Open Applied Calculus

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December 29, 2021

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First release, August 2019

Acknowledgements

I gratefully acknowledge support for the development of this book from the following supporters.

- Pennsylvania State System of Higher Education (PASSHE) Faculty Professional Development Committee Grant
- Dan Stafford, OER Librarian, Rohrbach Library, Kutztown University

I am grateful to the following who have contributed to the development of the text.

- Dr. Eric Bancroft, *Grove City College.*

I am grateful to the following reviewers for their many helpful suggestions and reviews.

- Dr. Carey Childers, *Clarion University of Pennsylvania.*
- Dr. Stephen Gendler, *Clarion University of Pennsylvania.*
- Dr. Kate Overmoyer, *Clarion University of Pennsylvania.*

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Contents

Chapter 0

What is Calculus?

"The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance." (John von Neumann)

A good speaker will often begin a talk with an outline of what will be discussed in the talk. In the same way, it is incumbent upon the author of a textbook to give the reader the big picture of the subject of the textbook. That is the purpose of this brief chapter.

Calculus is the study of change. Over time, the population of a city may grow. The price of a stock will fluctuate: sometimes increasing and sometimes decreasing. The cumulative profits of a company may grow or wane. These situations represent changing quantities. We may want to dig deeper and find out how quickly the city is growing or the rate at which a company's stock price or profits are growing. Those are questions that calculus attempts to answer.

Calculus was first developed in the late 1600s independently by Sir Isaac Newton and Gottfried Wilhelm Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic, and social phenomena and to describe and solve problems in those areas.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields.

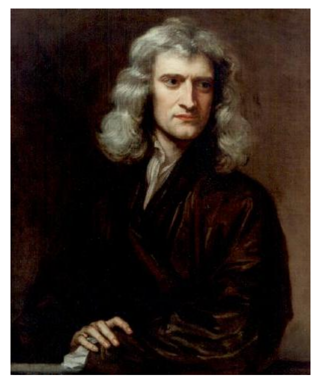


Figure 1: Isaac Newton in 1689

Figure 2: Gottfried Wilhelm Leibniz circa 1695

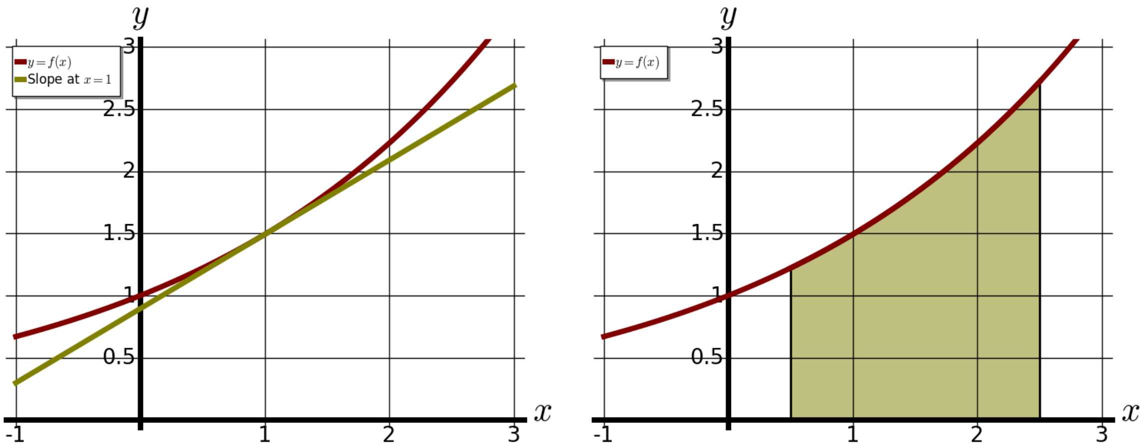
0.1 Two Problems

Calculus is the study of two seemingly different questions that are actually inverses of each other.

1. What is the **rate of change** of a function? How rapidly is some quantity changing, relative to another related quantity?

2. What is the **accumulation** of a function? Given the rate of change of some quantity, how much does the quantity accumulate over some span?

Geometrically, these questions can be visualized as the **slope of a curve** and the **area under a curve**, respectively, as seen in Figure 3. We start with these pictures because a conceptual understanding of calculus (and really, all of mathematics), and not merely as a set of rules and procedures,



(a) Slope of a Curve at a Point

(b) Area Under a Curve over an Interval

Figure 3: Visualizing the Two Problems of Calculus

is crucial to understanding the content of this course and being able to apply it to real-world problems, perhaps even in novel ways.

In this book, we will unpack these two problems and two pictures and show how they are useful to business, economics, finance, the social sciences, and the life sciences. We will take a data-driven and a problem-solving approach to this course, working with real and realistic data. Yes, all of calculus is derived from understanding those two pictures. Study those two pictures again.

Let's consider a pair of examples to illustrate these two concepts.

Example 0.1.1. Suppose that you're paid y dollars for working x hours. A graph of the relationship between the amount of time you work (in hours) and your pay (in dollars) is in Figure 4. The slope of the line is your hourly pay rate. In this case, the slope of the curve and the rate of change of the function is \$15 per hour. Note that the curve is a line, so we find the slope of the line by finding the slope between any two points on the line. Let's take the points $(0, 0)$ and $(20, 300)$.

$$\text{rate of change} = \text{slope} = \frac{300 - 0 \text{ dollars}}{20 - 0 \text{ hours}} = \frac{300 \text{ dollars}}{20 \text{ hours}} = \$15 \text{ per hour}$$

.

Example 0.1.2. Suppose that t hours after a snowstorm starts, the snowfall rate is $r(t)$ inches per hour. The shaded area in Figure 5 represents the total accumulation of the snow during the storm. How much snow accumulated during the storm?

Solution: To find the total accumulation of snow, we can break this image up in a natural way into four shapes: the triangle, trapezoid, square, and triangle over the intervals $[0, 4]$, $[4, 6]$, $[6, 8]$, and $[8, 12]$, respectively. Therefore, the total shaded area is (note the units):

$$\begin{aligned}
&\frac{1}{2}(4 \text{ hours})\left(1\frac{\text{inches}}{\text{hour}}\right) + (2 \text{ hours})\left(\frac{1+2}{2}\frac{\text{inches}}{\text{hour}}\right) + (2 \text{ hours})\left(2\frac{\text{inches}}{\text{hour}}\right) + \frac{1}{2}(4 \text{ hours})\left(2\frac{\text{inches}}{\text{hour}}\right) \\
&= 2 \text{ inches} + 3 \text{ inches} + 4 \text{ inches} + 4 \text{ inches} \\
&= 13 \text{ inches} \ .
\end{aligned}$$

The shaded area represents the total snow accumulation: 13 inches. Get out the shovels and sleds! ■

0.2 Problem Solving

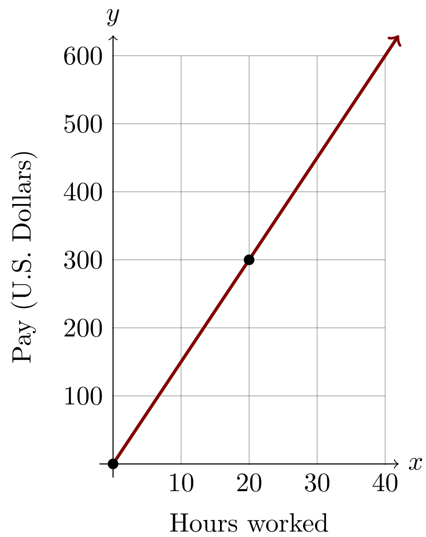


Figure 4: Pay as a function of time worked.

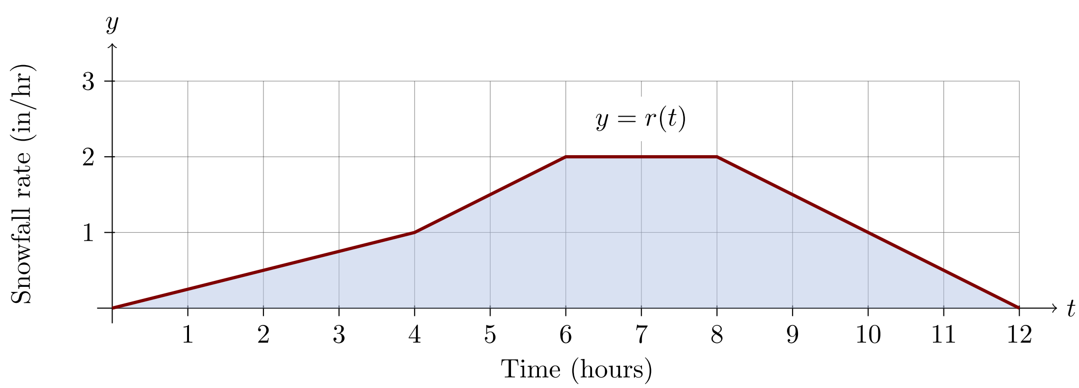


Figure 5: Snowfall rate during a snowstorm.

Figure 6: George Pólya

One day when my wife was pregnant with our second child, she was lying down on the couch with a bit of nausea, which is not uncommon in the first trimester. I offered her a ginger candy, which helps to alleviate nausea. Our son, who was a clever two years old at the time asked me, "Pwease I have ginger candy?" I replied, "No, you can't have ginger candy. It's for mommy. Mommy's tummy hurts." My son sat for a minute thinking. After a moment, he said. "My tummy hurts. Pwease I have ginger candy?" He still didn't get the candy; we don't reward dishonesty, but it was a humorous example of problem solving.

Have you ever looked at a math problem and said to yourself, "I don't know where to start!" No matter how you answered that question, this section will offer some tips.

One often thinks of the study of a mathematical subject as mere rote memorization of formulas and step-by-step procedures. On the contrary, the approach that mathematicians take to the subject is much more creative. We see mathematics as a way to apply abstract concepts to solve problems. Often there is just one correct answer. Often, however, the "correct" or "best" answer may be impossible to find, so we are instead interested in a solution that would be considered "good," "plausible," or "reasonable." That is the approach that will drive much of the content of this textbook. In short, one of our objectives is to retrain the way you view mathematics: it's not about knowing or not knowing the answer, it's about figuring out a solution. This textbook is not a book to tell you what to think, but how to think: how to think about quantitative problem solving.

In light of this, we introduce a well-known problem-solving process and thoughts on how to implement it in practice. In his famous book *How to Solve It*, mathematician George Pólya described his problem-solving process in four steps, which we have modified to follow the acronym "C.O.P.E."

C.O.P.E. Problem-Solving Process

1. **Comprehend** (Understand) the problem.
   - **What is the problem asking?**
   - It is impossible to solve a problem without this step, but it is often skipped.
   - Read and understand the instructions if there are any.
   - Read the problem over and over until you understand.
   - Look up words and symbols you don't understand.
   - Ask yourself and answer the following questions:
     - Is the problem assuming that I know something that I don't?
     - What knowledge gaps must I fill in order to really understand the problem?
     - Is there enough information given to solve the problem?
     - Do I need to look up information for the problem?
     - Is there any irrelevant information that can be ignored?
     - Do I need to make any assumptions? If so, are these assumptions reasonable?
   - It may help to draw a picture or diagram.
   - It may help to simplify the problem, i.e., work with a simpler "toy" problem.
   - Study examples similar to the problem.
   - It may help to compare the problem to a similar one.
   - Be patient with yourself. Sufficient understanding may take time.
   - What is a range of plausible solutions?
   - Often there is more than one level of comprehension to a problem or concept. A deeper understanding of a problem or concept results in a better or more advanced solution to the problem.
   - You may obtain a deeper understanding of the problem by pursuing the next step: "Observe."
2. **Observe** the information and applicable tools and devise a plan.
   - **What information is given in the problem and what tools do you have that apply?**
   - "Look" around. What (mathematical or other) tools, results, theorems, algorithms, etc. apply in this situation, based on the information given?
   - If you studied examples similar to the problem, do the same tools, results, theorems, or algorithms apply in this case? Why or why not?
   - Here are some ideas to work with the information.
     - Define variables to describe components of the problem and solution. Consider units.
     - Express the information we have mathematically.
     - Draw a picture or diagram. Label the picture with variables, numbers, etc.
     - Make a chart or list or plot some of the data or information.
     - Experiment. Play with the problem and the math until something works.
     - Guess and check: Trial and error helps to develop intuition.
     - Be systematic. Use an organized method to check all cases.
     - Look for patterns.
     - Work backwards.

   - Look for counterexamples.
   - Divide and conquer. Split the problem up into pieces and solve the individual pieces.

3. **Proceed** with the plan and get a solution.
   - **It often helps to work out a solution on scratch paper first.**
   - **Make your solution easy to follow for yourself and others.**
   - Check your work at each step.
   - If you get stuck or the solution doesn't work, go back to the planning stage and try something else. Thomas Edison tested thousands of designs before developing a practical working lightbulb.
   - Think about the problem during otherwise unproductive times such as when waiting in line, walking, driving, or falling asleep.
   - When stuck, take a break and let your brain work on the problem subconsciously.
   - Be patient. It may take a while for the solution to "click."
   - Work on the solution on scratch paper, and carefully write up your solution when you've solved it. The solution must clearly communicate the problem that is being solved and the solution itself. Your solution will be read by others and your future self.

4. **Evaluate** (in the sense of "Reflect on") your solution and check your work.
   - (1) **Does the solution answer the question or solve the problem?**
   - (2) **Does the solution make sense?**
   - Skipping this step often results in "stupid" mistakes that are easy to catch.
   - Are there any obvious errors or contradictions?
   - Does the solution fit within a range of plausible solutions?
   - Is there another, independent approach to validate the solution or confirm that it is plausible?
   - Do the units make sense?
   - Check your work. This is why organized solutions are important.
   - Proofread any written work, ideally at least a day after you wrote it.

5. **Extend** (Generalize) the solution or problem and reflect.
   - Is there another, possibly easier, way to solve the problem?
   - Can you generalize the problem?
   - Would you do something different to improve the solution?

My charge to you, the student of calculus, is to keep the C.O.P.E. problem solving method in mind throughout your course. **Comprehension** will often involve reviewing earlier chapters and sections, looking up content in the index, and in many cases, and reviewing material from earlier courses such as algebra. Be patient with yourself. When **observing** what tools apply, again, review of the text may be likely. Just as a hammer doesn't work well to put a nut on a bolt, make sure that the right mathematical tool applies to the problem at hand. If not, then don't use the tool. Be patient with yourself. When **proceeding** with the solution, keep in mind that the point of writing up a solution is to clearly communicate the solution to the intended audience. That audience may be your future self. Write your solution so as to make it impossible to misunderstand. Be sure that all graphics are clear and labeled and that all variables are defined. Be patient with yourself. When **evaluating** your solution, be critical with yourself. Expect there to be something wrong and scrutinize each step. Finally, if you approach this course with the intent to understand the concepts and the pictures behind each concept, then you will be in an excellent position to **extend** any solution and apply it in powerful ways in your particular field of study. I hope you enjoy this book and this course and that you find it useful throughout your life.

0.3 Exercises

1. Give an example of a rate of change that you have seen in your life (other than one of the examples in this chapter).
2. Give an example of an accumulation of a rate that you have seen in your life (other than one of the examples in this chapter).
3. Why would we call a solution to a problem "plausible," "good," or "reasonable," rather than "correct?"
4. Give an example of a time when you used the C.O.P.E. problem solving process in your life.
5. What does it mean to have a conceptual understanding of mathematical idea?
6. Why is it important to understand a mathematical principle conceptually, rather than merely memorizing rote procedures?

Chapter 1

Models, Graphs, and Functions

1.1 What is a Function?

1.1.1 Function Concepts

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or **independent variable**, and a corresponding output quantity, or **dependent variable**. From this, we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Definition 1.1.1. A **function** is a rule for a relationship between an **input** (or **independent**) quantity and an **output** (or **dependent**) quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1.1.1. In the height and age example above, is height a function of age? Is age a function of height?
Solution: In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. You cannot have two different heights at any one instant in time.

However, age is not a function of height, since one height input might correspond with more than one output age. Once you have stopped growing, your height essentially remains constant for the rest of your life. ■

1.1.2 Representing Functions

Functions can be represented in many ways:

1. A description of a relationship between variables,
2. Tables of values,
3. Graphs,
4. Formulas,
5. An action verb, and
6. A black box with input and output.

Example 1.1.1 represented a function in words, but it will be convenient to streamline the discussion of a function. To that end, we introduce notation of functions.

Function Notation. To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

If we name the function f we could write "height is a function of age" as "h is f of a," or more simply:

$$h = f(a) \ .$$

We could instead name the function h and write $h(a)$, which is read "h of a."

We can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value "a" must be put into the function "h" to get a result.

Remark 1.1.1. Be careful! The parentheses indicate that age is the input into the function. Do not confuse these parentheses with multiplication!

Example 1.1.2. A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ tell us?

Solution: When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function: the year (y). The output value is 300, the number of police officers (N), a value for the output quantity. Remember $N = f(y)$. This tells us that in the year 2005 there were 300 police officers in the town. ■

Tables as Functions. A table lists the input and corresponding output values of a function.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1.1 below represents the age of a child in years and his corresponding height. This represents just some of the data available for the age and height of the child.

(Input:) a, age (years)	4	5	6	7	8	9	10
(Output:) h, height (inches)	40	42	44	47	50	52	54

Table 1.1: Tabulating height as a function of age for a child.

From this, we can create equations such as $h(6) = 44$, meaning that when the child was 6 years old, he was 44 inches tall.

Example 1.1.3. Which of these tables define a function (if any)?

Table A.

Input	**Output**
2	1
5	3
8	6

Table B.

Input	**Output**
-3	5
0	1
4	5

Table C.

Input	**Output**
1	0
5	2
5	4

Solution: Tables A and B define functions. In both tables, each input corresponds to exactly one output. Table C does not define a function since the input value of 5 corresponds with two different output values: 2 and 4. ■

Graphs as Functions A function can often be represented as a **graph**, a set of **points** plotted on **coordinate axes**: the **horizontal axis** and the **vertical axis**. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical axis.

Points on the graph are represented by ordered pairs of the form (a, b). Beginning at the **origin** (the point $(0, 0)$), you move a units to the left and b units up to arrive at the point (a, b). If $a < 0$, then movement is to the left and if $b < 0$, then movement is down.

The horizontal and vertical axes are typically called the x-**axis** and the y-**axis**, but these axes can be labeled with any variable name, not just x and y. We say y is a function of x, or $y = f(x)$ when the function is named f. The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

As an example, Figure 1.1 is a plot of the data from Table 1.1.

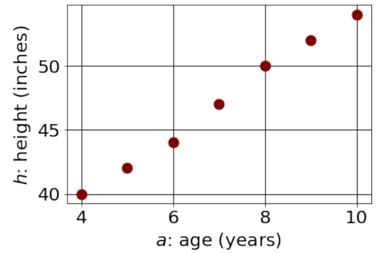
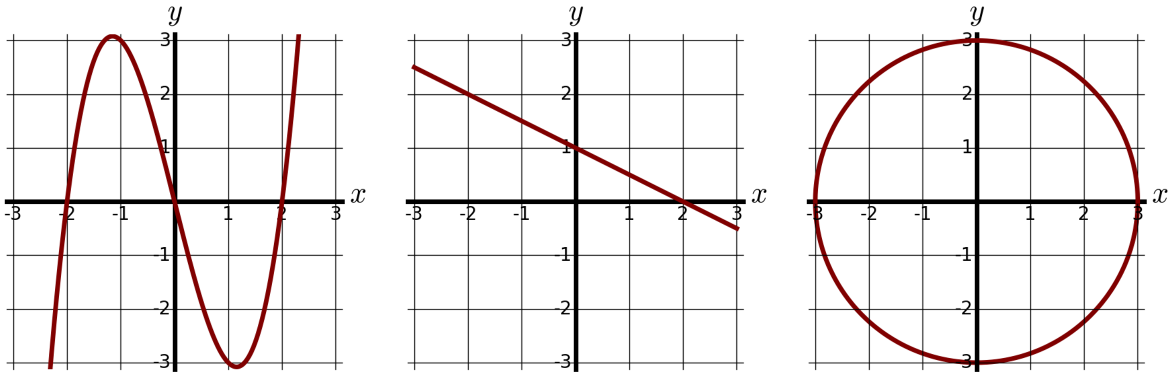


Figure 1.1: A plot of height data.

Example 1.1.4. Which of these graphs defines a function $y = f(x)$?



(a) Graph A (b) Graph B (c) Graph C

Solution: Looking at the graphs above, Graphs A and B define a function $y = f(x)$, since for each input value along the horizontal (x) axis, there is exactly one corresponding output value, determined by the y-value of the graph. Graph C does not define a function $y = f(x)$ since some input values, such as $x = 2$, correspond with more than one output value. ■

The **vertical line test** is an easy way to determine whether a graph defines a function or not. Imagine drawing vertical lines through the graph. Since a function has exactly one output for every input, if there is a vertical line that would cross the graph more than once, then the graph does not define a function.

Figure 1.3 illustrates how Graph C from Example 1.1.4 fails the vertical line test. The vertical line $x = 2$ (in gold) intersects Graph C (in maroon) at two points; there are two outputs for for the input $x = 2$.

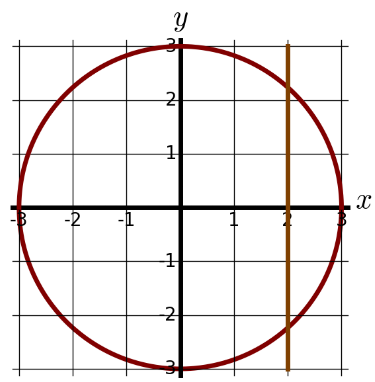


Figure 1.3: A graph failing the vertical line test.

Formulas as Functions When possible, it is very convenient to define relationships between quantities using a formula. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 1.1.5. Express the relationship $2N + 6p = 12$ as a function $p = f(N)$ if possible.

Solution: To express the relationship in this form, we need to be able to write the relationship where p is a function of N, which means writing it as $p =$ something involving N, or p in terms of N. We proceed using algebra.

$2N + 6p = 12$	Subtract $2N$ from both sides.
$6p = 12 - 2N$	Divide both sides by 6 and simplify.
$p = \dfrac{12 - 2N}{6}$	
$p = \dfrac{12}{6} - \dfrac{2N}{6}$	
$p = 2 - \dfrac{1}{3}N$	

We can now express p as a function of N:

$$p = f(N) = 2 - \frac{1}{3}N \ .$$

■

It is important to note that not every relationship can be expressed as a function with a formula. Consider the examples of the boy's height as a function of age in Table 1.1 or a company's stock price as a function of time. Neither situation allows one to define the relationship using a formula precisely, yet we still may want to analyze aspects of these relationships using tools of calculus. The rest of the chapter will introduce us to some tools to help us with that.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed.

