation, it is relatively easy to do it for a polynomial of any degree. To find **horizontal intercepts**, we need to solve for when the output will be 0. For general polynomials, this can be a challenging prospect. Consequently, we will limit ourselves to three cases:

- The polynomial can be factored using known methods: greatest common factor and trinomial factoring.

- The polynomial is given in factored form.

- Technology is used to determine the intercepts.

Example 1.4.6. Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution: We can attempt to factor this polynomial to find solutions for $f(x) = 0$:

$$x^6 - 3x^4 + 2x^2 = 0$$

$$x^2(x^4 - 3x^2 + 2) = 0 \qquad \text{Factor out the greatest common factor.}$$

$$x^2(x^2 - 1)(x^2 - 2) = 0 \qquad \text{Factor the inside as a quadratic in } x^2.$$

Then we break these factors apart to find all the solutions.

$$x^2 = 0 \qquad\qquad x^2 - 1 = 0 \qquad\qquad x^2 - 2 = 0$$

$$x = 0 \qquad\qquad x^2 = 1 \qquad\qquad x^2 = 2$$

$$x = 0 \qquad\qquad x = \pm 1 \qquad\qquad x = \pm\sqrt{2}$$

This gives us five horizontal intercepts: $x = 0, \pm 1, \pm\sqrt{2}$. ■

Example 1.4.7. Find the horizontal intercepts of $h(t) = t^3 + 4t^2 + t - 6$.

Solution: Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

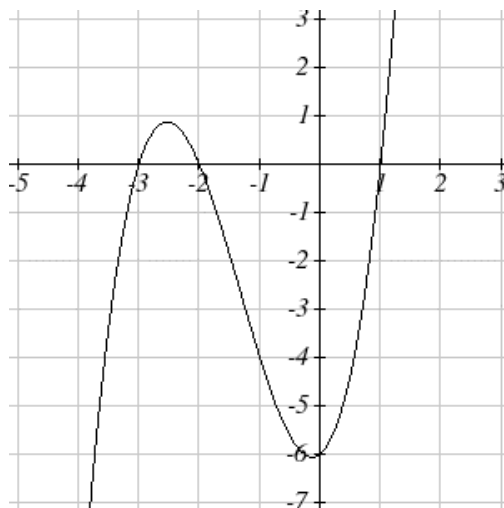Graphing this function, it appears there are horizontal intercepts at $t = -3, -2$, and 1.

Figure 1.21

We could check these are correct by plugging in these values for t and verifying that $h(-3) = h(-2) = h(1) = 0$. ■

Solving Polynomial Inequalities   One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative, and one we will use later in this course.

Example 1.4.8. Solve $(x + 3)(x + 1)^2(x - 4) > 0$.

Solution: As with all inequalities, we start by solving the equality $(x + 3)(x + 1)^2(x - 4) = 0$, which has solutions at $x = -3, -1$, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into four intervals.

We could choose a test value in each interval and evaluate the function $f(x) = (x + 3)(x + 1)^2(x - 4)$ at each test value to determine if the function is positive or negative in that interval:

On a number line this would look like:

From our test values, we can determine this function is positive when $x < -3$ or $x > 4$, or in interval notation, $(-\infty, -3) \cup (4, \infty)$. ■

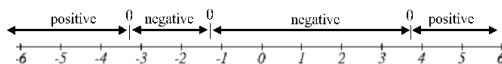| Interval	Test x in interval	$f(x)$	> 0 or < 0?
$x < -3$	-4	72	> 0
$-3 < x < -1$	-2	-6	< 0
$-1 < x < 4$	0	-12	< 0
$x > 4$	5	288	> 0

Figure 1.22

1.4.3   Rational Functions

Definition 1.4.3. A **rational function** is the ratio, or fraction, of two polynomials, $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p}{b_0 + b_1 x + b_2 x^2 + \cdots + b_q x^q}$$

.

Rational functions can arise from both simple and complex situations.

Example 1.4.9. You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.
Solution: You may recall that multiplying speed by time will give you distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which we drive, then $vt =$ distance. Since our distance is fixed at 100 miles, $vt = 100$. Solving this relationship for time gives us the function we desired:

$$t(v) = \frac{100}{v} \quad .$$

■

Notice that this is a transformation of the reciprocal toolkit function, $f(x) = \dfrac{1}{x}$. Several natural phenomena, such as gravitational force and volume of sound, behave in a manner inversely proportional to the square of another quantity. For example, the volume, V, of a sound heard at a distance d from the source would be related by $V = \dfrac{k}{d^2}$ for some constant value k. These functions are transformations of the reciprocal squared toolkit function $f(x) = \dfrac{1}{x^2}$.

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our review of toolkit functions. These graphs have several important features.

Let's begin by looking at the reciprocal function, $f(x) = \dfrac{1}{x}$. As you well know, dividing by 0 is not allowed and therefore 0 is not in the domain, and so the function is undefined at an input of 0.