1.1.3 Evaluating Functions.

The fundamental use of a function is to **evaluate** the function: "plugging in" some number into the function, or more precisely, to determine the corresponding output for a given input. In other words, we substitute or replace the input variable of the function with the input value. Evaluating will always produce one result, since each input of a function corresponds to exactly one output. We can evaluate a function from a table, graph, or formula.

Another related use is to determine the input or inputs of a function, given an output of the function. This is called **solving** an equation and could produce more than one solution, since different inputs can produce the same output.

Remark 1.1.2. The concepts of evaluating and solving often get confused. When we use the word *solve*, we will be *solving a problem* or *solving an equation for an unknown quantity or variable*. It does not make sense to solve a function, since a function is merely a mathematical expression and not an equation.

Example 1.1.6. Let $Q = g(n)$ and use the table shown.

$Q = g(n)$ as a table.

n	1	2	3	4	5
Q	8	6	7	6	8

(a) Evaluate $g(3)$.

Solution: Evaluating $g(3)$, read "g of 3," means that we need to determine the output value, Q, of the function g given the input value of $n = 3$. Looking at the table, we see the output corresponding to $n = 3$ is $Q = 7$, allowing us to conclude $g(3) = 7$. ■

(b) Solve $g(n) = 6$ for n.

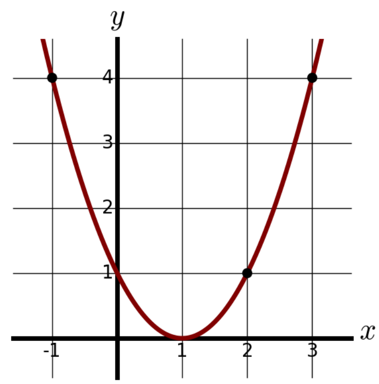
Solution: Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we evaluate $g(n)$ at 2, our output is $Q = 6$.

When we evaluate $g(n)$ at 4, our output is also $Q = 6$. ■

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 1.1.7. Consider the graph of a function $f(x)$ to the right.



(a) Evaluate $f(2)$.

Solution: To evaluate $f(2)$, we find the input of $x = 2$ on the horizontal (x) axis. Moving up to the graph gives the point $(2, 1)$, giving an output of $y = 1$. Therefore, $f(2) = 1$. ■

(b) Solve $f(x) = 4$ for x.

Solution: To solve $f(x) = 4$, we find the value 4 on the vertical (y) axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: $(-1, 4)$ and $(3, 4)$. These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$.

This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. ■

Example 1.1.8. Let $k(t) = t^3 + 2$.

(a) Evaluate $k(2)$.

Solution: To evaluate $k(2)$, we replace t with 2 in the expression $t^3 + 2$, then simplify.

$$\begin{aligned} k(2) &= 2^3 + 2 \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

So $k(2) = 10$. ■

(b) Solve $k(t) = 1$ for t.

Solution: To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output.

$$\begin{aligned} k(t) &= 1 && \text{Substitute the original formula } k(t) = t^3 + 2. \\ t^3 + 2 &= 1 && \text{Subtract 2 from each side.} \\ t^3 &= -1 && \text{Evaluate the cube root of each side.} \\ t &= \sqrt[3]{-1} = -1 \end{aligned}$$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

To check our work, we want to know if $k(t) = 1$ is a true statement when $t = -1$.

$$\begin{aligned} k(-1) &= (-1)^3 + 2 \\ &= -1 + 2 \\ &= 1 \ , \end{aligned}$$

which was the desired result. ■

1.1.4 Cost, Revenue, Profit, and Demand

Suppose that your club wants to raise funds by selling T-shirts. The screen printing shop that will make the shirts will charge your club \$50 to cover overhead costs and \$5 per shirt for the shirts themselves. You decide to charge \$15 per shirt. Some questions naturally arise. How many shirts need to be sold to **break even**? How much **profit** can be expected?

We can describe situations such as this with functions. The **total cost** to produce these shirts combines **fixed costs** and **variable costs**. The fixed costs are also called **overhead costs** and do not depend on the number of T-shirts made, while the variable costs are the per item cost. If n is the number of T-shirts the screen printing shop will make, $F(n)$ is the fixed costs and $V(n)$ is the variable costs, then the **cost function** to produce n T-shirts is

$$\begin{aligned}\text{total cost} &= \text{fixed costs} + \text{variable costs}\\ C(n) &= F(n) + V(n)\\ &= \$50 + (\$5 \text{ per shirt})\,(n \text{ shirts})\\ &= 50 + 5n \text{ dollars}\end{aligned}$$

If you sell each shirt for \$15 and you sell n shirts, then your **revenue** from selling n shirts will be $15n$ dollars. This is your **revenue function**: $R(n) = 15n$ dollars.

Finally, the **profit** that your club will earn from selling n shirts is the revenue minus the total costs. If $P(n)$ is the profit from selling n items, then the **profit function** is

$$P(n) = R(n) - C(n)\ .$$

In this example, we have

$$\begin{aligned}P(n) &= R(n) - C(n)\\ &= 15n - (50 + 5n) \text{ dollars}\\ &= 15n - 50 - 5n \text{ dollars}\\ &= 10n - 50 \text{ dollars}\end{aligned}$$

Example 1.1.9. Consider the T-shirt scenario above.

(a) What is the profit from selling 30 shirts?

Solution: $P(30) = 10 \cdot 30 - 50 = 300 - 50 = \$250$.

If you sell 30 T-shirts, then you profit \$250. ■

(b) What is the profit from selling 3 shirts?

Solution: $P(3) = 10 \cdot 3 - 50 = 30 - 50 = \$(-20)$.

If you only sell 3 T-shirts, then you lose \$20; your profit is negative. ■

(c) What is the profit from selling 0 shirts?

Solution: $P(0) = 10 \cdot 0 - 50 = 0 - 50 = \$(-50)$.

If you don't sell any T-shirts, then you lose \$50; your profit is negative. ■

The **break-even point** in this context is the minimum number of T-shirts that must be sold in order to have a profit of at least \$0. How do we find this? If our profit is \$0, then we turn that sentence into a mathematical equation and solve for the number of shirts. The profit from selling n shirts is $P(n)$, "is" is "=", and 0 is 0.

$$\begin{aligned}P(n) &= 0\\ 10n - 50 &= 0\\ 10n &= 50\\ n &= \frac{50}{10} = 5\end{aligned}$$

So you need to sell at least five T-shirts in order to break even.

Remark 1.1.3. Note the distinction between $P(0)$ and $P(n) = 0$. $P(0)$ is the profit from selling 0 items, while $P(n) = 0$ is an equation whose solution tells you the number of items that must be sold in order to have no profit.

Definition 1.1.2. In summary, the **total cost** to produce n items, $C(n)$, is the combination of **fixed costs**, $F(n)$, and **variable costs**, $V(n)$. The fixed costs are also called **overhead costs** and is constant regardless of the number of items made, while the variable costs depend on the number of items being made. So

$$C(n) = F(n) + V(n) \ .$$

The **revenue** function, $R(n)$ gives the amount of money brought in from selling n items. The difference between revenue and total costs is **profit** ($P(n)$) from selling n items:

$$P(n) = R(n) - C(n) \ .$$

The **break-even point** is the fewest number of items, n, that must be sold in order for $P(n) \geq 0$, in other words, the fewest number of items to guarantee that you won't have negative profit and be losing money.

Definition 1.1.3. Demand is the functional relationship between the price p and the quantity q of an item that can be sold (that is demanded). Depending on your situation, you might think of p as a function of q, or of q as a function of p. If the quantity of an item that is demanded depends on the price it is sold at, then we would write $q = D(p)$.

Definition 1.1.4. The **supply** of an item is the quantity q that is available for sale (that is supplied) at a price p. If the supply of an item for sale depends on the price it is sold at, then we would write $q = S(p)$.

Definition 1.1.5. The **average cost** to produce n items is

$$A(n) = \frac{C(n)}{n} \ .$$

Example 1.1.10. Table 1.2 shows the total cost ($C(n)$) of producing n items.

Items (n)	Cost $C(n)$
0	\$20,000
100	\$35,000
200	\$45,000
300	\$53,000

Table 1.2: Cost to produce n items.

(a) What is $F(n)$, the fixed cost for producing n items?

Solution: The fixed cost is $F(n) = C(0) = \$20,000$, the cost even when no items (i.e., when $n = 0$ items) are made. ■

(b) When 200 items are made, what is the variable cost?

Solution: The fixed cost is \$20,000, and when 200 items are made, the total cost is $C(200) = \$45,000$. Subtracting the fixed cost, the total variable cost is $V(200) = C(200) - F(n) = C(200) - C(0) = \$45,000 - \$20,000 = \$25,000$. ■

(c) When 200 items are made, what is the average cost?

Solution: The average cost is the total cost divided by the number of items: $A(200) = \frac{C(200)}{200 \text{ items}} = \frac{\$45,000}{200 \text{ items}} = \$225$ per item. So, on average, each item had a cost of \$225. ■

(d) When 200 items are made, what is the average **variable** cost?

Solution: The average variable cost would be the total variable cost divided by the number of items: $\frac{V(200)}{200 \text{ items}} = \frac{\$25,000}{200 \text{ items}} = \$125$ per item. So, on average, each item had a variable cost of \$125. ■

1.1.5 Domain and Range.

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of the models we create. In our most recent example, it wouldn't make sense to sell -4 T-shirts or 4.4 T-shirts. We also wouldn't expect to sell a billion T-shirts for a university club fund raiser. When using a function to describe a real-world scenario, we need to set common sense boundaries for the input and output. Here is another example.

Table 1.3 shows a relationship between the circumference, c, and height, h, of a tree as it grows. In this table, we would consider height as a function of circumference: $h = h(c)$.

Circumference: c (feet)	1.7	2.5	5.5	8.2	13.7
Height: h (feet)	24.5	31.0	45.2	54.6	92.1

Table 1.3: Height of a tree as a function of its circumference.

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the **domain** and **range** of the function.

Definition 1.1.6. The **domain** of a function is the set of possible input values to the function.

The **range** of a function is the set of possible output values of the function.

Example 1.1.11. Using Table 1.3 above, determine a reasonable domain and range of the function h.

Solution: We can combine the data provided with additional research and our own reason to determine an appropriate domain and range of the function $h = h(c)$. For the domain, it doesn't make sense for the circumference (input) to be negative, so $c \geq 0$. For a maximum circumference, we could make an educated guess at a reasonable value, or look up that the maximum recorded circumference is about 119 feet[1]. With this information, we would say a reasonable domain is $0 \leq c \leq 119$ feet.

Similarly for the range, if we only consider the tree when the sprout has broken through the ground, it doesn't make sense to have negative heights. The maximum recorded height of a tree could be looked up to be 379 feet, so a reasonable range is $0 \leq h \leq 379$ feet. ■

Interval Notation. A convenient alternative to the notation using inequalities is **interval notation**, in which intervals of values are referred to by the starting and ending values. Parentheses "()" are used for "strictly less than," and square brackets "[]" are used for "less than or equal to." Since infinity, ∞, is not a number, neither $-\infty$ nor ∞ are included in the domain and range of a function, so we always use curved parentheses with $\pm\infty$. Table 1.4 shows how inequalities correspond to interval notation for an arbitrary variable x.

To combine two intervals together, we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Inequality	**Interval Notation**
$5 \leq x \leq 10$	$[5, 10]$
$5 < x \leq 10$	$(5, 10]$
$5 \leq x < 10$	$[5, 10)$
$5 < x < 10$	$(5, 10)$
$x < 10$	$(-\infty, 10)$
$5 \leq x$	$[5, \infty)$
All real numbers	$(-\infty, \infty)$

Table 1.4: Interval Notation

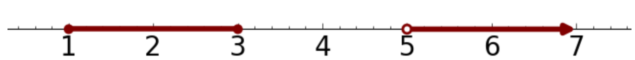


Figure 1.4: A union of intervals.

Example 1.1.12. Describe the intervals of values shown in Figure 1.4 using inequalities and using interval notation.

[1] http://en.wikipedia.org/wiki/Tree, retrieved July 19, 2010

Solution: To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

As an inequality it is: $1 \leq x \leq 3$ or $x > 5$.

In interval notation: $[1, 3] \cup (5, \infty)$. ■

Example 1.1.13. Find the domain of each function.

(a) $f(x) = 2\sqrt{x+4}$

Solution: Since we cannot evaluate the square root of a negative number, we need the inside of the square root to be non-negative. $x+4 \geq 0$ when $x \geq -4$. (Subtract 4 from both sides of the inequality.) Therefore, the domain of $f(x)$ is $[-4, \infty)$. ■

(b) $g(x) = \frac{3}{6-3x}$

Solution: We cannot divide by zero, so we need the denominator to be non-zero. Solving $6 - 3x = 0$ for x, we have $x = 2$, so we must exclude 2 from the domain. Therefore, the domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$. ■

1.1.6 Exercises

1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

   (a) The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.

   (b) Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

   (a) A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.

   (b) Explain the meaning of the statement $g(100) = 1$.

3. Let $n(t)$ be the number of subscribers to a YouTube channel t years after 2005. Explain the meaning of each statement.

   (a) $n(5) = 300$

   (b) $n(10) = 4000$

4. Let $p(t)$ be the stock price, in dollars, of Valvoline (VVV) t years after its Initial Public Offering (IPO) on September 23, 2016. Explain the meaning of each statement.

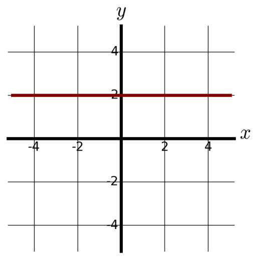
   (a) $p(0) = 23.10$

   (b) $p(1) = 23.27$

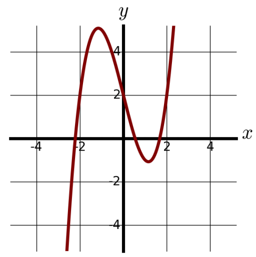
   (c) $p(2) = 21.47$

5. Select all of the following graphs which represent y as a function of x.

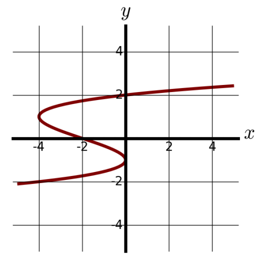
Graph A.



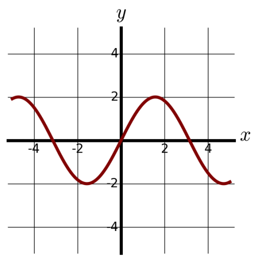
Graph B.



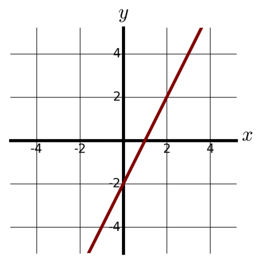
Graph C.



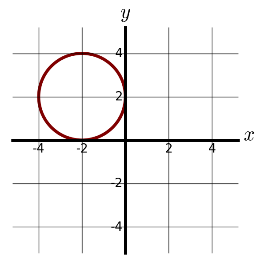
Graph D.



Graph E.

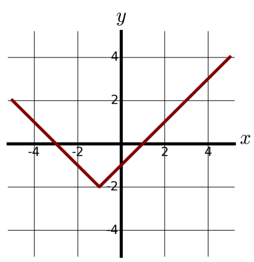


Graph F.

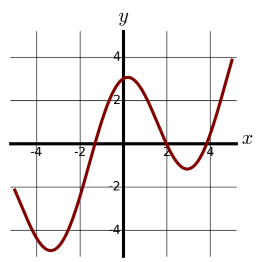


6. Select all of the following graphs which represent y as a function of x.

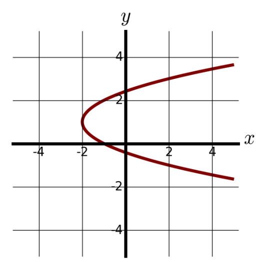
Graph A.



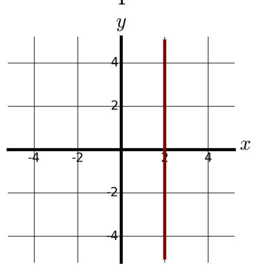
Graph B.



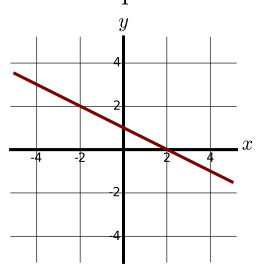
Graph C.



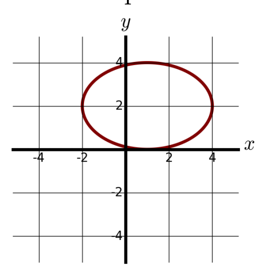
Graph D.



Graph E.



Graph F.



7. Select all of the following tables which represent y as a function of x.

Table A.

x	5	10	15
y	3	8	14

Table B.

x	5	10	15
y	3	8	8

Table C.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent y as a function of x.

Table A.

x	2	6	13
y	3	10	10

Table B.

x	2	6	6
y	3	10	14

Table C.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent y as a function of x.

Table A.

x	y
0	-2
3	1
4	6
8	9
3	1

Table B.

x	y
-1	-4
2	3
5	4
8	7
12	11

Table C.

x	y
0	-5
3	1
3	4
9	8
16	13

Table D.

x	y
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent y as a function of x.

Table A.

x	y
-4	-2
3	2
6	4
9	7
12	16

Table B.

x	y
-5	-3
2	1
2	4
7	9
11	10

Table C.

x	y
-1	-3
1	2
5	4
9	8
1	2

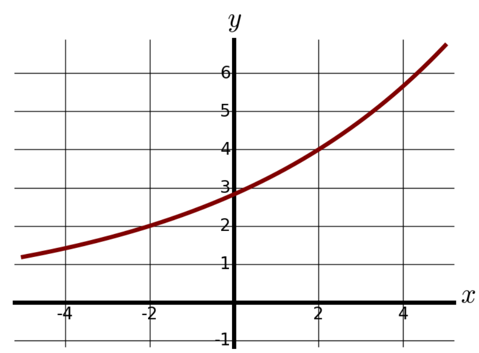
Table D.

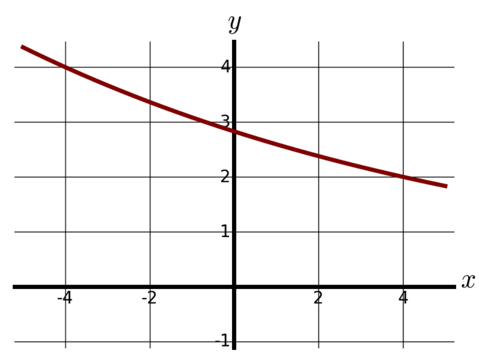
x	y
-1	-5
3	1
5	1
8	7
14	12

11. Let $g(x)$ be the function graphed on the right.

(a) Evaluate $g(2)$.

(b) Solve $g(x) = 2$ for x.





12. Let $f(x)$ be the function graphed on the right.

(a) Evaluate $f(4)$.

(b) Solve $f(x) = 4$ for x.

13. Consider the table at right.

(a) Evaluate $f(3)$.

(b) Solve $f(x) = 1$ for x.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

14. Consider the table at right.

    (a) Evaluate $g(8)$.

    (b) Solve $g(x) = 7$ for x.

x	0	1	2	3	4	5	6	7	8	9
$g(x)$	62	8	7	38	86	73	70	39	75	34

For Exercises 15-22, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$, if possible.

15. $f(x) = 4 - 2x$

16. $f(x) = 8 - 3x$

17. $f(x) = 8x^2 - 7x + 3$

18. $f(x) = 6x^2 - 7x + 4$

19. $f(x) = 3 + \sqrt{x+3}$

20. $f(x) = 4 - \sqrt[3]{x-2}$

21. $f(x) = \frac{x-3}{x+1}$

22. $f(x) = \frac{x-2}{x+2}$

23. Let $f(t) = 3t + 5$.

    (a) Evaluate $f(0)$.

    (b) Solve $f(t) = 0$ for t.

24. Let $g(p) = 6 - 2p$.

    (a) Evaluate $g(0)$.

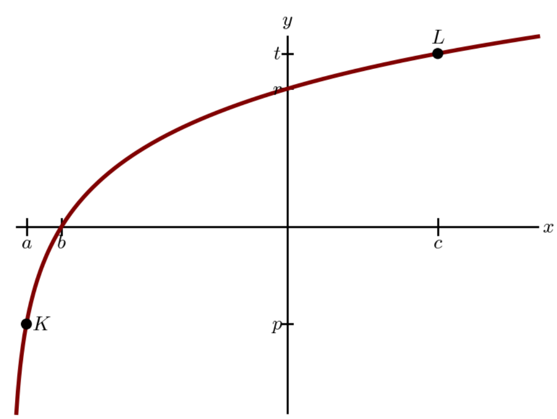
    (b) Solve $g(p) = 0$ for t.

25. Consider the graph of $f(x)$ on the right.

    (a) Evaluate $f(c)$.

    (b) Solve $f(x) = p$ for x.

    (c) Find the coordinates of points L and K.



1.2 Models

"All models are wrong; some models are useful." (George E. P. Box)

In the real world, we will, in all likelihood, have to work with data. "Data-driven decisions" is a common phrase heard today. One thing that can be said is that real world data is messy and complex. Countless factors influence every phenomenon. Some factors are designed and some are purely random. Sometimes a data set is incomplete, missing pieces of data. Yet we may have to work with that data set as if it was complete. Sometimes we are tasked with **forecasting**, which is predicting future data points, given the ones that we have.

1.2.1 What is a Mathematical Model?

In many different areas, we need tools to simplify a set of data and work with that simplified version of the data. These simplifications must be based on reasonable assumptions that connect to the larger context of the data. Simplifications serve two purposes. First, it may be impossible to take every variable into account. Second, models must often be communicated to others and a simple model is generally more clear and therefore much easier to communicate. This is the essence of what is called **mathematical modeling**, or simply **modeling**.

Definition 1.2.1. A **model** or a **mathematical model** is a mathematical framework to help describe some phenomenon, specifically how an input or quantity affects or relates to some output. A model has three main components:

1. One or more input variables, with specific descriptions of what these variables represent, including units.
2. One or more functions, with specific descriptions of what the output of the functions represent, including units.
3. A domain and/or range over which the function(s) make sense to use.

Additionally, we may want to consider the following when creating a model.

- Identify underlying assumptions that were used to simplify the situation.
- Perform a sensitivity analysis to determine if the model is relatively unchanged if the data varies slightly.

For example, one could construct a model to speculate how the price of an item affects the demand for that item, and from that, predict revenue from sales of that item.

It is crucial to state that a model is used to simplify reality and does not dictate or reflect past, present, or future reality with absolute precision. Although good models can be useful for forecasting, decision-making, and filling in missing data. That is the essence of the quote at the beginning of the section by the late George Box, a famous British statistician.

The following is an example of a mathematical model, based on an actual data set from "Plant W." We will analyze this scenario in several places in this text and.