Short Run Behavior.   As the input values approach 0 from the left side (taking on very small, negative values), the function values become very large in the negative direction (in other words, they approach negative infinity). We write: $x \to 0^-$, $f(x) \to -\infty$.

As we approach 0 from the right side (small, positive input values), the function values become very large in the positive direction (approaching infinity). We write: as $x \to 0^+$, $f(x) \to \infty$.

This behavior creates a vertical asymptote. An asymptote is a line that the graph approaches. In this case the graph is approaching the vertical line $x = 0$ as the input becomes close to 0.

Long Run Behavior.   As the values of x approach infinity, the function values approach 0. Also, as the values of x approach negative infinity, the function values approach 0. Symbolically: as $x \to \pm\infty$, $f(x) \to 0$.
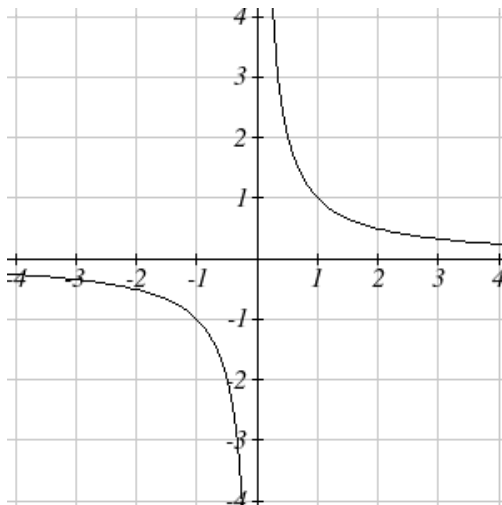
Figure 1.23: $f(x) = \dfrac{1}{x}$
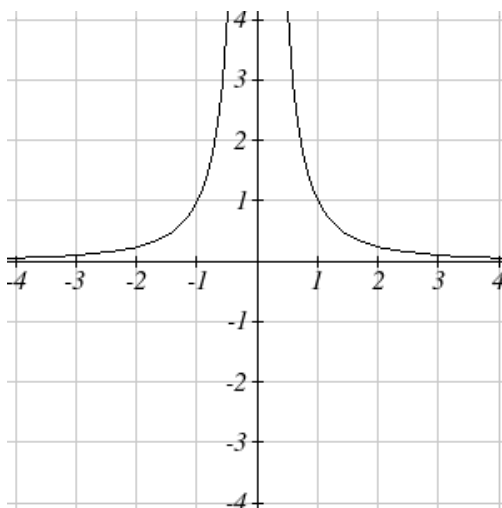
Figure 1.24: $f(x) = \dfrac{1}{x^2}$

Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a horizontal asymptote. In this case the graph is approaching the horizontal line $f(x) = 0$ as the input becomes very large in the negative and positive directions.

Definition 1.4.4 (Vertical and Horizontal Asymptotes)**.** A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach a. As $x \to a$, $f(x) \to \pm\infty$.

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs get large. As $x \to \pm\infty$, $f(x) \to b$.

Example 1.4.10. Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution: Transforming the graph left 2 and up 3 would result in the function $f(x) = \dfrac{1}{x+2} + 3$, or

equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x + 7}{x + 2} \ .$$

Shifting the toolkit function would give us this graph. Notice that this equation is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$. As $x \to -2^-$, $f(x) \to -\infty$, and as $x \to -2^+$, $f(x) \to \infty$.
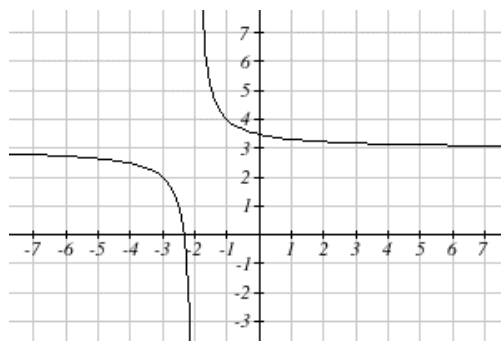

Figure 1.25

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$. As $x \to \pm\infty$, $f(x) \to 3$ Notice that horizontal and vertical asymptotes get shifted left 2 and up 3 along with the function.   ∎

Example 1.4.11. A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.
Solution: Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water} = 100 + 10t \qquad \text{sugar} = 5 + 1t \ .$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water:

$$C(t) = \frac{5 + t}{100 + 10t} \ .$$

                                                                                            ∎

Vertical and Horizontal Asymptotes of Rational Functions.   The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to 0 and the numerator is not 0.

The **horizontal asymptote** of a rational function, $\frac{P(x)}{Q(x)}$ can be determined by looking at the degrees of the numerator, $P(x)$, and the denominator, $Q(x)$.

- If $\deg(Q) > \deg(P)$, then the horizontal asymptote is $y = 0$.

- If $\deg(Q) < \deg(P)$, then there is no horizontal asymptote.

- If $\deg(Q) = \deg(P)$, then the horizontal asymptote is $y = \frac{a_p}{b_q}$ (p and q are equal in this case).

Example 1.4.12. In the sugar concentration problem from earlier, we created the equation $C(t) = \frac{5 + t}{100 + 10t}$. Find the horizontal asymptote and interpret it in context of the scenario.

Solution: Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: as $t \to \infty$, $C(t) \to \dfrac{1}{10}$. This function will have a horizontal asymptote of $y = \dfrac{1}{10}$.

This tells us that as the input gets large, the output values will approach $\dfrac{1}{10}$. In context, this means that as more time goes by, the concentration of sugar in the tank will approach 0.1 lb. of sugar per gallon of water or $\dfrac{1}{10}$ pounds per gallon.   ∎

Example 1.4.13. Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \ .$$

Solution: First, note this function has no inputs that make both the numerator and denominator 0, so there are no potential holes. The function will have vertical asymptotes when the denominator is 0, causing the function to be undefined. The denominator will be 0 at $x = 1, -2$, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards 0 as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote of $y = 0$.   ∎

As with all functions, a rational function will have a vertical intercept when the input is 0, if the function is defined at 0. It is possible for a rational function to not have a vertical intercept if the function is undefined at 0.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be 0 (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to 0 when the numerator is 0, horizontal intercepts will occur when the numerator of the rational function is equal to 0.

Example 1.4.14. Find the intercepts of

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \ .$$

Solution: We can find the vertical intercept by evaluating the function at 0:

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = \frac{-3}{5} \ .$$

The horizontal intercepts will occur when the function is equal to 0:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \qquad \text{(This is zero when the numerator is zero.)}$$
$$0 = (x-2)(x+3)$$
$$x = 2, -3.$$

∎

1.5   Exponential Functions

1.5.1   Laws of Exponents

The Laws of Exponents let you rewrite algebraic expressions that involve exponents. The last three listed here are really definitions rather than rules.

Theorem 1.5.1 (Laws of Exponents). All variables here represent real numbers and all variables in denominators are nonzero.

1. $x^a \cdot x^b = x^{a+b}$

2. $\dfrac{x^a}{x^b} = x^{a-b}$

3. $(x^a)^b = x^{ab}$

4. $(xy)^a = x^a y^a$

5. $\left(\dfrac{x}{y}\right)^b = \dfrac{x^b}{y^b}$

6. $x^0 = 1$, provided $x \neq 0$, although in some contexts, $0^0 = 1$.

7. $x^{-n} = \dfrac{1}{x^n}$, provided $x \neq 0$.

8. $x^{1/n} = \sqrt[n]{x}$, provided $x \neq 0$.

Example 1.5.1. Simplify $\left(2x^2\right)^3 (4x)$.

Solution: We'll begin by simplifying the $\left(2x^2\right)^3$ portion. Using Property 4, we can write

$$
\begin{aligned}
\left(2x^2\right)^3 &= 2^3 \left(x^2\right)^3 (4x) && \text{Use Property 4.} \\
&= 8x^6(4x) && \text{Evaluate } 2^3 = 8, \text{ and use Property 3.} \\
&= 32x^7 && \text{Multiply the constants, and use Property 1, recalling } x = x^1.
\end{aligned}
$$

Being able to work with negative and fractional exponents will be very important later in this course.

Example 1.5.2. Rewrite $\dfrac{5}{x^3}$ using negative exponents.

Solution: Since $x^{-n} = \dfrac{1}{x^n}$, then $x^{-3} = \dfrac{1}{x^3}$ and thus

$$\frac{5}{x^3} = 5x^{-3}.$$

Example 1.5.3. Simplify $\left(\dfrac{x^{-2}}{y^{-3}}\right)^2$ as much as possible and write your answer using only positive exponents.

Solution:

$$
\begin{aligned}
\left(\frac{x^{-2}}{y^{-3}}\right)^2 &= \frac{\left(x^{-2}\right)^2}{\left(y^{-3}\right)^2} \\
&= \frac{x^{-4}}{y^{-6}} \\
&= \frac{y^6}{x^4}
\end{aligned}
$$

Example 1.5.4. Rewrite $\left(\sqrt{p^5}\right)^{-1/3}$ using exponents.

Solution: A square root is a radical with index of two. In other words, $\sqrt{x} = \sqrt[2]{x}$. Using the exponent rule above, $\sqrt{x} = \sqrt[2]{x} = x^{1/2}$. Rewriting the square roots using the fractional exponent,

$$4\sqrt{x} - \frac{3}{\sqrt{x}} = 4x^{1/2} - \frac{3}{x^{1/2}}.$$

Now we can use the negative exponent rule to rewrite the second term in the expression:

$$4x^{1/2} - \frac{3}{x^{1/2}} = 4x^{1/2} - 3x^{-1/2}.$$

Example 1.5.5. Rewrite $\left(\sqrt{p^5}\right)^{-1/3}$ using only positive exponents.
Solution:

$$\left(\sqrt{p^5}\right)^{-1/3} = \left(\left(p^5\right)^{1/2}\right)^{-1/3}$$
$$= p^{-5/6}$$
$$= \frac{1}{p^{5/6}}$$