Model: Plant W heats an external tank and powers their operations by burning fuel oil. The rate at which they burned this oil in 2016 can be described by the following model. Let m be the month of the year, with 1 referring to January 1, and 12 referring to December 1, so that $1 \leq m < 13$. Let $f(m)$ describe the rate at which the fuel burns, in gallons per month. A model for $f(m)$ is

$$f(m) = 5.76m^3 - 109.98m^2 + 532.58m + 70.17 \text{ gallons per month.}$$

Figure 1.5 plots this curve over the given domain.

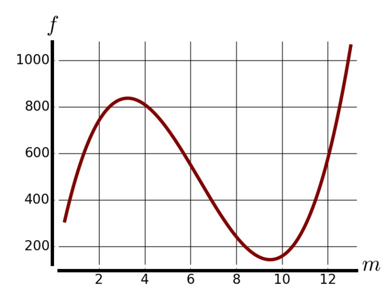


Figure 1.5: Fuel Oil Usage by Plant W in 2016.

Remark 1.2.1. Note that in this example, we clearly described the input variable, what it represented, a domain that it makes sense over, and gave the units (months). We clearly described the function (the output), what it modeled, and gave the units.

Examining the graph of the function, we can see that the model makes sense, based on the context. If Plant W is heating a large external tank, then in the winter and early spring, we expect more oil to be used to heat it. This tank may also have retained some heat, as fluids tend to do, from the late summer and fall. We expect a significant drop in oil usage through the late spring and summer months and finally a significant increase in oil usage during the fall and early winter.

Is the model precise? Of course not. First, we know that in 2016 (a leap year), there were months with 29, 30, or 31 days, yet we treated each month as equal. Perhaps a better model would have modeled fuel usage per day. For the sake of clarity, however, we aggregated the days into months; we know immediately that month 6 is June, but do not immediately know which month is day 276, for example.

A second reason that we know that the model is not accurate is that we can't be certain that on January 1, 2016 that Plant W burned oil at a rate of $f(1) = 498.53$ gallons per month. In other words, on January 1, 2016, did Plant W burn $f(1)/31 = 16.08$ gallons of fuel oil? We can be almost certain that they did not, but the actual amount may have been somewhat close to that. Other factors that we didn't consider were the range of temperatures that day and the percentage of the tank that was full. Considering these factors would make the model more accurate, but more complex.

A third item to consider is that in this case, we are dealing with a yearly cycle. We should expect $f(1)$ to be really close to $f(13)$, since both input values would refer to January 1 in 2016 and 2017, respectively. However, we have

$$f(1) = 498.53 \quad \text{and} \quad f(13) = 1061.81 \ ,$$

which is a significant difference. Perhaps a better model would attempt to get these two points either closer to each other or make them exactly equal to each other. Section 1.8 will describe how to work with periodic or seasonal data such as this. ■

1.2.2 Scatter Plots: Plotting Data in a Spreadsheet

Throughout this book, we will use a spreadsheet to plot and analyze data. In the spirit of using free and open-source resources, our examples will use the well-known office suite *LibreOffice*, specifically its spreadsheet software *Calc*. LibreOffice is compatible with other office suites and the various menus and wizards are very similar to that of other office suites. *LibreOffice* is installed by default in most distributions of the Linux operating system, but the most current version is available for free at `https://www.libreoffice.org/` for Linux and other major operating systems.

In this section, we will see how to create two different kinds of **scatter plots** in *Calc*. A scatter plot is simply a representation of data in which each element of a data set can be represented as a point on a set of coordinate axes. A scatter plot helps to visualize the relationship between variables in a data set.

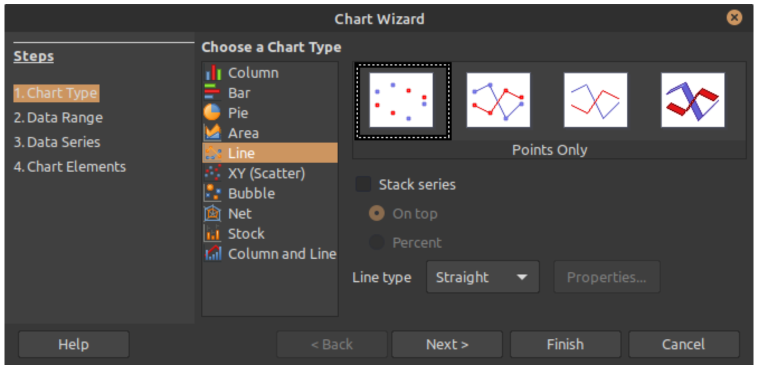
The first kind of scatter plot is ideal for **time series**, a data set that is in chronological order and there is at most one data point for every unit of time. Examples include economic data, such as a country's gross domestic product (GDP) over time; financial data, such as the price of a stock or commodity; biological data, such as the population of a species in a region over time; business data, such as a company's sales over time; and so on. Such plots help to identify trends and the strength of that trend.

The following spreadsheet shows the oil usage data of Plant W in 2016.

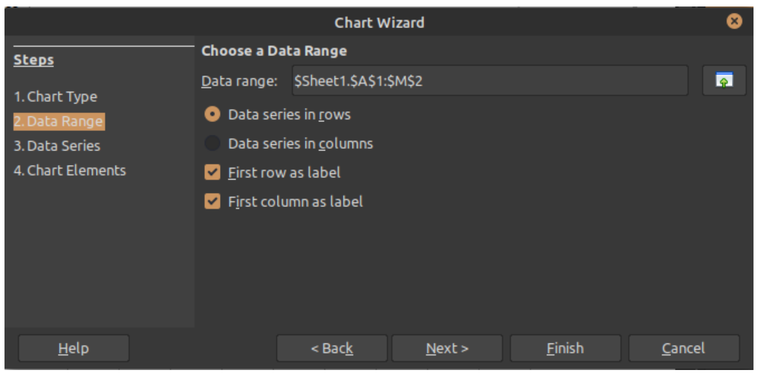
	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Month**	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
2	**Oil (Gal.)**	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0		638.0

To make a scatter plot of this data:

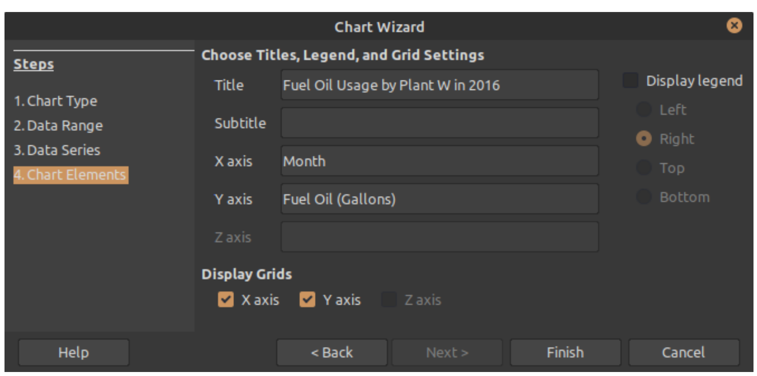
- Highlight the data (including the labels).
- Select `Insert` from the `LibreOffice Calc` menu.
- Select `Chart ...` to bring up the **Chart Wizard**.
- Select `Line` and the **Points Only** plot.



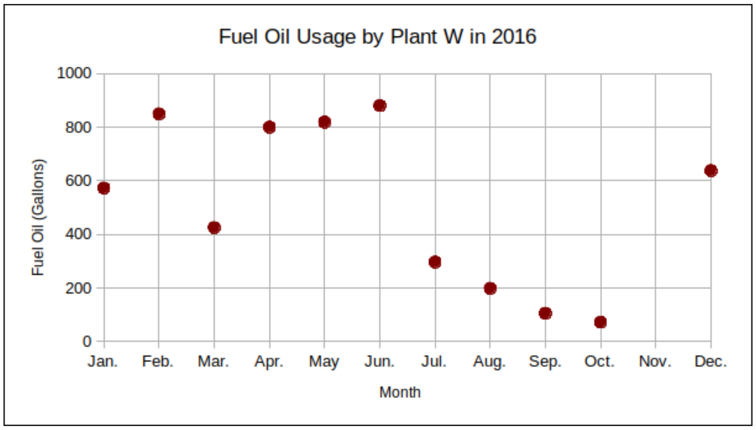
- Click on `Next >`. If you've already highlighted the data, then the wizard should have the correct range.
- Select `Data series in rows` and check `First column as label` since the first column are the axis labels of the data.



- Select `4. Chart Elements` on the left panel of the wizard.
- Put in appropriate titles and axis labels for the plot.
- Other options in this part of the wizard will make the chart more readable.
- Select `Finish`.



- Once the wizard is complete, we can make other changes to the chart.
- Double-clicking on the plotted points allows you to change the color and shape of the plotted points.
- Double-click on the y-axis and select the `Scale` tab. Changing the `Major interval` to 200 makes the axis a little more readable.



The second type of scatter plot is more general and seeks to understand the relationship between two variables in a particular phenomenon.

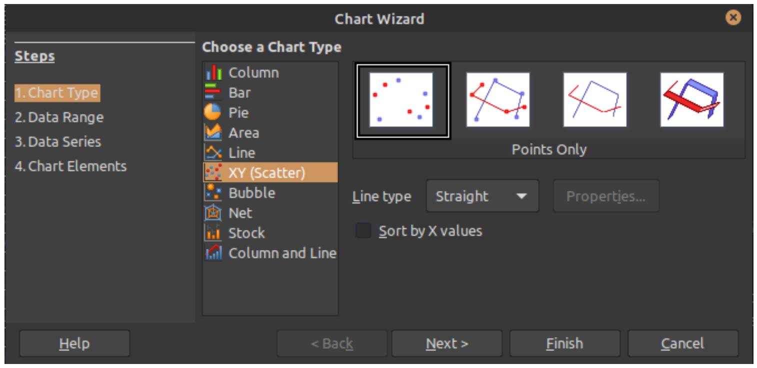
Suppose that you are a new realator in King County, Washington. You want to understand the housing market in the county in order to advise potential home buyers and sellers on reasonable prices for homes. You acquire a data set giving information on 21,613 home sales in King County, WA between May 2, 2014 and May 27, 2015. The following spreadsheet is a very small portion of this data set. To get started, you want to understand the relationship between living space area and the sale price of a home, though there are certainly more variables to consider. It makes more sense to think of the sale price as a function of the living space area, so the x-axis will be living space area in square feet and the sale price (in U.S. dollars) will be along the y-axis.

	A	B
1	**Price ($)**	**Living Area (ft^2)**
2	313000	1340
3	2384000	3650
4	342000	1930
...	...	...
21613	445500	1390
21614	1310000	3750

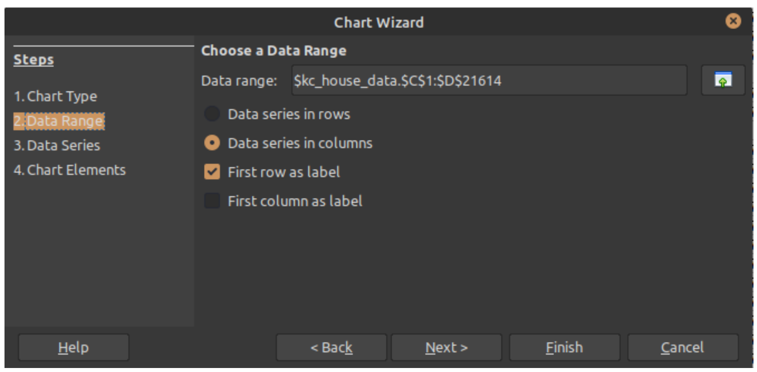
Table 1.7: King County Home Sales from May 2, 2014 to May 27, 2015

To make a scatter plot of this data:

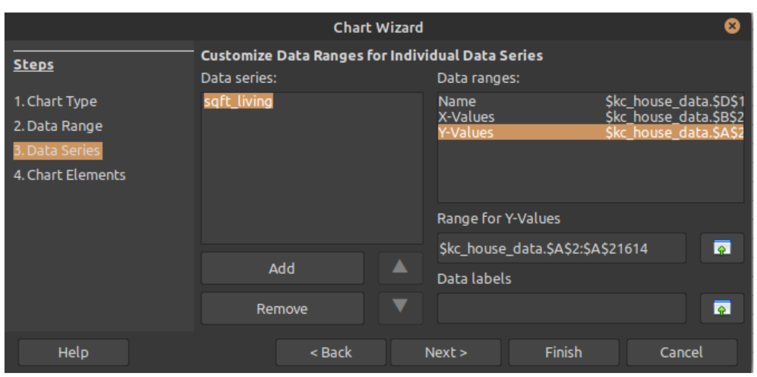
- Highlight the data (including the labels).
- Select `Insert` from the `LibreOffice Calc` menu.
- Select `Chart ...` to bring up the **Chart Wizard**.
- Select `XY (Scatter)` and the **Points Only** plot.



- Click on `Next >`. If you've already highlighted the data, then the wizard should have the correct range.
- Select `Data series in columns` and check `First row as label` since the first row are the axis labels of the data.

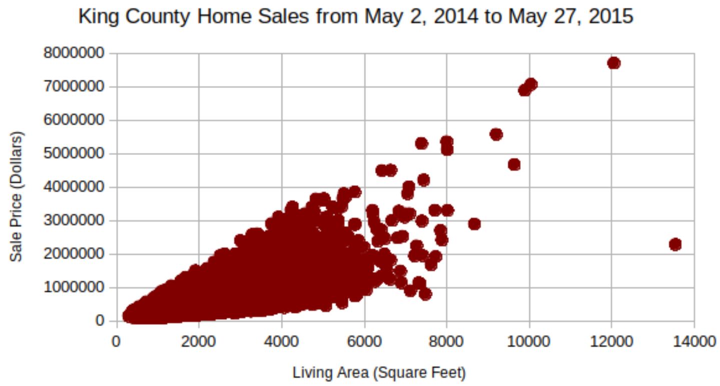


- Click on `Next >`.
- Notice that the x-axis data is in Column B and the y-axis data is in Column A. In this `Data Series` window, manually adjust this.
- In the `Data ranges:` area, click on `X-Values` and `Y-Values` and change the `Range for X-Values` and `Range for Y-Values`, respectively.



- Click on `Next >` to go to `4. Chart Elements`.
- Put in appropriate titles and axis labels for the plot.
- Other options in this part of the wizard will make the chart more readable.
- Select `Finish`.

- We will adjust the chart as we did above to make it more readable.

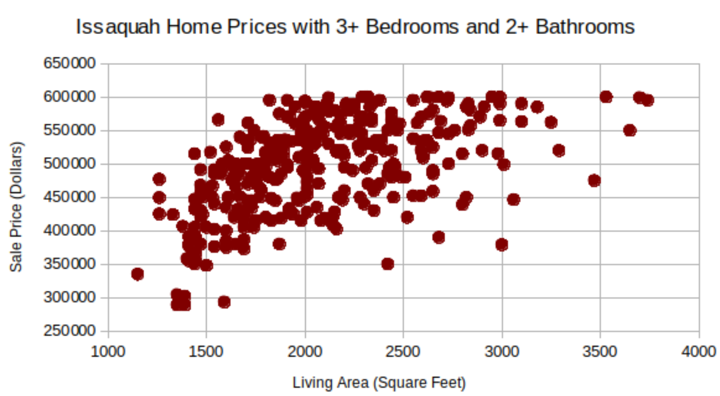


We immediately notice a number of things. First, with such a large data set, it is apparent that data is messy. There appears to be a predictable trend but for any living area, there is a very wide price range. Most people looking to buy or sell a house will also not be interested in the data of the very expensive houses, so the data on the large and expensive houses could be ignored. Also, King County covers a large area: Seattle, many of its suburbs, and areas within the Cascade Range. Houses in the city and upscale areas of the suburbs will sell for much more than a similarly sized house in the mountains. For this reason, data and models need to be considered in context.

Now suppose that you are helping a family with three children find a house in Issaquah. The father is a software engineer and the mother homeschools the children and based on their budget, they would consider a house no more than \$400,000. Would you show them this chart? Of course not. We will sort the data and consider homes in Issaquah (ZIP codes 98027, 98029, and 98075) that have sold for \$400,000 or less. However, to attempt to establish a more accurate trend, we will plot data on house prices up to \$600,000.

- Highlight the spreadsheet by clicking on the empty box next to the `A` column designation.
- Click on `Data` in the menu.
- Select `Sort...`
- On the `Options` tab, select `Range contains column labels`.
- Go back to `Sort Criteria`. Under `Sort Key 1`, select the ZIP code and `Ascending`.
- Under `Sort Key 2`, select the price and `ascending`.

Gathering this data into a separate spreadsheet, we now have 464 entries. The family wants at least three bedrooms and at least two bathrooms. This narrows our data set to 345 entries, which is more readable.



Given this level of context, you, as the real estate agent, and the family have a more accurate picture of the relationship between living area and home price. This would better equip you to make a reasonable offer on a house that comes up for sale.

The data is still quite messy. It's called a scatter plot for a reason! Though we have elimined much of the data by drilling down and focusing on certain values for some variables, there are more variables to consider. Throughout the book, we will explore various ways to model this data with a curve.

1.2.3 Curve Fitting, Interpolation, and Extrapolation

A common and simple way to create a mathematical model from a scatter plot of a data set is to determine a function that is a reasonable fit to the curve, given the scenario. Figure 1.6 gives an example of a scatter plot of data from Table 1.8 that will be used in Example 1.2.1. **Curve fitting** is a technique in which one creates a continuous function to smooth out discrete data. The curve will generally not "connect the dots" of a scatter plot of the data, but will give the general behavior of the data. The most common and well-known means to fit a curve to data is by creating a **regression curve** of the data. The mathematical details of how to make these curves is outside the scope of this text. It is sufficient to understand that regression curves are **curves of best fit** or **best fit curves**.

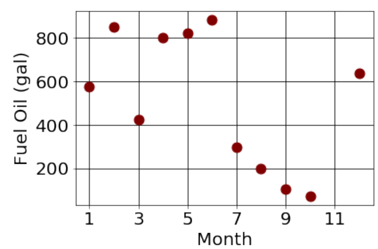


Figure 1.6: Fuel Oil Usage by Plant W in 2016.

With an appropriate curve fit to data, we can **interpolate** and **extrapolate** the data.

Definition 1.2.2. Using the graph of a function to estimate values between known data points (i.e., within the domain) is called **interpolation**. Making predictions beyond the domain of known data is called **extrapolation**. When a model no longer applies after some point, and extrapolation is unreasonable it is sometimes called **model breakdown**.

Interpolation. In cases in which there is missing data, we can use interpolation techniques to make educated guesses for the actual data. This is just one example in which interpolation is used. The following example uses regression curves and another simpler technique.

Example 1.2.1. Plant W, mentioned earlier, powers their operations by burning fuel oil. Table 1.8 below shows how much oil they burned in 2016, but they are missing data from November. (Figure 1.6 gave a scatter plot of this data.) What are some reasonable values for the amount of fuel oil that they burned in November 2016?

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Oil (gal)	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0	??	638.0

Table 1.8: Fuel Oil Usage at Plant W in 2016

Solution: We will show two ways to estimate the missing data point. The first is straightforward. The other will use a model developed from a regression curve, applying concepts and techniques that we will learn more about in Section 1.4.

A simple way to interpolate a missing data point is to find the average (or mean) of the data point before and the data point after the missing point. With this approach, we have an estimate for the November 2016 fuel oil consumption of

$$\frac{72.0 + 638.0}{2} \text{ gallons} = \frac{710.0}{2} \text{ gallons} = 355 \text{ gallons}.$$

The second method creates a model by finding a curve the best fits the data in Table 1.8. The volume of fuel oil burned by Plant W in month m of 2016 is

$$f(m) = 5.76m^3 - 109.98m^2 + 532.58m + 70.17 \text{ gallons}.$$

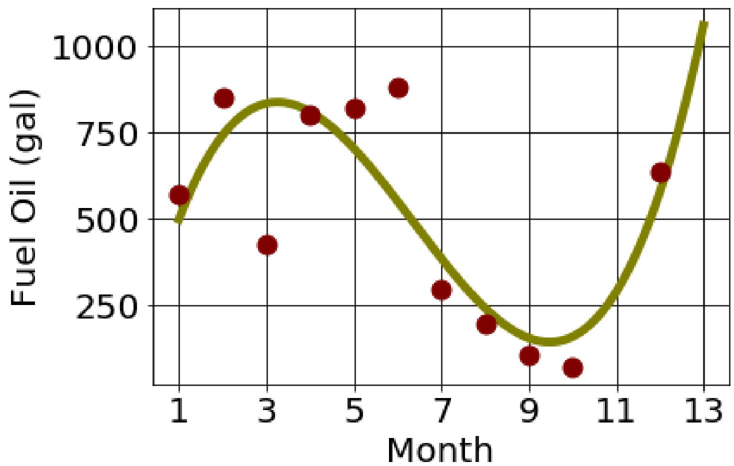


Figure 1.7: Fuel Oil Usage by Plant W in 2016.

This model is plotted with the data in Figure 1.7, showing that the model is reasonable.

From this model, we can estimate that Plant W burned

$$f(11) = 5.76 \cdot 11^3 - 109.98 \cdot 11^2 + 532.58 \cdot 11 + 70.17 \text{ gallons} = 287.53 \text{ gallons}$$

in November (month $m = 11$). ■

Extrapolation. In contrast to interpolation, extrapolation is more difficult because it involves predicting data points beyond the domain of the data using the trends that currently exist. Extra care must be used to create and justify assumptions used to develop the model that is used to extrapolate.

Let's discuss the following table and plot of winning Men's and Women's 100 meter dash times in the Olympics. As of this writing, the next summer Olympics will be in 2020, so we can extrapolate from the given data to predict the winning times in the next Olympics. When we plot the data, however, many more questions will naturally arise and the answers to those questions will vary depending on the model that we use to describe the data.

Year	1928	1932	1936	1948	1952	1956	1960	1964	1968	1972	1976
Time (M, s)	10.8	10.3	10.3	10.3	10.4	10.5	10.2	10.0	9.95	10.14	10.06
Time (W, s)	12.2	11.9	11.5	11.9	11.5	11.5	11.0	11.4	11.0	11.07	11.08
Year	1980	1984	1988	1992	1996	2000	2004	2008	2012	2016	2020
Time (M, s)	10.25	9.99	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81	????
Time (W, s)	11.06	10.97	10.54	10.82	10.94	11.12	10.93	10.78	10.75	10.71	????

Table 1.9: Winning Men's and Women's 100 Meter Dash Times in the Olympics

Figure 1.8 gives a scatter plot of the data. It's clear that there has been a downward trend in the gold medal times over the past century, but it's not a smooth trend. The data is choppy. The simplest curve-fitting model to smooth out the data is to use a best fit line. In the next section, we will learn how to find best fit lines, but for now, we will describe the model and discuss the model.