Example 1.5.6. Rewrite $x^{-4/3}$ as a radical.
Solution:

$$x^{-4/3} = \frac{1}{x^{4/3}}$$
$$= \frac{1}{\left(x^{1/3}\right)^4} \quad \left(\text{since } \frac{4}{3} = 4 \cdot \frac{1}{3}\right)$$
$$= \frac{1}{\left(\sqrt[3]{x}\right)^4} \quad \text{(using the radical equivalence)}$$

1.5.2   Exponential Models

Consider these two companies:

- Company A has 100 stores, and expands by opening 50 new stores a year

- Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Company A is exhibiting linear growth. In linear growth, we have a constant rate of change – a constant number that the output increased for each increase in input. For company A, the number of new stores per year is the same each year.

Company B is different – we have a percent rate of change rather than a constant number of stores/year as our rate of change. To see the significance of this difference compare a 50% increase when there are 100 stores to a 50% increase when there are 1000 stores:

- 100 stores, a 50% increase is 50 stores in that year.

- 1000 stores, a 50% increase is 500 stores in that year.

Calculating the number of stores after several years, we can clearly see the difference in results.

Years	Company A	Company B
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767

This percent growth can be modeled with an **exponential function**.

Definition 1.5.1 (Exponential Function). An **exponential growth** or **decay** function is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$f(x) = a \cdot (1 + r)^x$$

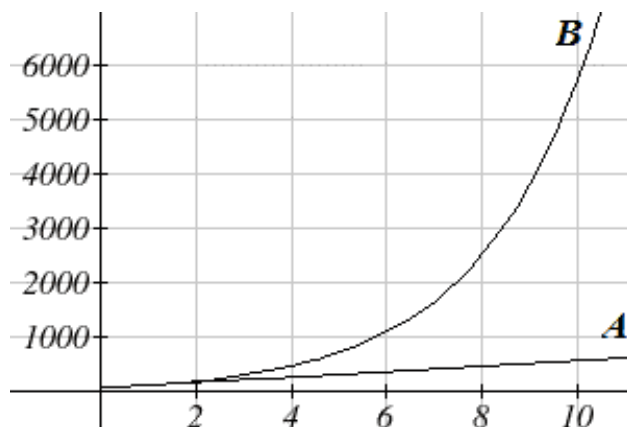or

$$f(x) = a \cdot b^x \ ,$$

where

Figure 1.26: Graphs of data from A and B, with B fit to a curve.

a is the **initial or starting value** of the function,

r is the **percent growth or decay rate**, written as a decimal,

b is the **growth factor** or **growth multiplier**: $b = 1 + r$.

Since powers of negative numbers behave strangely, we must have $b > 0$.

Example 1.5.7. India's population was 1.14 billion in the year 2008 and is growing by about 1.34% each year. Write an exponential function for India's population, and use it to predict the population in 2020.
Solution: Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for r is 0.0134.
   Using the basic formula for exponential growth $f(x) = a(1 + r)^x x$, we can write the formula,

$$f(t) = 1.14(1 + 0.0134)^t$$

To estimate the population in 2020, we evaluate the function at $t = 12$, since 2020 is 12 years after 2008:

$$f(t) = 1.14(1 + 0.0134)^{12} \approx 1.337 \text{ billion people in 2020.}$$

Example 1.5.8. A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a $1000 investment grow to over those 24 months? What is the equivalent annual percentage yield (APY)?
Solution: First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly. To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year: $\frac{1.2\%}{12} = 0.1\%$. Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of $1000 and a growth rate of $r = 0.001$, and our input m measured in months:

$$f(m) = 1000 \left(1 + \frac{0.012}{12}\right)^m = 1000(1.001)^m \ .$$

After 24 months, the account will have grown to $f(24) = 1000(1.001)^{24} \approx \$1024.28$.
   The annual percentage yield (APY) is the interest rate that is equivalent to 1.2% compounded quarterly if we were to compound annually instead. So we will calculate how much our investment grows in one year (expressed as a percentage of the original amount), which means using $m = 12$ months:

$$f(12) = 1000(1.001)^{12} \approx \$1012.07 \ ,$$

so we have

$$\frac{\text{Final value}}{\text{Initial value}} = \frac{\$1012.07}{\$1000} = 1.01207$$

or 101.207% of what we started with, which is a 1.207% gain. Thus our APY is 1.207%. This answer seems reasonable since when compounding quarterly we earn interest on interest from the previous quarters, so to compensate for the fact that we are not compounding as often we need a slightly higher rate to earn an equal amount.

Example 1.5.9. Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (Polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?
Solution: With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is $a = 100$ mg, and our growth rate will be negative 13%, since we are decreasing: $r = -0.13$. This gives the equation

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d \ .$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be $Q(7) = 100(0.87)^7 = 37.73$ mg of Bismuth-210.

Example 1.5.10. $T(q)$ represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010. Interpret all of the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$.
Solution: Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$ means that in the second quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.