Model: Let y be the year and $1928 \le y \le 2016$. Then the winning men's and women's 100-meter dash times in the Olympics in year y can be described by $m(y)$ and $w(y)$, respectively.

$$\begin{aligned} m(y) &= -0.009498y + 28.841 \text{ seconds} \qquad (1.1) \\ w(y) &= -0.014185y + 39.188 \text{ seconds} \end{aligned}$$

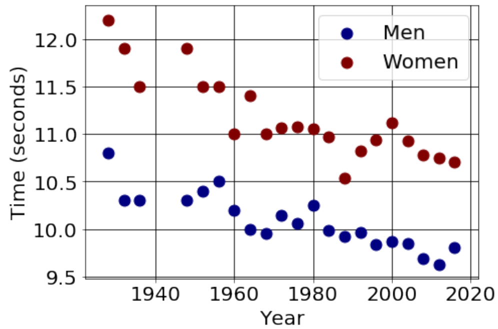


Figure 1.8: Winning Olympic 100-Meter Dash Times

Note that we have all the necessary components of a model: (1) a description of the input variable (y), with the units (years), (2) a description of the functions (m and w) and their units (seconds), and (3) a domain over which the models make sense. A plot of the models with the scatter plot of the data shows that the models make sense.

Example 1.2.2. Given the data in Table 1.9 and the models in (1.1), what are reasonable predictions for the winning times in the 100-meter dash at the 2020 Olympics?
Solution: Since y is the year of the Olympics, then we predict the winning time for the men's 100-meter dash to be

$$m(2020) = -0.009498 \cdot 2020 + 28.841 \text{ seconds} = 9.66 \text{ seconds}$$

and for the women's 100-dash:

$$w(2020) = -0.014185 \cdot 2020 + 39.188 \text{ seconds} = 10.53 \text{ seconds}.$$

These times aren't completely unreasonable, given the data, but taken in context, one might be a bit skeptical of these predictions. For the men's 100-meter dash, the time of 9.66 would be only 0.03 slower than the Olympic record, held by Usain Bolt of Jamaica. Bolt ran the three fastest Olympic 100-meter dashes at the 2008, 2012, and 2016 Olympics, but has since retired from sprinting. The forecast women's time would beat Florence Griffith-Joyner's world record, set in 1988, by 0.01 second. ■

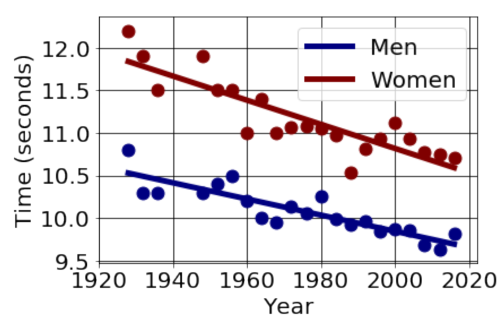


Figure 1.9: Winning Olympic 100-Meter Dash Times with Models

To create a better model, we could include data from other races, not just the Olympics. We could also consider curves other than lines. To see why extrapolation has it's limitations with the linear model, consider the predicted winning times for the Olympics in the year 3000. For the men:

$$m(3000) = -0.009498 \cdot 3000 + 28.841 \text{ seconds} = 0.347 \text{ seconds}$$

and for the women's 100-meter dash:

$$w(3000) = -0.014185 \cdot 3000 + 39.188 \text{ seconds} = -3.37 \text{ seconds}.$$

These times are clearly absurd. The men's time would require a runner to run faster than a race car and for the women, no one can run a race in negative time. Therefore, the further out we attempt to extrapolate, the less plausible and more uncertain the results are.

The plot also gives us a question to consider. Notice that the women's times are decreasing more rapidly than the men's times. Will a woman ever run the 100 meter dash faster than a man in a single Olympics? The model predicts that this could happen, but the winning times would again be absurd.

1.3 Linear Functions

1.3.1 Linear Function Basics

In Section 1.1.4, we had the example of a screen printing shop that charges \$50 to cover the overhead costs of setting the screen to make T-shirts and \$5 per shirt for the shirts themselves. Using descriptive variables, we chose n for the number of T-shirts and C for the cost in dollars as a function of the number of shirts: $C(n)$.

We know that $C(0) = 50$, since the overhead, or fixed, costs are charged regardless of the number of T-shirts made. Since \$5 is added for each T-shirt ordered, then $C(1) = \$50 + \$5 = \$55$, $C(2) = \$55 + \$5 = \$60$, and so on.

If n T-shirts are ordered, then

$$C(n) = \$50 + \left(\frac{\$5}{\text{T-shirt}}\right)(n \text{ T-shirts}) = 50 + 5n \text{ dollars} \ .$$

Notice how the units (or dimensions) make sense in this equation. Each term has units of dollars, after cancelling the unit of “T-shirt” in the second term.

Notice this equation consisted of two quantities. The first is the fixed \$50 charge which does not change based on the value of the input. When no T-shirts are ordered, the cost is \$50, giving the point $(0, 50)$ on the graph of the function. This is the **vertical** or y-intercept. The second is the \$5 per T-shirt, which is a **rate of change**. In $C(n)$, this rate of change is multiplied by the input value. It is important here to note that in this equation, the rate of change is **constant**; over any interval, the rate of change is the same.

Looking at $C(n)$ as a table, we can also see the cost changes by \$5 for each additional T-shirt.

n	0	1	2	3
$C(n)$	50	55	60	65

(Even though we cannot order a fractional number of T-shirts, we still graph $y = C(n)$ on all $n \geq 0$.)

The graph is increasing in a straight line from left to right because for each T-shirt, the cost goes up by \$5; this rate remains consistent.

In this example, you have seen the T-shirt cost modeled in words, an equation, a table, and as a graph. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all four representations for a problem and so being able to work with all four forms is very important.

The function $C(n)$ is an example of a **linear function**. This name comes from the fact that a graph of a linear function is a line.

Definition 1.3.1. A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$f(x) = mx + b \ ,$$

where:

- b is the initial or starting value of the function (when the input, $x = 0$), and
- m is the constant rate of change of the function.

This form of the line is called the **slope-intercept form**.

Many people like to write linear functions in the form $y = mx + b$ because it corresponds to the way we tend to speak: “The output starts at b and increases at a rate of m.”

Slope and Increasing/Decreasing The constant rate of change of a linear function, m, is also called the **slope** of the function. The slope determines if the function is an increasing function or a decreasing function.

- $f(x) = mx + b$ is an **increasing** function if $m > 0$.
- $f(x) = mx + b$ is a **decreasing** function if $m < 0$.

If $m = 0$, then the rate of change of $f(x) = mx + b$ is zero, and $f(x) = 0 \cdot x + b = b$, so its graph is just the horizontal line passing through the point $(0, b)$, neither increasing nor decreasing.

The concepts of slope and rate of change are major component of calculus. It is crucial to understand each facet of these two concpets that we will discuss in this section since these conepts will be generalized from linear functions to a wide array of functions in Chapter 2.

Example 1.3.1. Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year?
Solution: The initial value for this function is 200, since he currently owns 200 songs, $N(0) = 200$ songs. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula:

$$N(m) = 200 \text{ songs} + \left(\frac{15 \text{ songs}}{\text{month}}\right)(m \text{ months}) = 200 + 15m \text{ songs} \enspace .$$

With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 200 + 15 \cdot 12 \text{ songs} = 200 + 180 = 380 \text{ songs} \enspace .$$

Marcus will have 380 songs in a year. ■

Calculating Rate of Change Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, (x_1, y_1) and (x_2, y_2), we can find a linear function that contains both points. First we calculate the rate of change, m, of the function.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$$

It is also customary to write Δv for a change (or difference) in the variable v. We read this "delta v". With this notation, we can write

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \enspace .$$

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \enspace .$$

Once we have computed m, we can use either of the given points to find b using algebra.

$$\begin{aligned} y_1 &= mx_1 + b \\ b &= y_1 - mx_1 \end{aligned}$$

Remark 1.3.1. It is a waste of your time to make a special effort to memorize this following formula. This is a concept to understand.

Example 1.3.2. The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

Solution: The rate of change will relate the change in population to the change in time. The population increased by $27800 - 23400 = 4400$ people over the four-year time interval. To find the rate of change, the number of people per year the population changed by:

$$m = \frac{27800 - 23400 \text{ people}}{2006 - 2002 \text{ years}} = \frac{4400 \text{ people}}{4 \text{ years}} = 1100 \text{ people per year} \ .$$

■

Notice that we knew the population was increasing, so we expected $m > 0$. This is a quick way to check if the solution is reasonable.

Example 1.3.3. The pressure, P, in pounds per square inch (psi) on a diver depends upon his depth below the water surface, d, in feet, following the equation

$$P(d) = 14.696 + 0.434d \ .$$

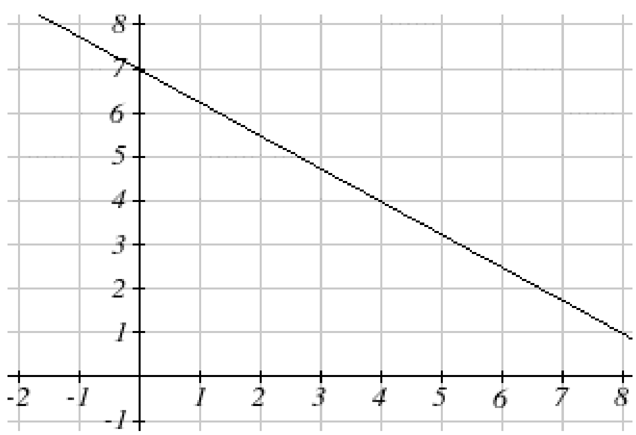
Interpret the components of this function.

Solution: The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{psi}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 psi for each foot his depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 psi. ■

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

Example 1.3.4. Write an equation for the linear function graphed below.



Solution: Looking at the graph, we might notice that it passes through the points $(0, 7)$ and $(4, 4)$. From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4 - 7}{4 - 0} = \frac{-3}{4} \ .$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x \ .$$

■

Example 1.3.5. If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.

Solution: The rate of change (or slope) of the function is $m = \frac{1-(-2)}{8-3} = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3:

$$f(3) = b + \frac{3}{5} \cdot 3 \ .$$

Since we know that $f(3) = -2$, we can substitute on the left side

$$-2 = b + \frac{3}{5} \cdot 3 \ .$$

This leaves us with an equation we can solve for the initial value

$$b = -2 - \frac{9}{5} = -\frac{19}{5} \ .$$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = -\frac{19}{5} + \frac{3}{5}x \ .$$

■

Example 1.3.6. Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so his weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.
Solution: The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change of $I(n)$.

$$m = \frac{920 - 760 \text{ dollars}}{5 - 3 \text{ policies}} = \frac{160}{2} \text{ dollars per policy} = \$80 \text{ per policy} \ .$$

Keeping track of units, as we did above, can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

Now let's solve for the initial value, b. We have $I(n) = b + 80n$ and when $n = 3$, $I(3) = 760$, so

$$760 = b + 80 \cdot 3 = b + 240 \ .$$

This allows us to solve for b:

$$b = 760 - 240 = 520 \ .$$

This form allows us to see the starting value for the function: 520. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week. ■

1.3.2 Graphs of Linear Functions

When we are working with a new function, it is useful to know as much as we can about the function: its graph, where the function is zero, and any other special behaviors of the function. We will begin this exploration of linear functions with a look at graphs.

When graphing a linear function, there are two basic ways to graph it.

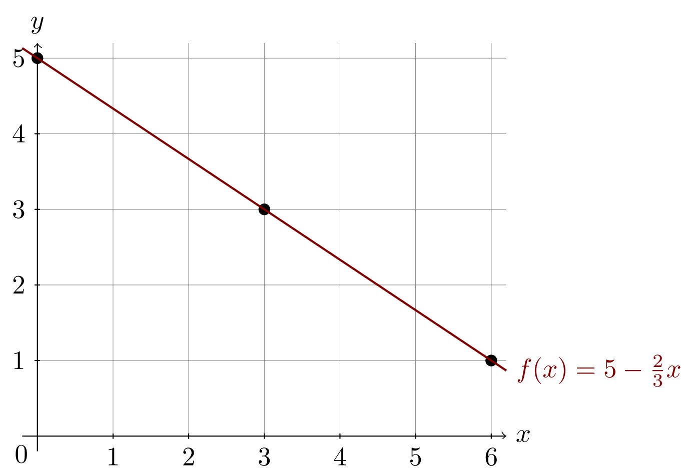
1. Plot at least two points and draw a line through the points.

2. Use the initial value (output when $x = 0$) and the rate of change (slope).

Example 1.3.7. Graph $f(x) = 5 - \frac{2}{3}x$ by plotting points.
Solution: In general, we evaluate the function at two or more inputs to find at least two points on the graph. Usually it is best to pick input values that will "work nicely" in the equation. In this equation, multiples of 3 will work nicely due to the in the equation, and of course using $x = 0$ to get the vertical intercept. Evaluating $f(x)$ at $x = 0, 3,$ and 6:

$$f(0) = 5 - \frac{2}{3} \cdot 0 = 5$$
$$f(3) = 5 - \frac{2}{3} \cdot 3 = 3$$
$$f(6) = 5 - \frac{2}{3} \cdot 6 = 1$$

These evaluations tell us that the points $(0, 5)$, $(3, 3)$, and $(6, 1)$ lie on the graph of the line. Plotting these points and drawing a line through them gives us the graph.



■

When using the initial value and rate of change to graph, we need to consider the graphical interpretation of these values. Remember the initial value of the function is the output when the input is 0, so in the equation $f(x) = b + mx$, the graph includes the point $(0, b)$. On the graph, this is the vertical intercept – the point where the graph crosses the vertical axis.

For the rate of change, it is helpful to recall that we calculated this value as

$$m = \frac{\text{change of output}}{\text{change of input}}$$

From a graph of a line, this tells us that if we divide the vertical difference, or rise, of the function outputs by the horizontal difference, or run, of the inputs, we will obtain the rate of change, also called slope of the line.

$$m = \frac{\text{change of output}}{\text{change of input}} = \frac{\text{rise}}{\text{run}}$$

Notice that this ratio is the same regardless of which two points we use.

Graphical Interpretation of a Linear Equation. Graphically, in the equation, $f(x) = b + mx$,

- b is the **vertical intercept** of the graph and tells us we can start our graph at $(0, b)$
- m is the **slope of the line** and tells us how far to rise and run to get to the next point.

Once we have at least two points, we can extend the graph of the line to the left and right.

Example 1.3.8. Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope.
Solution: The vertical intercept of the function is $(0, 5)$, giving us a point on the graph of the line. The slope is $\dfrac{-2}{3}$. This tells us that for every 3 units the graph "runs" in the horizontal, the vertical "rise" decreases by 2 units. In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value $(0, 5)$, the slope tells us that if we move to the right 3 units, we will move down 2 units, moving us to the point $(3, 3)$. We can continue this again to find a third point at $(6, 1)$. Finally, extend the line to the left and right, containing these points.

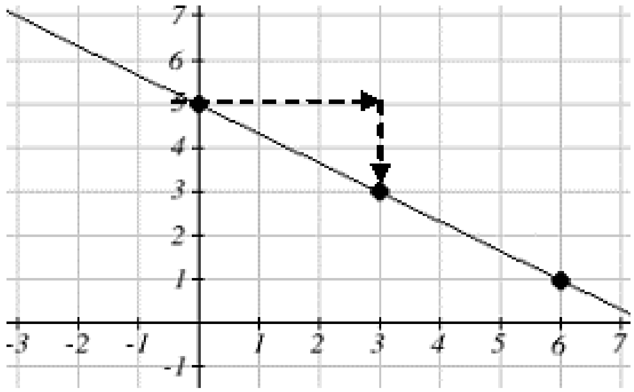


Figure 1.10

■

Figure 1.11 below gives some examples of the graph of $f(x) = mx$ to show the effect of the slope, m, on the shape of the graph.

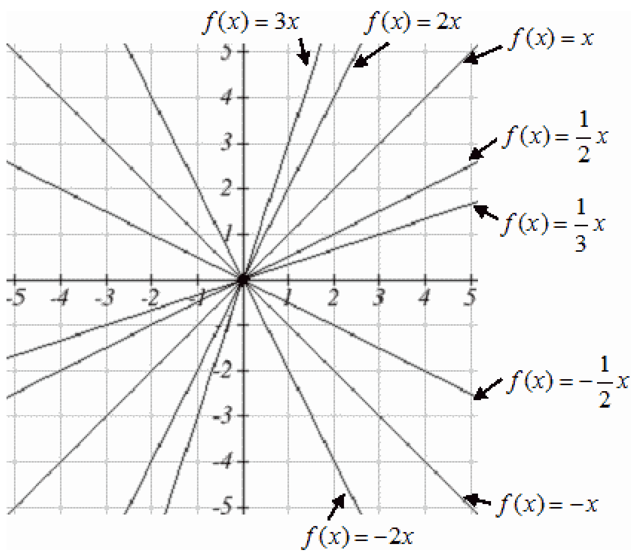


Figure 1.11: $f(x) = mx$ for several values of m.

In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Figure 1.12 gives some examples.

Example 1.3.9. Match each function with one of the lines in Figure 1.13.

$$f(x) = 2x + 3 \qquad g(x) = 2x - 3 \qquad h(x) = -2x + 3 \qquad j(x) = \frac{1}{2}x + 3$$

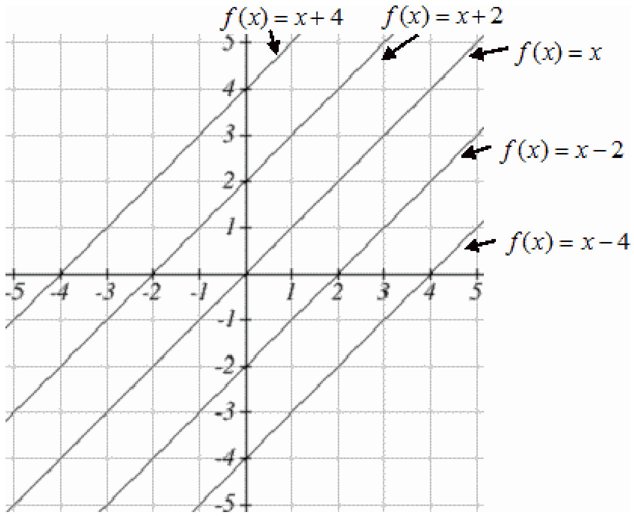


Figure 1.12: $f(x) = mx$ for several values of b.

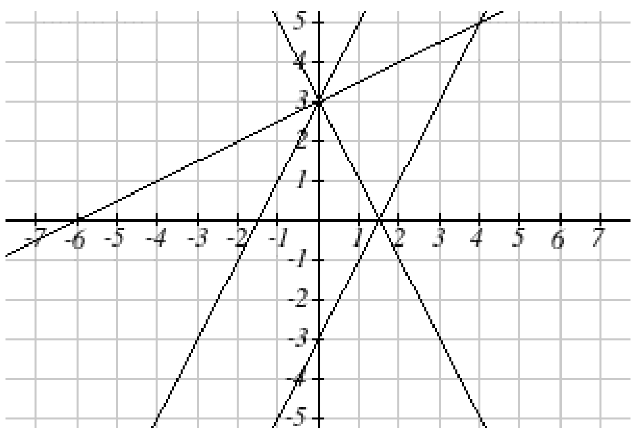


Figure 1.13

Solution: Only one graph has a vertical intercept of -3, so we can immediately match that graph with $g(x)$. For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with $h(x)$. Of the other two, the steeper line would have a larger slope, so we can match that graph with $f(x)$, and the flatter line with $j(x)$.

■

In addition to understanding the basic behavior of a linear function (increasing or decreasing, recognizing the slope and vertical intercept), it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

Finding Horizontal Intercepts

Definition 1.3.2. The **horizontal intercept** of the function is where the graph crosses the horizontal axis. If a function has a horizontal intercept, you can always find it by solving $f(x) = 0$ for x.

Example 1.3.10. Find the horizontal intercept of $f(x) = -3 + \frac{1}{2}x$

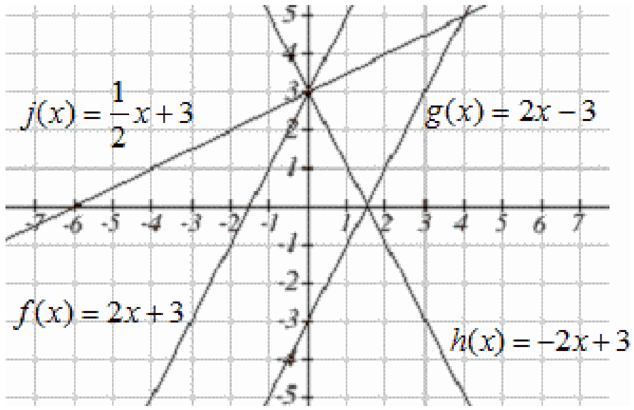


Figure 1.14

Solution: Setting the function equal to zero to find what input will put us on the horizontal axis,

$$\begin{aligned} 0 &= -3 + \frac{1}{2}x \\ 3 &= \frac{1}{2}x \\ x &= 6 \ . \end{aligned}$$

The graph crosses the horizontal axis at the point $(6, 0)$. ■

Intersections of Lines The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

Economics tells us that in a free market, the price for an item is related to the quantity that producers will supply and the quantity that consumers will demand. Increases in prices will decrease demand, while supply tends to increase with prices. Sometimes supply and demand are modeled with linear functions.

Example 1.3.11. The supply, in thousands of items, for custom phone cases can be modeled by the equation $s(p) = 0.5 + 1.2p$ while the demand can be modeled by $d(p) = 8.7 - 0.7p$, where p is in dollars. Find the equilibrium price and quantity, the intersection of the supply and demand curves.
Solution: Setting $s(p) = d(p)$, we find

$$\begin{aligned} 0.5 + 1.2p &= 8.7 - 0.7p \\ 1.9p &= 8.2 \\ p &= \frac{8.2}{1.9} \approx 4.32 \end{aligned}$$

We can find the output value of the intersection point by evaluating either function at this input:

$$s(4.32) = 0.5 + 1.2(4.32) \approx 5.68 \ .$$

These lines intersect at the point $(4.32, 5.68)$. Therefore, the equilibrium price is \$4.32 and the equilibrium quantity is 5,680 items. Looking at the graph in Figure 1.15, this result seems reasonable. ■

1.3.3 Modeling with Linear Functions

Here are a number of examples of modeling with linear functions.

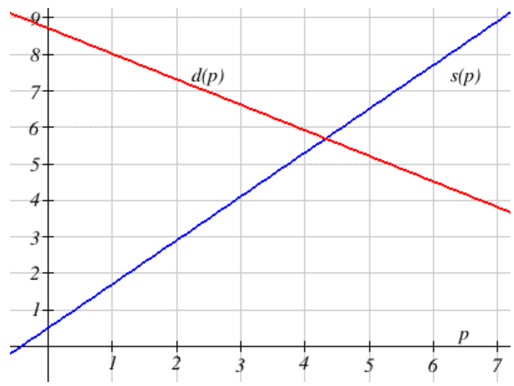


Figure 1.15

Example 1.3.12. Emily saved up \$3500 for her summer visit to Seattle. She anticipates spending \$400 each week on rent, food, and fun. Find and interpret the horizontal intercept and determine a reasonable domain and range for this function.
Solution: In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.