When working with exponentials, there is a special constant we must talk about. It arises when we talk about things growing continuously, such as continuous compounding, or natural phenomena like radioactive decay that happen continuously.

Definition 1.5.2 (Euler's Number: e).
$$e \approx 2.718282$$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function $\exp(x)$, where $\exp(x) = e^x$. Since calculus studies continuous change, we will almost always use the e-based form of exponential equations in this course.

Definition 1.5.3 (Continuous Growth Formula). **Continuous growth** can be calculated using the formula

$$f(x) = a \cdot e^{rx} \ ,$$

where

  a is the **starting amount** or **initial quantity**,

  r is the **continuous growth rate**.

Example 1.5.11. Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?
Solution: Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: $r = -0.173$, $f(3) = 100e^{-0.173 \cdot 3} \approx 59.512$ mg of Radon-222 will remain.

1.5.3   Graphs of Exponential Functions

Theorem 1.5.2 (Graphical Features of Exponential Functions). In the graph of the function $f(x) = a \cdot b^x$, we have the following.

- a is the **vertical intercept** of the graph.

- b determines the **rate** at which the graph grows:

  - the function will **increase** if $b > 1$,
  - the function will **decrease** if $0 < b < 1$.

- The graph will have a **horizontal asymptote** at $y = 0$.

- The graph will be:

  - **concave up** if $a > 0$
  - **concave down** if $a < 0$.

- The **domain** of the function is all real numbers, $\mathbb{R}$.

- The **range** of the function is $(0, \infty)$ if $a > 0$, and $(-\infty, 0)$ if $a < 0$.

When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points $(0, a)$ and $(1, ab)$.

Theorem 1.5.3. The value b will determine the function's long run behavior. The notation $x \to \infty$ means "as the value of x grows (without bound) to infinity (∞)," or, more briefly, "as x goes to ∞."

- If $b > 1$:

  - As $x \to \infty$, $f(x) \to \infty$.
  - As $x \to -\infty$, $f(x) \to 0$.

- If $0 < b < 1$:

  - As $x \to \infty$, $f(x) \to 0$.
  - As $x \to -\infty$, $f(x) \to \infty$.

Example 1.5.12. Sketch a graph of $f(x) = 4 \cdot \left(\frac{1}{3}\right)^x$.
Solution: This graph will have a vertical intercept at the point $(0, 4)$, and passes through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the graph will be decreasing towards zero. Since $a > 0$, the graph will be concave up.
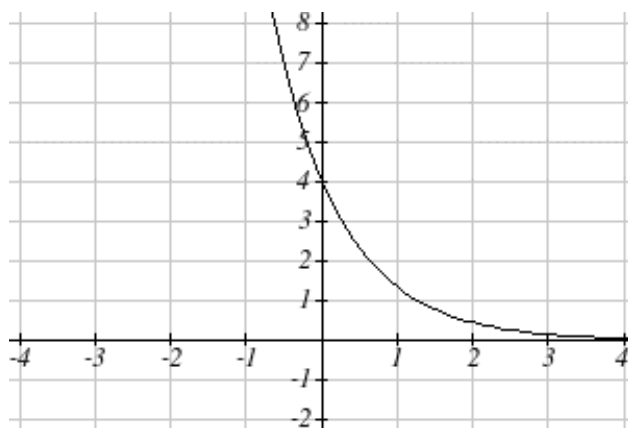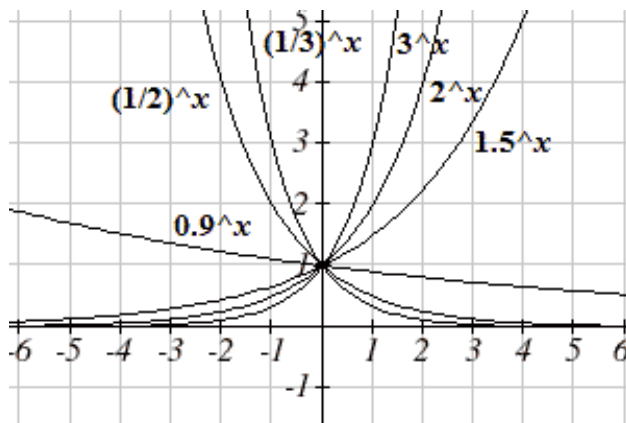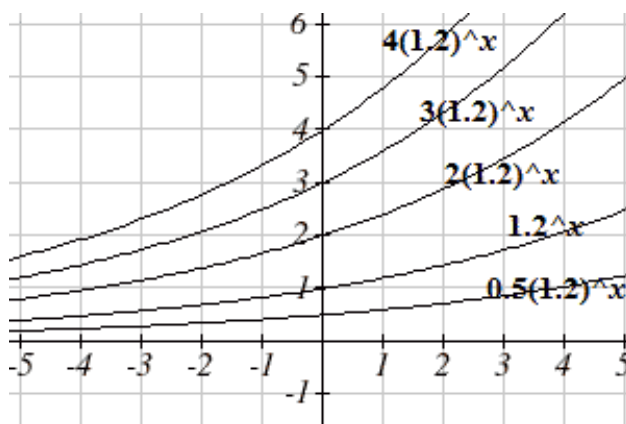

Figure 1.27

We can also see from the graph the long run behavior: as $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to \infty$.

Figure 1.28: Changing the value of b.

Figure 1.29: Changing the value of a.

To get a better feeling for the effect of the coefficients a and b on the graph, examine the sets of graphs below. The first set shows various graphs, where a remains the same and we only change the value for b. Notice that the closer the value of b is to 1, the less steep the graph will be.

In the next set of graphs, a is altered and our value for b remains the same.

Notice that changing the value for a changes the vertical intercept. Since a is multiplying the b^x term, a acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these a values introduced a vertical flip.

Example 1.5.13. Match each equation with its graph.

$$f(x) = 2(1.3)^x$$

$$g(x) = 2(1.8)^x$$

$$h(x) = 4(1.3)^x$$

$$k(x) = 4(0.7)^x$$

Solution: The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor less than 1, which will produce a decreasing graph. The graph of $h(x)$ can be identified as the only growing exponential function with a vertical intercept at the point $(0, 4)$. The graphs of $f(x)$ and $g(x)$ both have a vertical intercept at $(0, 2)$, but since $g(x)$ has a larger growth factor, we can identify it as the graph that is increasing faster.
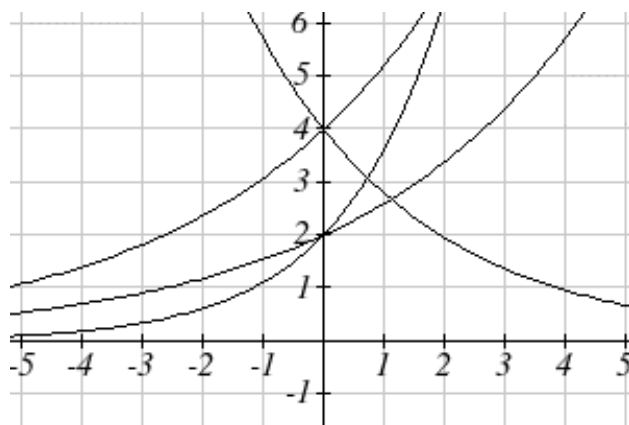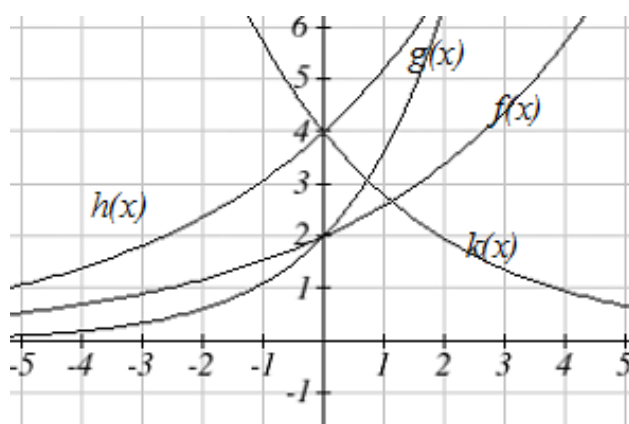
Figure 1.30

Figure 1.31

1.5.4 Exercises

Simplify each expression.

1. $x^3 x^5$

2. $x^4 x^2$

3. $(x^3)^4$

4. $(x^7)^2$

5. $(2x^2)^3 x^4$

6. $(5x^4)^2 x^5$

7. $\frac{(3x^2)^2}{6x^3}$

8. $\frac{5x(4x)^2}{2x^2}$

Simplify, and rewrite without negative exponents.

9. $4x^{-3}$

10. $2x^{-5}$

11. $x^{-4}x^2$

12. $x^{-2}x$

13. $\frac{5x^{-3}}{2x^{-6}}$

14. $\frac{2x^{-4}}{6x^{-2}}$

Rewrite using negative or fractional exponents.

15. $\frac{4}{x^5}$

16. $\frac{4}{x^3}$

17. $3\sqrt{x}$

18. $\sqrt[4]{x}$

19. $\frac{4}{\sqrt[3]{x}}$

20. $\frac{1}{5\sqrt{x}}$

Rewrite as a radical.

21. $4x^{-1/2}$

22. $5x^{-1/3}$

23. $2x^{1/3}$

24. $5x^{3/2}$

25. A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

26. A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

27. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

28. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

29. If $4,000 is invested in a bank account at an interest rate of 7 per cent per year, find the amount in the bank after 9 years if interest is compounded annually, quarterly, monthly, and continuously.

30. If $6,000 is invested in a bank account at an interest rate of 9 per cent per year, find the amount in the bank after 5 years if interest is compounded annually, quarterly, monthly, and continuously. Match each function with one of the graphs below.