Input: t, time, in weeks

Output: M, money remaining, in dollars

Reading the problem, we identify two important values. The first, \$3500, is the initial value for M. The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

To answer the first question, looking for the horizontal intercept, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation: $M(t) = 3500 - 400t$.

To find the horizontal intercept, we set the output to zero, and solve for the input, t:

$$\begin{aligned} 0 &= 3500 - 400t \\ 400t &= 3500 \\ t &= \frac{3500}{400} = 8.75 \end{aligned}$$

The horizontal intercept is 8.75 weeks. Since this represents the input value where the output will be 0, interpreting this, we could say: "Emily will have no money left after 8.75 weeks."

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than 0. It is also likely that this model is not valid after the horizontal intercept (unless Emily's going to start using a credit card and go into debt).

The domain represents the set of input values and so the reasonable domain for this function is $0 \leq t \leq 8.75$.

However, in a real world scenario, the rental might be weekly or nightly. She may not be able to stay a partial week and so all options should be considered. Emily could stay in Seattle for 0 to 8 full weeks (and a couple of days), but would have to go into debt to stay 9 full weeks, so restricted to whole weeks, a reasonable domain without going in to debt would be $0 \leq t \leq 8$, or $0 \leq t \leq 9$ if she went into debt to finish out the last week.

The range represents the set of output values and she starts with \$3500 and ends with \$0 after 8.75 weeks so the corresponding range is $0 \leq M(t) \leq 3500$. If we limit the rental to whole weeks, however, the range would change. If she left after 8 weeks because she didn't have enough to stay for a full 9 weeks, she would have $M(8) = 3500 - 400 \cdot 8 = \$300$ left after 8 weeks, giving a range of $300 \leq M(t) \leq 3500$. If she wanted to stay the full 9 weeks she would be \$100 in debt giving a range of $-100 \leq M(t) \leq 3500$. ∎

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

Example 1.3.13. Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of \$20, then 59 cents per mile. The second, Budget, charges an up-front fee of \$16, then 63 cents per mile.[2] When will U-Haul be the better choice for Jamal?

Solution: The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input: m, miles driven

Outputs:

$Y(m)$: cost, in dollars, for renting from U-Haul

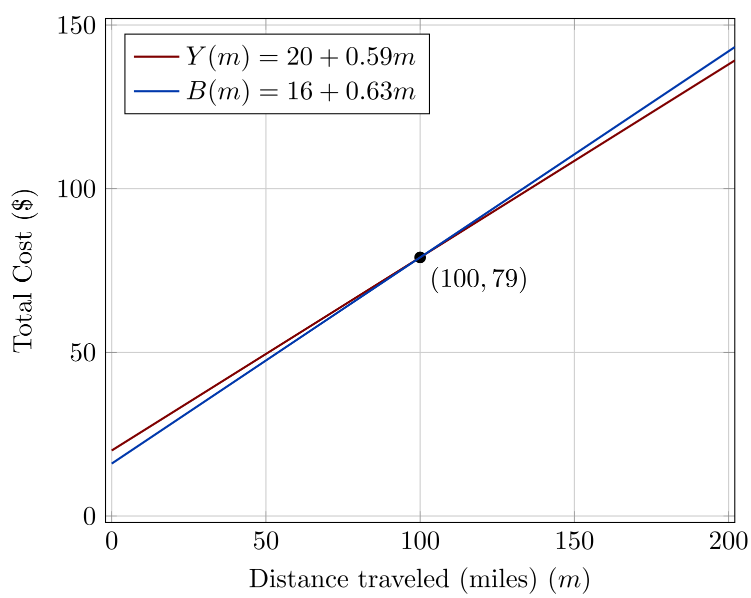
$B(m)$: cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: \$0.59 per mile for U-Haul, and \$0.63 per mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where $Y(m)$ is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$
$$B(m) = 16 + 0.63m$$

These graphs are plotted below.



[2]Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/.

To find the intersection, we set the equations equal to each other and solve for m.

$$\begin{aligned} Y(m) &= B(m) \\ 20 + 0.59m &= 16 + 0.63m \\ 4 &= 0.04m \\ m &= \frac{4}{0.04} = 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if you drive 100 miles. Either by looking at the graph, or noting that $Y(m)$ is growing at a slower rate, we can conclude that U-Haul would be the cheaper option if you drive more than 100 miles. ■

Example 1.3.14. A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. If this trend continues,

a) Predict the population in 2013.

b) When will the population reach 15,000?

Solution: The two changing quantities are the population and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2004.

Input: t, years since 2004

Output: $P(t)$, the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2009 would correspond to $t = 5$, giving us the point $(5, 8100)$.

a) To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15000, we would need to solve for the input that would provide an output of 15000. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{8100 - 6200}{5 - 0} = \frac{1900}{5} = 380 \text{ people per year} .$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$P(t) = 6200 + 380t \ .$$

To predict the population in 2013, we evaluate our function at $t = 9$:

$$P(9) = 6200 + 380 \cdot 9 = 9620 \ .$$

If the trend continues, our model predicts a population of 9,620 in 2013.

b) To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} P(t) &= 15000 \\ 6200 + 380t &= 15000 \\ 380t &= 8800 \\ t &= \frac{8800}{380} \approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

■

1.3.4 Fitting Linear Models to Data

In the real world, things rarely follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use a linear model to make predictions about the data, inside and outside of the data range.

Example 1.3.15. The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit[3]. Plot this data, and determine whether the data appears to be linearly related.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temp. (°F)	80.5	70.5	57	66	68	72	52	73.5	53

Table 1.10: Cricket Chirps in 15 Seconds Versus Temperature.

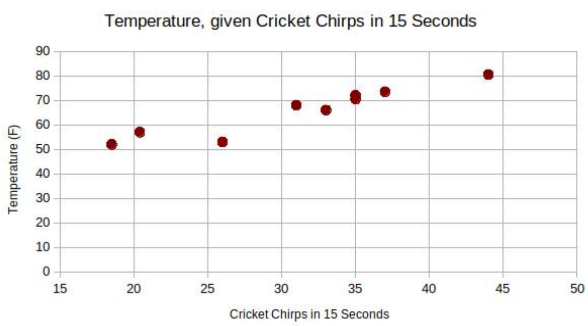
Solution: Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so. We will plot this data in a spreadsheet. First, we put the data in a spreadsheet.

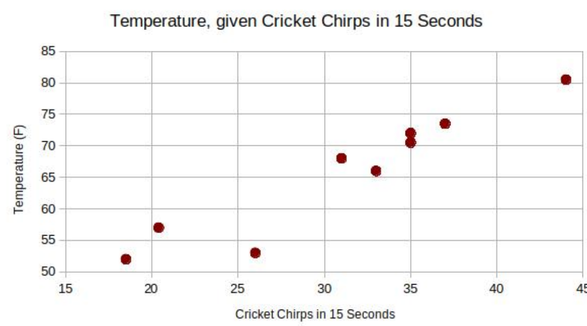
	A	B	C	D	E	F	G	H	I	J
1	**Chirps**	44	35	20.4	33	31	35	18.5	37	26
2	**Temp. (°F)**	80.5	70.5	57	66	68	72	52	73.5	53

Table 1.11

First we make a scatter plot. See Section 1.2.2 for the details on how we create a scatterplot in *LibreOffice Calc*.

Remark 1.3.2. The default scatter plot is on the left and a revised plot is on the right. Notice that the default plot includes a lot of unnecessary "white space," with a range of 0°F to 90°F. A more reasonable range is 50°F to 85°F. Within *Calc*, clicking on the y-axis will allow us to adjust the chart to make it more presentable. We can likewise trim the domain slightly.





■

The simplest way to find an equation to approximate this data is to try to "eyeball" a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

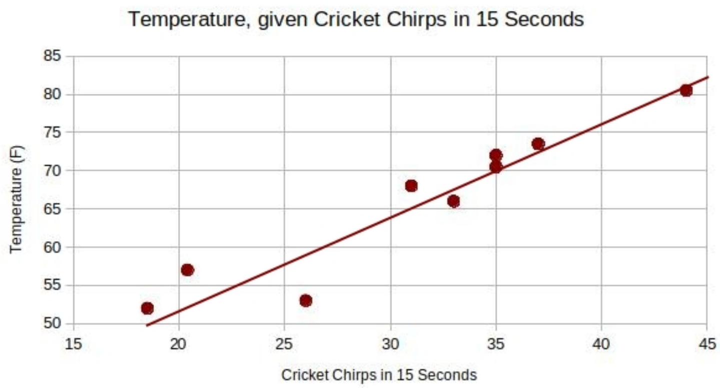
You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

Questions to Consider:

[3]Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010

- What descriptive variables would you choose to represent temperature and chirps?
- Which variable is the independent variable and which is the dependent variable?
- Based on this data and the graph, what is a reasonable domain and range?

Example 1.3.16. Using the table of values from the previous example, find a linear function that fits the data by "eyeballing" a line that seems to fit.
Solution: On a graph, we could try sketching in a line. The figure below has a rough line that approximates the trend of the data.



To find an equation of the line, we pick two points on the line. The best places to pick points are he intersection of grid lines, such as $(35, 70)$. Others would be points near extreme points of the line on a grid line, such as $(20, 52)$ and $(45, 82)$; 52 and 82 are estimates. Based on these latter two points, the line has a slope of $m = \dfrac{82-52}{45-20} = \dfrac{30}{25} = 1.2$. Now we find the vertical intercept at 30. Since the line has an equation of the form $y = 1.2x + b$ and one point has $x = 35$ and $y = 70$, we have

$$\begin{aligned} 70 &= 1.2 \cdot 35 + b \\ &= 42 + b \\ b &= 70 - 42 = 28 \ . \end{aligned}$$

This gives us the model

$$T(c) = 28 + 1.2c \ ,$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. ■

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. Recall the notions of interpolation and extrapolation.

Example 1.3.17. Consider the cricket chirp data in Table 1.11.

a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

Solution:

a) With our cricket data, the number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so this would be a case of interpolation. Using our model:
$$T(28) = 28 + 1.2 \cdot 30 = 64°\text{F} \ .$$
Based on the data we have, this value seems reasonable.

b) The temperature values varied from 52°F to 80.5°F. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$\begin{aligned} 40 &= 28 + 1.2c \\ 12 &= 1.2c \\ c &= \frac{12}{1.2} \approx 10 \end{aligned}$$

Our model predicts the crickets would chirp 10 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, crickets generally stop chirping altogether below around 50°F. Therefore, our prediction is likely unreasonable.

■

1.3.5 Fitting Lines with a Spreadsheet.

In this section, we will use *LibreOffice Calc* to compute the **line of best fit** or **regression line** for various data sets.

Example 1.3.18. Find the least-squares regression line using the cricket chirp data from Table 1.11.
Solution: Double-click on the chart made from the data in Table 1.11.

- Click on a data point so that the points turn green.
- Select the menu item `Trend Line...`. This will bring up a regression curve wizard.
- Click on the `Type` tab.
- Make the selections given in Figure 1.16.

The new scatter plot in Figure 1.17 now has the linear regression line. By default, the regression line has far too much precision than we need for this context. Two decimal place of precision is enough. We can make that change by clicking on the equation for the line of best fit and going to the `Numbers` tab. Therefore, the best-fit line is

$$T(c) = 1.14c + 30.28 \text{ degrees Fahrenheit.}$$

This line is very similar to the equation we "eyeballed," but it fits the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$T(30) = 1.14 \cdot 30 + 30.28 = 64.48°\text{F} \;.$$

■

Notice that the plot also includes another number: $R^2 = 0.90$. Most calculators and computer software will also provide you with this number, called the **coefficient of determination**, or the related **correlation coefficient**. These numbers measure how well a function models a set of data.

Definition 1.3.3 (Correlation Coefficient)**.** The **correlation coefficient** is a value, $-1 \leq r \leq 1$ that measures the strength of the relationship between two variables, x and y. $|r|$ measures the proportion of the output variable, y, can be explained by the input variable, x.

- $r > 0$ suggests a positive (increasing) relationship between x and y.
- $r < 0$ suggests a negative (decreasing) relationship between x and y.
- The closer r is to 0, the more scattered or **noncorrelated** the data.
- The closer r is to 1 or -1, the stronger the relationship the data is.

The **coefficient of determination** is the value $0 \leq r^2 \leq 1$, which measures the proportion of the variable y that is determined, or predicted, by x.

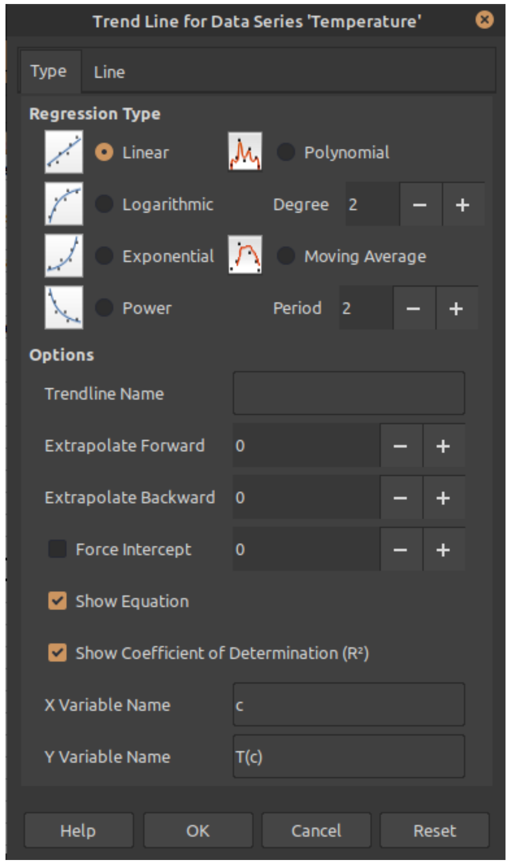


Figure 1.16

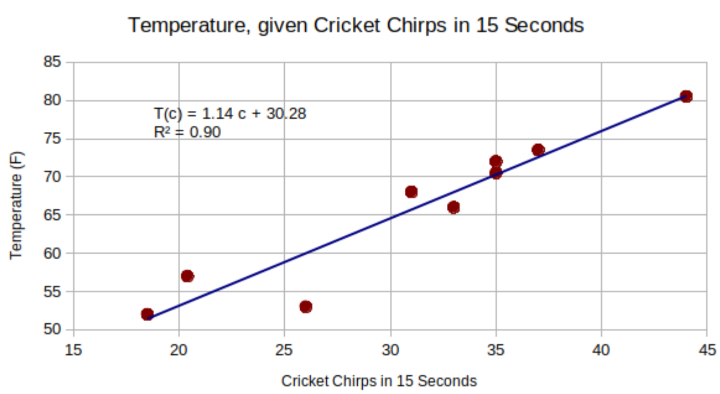


Figure 1.17

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of r and the graph of the data, here are some large data sets with their correlation coefficients:

Example 1.3.19. The coefficient of determination of the cricket chirp data is $r^2 = 0.90$. Since the linear regression line has a positive slope, the correlation coefficient is $r = \sqrt{0.9} = 0.95$. This is a very strong relationship between a cricket's chirp rate and the temperature. ■

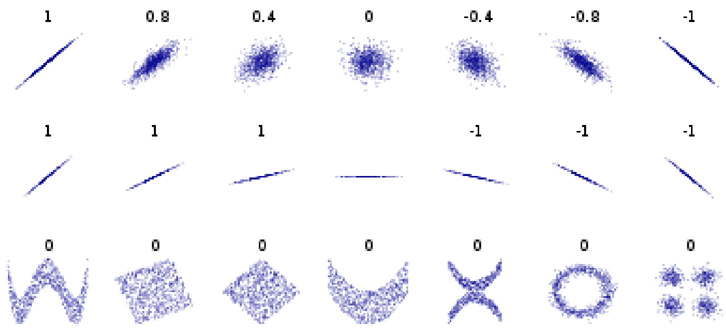


Figure 1.18: Comparing data sets with correlation coefficients [4]

We can compute the correlation coefficient directly in *Calc*.

Example 1.3.20. Compute the correlation coefficient of the following data, repeated from Table 1.11.
Solution: In a new box, type the following command and hit `Enter`.

	A	B	C	D	E	F	G	H	I	J
1	**Chirps**	44	35	20.4	33	31	35	18.5	37	26
2	**Temp. (°F)**	80.5	70.5	57	66	68	72	52	73.5	53

```
=CORREL(B1:J1, B2:J2)
```

We find $r = 0.9509$. ■

Example 1.3.21. Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below.[5] Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Gasoline consumption (billion gallons)	113	116	118	119	123	125	128	126	131	133	136

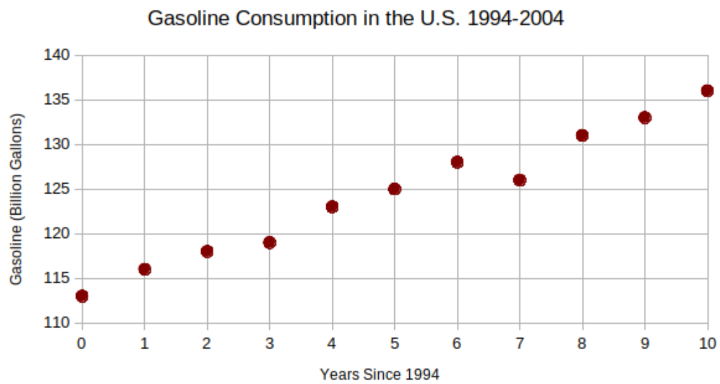
Table 1.12: Gasoline consumption in the United States from 1994 to 2004.

Solution: To make things simpler, a new input variable is introduced, t, representing years since 1994. We make the following spreadsheet.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Year**	0	1	2	3	4	5	6	7	8	9	10
2	**Gas Consumption**	113	116	118	119	123	125	128	126	131	133	136

The following is a scatterplot of the data.

[5] http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html



The data appears linear. Let's compute the correlation coefficient.

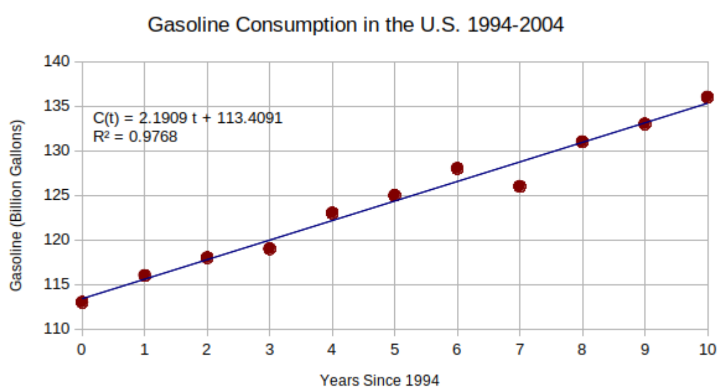
```
=CORREL(B1:L1, B2:L2)
```

We have $r = 0.9883$, suggesting a very strong increasing linear trend.

Let t be the number of years since 1994 and $C(t)$ be the volume of gasoline consumed in the United States, in billions of gallons. Then the least-squares regression equation is:

$$C(t) = 113.41 + 2.19t \ .$$

Here is the scatter plot with the best fit line.



Using this to predict gasoline consumption in 2008 ($t = 14$), we have

$$C(14) = 113.41 + 2.19 \cdot 14 = 144.07 \text{ billion gallons.}$$

■

1.3.6 Exercises

1. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation for the population t years after 2003.

2. A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation for the population t years after 2005.

3. Sonya is currently 10 miles from home, and is walking away from home at 3 miles per hour. Write an equation for her distance from home t hours from now.

4. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.
5. Timmy goes to the fair with \$40. Each ride costs \$2. How much money will he have left after riding n rides?
6. At noon, a barista notices she has \$20 in her tip jar. If she makes an average of \$0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

1.4 Polynomial Functions

Example 1.4.1. Suppose you ran a pizza shop. You make pizzas in sizes small (10 inches in diameter), medium (12"), large (14"), and extra large (16"). The prices for a cheese pizza are in the table below:

Size	S (10")	M (12")	L (14")	XL (16")
Price	\$7	\$9	\$12	\$16

There is demand for a "personal size" (extra small, XS) pizza with diameter 8" and a jumbo size (J) with diameter 18". Based on the prices above, what should you price these pizzas? **Solution:** Figure 1.19 plots the price data for the pizza. A quick look at the data might make you think that this is linear. However,

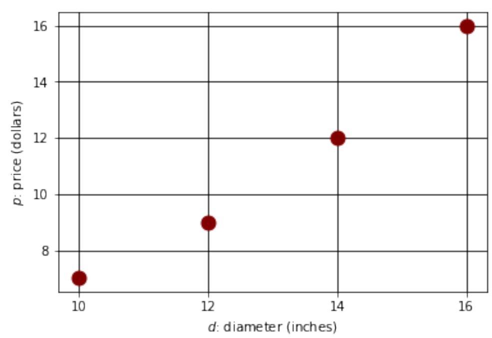


Figure 1.19: The price of a pizza with diameter d.

the formula for the area of a circle (the shape of a pizza) is $A = \pi r^2$, so the area of a pizza does not increase linearly. It makes the most sense to use a **quadratic** model to price the pizza in order to have a relatively constant price per square inch.

A **best fit curve** can be used to find the best quadratic function for this data. If d is the diameter of a pizza, then the price, $p(d)$ is

$$p(d) = 0.125d^2 - 1.75d + 12 \text{ dollars}$$

and fits the data perfectly. With this model, we have $p(8) = \$6$ and $p(18) = \$21$. ■

1.4.1 Quadratic Functions

Quadratics are transformations of the function $f(x) = x^2$. Quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 1.4.2. A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose three sides, and will put the fourth side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

Solution: In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the fourth side or

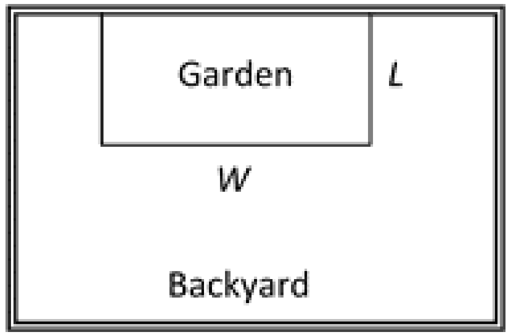


Figure 1.20

backyard fence. Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L: $W = 80 - 2L$.

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so $A = LW = L(80 - 2L)$, so

$$A(L) = 80L - 2L^2 \ .$$

This formula represents the area of the fence in terms of the variable length L. ■

Definition 1.4.1 (Forms of Quadratic Functions)**.** The **standard form** of a **quadratic function** is $f(x) = ax^2 + bx + c$.