31. $f(x) = 2(0.69)^x$

32. $f(x) = 2(1.28)^x$

33. $f(x) = 2(0.81)^x$

34. $f(x) = 4(1.28)^x$

35. $f(x) = 2(1.59)^x$

36. $f(x) = 4(0.69)^x$

If all the graphs to the right have equations with form $f(t) = a \cdot e^{kt}$,

37. Which graph has the largest value for k?

38. Which graph has the smallest value for k?

39. Which graph has the largest value for a?

40. Which graph has the smallest value for a?

1.6 Logistic Functions

Definition 1.6.1 (Logistic Function). A **logistic function** is a function of the form

$$f(x) = \frac{L}{1 + ae^{bx}} \ .$$

1.7 Logarithmic Functions

Logarithms are the inverse of exponential functions they allow us to undo exponential functions and solve for the exponent. They are also commonly used to express quantities that vary widely in size.

Definition 1.7.1 (Logarithms and Exponentials). The **logarithm function** of **base** b, written $\log_b(x)$, is the inverse of the **exponential function** (base b), b^x.

This means the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Theorem 1.7.1 (Properties of Logarithms: Inverse Properties). If $b > 0$ and $b \neq 1$, then:

- $\log_b(b^x) = x$ for all $x \in \mathbb{R}$;

- $b^{\log_b(x)} = x$ for all $x > 0$.

Example 1.7.1. Write these exponential equations as logarithmic equations:

a) $2^3 = 8$

b) $5^2 = 25$

c) $10^{-4} = \dfrac{1}{10000}$

Solution:

a) $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b) $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c) $10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$.

■

Example 1.7.2. Solve $2^x = 10$ for x.
Solution: By rewriting this expression as a logarithm, we get $x = \log_2(10)$. ■

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always usefuloften we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as well see briefly.

Definition 1.7.2 (Common and Natural Logarithms). These are by far the most frequently used logarithms.

- The **common logarithm** is the logarithm with base 10, and is typically written $\log(x)$.

- The **natural logarithm** is the logarithm with base e, and is typically written $\ln(x)$.

Example 1.7.3. Evaluate $\log(1000)$ using the definition of the common logarithm.
Solution:
$$10^x = 1000 \ .$$
From this, we might recognize that 1000 is the cube of 10, i.e., 10^3, so $x = 3$.

We also can use the inverse property of logarithms to write $\log_{10}(10^3) = 3$.   ∎

Table 1.12: Values of the common logarithm

x	x as an exponential	$\log(x)$
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Example 1.7.4. Evaluate $\log(500)$ using your calculator or computer.
Solution: Using a computer or calculator, we can evaluate and find that $\log(500) \approx 2.69897$.   ∎

Another property provides the basis for solving exponential equations.

Theorem 1.7.2 (Properties of Logarithms: Exponent Property)**.**
$$\log_b(A^r) = r \log_b(A)$$

Remark 1.7.1 (Solving exponential equations:)**.** Here are steps to help you solve exponential equations by hand.

- Isolate the exponential expressions when possible.

- Take the logarithm of both sides.

- Utilize the exponent property for logarithms to pull the variable out of the exponent.

- Use algebra to solve for the variable.

Example 1.7.5. In the last section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1 + 0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?
Solution: We need to solve for t so that $f(t) = 2$.

$$2 = 1.14(1.0134)^t \qquad\qquad \text{Initial equation.}$$

$$\frac{2}{1.14} = 1.0134^t \qquad\qquad \text{Divide by 1.14 to isolate the exponential expression.}$$

$$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right) \qquad\qquad \text{Take the logarithm of both sides of the equation.}$$

$$\ln\left(\frac{2}{1.14}\right) = t \cdot \ln(1.0134) \qquad\qquad \text{Apply the exponent property on the right side.}$$

$$t = \frac{\ln\left(\dfrac{2}{1.14}\right)}{\ln(1.0134)} \qquad\qquad \text{Divide both sides by } \ln(1.0134).$$

$$t \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.   ∎

Example 1.7.6. Solve $5e^{-0.3t} = 2$ for t.
Solution: First we divide by 5 to isolate the exponential:

$$e^{-0.3t} = \frac{2}{5} \ .$$

Since this equation involves e, it makes sense to use the natural logarithm:

$$\ln\left(e^{-0.3t}\right) = \ln\left(\frac{2}{5}\right) \qquad \text{Take the natural logarithm of both sides.}$$

$$-0.3t = \ln\left(\frac{2}{5}\right) \qquad \text{Utilize the inverse property of logarithms.}$$

$$t = \frac{\ln\left(\frac{2}{5}\right)}{-0.3} \qquad \text{Now divide by } -0.3.$$

$$t \approx 3.054$$

∎

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 1.7.7. In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation

$$pH = -\log([H^+])$$

If a liquid has concentration of 0.0001 moles per liter, determine the pH. Determine the hydrogen ion concentration of a liquid with pH of 7.
Solution: To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logarithms:

$$-\log(0.0001) = -\log(10^{-4}) = -(-4) = 4 \ .$$

To answer the second question, we need to solve the equation $7 = -\log([H^+])$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1: $-7 = \log([H^+])$. Rewriting into exponential form yields the answer

$$[H^+]^{-7} = 0.0000001 \text{ moles per liter.}$$

∎

While we don't often need to sketch the graph of a logarithm, it is helpful to understand the basic shape.