The **transformation form** of a quadratic function is $f(x) = a(x - h)^2 + k$.

The **vertex** of the quadratic function is located at the point (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

Example 1.4.3. Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$.

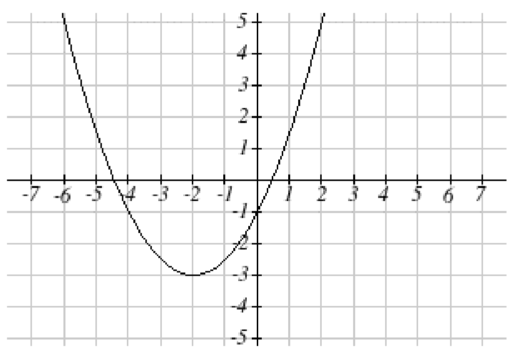


Figure 1.21

Solution: We can see the graph is the basic quadratic shifted to the left 2 and down 3, putting the vertex at the point $(-2, -3)$, giving an equation of the form $y = a(x + 2)^2 - 3$. By plugging in a point that falls on the grid, such as $(0, -1)$, we can solve for the stretch factor:

$$\begin{aligned} -1 &= a(0 + 2)^2 - 3 \\ 2 &= 4a \\ a &= \frac{1}{2} \ . \end{aligned}$$

Therefore, the equation that describes the graph is

$$y = \frac{1}{2}(x + 2)^2 - 3 \ .$$

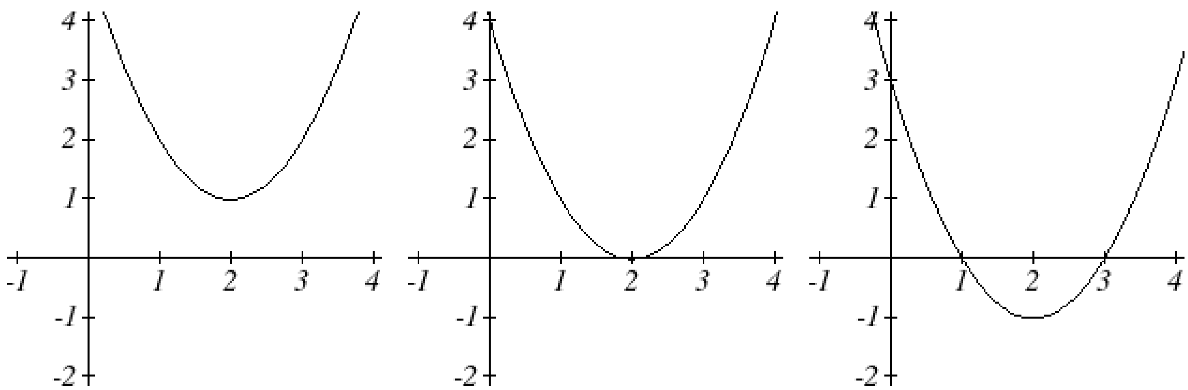
■

Short Run Behavior: Intercepts. As with any function, we can find the **vertical intercepts** of a quadratic function by evaluating the function at an input of 0, and we can find the **horizontal intercepts** by determining where the output is 0.

Definition 1.4.2. Let $f(x)$ be a function.

- The **vertical intercept** of the graph of $f(x)$ is the point $(0, f(0))$.
- A **horizontal intercept** of the graph of $f(x)$ is a point $(a, 0)$ such that $f(a) = 0$.
- A number a such that $f(a) = 0$ is called a **zero** or a **root** of $f(x)$.

Notice that a quadratic function can have zero, one, or two horizontal intercepts (or roots).



(a) No horizontal intercepts (b) One horizontal intercept (c) Two horizontal intercepts

Notice that in the standard form of a quadratic, the constant term c reveals the vertical intercept of the graph, since $f(0) = a(0)^2 + b(0) + c = c$.

Example 1.4.4. Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.
Solution: We can find the vertical intercept by evaluating the function at an input of 0:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2 \ .$$

So the vertical intercept is the point $(0, -2)$.
To find the horizontal intercepts of $y = f(x)$, we solve $f(x) = 0$ for x:

$$0 = 3x^2 + 5x - 2 \ .$$

In this case, the quadratic can be factored easily, providing the simplest method for solution:

$$0 = (3x - 1)(x + 2) \ ,$$

so either

$$\begin{array}{lcl} 0 = 3x - 1 & \text{or} & 0 = x + 2 \\ x = \dfrac{1}{3} & \text{or} & x = -2 \end{array}$$

So the horizontal intercepts are the points $\left(\dfrac{1}{3}, 0\right)$ and $(-2, 0)$. ■

When a quadratic is not factorable or is hard to factor, we can turn to the quadratic formula.

Theorem 1.4.1 (Quadratic Formula)**.** If $f(x) = ax^2 + bx + c$, then the **quadratic formula** gives the roots of $f(x)$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

Example 1.4.5. If a ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second, then the ball's height above ground after t seconds can be modeled by:

$$H(t) = -16t^2 + 80t + 40 \text{ feet.}$$

When does the ball hit the ground?
Solution: To find when the ball hits the ground, we need to determine when the height is 0, i.e., when $H(t) = 0$. Since $H(t)$ is in standard form, with $a = -16$, $b = 80$, and $c = 40$, we use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32} \ .$$

Since $\sqrt{8960}$ does not simplify nicely, we can use a calculator to approximate these roots.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second root is outside the reasonable domain of our model, so the ball will hit the ground after about 5.458 seconds. ■

1.4.2 Polynomial Functions

Definition 1.4.3 (Terminology of Polynomial Functions)**.** A **polynomial** is a function that can be written as

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \ .$$

Each of the a_i are called **coefficients** and can be any real number.

A **term** of the polynomial is any one piece of the sum, that is any a_ix_i.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial. We often write $\deg(f(x)) = n$.

The **leading term** is the term of highest degree: a_nx^n.

The **leading coefficient** is a_n, the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending:

$$f(x) = a_nx^n + a_{n-1}x^{n-1} \ldots a_2x^2 + a_1x + a_0 \ .$$

Example 1.4.6. Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 3 + 2x^2 - 4x^3$.
Solution: The degree is 3, the highest power of x. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4. ■

Short Run Behavior: Intercepts As with any function, the **vertical intercept** of a polynomial $f(x)$ is the point $(0, f(0))$. Again, to find the **horizontal intercepts** of $f(x)$, we need to solve $f(x) = 0$ for x. While there are formulas to find all roots for degree 3 and 4 polynomials, there is no such formula to find all roots of polynomials of degree 5 or higher. Consequently, we will limit ourselves to three cases.

- The polynomial can be factored using known methods.
- The polynomial is given in factored form.
- Technology is used to determine the roots.

Example 1.4.7. Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.
Solution: We will factor this polynomial to solve $f(x) = 0$ for x.

$$x^6 - 3x^4 + 2x^2 = 0$$
$$x^2(x^4 - 3x^2 + 2) = 0 \qquad \text{Factor out the greatest common factor.}$$
$$x^2(x^2 - 1)(x^2 - 2) = 0 \qquad \text{Factor the inside as a quadratic in } x^2.$$

Then we break these factors apart to find all the solutions.

$$x^2 = 0 \qquad x^2 - 1 = 0 \qquad x^2 - 2 = 0$$
$$x = 0 \qquad x^2 = 1 \qquad x^2 = 2$$
$$x = 0 \qquad x = \pm 1 \qquad x = \pm\sqrt{2}$$

This gives us five horizontal intercepts: $(0,0), (\pm 1, 0), \left(\pm\sqrt{2}, 0\right)$. ■

Example 1.4.8. Find the horizontal intercepts of $h(t) = t^3 + 4t^2 + t - 6$.
Solution: Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

Graphing this function, it appears there are horizontal intercepts at $t = -3, -2$, and 1.

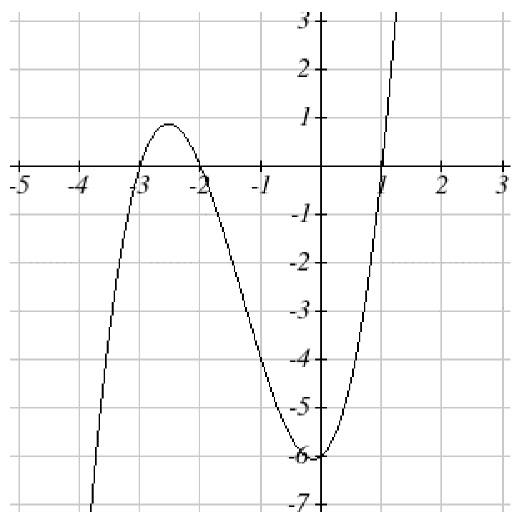


Figure 1.23

We verify that these are the roots by plugging in these values for t: $h(-3) = h(-2) = h(1) = 0$. ■

Solving Polynomial Inequalities One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative, and one we will use later in this course.

Example 1.4.9. Solve $(x+3)(x+1)^2(x-4) > 0$.
Solution: As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4) = 0$, which has solutions at $x = -3, -1$, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into four intervals.

We then pick a number out of each interval and evaluate the function $f(x) = (x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval.

On a number line this would look like:

From our test values, we can determine this function is positive when $x < -3$ or $x > 4$, or in interval notation, $(-\infty, -3) \cup (4, \infty)$. ■

Interval	Test x in interval	$f(x)$	> 0 or < 0?
$x < -3$	-4	72	> 0
$-3 < x < -1$	-2	-6	< 0
$-1 < x < 4$	0	-12	< 0
$x > 4$	5	288	> 0

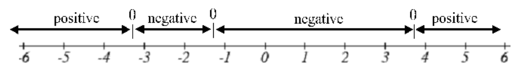


Figure 1.24

1.4.3 Rational Functions

Definition 1.4.4. A **rational function** is the ratio, or fraction, of two polynomials, $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_px^p}{b_0 + b_1x + b_2x^2 + \cdots + b_qx^q}$$

.

Rational functions can arise from both simple and complex situations.

Example 1.4.10. You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.
Solution: You may recall that multiplying speed by time will give you distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which we drive, then $vt =$ distance. Since our distance is fixed at 100 miles, $vt = 100$. Solving this relationship for time gives us the function we desired:

$$t(v) = \frac{100}{v} \ .$$

■

Several natural phenomena, such as gravitational force and volume of sound, behave in a manner inversely proportional to the square of another quantity. For example, the volume, V, of a sound heard at a distance d from the source would be related by $V = \frac{k}{d^2}$ for some constant value k.

Here are the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$. These graphs have several important features.

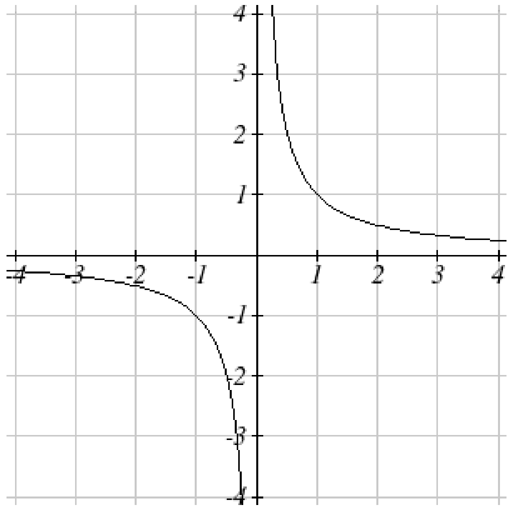
Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. As you well know, dividing by 0 is not allowed and therefore 0 is not in the domain, so the function is undefined at an input of 0.

Short Run Behavior of $\frac{1}{x}$. As the input values approach 0 from the left side (taking on very small, negative values), the function values become very large in the negative direction (in other words, they approach negative infinity). We write: as $x \to 0^-$, $f(x) \to -\infty$.

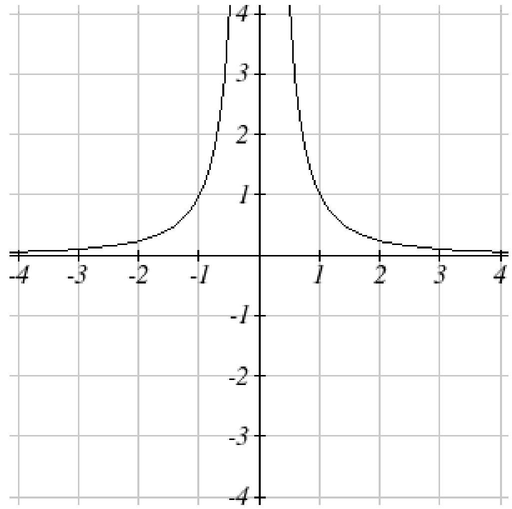
As we approach 0 from the right side (small, positive input values), the function values become very large in the positive direction (approaching infinity). We write: as $x \to 0^+$, $f(x) \to \infty$.

This behavior creates a **vertical asymptote**. An **asymptote** is a line that the graph approaches. Both graphs in Figure 1.25 approach the vertical line $x = 0$ as the input becomes close to 0; $x = 0$ is the vertical asymptote of $y = \frac{1}{x}$.

Long Run Behavior of $\frac{1}{x}$. As the values of x approach infinity, the function values approach 0. Also, as the values of x approach negative infinity, the function values approach 0. Symbolically: as $x \to \pm\infty$, $f(x) \to 0$.



(a) The graph of $f(x) = \dfrac{1}{x}$.



(b) The graph of $f(x) = \dfrac{1}{x^2}$.

Figure 1.25

Based on this long run behavior and the graph, we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a horizontal asymptote. In this case the graph is approaching the horizontal line $y = 0$ as the input becomes very large in the negative and positive directions; $y = 0$ is the horizontal asymptote of $y = \dfrac{1}{x}$.

Definition 1.4.5 (Vertical and Horizontal Asymptotes)**.** A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach a. As $x \to a$, $f(x) \to -\infty$ or $f(x) \to \infty$.

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs get large. As $x \to -\infty$ or $x \to \infty$, $f(x) \to b$.

Example 1.4.11. Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution: Transforming the graph left 2 and up 3 would result in the function $f(x) = \dfrac{1}{x+2} + 3$, or equivalently, by giving the terms a common denominator,

$$\begin{aligned} f(x) &= \frac{1}{x+2} + 3 \\ &= \frac{1}{x+2} + \frac{3(x+2)}{x+2} \\ &= \frac{1}{x+2} + \frac{3x+6}{x+2} \\ &= \frac{3x+7}{x+2} \quad . \end{aligned}$$

Shifting the graph of $y = \dfrac{1}{x}$ would give us this graph. Notice that $f(x) = \dfrac{3x+7}{x+2}$ is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$. As $x \to -2^-$, $f(x) \to -\infty$, and as $x \to -2^+$, $f(x) \to \infty$.

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$. As $x \to \pm\infty$, $f(x) \to 3$. Notice that horizontal and vertical asymptotes get shifted left 2 and up 3 along with the function. ∎

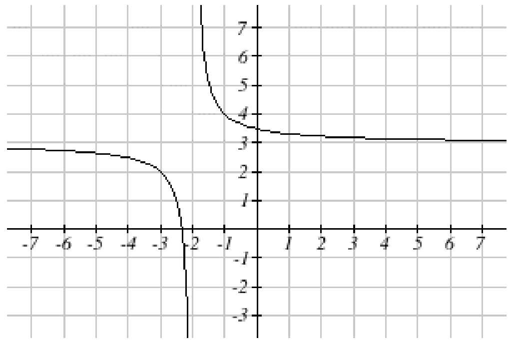


Figure 1.26

Example 1.4.12. A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Solution: Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water: } W(t) = 100 + 10t \text{ gallons} \qquad \text{sugar: } S(t) = 5 + t \text{ pounds.}$$

The concentration, C, will be the ratio of sugar to water:

$$C(t) = \frac{S(t)}{W(t)} = \frac{5+t}{100+10t} \text{ pounds per gallon.}$$

■

Vertical and Horizontal Asymptotes of Rational Functions. The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to 0 and the numerator is not 0.

The **horizontal asymptote** of a rational function, $\dfrac{P(x)}{Q(x)}$ can be determined by looking at the degrees of the numerator, $P(x)$, and the denominator, $Q(x)$.

- If $\deg(Q) > \deg(P)$, then the horizontal asymptote is $y = 0$.
- If $\deg(Q) < \deg(P)$, then there is no horizontal asymptote.
- If $\deg(Q) = \deg(P)$, then the horizontal asymptote is $y = \dfrac{a_p}{b_q}$ ($p = q$ in this case).

Example 1.4.13. In Example 1.4.12, we developed the model $C(t) = \dfrac{5+t}{100+10t}$. Find the horizontal asymptote of $y = C(t)$ and interpret it in context of the scenario.

Solution: Both the numerator and denominator are linear (degree 1), so the horizontal asymptote will be at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: as $t \to \infty$, $C(t) \to \dfrac{1}{10}$. This function will have a horizontal asymptote of $y = \dfrac{1}{10}$.

This tells us that as the input gets large, the output values will approach $\dfrac{1}{10}$. In context, this means that over time, the concentration of sugar in the tank will approach 0.1 lb. per gallon of water or $\dfrac{1}{10}$ pounds per gallon. ■

Example 1.4.14. Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \ .$$

Solution: First, note this function has no inputs that make both the numerator and denominator 0, so there are no potential holes. The function has vertical asymptotes when the denominator is 0, causing the function to be undefined. Since the denominator is 0 at $x = 1, -2$, and 5, the vertical asymptotes are $x = 1$, $x = -2$, and $x = 5$.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards 0 as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote of $y = 0$. ■

As with all functions, a rational function will have a vertical intercept when the input is 0, if the function is defined at 0. It is possible for a rational function to not have a vertical intercept if the function is undefined at 0.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be 0 (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to 0 when the numerator is 0, horizontal intercepts will occur when the numerator of the rational function is equal to 0.

Example 1.4.15. Find the intercepts of

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \ .$$

Solution: We can find the vertical intercept by evaluating the function at 0:

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = \frac{-3}{5} \ .$$

So the vertical intercept is the point $\left(0, \dfrac{-3}{5}\right)$. The horizontal intercepts will occur when the function is equal to 0:

$$\begin{aligned}
0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} && \text{(This is zero when the numerator is zero.)} \\
0 &= (x-2)(x+3) \\
x &= 2, -3.
\end{aligned}$$

So the horizontal intercepts are the points $(2, 0)$ and $(-3, 0)$. ■

1.5 Exponential Functions

1.5.1 Laws of Exponents

The Laws of Exponents let you rewrite algebraic expressions that involve exponents. The last three listed here are really definitions rather than rules.

Theorem 1.5.1 (Laws of Exponents)**.** All variables here represent real numbers and all variables in denominators are nonzero.

1. $x^a \cdot x^b = x^{a+b}$
2. $\dfrac{x^a}{x^b} = x^{a-b}$

3. $(x^a)^b = x^{ab}$
4. $(xy)^a = x^a y^a$
5. $\left(\dfrac{x}{y}\right)^b = \dfrac{x^b}{y^b}$
6. $x^0 = 1$, provided $x \neq 0$, although in some contexts, $0^0 = 1$.
7. $x^{-n} = \dfrac{1}{x^n}$, provided $x \neq 0$.
8. $x^{1/n} = \sqrt[n]{x}$, provided $x \neq 0$.

Example 1.5.1. Simplify $\left(2x^2\right)^3(4x)$.

Solution: We'll begin by simplifying the $\left(2x^2\right)^3$ portion. Using Property 4, we can write

$$\begin{aligned}
\left(2x^2\right)^3 &= 2^3\left(x^2\right)^3(4x) && \text{Use Property 4.}\\
&= 8x^6(4x) && \text{Evaluate } 2^3 = 8\text{, and use Property 3.}\\
&= 32x^7 && \text{Multiply the constants, and use Property 1, recalling } x = x^1.
\end{aligned}$$

■

Being able to work with negative and fractional exponents will be very important later in this course.

Example 1.5.2. Rewrite $\dfrac{5}{x^3}$ using negative exponents.

Solution: Since $x^{-n} = \dfrac{1}{x^n}$, then $x^{-3} = \dfrac{1}{x^3}$ and thus

$$\frac{5}{x^3} = 5x^{-3}.$$

■

Example 1.5.3. Simplify $\left(\dfrac{x^{-2}}{y^{-3}}\right)^2$ as much as possible and write your answer using only positive exponents.

Solution:

$$\begin{aligned}
\left(\frac{x^{-2}}{y^{-3}}\right)^2 &= \frac{\left(x^{-2}\right)^2}{\left(y^{-3}\right)^2}\\
&= \frac{x^{-4}}{y^{-6}}\\
&= \frac{y^6}{x^4}
\end{aligned}$$

■

Example 1.5.4. Rewrite $4\sqrt{x} - \dfrac{3}{\sqrt{x}}$ using exponents.

Solution: A square root is a radical with index of two. In other words, $\sqrt{x} = \sqrt[2]{x}$. Using the exponent rule above, $\sqrt{x} = \sqrt[2]{x} = x^{1/2}$. Rewriting the square roots using the fractional exponent,

$$4\sqrt{x} - \frac{3}{\sqrt{x}} = 4x^{1/2} - \frac{3}{x^{1/2}}.$$

Now we can use the negative exponent rule to rewrite the second term in the expression:

$$4x^{1/2} - \frac{3}{x^{1/2}} = 4x^{1/2} - 3x^{-1/2}.$$

■

Example 1.5.5. Rewrite $\left(\sqrt{p^5}\right)^{-1/3}$ using only positive exponents.
Solution:

$$\begin{aligned}\left(\sqrt{p^5}\right)^{-1/3} &= \left(\left(p^5\right)^{1/2}\right)^{-1/3} \\ &= p^{-5/6} \\ &= \frac{1}{p^{5/6}}\end{aligned}$$

■

Example 1.5.6. Rewrite $x^{-4/3}$ as a radical.
Solution:

$$\begin{aligned}x^{-4/3} &= \frac{1}{x^{4/3}} \\ &= \frac{1}{\left(x^{1/3}\right)^4} \quad (\text{since } \frac{4}{3} = 4 \cdot \frac{1}{3}) \\ &= \frac{1}{(\sqrt[3]{x})^4} \quad (\text{using the radical equivalence})\end{aligned}$$

■

1.5.2 Exponential Models

Consider these two companies:

- Company A has 100 stores, and expands by opening 50 new stores a year
- Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Company A is exhibiting linear growth. In linear growth, we have a constant rate of change – a constant number that the output increased for each increase in input. For company A, the number of new stores per year is the same each year.

Company B is different – we have a percent rate of change rather than a constant number of stores per year as our rate of change. To see the significance of this difference compare a 50% increase when there are 100 stores to a 50% increase when there are 1000 stores:

- 100 stores, a 50% increase is 50 stores in that year.
- 1000 stores, a 50% increase is 500 stores in that year.

Calculating the number of stores after several years, we can clearly see the difference in results.

Years	Company A	Company B
0	100	100
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767

This percent growth can be modeled with an **exponential function**.

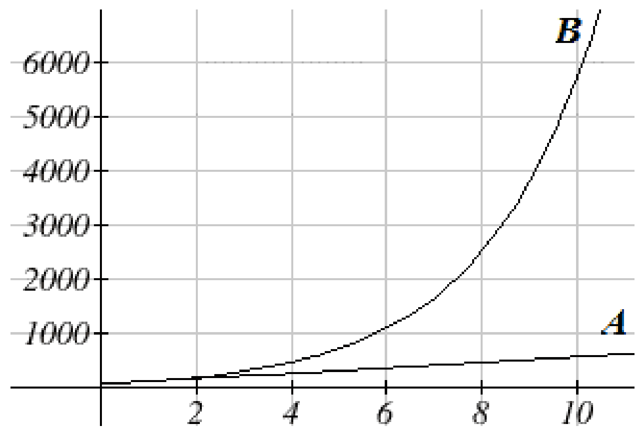


Figure 1.27: Graphs of data from A and B, with B fit to a curve.

Definition 1.5.1 (Exponential Function)**.** An **exponential growth** or **decay** function is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$f(x) = a \cdot (1 + r)^x$$

or

$$f(x) = a \cdot b^x \ ,$$

where

a is the **initial or starting value** of the function,

r is the **percent growth or decay rate**, written as a decimal,

b is the **growth factor** or **growth multiplier**: $b = 1 + r$.

Since powers of negative numbers behave strangely, we must have $b > 0$.

Remark 1.5.1. Exponential functions grow very fast. Because of this, exponential models should use relatively small values for the input and domain.

Example 1.5.7. India's population was 1.14 billion in the year 2008 and is growing by about 1.34% each year. Write an exponential function for India's population, and use it to predict the population in 2020.
Solution: While it is tempting to use the year to define the input variable, Remark 1.5.1 suggests using smaller values for the input. Based on this, we will let t be the number of years since 2008, so that $t = 0$ corresponds with the year 2008. Let $p(t)$ be the population of India in billions. Since the initial population is 1.14 billion, $p(0) = 1.14$. Since the percent growth rate is 1.34%, $r = 0.0134$.

Using the basic formula for exponential growth $f(x) = a(1 + r)^x$, we have the model:

$$p(t) = 1.14(1 + 0.0134)^t = 1.14 \cdot 1.0134^t \text{ billion people.}$$

To estimate the population in 2020, we evaluate $p(t)$ at $t = 12$, since 2020 is 12 years after 2008:

$$p(12) = 1.14 \cdot 1.0134^{12} \approx 1.337 \text{ billion people in 2020.}$$

■

Compound Interest. An exponential model that we see in our everyday lives is **compound interest**. Suppose that you put some money into a bank account. You gain interest on that money at some rate and typically interest is calculated monthly. We call the initial deposit, P, the **principal** and the interest rate, r, the **annual percentate rate**, or **APR**. If interest is applied monthly, then each month, your account increases by $\frac{r}{12} \cdot 100\%$. In general, if interest is calculated and applied k times per year, then after t years, the amount in your account would be

$$A(t) = P \cdot \left(1 + \frac{r}{k}\right)^{kt} \quad .$$

The **annual percentage yield**, or **APY**, is the actual percentage change in the account after one year. Therefore,

$$\text{APY} \ = \frac{A(1) - A(0)}{A(0)} \cdot 100\% \ .$$

Example 1.5.8. A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a \$1000 investment grow to over those 24 months? What is the equivalent annual percentage yield (APY)?

Solution: First, the initial investment is $P = \$1000$. The interest rate is 1.2%, so $r = 0.012$. This is compounded monthly, so $k = 12$. In other words, each month we will earn $\dfrac{1.2\%}{12} = 0.1\%$ interest. Therefore,

$$A(t) = 1000 \cdot \left(1 + \frac{0.012}{12}\right)^{12t} = 1000 \cdot (1.001)^{12t} \ .$$

After 24 months, or two years, the account will have grown to $A(2) = 1000 \cdot (1.001)^{12 \cdot 2} \approx \$1024.28$.

The annual percentage yield (APY) is the percentage change in the account after one year.

$$\text{APY} \ = \frac{A(1) - A(0)}{A(0)} \cdot 100\% = \frac{1000(1.001)^{12} - 1000}{1000} \cdot 100\% \approx \frac{1012.07 - 1000}{1000} \cdot 100\% = 1.207\% \ .$$

Thus our APY is 1.207%. This answer is reasonable because we expect the APY to be close to, but slightly larger than the APR or 1.2%. ■

Example 1.5.9. Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (Polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution: With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is $a = 100$ mg, and our growth rate will be negative 13%, since we are decreasing: $r = -0.13$. This gives the equation

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d \ .$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be $Q(7) = 100(0.87)^7 = 37.73$ mg of Bismuth-210. ■

Example 1.5.10. $T(q)$ represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010. Interpret all of the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$.

Solution: Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$ means that in the second quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts. ■

When working with exponentials, there is a special constant we must talk about. It arises when we talk about things growing continuously, such as continuous compounding, or natural phenomena like radioactive decay that happen continuously.

Definition 1.5.2 (Euler's Number: e)**.**

$$e \approx 2.718282$$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function $\exp(x)$, where $\exp(x) = e^x$. Since calculus studies continuous change, we will almost always use the e-based form of exponential equations in this course.

Definition 1.5.3 (Continuous Growth Formula)**. Continuous growth** can be calculated using the formula

$$f(x) = a \cdot e^{rx} \quad ,$$

where

a is the **starting amount** or **initial quantity**,

r is the **continuous growth rate**.

Example 1.5.11. Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?
Solution: Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: $r = -0.173$, $f(3) = 100e^{-0.173 \cdot 3} \approx 59.512$ mg of Radon-222 will remain. ■

1.5.3 Graphs of Exponential Functions

Theorem 1.5.2 (Graphical Features of Exponential Functions)**.** In the graph of the function $f(x) = a \cdot b^x$, we have the following.

- a is the **vertical intercept** of the graph.
- b determines the **rate** at which the graph grows:
  - the function will **increase** if $b > 1$,
  - the function will **decrease** if $0 < b < 1$.
- The graph will have a **horizontal asymptote** at $y = 0$.
- The graph will be:
  - **concave up** if $a > 0$
  - **concave down** if $a < 0$.
- The **domain** of the function is all real numbers, $\mathbb{R}$.
- The **range** of the function is $(0, \infty)$ if $a > 0$, and $(-\infty, 0)$ if $a < 0$.

When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points $(0, a)$ and $(1, ab)$.

Theorem 1.5.3. The value b will determine the function's long run behavior. The notation $x \to \infty$ means "as the value of x grows (without bound) to infinity (∞)," or, more briefly, "as x goes to ∞."

- If $b > 1$:
  - As $x \to \infty$, $f(x) \to \infty$.
  - As $x \to -\infty$, $f(x) \to 0$.
- If $0 < b < 1$:
  - As $x \to \infty$, $f(x) \to 0$.
  - As $x \to -\infty$, $f(x) \to \infty$.

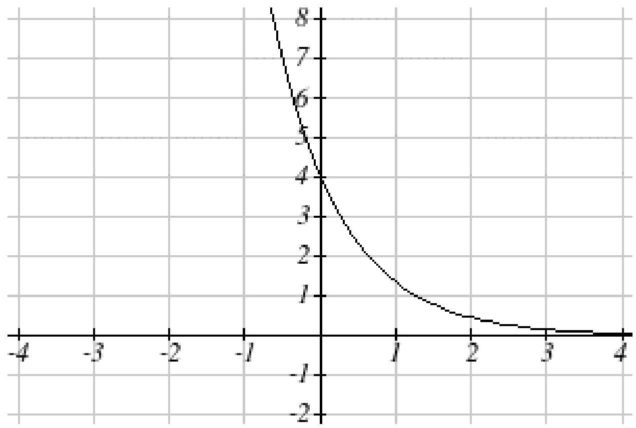


Figure 1.28

Example 1.5.12. Sketch a graph of $f(x) = 4 \cdot \left(\frac{1}{3}\right)^x$.
Solution: This graph will have a vertical intercept at the point $(0, 4)$, and passes through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the graph will be decreasing towards zero. Since $a > 0$, the graph will be concave up.

We can also see from the graph the long run behavior: as $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to \infty$.
■

To get a better feeling for the effect of the coefficients a and b on the graph, examine the sets of graphs below. The first set shows various graphs, where a remains the same and we only change the value for b. Notice that the closer the value of b is to 1, the less steep the graph will be.

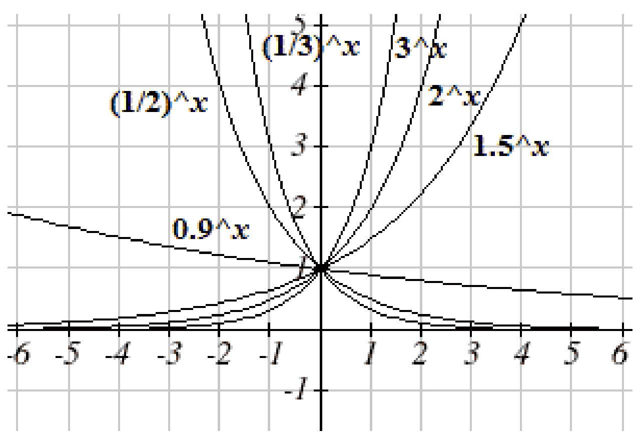


Figure 1.29: Changing the value of b.

In the next set of graphs, a is altered and our value for b remains the same.

Notice that changing the value for a changes the vertical intercept. Since a is multiplying the b^x term, a acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these a values introduced a vertical flip.

Example 1.5.13. Match each equation with its graph.

$f(x) = 2(1.3)^x$

$g(x) = 2(1.8)^x$

$h(x) = 4(1.3)^x$

$k(x) = 4(0.7)^x$

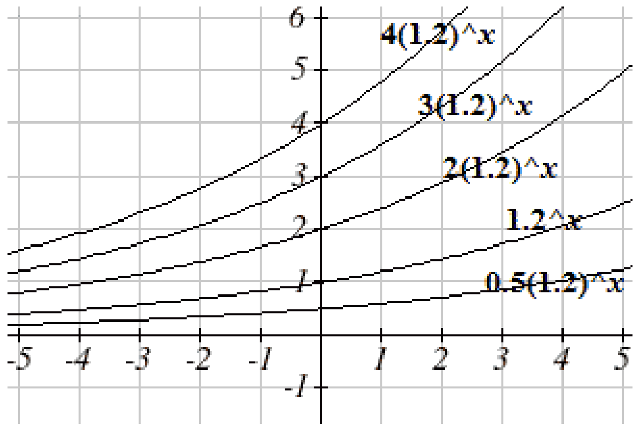


Figure 1.30: Changing the value of a.

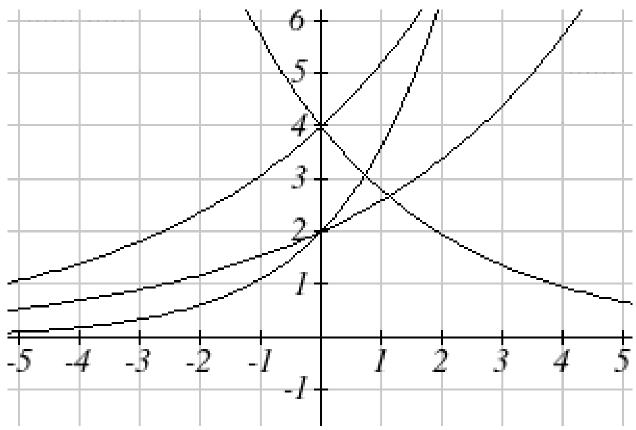


Figure 1.31

Solution: The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor less than 1, which will produce a decreasing graph. The graph of $h(x)$ can be identified as the only growing exponential function with a vertical intercept at the point $(0, 4)$. The graphs of $f(x)$ and $g(x)$ both have a vertical intercept at $(0, 2)$, but since $g(x)$ has a larger growth factor, we can identify it as the graph that is increasing faster.

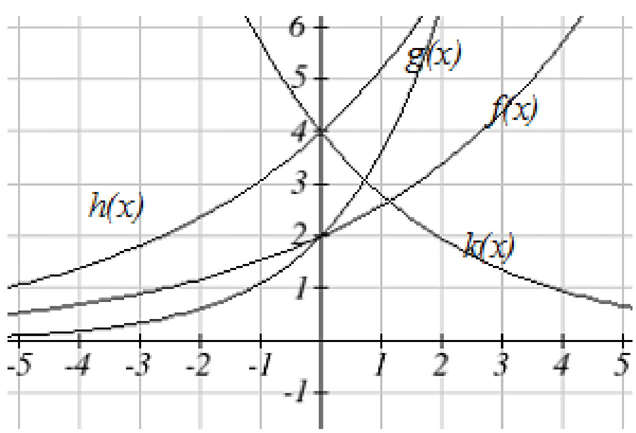


Figure 1.32

■

1.5.4 Exercises

Simplify each expression.

1. x^3x^5
2. x^4x^2
3. $(x^3)^4$
4. $(x^7)^2$
5. $(2x^2)^3x^4$
6. $(5x^4)^2x^5$
7. $\frac{(3x^2)^2}{6x^3}$
8. $\frac{5x(4x)^2}{2x^2}$

Simplify, and rewrite without negative exponents.

9. $4x^{-3}$
10. $2x^{-5}$
11. $x^{-4}x^2$
12. $x^{-2}x$
13. $\frac{5x^{-3}}{2x^{-6}}$
14. $\frac{2x^{-4}}{6x^{-2}}$

Rewrite using negative or fractional exponents.

15. $\frac{4}{x^5}$
16. $\frac{4}{x^3}$
17. $3\sqrt{x}$
18. $\sqrt[4]{x}$
19. $\frac{4}{\sqrt[3]{x}}$
20. $\frac{1}{5\sqrt{x}}$

Rewrite as a radical.

21. $4x^{-1/2}$
22. $5x^{-1/3}$
23. $2x^{1/3}$
24. $5x^{3/2}$
25. A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

26. A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

27. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

28. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

29. If $4,000 is invested in a bank account at an interest rate of 7 per cent per year, find the amount in the bank after 9 years if interest is compounded annually, quarterly, monthly, and continuously.

30. If $6,000 is invested in a bank account at an interest rate of 9 per cent per year, find the amount in the bank after 5 years if interest is compounded annually, quarterly, monthly, and continuously. Match each function with one of the graphs below.

31. $f(x) = 2(0.69)^x$

32. $f(x) = 2(1.28)^x$

33. $f(x) = 2(0.81)^x$

34. $f(x) = 4(1.28)^x$

35. $f(x) = 2(1.59)^x$

36. $f(x) = 4(0.69)^x$

    If all the graphs to the right have equations with form $f(t) = a \cdot e^{kt}$,

37. Which graph has the largest value for k?

38. Which graph has the smallest value for k?

39. Which graph has the largest value for a?

40. Which graph has the smallest value for a?

1.6 Logistic Functions

The (fictional) town of Sigmoidville has 5000 people. You and some friends design an app for residents of the town. Table 1.13 shows the number of downloads of the app over a span of 90 days.

Day (t)	0	10	20	30	40	50	60	70	80	90
Downloads (D)	16	52	166	491	1201	2127	2771	3049	3144	3174

Table 1.13: Total downloads, D, of the Sigmoidville app after t days.

You notice that the download rate is slow, then rapid, then slow again. A plot of the data shows that this is indeed the case.

Why does this make sense? At first, only a few of you know about the app. As you tell others about the app, more will download it and they may tell others about the app as well, so download rates accelerate. However, there are only 5000 people in town and you don't expect anyone outside the town to download it, so eventually the market will be **saturated**. Also, not everyone in town would want the app, so you may not get 5000 total downloads.

This situation is an example of **bounded exponential growth**. In general, a curve called a **logistic curve** models **bounded exponential growth or decay**:

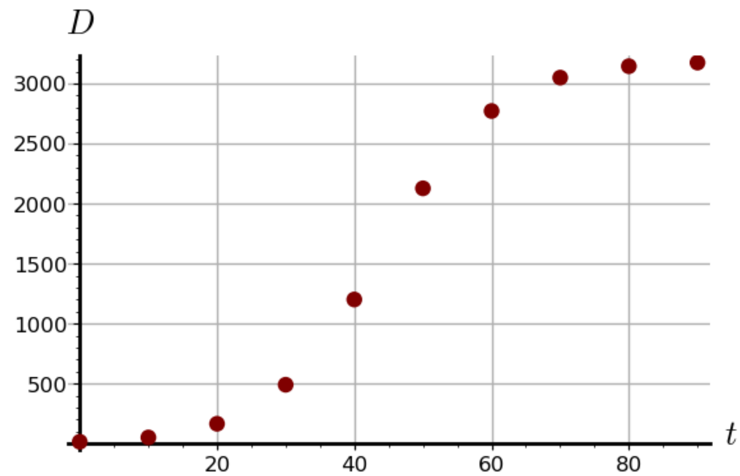


Figure 1.33: Total downloads, D, of the Sigmoidville app after t days.

Definition 1.6.1 (Logistic Function)**.** A **logistic function** is a function of the form

$$f(x) = \frac{L}{1 + ae^{-bx}} \ .$$

- If $b > 0$, then $f(x)$ increases to the **limiting value** L.
- If $b < 0$, then $f(x)$ decreases to 0.

Logistic models apply when there is a limit to the growth of some phenomenon. Examples of logistic models include the following.

- Product sales: an astute inventory manager will note when the total sales of an item have reached an **inflection point** and will stop ordering that item to stock inventory.
- Population growth, where limited resources define a **carrying capacity** of the population
- The spread of an infectious disease.

Example 1.6.1. A (slightly simplified) **best fit curve** can be found for the number of downloads of the Sigmoidville app after t days:

$$D(t) = \frac{3187}{1 + 201 \cdot e^{-0.12t}} \text{ downloads after } t \text{ days.}$$

This tells us that we can expect a total of 3187 app downloads, so you have effectively reached the point of market saturation after 90 days and you can only expect a dozen or so more downloads.

You can model logistic growth with a group of people. One or two people are secretly, possibly randomly, chosen to be zombies. There are a number of "rounds" in which during each round, everyone shakes hands with another person. Zombies will give an extra squeeze or blink their eyes to indicate they are a zombie. If you shake hands with a zombie, you become a zombie yourself. Everyone must remember which round they became a zombie. Plot the cumulative number of zombies in each round. A plot of the data should look logistic, since there will be rapid growth initially, but as the proportion of zombies increases, there will be more occurrences of zombies shaking hands with other zombies, which does not increase the total number of zombies. Eventually, the whole group of people will be zombies.

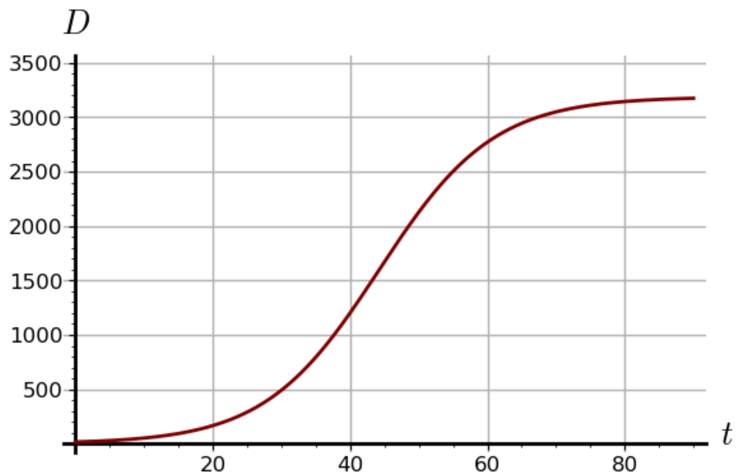


Figure 1.34: Modeling the total downloads, D, of the Sigmoidville app after t days.

1.7 Logarithmic Functions

Logarithms are the inverse of exponential functions – they allow us to undo exponential functions and solve for the exponent. They are also commonly used to express quantities that vary widely in size.

Definition 1.7.1 (Logarithms and Exponentials)**.** The **logarithm function** of **base** b, written $\log_b(x)$, is the inverse of the **exponential function** (base b), b^x.

This means the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Theorem 1.7.1 (Properties of Logarithms: Inverse Properties)**.** If $b > 0$ and $b \neq 1$, then:

- $\log_b(b^x) = x$ for all $x \in \mathbb{R}$;
- $b^{\log_b(x)} = x$ for all $x > 0$.

Example 1.7.1. Write these exponential equations as logarithmic equations:

a) $2^3 = 8$

b) $5^2 = 25$

c) $10^{-4} = \dfrac{1}{10000}$

Solution:

a) $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b) $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c) $10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$.

■

Example 1.7.2. Solve $2^x = 10$ for x.
Solution: By rewriting this expression as a logarithm, we get $x = \log_2(10)$. ■

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful–often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Definition 1.7.2 (Common and Natural Logarithms)**.** These are by far the most frequently used logarithms.

- The **common logarithm** is the logarithm with base 10, and is typically written $\log(x)$.
- The **natural logarithm** is the logarithm with base e, and is typically written $\ln(x)$.

Example 1.7.3. Evaluate $\log(1000)$ using the definition of the common logarithm.
Solution:

$$10^x = 1000 \ .$$

From this, we might recognize that 1000 is the cube of 10, i.e., 10^3, so $x = 3$.

We also can use the inverse property of logarithms to write $\log_{10}(10^3) = 3$. ■

Table 1.14: Values of the common logarithm

x	x as an exponential	$\log(x)$
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Example 1.7.4. Evaluate $\log(500)$ using your calculator or computer.
Solution: Using a computer or calculator, we can evaluate and find that $\log(500) \approx 2.69897$. ■

Another property provides the basis for solving exponential equations.

Theorem 1.7.2 (Properties of Logarithms: Exponent Property)**.**

$$\log_b(A^r) = r\log_b(A)$$

Remark 1.7.1 (Solving exponential equations:)**.** Here are steps to help you solve exponential equations by hand.

- Isolate the exponential expressions when possible.
- Take the logarithm of both sides.
- Utilize the exponent property for logarithms to pull the variable out of the exponent.
- Use algebra to solve for the variable.