Theorem 1.7.3 (Graphical Features of the Logarithm). Graphically, given the function $g(x) = \log_b(x)$:

- The graph has a horizontal intercept at the point $(1,0)$.

- The graph has a vertical asymptote at $x = 0$.

- The graph is increasing and concave down.

- The domain of the function is $x > 0$, or $(0, \infty)$ in interval notation.
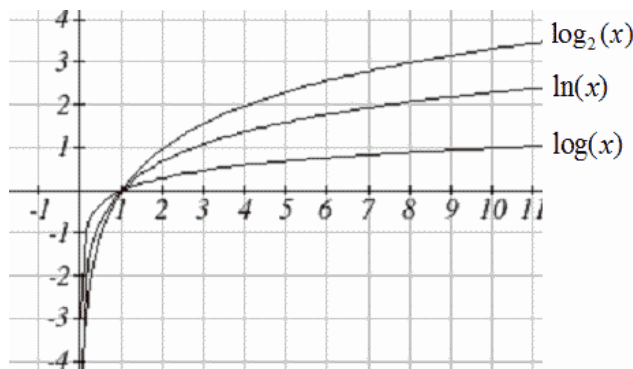
Figure 1.32: Graphs of the Binary, Natural, and Common Logarithms

- The range of the function is all real numbers ($\mathbb{R}$) or $(-\infty, \infty)$ in interval notation.

When sketching a general logarithm with base b, it can be helpful to remember that the graph will pass through the points $\left(\frac{1}{b}, -1\right)$, $(1, 0)$, and $(b, 1)$.

To get a feeling for how the base affects the shape of the graph, examine the graphs below:

Another important observation made was the domain of the logarithm: $x > 0$. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a logarithm.

Example 1.7.8. Find the domain of the function $f(x) = \log(5 - 2x)$.
Solution: The logarithm is only defined when the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality, $-2x > -5$, so $x < \frac{5}{2}$.

The domain of this function is $x < \frac{5}{2}$, or, in interval notation, $\left(-\infty, \frac{5}{2}\right)$.   ■

1.8   Trigonometric Functions

Most applied calculus texts skip **trigonometric** functions, but they are extremely powerful for modeling periodic or seasonal data. We will keep the treatment of these models very basic and intuitive, but independent from the rest of the text; it is optional. For the scientist, trigonometric functions are indispensable. However, for our purposes, we will focus our attention exclusively on two related functions: the **sine** and **cosine** functions.

1.8.1   Definitions

A **sine wave** is of the form

$$f(x) = v + a \cdot \sin\left(\frac{2\pi(x - h)}{p}\right) \ . \tag{1.2}$$

where v is the **vertical shift** (mean of the data), a is the **amplitude** of the wave (roughly the difference between the maximum and the mean of the data), h is the **horizontal shift** (where the data is at the mean and begins to rise to the maximum), and p is the **period**.

1.8.2   Modeling with Sine Waves

Consider the following table that we saw in Sections 1.2 and 1.4 on the fuel oil usage of Plant W in 2016. One task that we were unable to complete before was to determine a model that would reflect the seasonality of the data. In other words, since the data represents a calendar year, we want a model that will oscillate and finish at the same output value as the output value of the beginning point. A **trigonometric model** serves that very purpose.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Oil (gal)	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0	??	638.0

Table 1.13: Fuel Oil Usage at Plant W in 2016

Given our data, we don't know the vertical shift, the amplitude, or the horizontal shift, but we do know the period. Since the input is in months and there are 12 months in a year, we choose $p = 12$ from 1.2. We then fit a sine wave to the data and get the following model.

If m is the mth month of the year, then Plant W burned oil at a rate of

$$f(m) = 492.82 - 334.33 \sin\left(\frac{2\pi(m + 29.49)}{12}\right) \text{ gallons per month}$$

in 2016.

Graphing $y = f(m)$ with the scatter plot of Table 1.13 gives us the following plot to verify that the model makes sense.
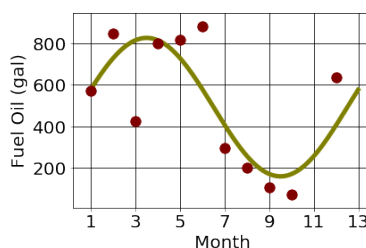
Figure 1.33: Fuel Oil Usage by Plant W in 2016 Fitted by a Sine Wave.

1.9   Combining Functions

Chapter 2

The Derivative: Rates of Change of a Function

Chapter 3

Applications of Derivatives

Chapter 4

Functions of Many Variables

Chapter 5

The Integral: Accumulation of a Rate Function

The previous chapters dealt with Differential Calculus. We started with the simple geometrical idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter deals with Integral Calculus and starts with the simple geometric idea of area. This idea will be developed into another combination of theory, techniques, and applications.

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