Example 1.7.5. In the last section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1 + 0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?
Solution: We need to solve for t so that $f(t) = 2$.

$$\begin{aligned}
2 &= 1.14(1.0134)^t && \text{Initial equation.}\\
\frac{2}{1.14} &= 1.0134^t && \text{Divide by 1.14 to isolate the exponential expression.}\\
\ln\left(\frac{2}{1.14}\right) &= \ln\left(1.0134^t\right) && \text{Take the logarithm of both sides of the equation.}\\
\ln\left(\frac{2}{1.14}\right) &= t\cdot\ln(1.0134) && \text{Apply the exponent property on the right side.}\\
t &= \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)} && \text{Divide both sides by } \ln(1.0134).\\
t &\approx 42.23 \text{ years}
\end{aligned}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050. ■

Example 1.7.6. Solve $5e^{-0.3t} = 2$ for t.
Solution: First we divide by 5 to isolate the exponential:

$$e^{-0.3t} = \frac{2}{5}\ .$$

Since this equation involves e, it makes sense to use the natural logarithm:

$$\begin{aligned} \ln\left(e^{-0.3t}\right) &= \ln\left(\frac{2}{5}\right) && \text{Take the natural logarithm of both sides.} \\ -0.3t &= \ln\left(\frac{2}{5}\right) && \text{Utilize the inverse property of logarithms.} \\ t &= \frac{\ln\left(\frac{2}{5}\right)}{-0.3} && \text{Now divide by } -0.3. \\ t &\approx 3.054 \end{aligned}$$

■

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 1.7.7. In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation

$$pH = -\log([H^+])$$

If a liquid has concentration of 0.0001 moles per liter, determine the pH. Determine the hydrogen ion concentration of a liquid with pH of 7.
Solution: To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logarithms:

$$-\log(0.0001) = -\log(10^{-4}) = -(-4) = 4\ .$$

To answer the second question, we need to solve the equation $7 = -\log([H^+])$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1: $-7 = \log([H^+])$. Rewriting into exponential form yields the answer

$$[H^+]^{-7} = 0.0000001 \text{ moles per liter.}$$

■

While we don't often need to sketch the graph of a logarithm, it is helpful to understand the basic shape.

Theorem 1.7.3 (Graphical Features of the Logarithm)**.** Graphically, given the function $g(x) = \log_b(x)$:

- The graph has a horizontal intercept at the point $(1, 0)$.
- The graph has a vertical asymptote at $x = 0$.
- The graph is increasing and concave down.
- The domain of the function is $x > 0$, or $(0, \infty)$ in interval notation.

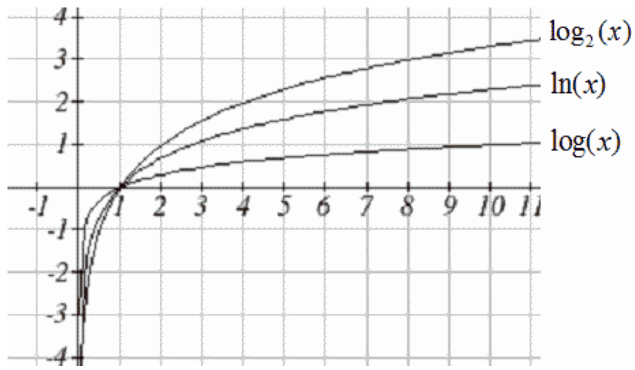


Figure 1.35: Graphs of the Binary, Natural, and Common Logarithms

- The range of the function is all real numbers ($\mathbb{R}$) or $(-\infty, \infty)$ in interval notation.

When sketching a general logarithm with base b, it can be helpful to remember that the graph will pass through the points $\left(\frac{1}{b}, -1\right)$, $(1, 0)$, and $(b, 1)$.

To get a feeling for how the base affects the shape of the graph, examine the graphs below:

Another important observation made was the domain of the logarithm: $x > 0$. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a logarithm.

Example 1.7.8. Find the domain of the function $f(x) = \log(5 - 2x)$.
Solution: The logarithm is only defined when the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality, $-2x > -5$, so $x < \frac{5}{2}$.

The domain of this function is $x < \frac{5}{2}$, or, in interval notation, $\left(-\infty, \frac{5}{2}\right)$. ■

1.8 Trigonometric Functions

Most applied calculus texts skip **trigonometric** functions, but they are extremely powerful for modeling periodic or seasonal data. We will keep the treatment of these models very basic and intuitive, but independent from the rest of the text; it is optional. For the scientist, trigonometric functions are indispensable. However, for our purposes, we will focus our attention exclusively on two related functions: the **sine** and **cosine** functions.

1.8.1 Definitions

A **sine wave** is of the form

$$f(x) = v + a \cdot \sin\left(\frac{2\pi(x-h)}{p}\right) \quad . \tag{1.2}$$

where v is the **vertical shift** (mean of the data), a is the **amplitude** of the wave (roughly the difference between the maximum and the mean of the data), h is the **horizontal shift** (where the data is at the mean and begins to rise to the maximum), and p is the **period**.

1.8.2 Modeling with Sine Waves

Consider the following table that we saw in Sections 1.2 and 1.4 on the fuel oil usage of Plant W in 2016. One task that we were unable to complete before was to determine a model that would reflect the seasonality of the data. In other words, since the data represents a calendar year, we want a model that will oscillate and finish at the same output value as the output value of the beginning point. A **trigonometric model** serves that very purpose.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Oil (gal)	573.0	850.0	425.3	800.1	818.9	880.9	296.5	198.7	105.4	72.0	??	638.0

Table 1.15: Fuel Oil Usage at Plant W in 2016

Given our data, we don't know the vertical shift, the amplitude, or the horizontal shift, but we do know the period. Since the input is in months and there are 12 months in a year, we choose $p = 12$ from 1.2. We then fit a sine wave to the data and get the following model.

If m is the mth month of the year, then Plant W burned oil at a rate of

$$f(m) = 492.82 - 334.33 \sin\left(\frac{2\pi(m + 29.49)}{12}\right) \text{ gallons per month}$$

in 2016.

Graphing $y = f(m)$ with the scatter plot of Table 1.15 gives us the following plot to verify that the model makes sense.

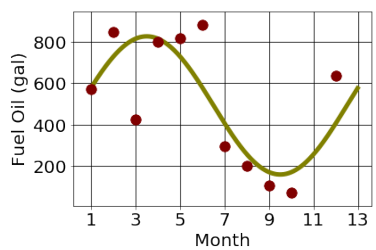


Figure 1.36: Fuel Oil Usage by Plant W in 2016 Fitted by a Sine Wave.

1.9 Combining Functions

Chapter 2

The Derivative: Rates of Change of a Function

2.1 Slopes and Average Rates of Change

2.1.1 Precalculus Idea: Slope and Rate of Change

The price of an asset, such as a stock, commodity, or currency, relative to some other asset typically fluctuates in a chaotic manner. We can see if the price is increasing, decreasing, or relatively stable, in a qualitatative sense, by looking at a plot of the price of an asset over time. An example is the graph of the price of bitcoin during 2020. Figure 2.1 plots the price of bitcoin at the end of each month from December 2019 to December 2020. Suppose that we are interested in getting the average rate of change of this price over the course of the year. This calculation ignores are the intermediate fluctuations in price and just focuses on two values: the starting and ending prices. Note that price fluctuates by the second; we just plotted monthly data.

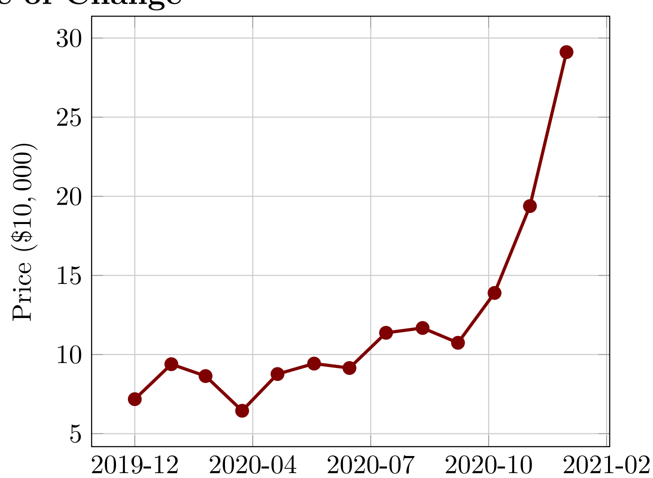


Figure 2.1: Price of Bitcoin in 2020, in Thousands of Dollars. Source: Coindesk; https://www.coindesk.com/price/bitcoin

Example 2.1.1. Given that the price of one bitcoin was \$7179.96 at the end of 2019 and \$29, 111.52 at the end of 2020, what was the average rate of change in the price of one bitcoin during 2020?

Solution: In this context, the rate of change of the price is $\frac{\text{change in price}}{\text{change in time}}$. In many cases, the unit in the denominator is clear from context. In this case, the only unit of time that is given is years, but it would not be very illustrative to talk about the rate of change of the price in dollars per year if we only have a year's worth of data. In context, it makes more sense to talk about the rate of change per month or per day, for example. Recall that 2020 was a leap year, so there were 366 days.

$$\begin{aligned}\frac{\text{change in price}}{\text{change in time}} &= \frac{29111.52 - 7179.96 \text{ dollars}}{12 \text{ months}} = \$1827.63 \text{ per month}\\ &= \frac{29111.52 - 7179.96 \text{ dollars}}{366 \text{ days}} = \$59.9223 \text{ per day}\end{aligned}$$

Therefore, the price of one bitcoin changed at a rate of \$1827.63 per month, on average during 2020. We could also conclude that the price of one bitcoin increased at an average rate of \$59.92 per day in 2020.

The **slope** of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the **rate of change** of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance an object traveled over time, for example, then the slope of the line represents the velocity of the object. In Figure 2.2, you can remind yourself how we calculate slope using two points on the line.

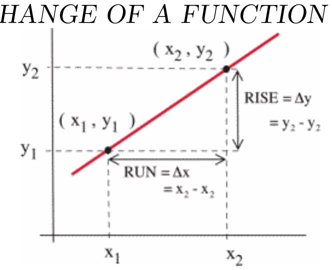


Figure 2.2: Slope between two points.

$$m = \text{Slope from } P \text{ to } Q = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We would like to get that same kind of information (how fast the curve rises or falls, velocity from distance) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in Figure 2.3?

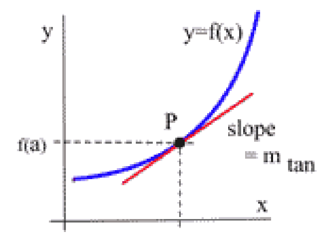


Figure 2.3: Tangent Line

2.1.2 Secant Lines

We need two points in order to determine the slope of a line. How can we find a slope of a curve, at just one point? The answer, as suggested in Figure 2.3, is to find the slope of the **tangent line** to the curve at that point. Most of us have an intuitive idea of what a tangent line is. Unfortunately, "tangent line" is hard to define precisely. We need to develop more tools for our mathematical toolbox in order to define it precisely. but we will build that definition from the concept of a secant line of a curve.

Definition 2.1.1 (Secant Line)**.** A **secant line** is a line through two points on a curve. In Figure 2.4, the red line is the secant line of the blue curve between the points $P = (a, f(a))$ and $Q = (b, f(b))$.

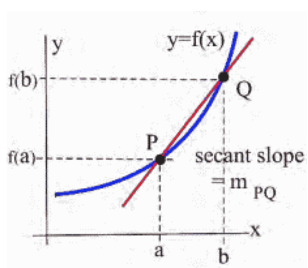


Figure 2.4: Secant line through points P and Q

Definition 2.1.2. The **rate of change** of a function $f(x)$ between two points, $P = (a, f(a))$ and $Q = (b, f(b))$, on the curve $y = f(x)$ is the slope of the secant line of the curve $y = f(x)$ between the points P and Q:

$$\text{Rate of change of } f(x) \text{ from } a \text{ to } b = \text{Slope from } P \text{ to } Q = \frac{f(b) - f(a)}{b - a} \ .$$

2.1.3 Exercises

1. What is the slope of the line between the points $(4, -2)$ and $(7, 10)$?
2. What is the slope of the line between the points $(-1, 6)$ and $(3, -10)$?
3. The S&P 500 rose from \$3244.67 to \$3756.07 in 2020. What is the average rate of change of the S&P 500 in 2020?
4. The price of a share of the S&P 500 ETF SPY rose from \$323.54 to \$373.8 in 2020. What is the average rate of change of a share price of SPY in 2020?

2.2 Tangent Lines and Instantaneous Rates of Change

2.2.1 Tangent Line Concepts

Example 2.2.1. Suppose that Figure 2.5 plots the population of a bacteria culture t hours after an antibiotic is added to the culture. What is the rate at which the population is changing eight hours after the bacteria is added to the culture?

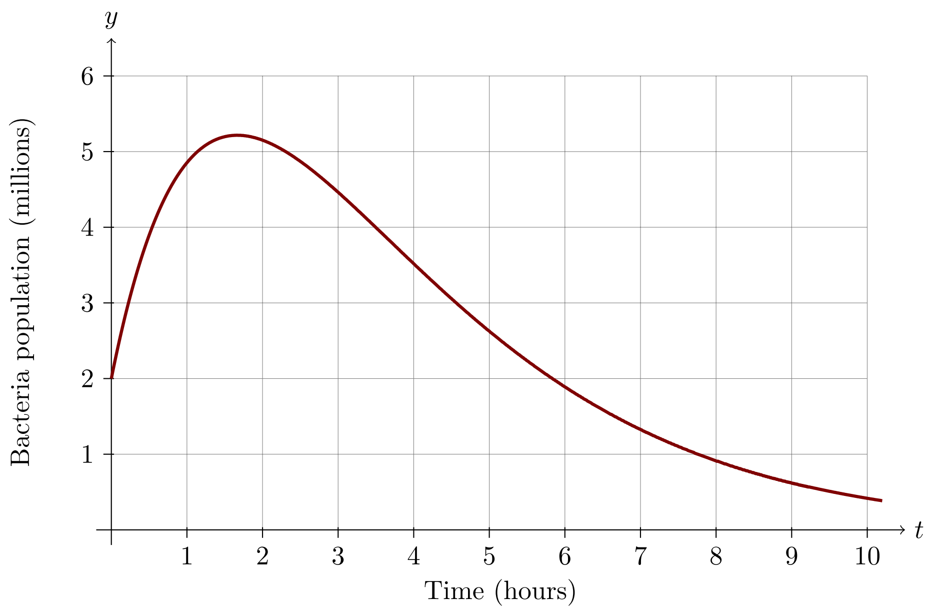


Figure 2.5: Population of a bacteria culture t hours after an antibiotic is added.

If we wanted to know the rate of change of the culture between two distinct points in time, then we could apply the techniques of the last section. Here, however, we want to know the rate of change at an instant in time, or the **instantaneous rate of change** of the culture at a single point in time. We found the average rate of change by computing the slope of a secant line. Here, we can estimate the instantaneous rate of change by approximating the slope of a certain line called a **tangent line** of the curve at a point.

An informal definition of a tangent line of a curve is that a **tangent line** of a curve is a line that touches one point on the curve in such a way that the curve and line would be indistinguishable if you zoomed in closely enough. In other words, the slope of the curve is the same as the slope of the line at that point.

Solution: Figure 2.6 shows the tangent line of the curve in blue. We can approximate the slope of the tangent line by computing the slope between any two points on the line. Pick any two. We will pick the two black points and estimate their coordinates: roughly $(0, 3.7)$ and $(8, 1)$. Keeping the units in mind, the population of bacteria is changing at a rate of roughly

$$\frac{1-3.7}{8-0}\frac{\text{million baceria}}{\text{hour}} \approx -0.34 \text{ million bacteria per hour} \;.$$

Since this value is negative, the population is decreasing. Note the negative slope of the tangent line. In conclusion, eight hours after the antibiotic is added to the bateria culture, the bacteria population is decreasing at a rate of roughly 340,000 bacteria per hour. Practically, this means that the antibiotic is successfully killing off the bacteria. ■

2.2.2 Tangent Lines, More Formally

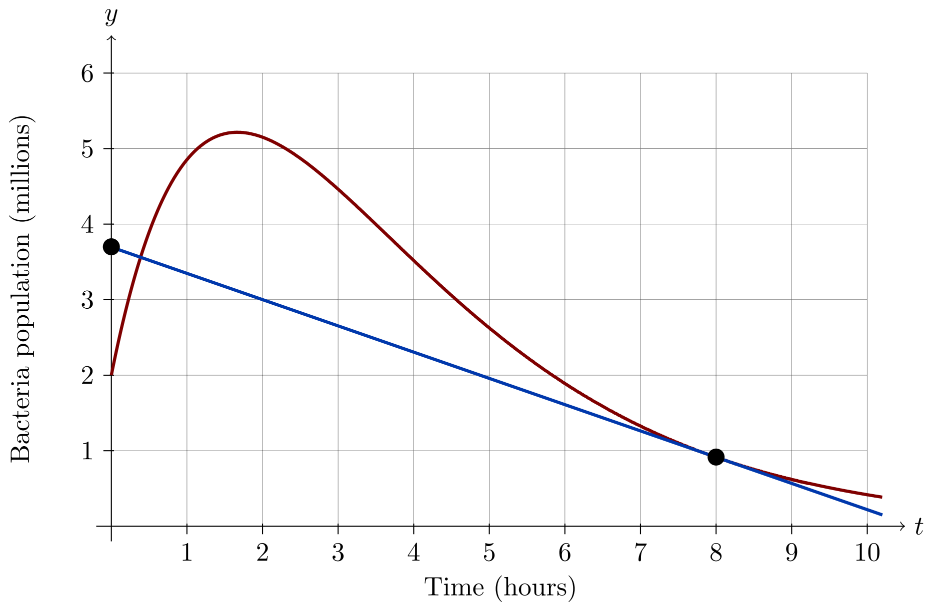


Figure 2.6: Bacteria culture population with a tangent line at $t = 8$ hours.

To develop our formal definition of a tangent line, we consider the plot in Figure 2.7. In the plot, the red line is the tangent line of the curve $y = f(x)$ at the point $P = (a, f(a))$ and the green lines are secant lines that go through P. The closer the point Q is to the point P, the closer the secant line slope gets to the tangent line slope and the closer the secant line gets to the tangent line. This will be key to finding the tangent slope, but first we need to more carefully define the idea of "getting closer to." This will be done in Sections 2.4 and 2.5.

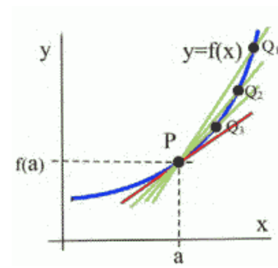


Figure 2.7: Secant lines approaching a tangent line.

This is an example for the reader to try.

Example 2.2.2. The graph in Figure 2.8 is the graph of $y = f(x)$. We want to find the slope of the tangent line of this curve at the point $(1, 2)$.

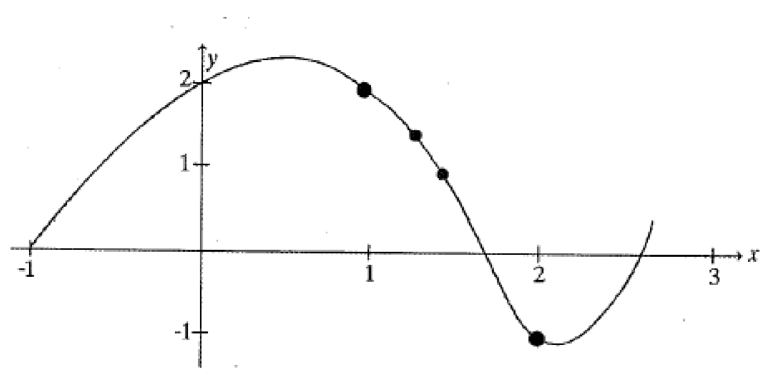


Figure 2.8: $y = f(x)$

Solution: First, draw the secant line between the points $(1, 2)$ and $(2, -1)$ and compute its slope. Now draw the secant line between the points $(1, 2)$ and $(1.5, 1)$ and compute its slope. Compare the two lines you have drawn. Which would be a better approximation of the tangent line to the curve at the point $(1, 2)$?

Now draw the secant line between the points $(1, 2)$ and $(1.3, 1.5)$ and compute its slope. Is this line an even better approximation of the tangent line? Now draw your best guess for the tangent line and measure its slope. Do you see a pattern in the slopes?

You should have noticed that as the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. That's good news! We know how to find the slope of a secant line. ■

In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines, that is, where the slope of the curve, or the rate of change of the function, is 0.

Example 2.2.3. The graph of $y = g(x)$ is in Figure 2.9. At what values of x does the graph of $g(x)$ have horizontal tangent lines?

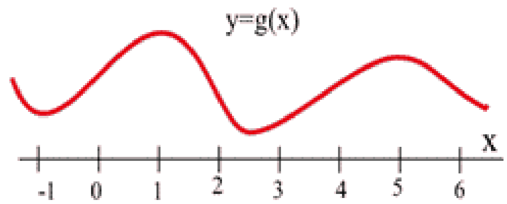


Figure 2.9: $y = g(x)$

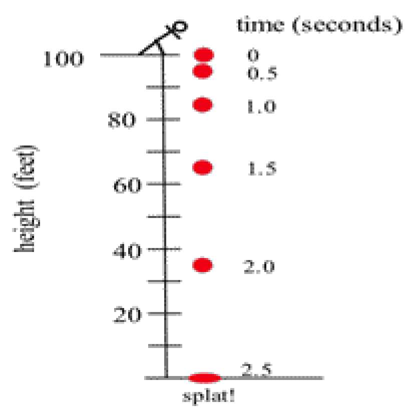
Solution: The tangent lines to the graph of $g(x)$ are horizontal when $x \approx -1, 1, 2.5$, and 5. ■

2.2.3 Example: Instantaneous Velocity

To give another example of the concept of instantaneous rate of change, we consider the well-known and well-understood example of velocity. In fact, it was the problem of computing instantaneous velocity that led to the invention of calculus.

Suppose we drop a tomato from the top of a 100 foot building and time its fall.

Some questions are easy to answer directly from the table.



time (s)	height (ft)
0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

Figure 2.10: The height of a dropped tomato versus time.

Example 2.2.4. (a) How long did it take for the tomato to drop 100 feet?

Solution: It took 2.5 seconds for the tomato to drop 100 feet. ■

(b) How far did the tomato fall during the first second?

Solution: The tomato fell $100 - 84 = 16$ feet during the first second. ■

(c) How far did the tomato fall during the last second?

Solution: The tomato fell $64 - 0 = 64$ feet during the last second. ■

(d) How far did the tomato fall between $t = 0.5$ and $t = 1$ seconds?

Solution: The tomato fell $96 - 84 = 12$ feet between $t = 0.5$ and $t = 1$ seconds. ■

Some questions require a little calculation.

Example 2.2.5. (e) What was the average velocity of the tomato during its fall?

Solution: Average velocity $= \dfrac{\text{change in position}}{\text{change in time}} = \frac{0-100 \text{ ft}}{2.5-0 \text{ s}} = -40$ ft/s. ■

(f) What was the average velocity between $t = 1$ and $t = 2$ seconds?

Solution: Average velocity $= \dfrac{\text{change in position}}{\text{change in time}} = \frac{36-84 \text{ ft}}{2-1 \text{ s}} = -48$ ft/s. ■

Now, we consider a different question that is an example of an instantaneous rate of change, and an example of a question that is significantly different from the previous example about average velocity.

Example 2.2.6. How fast was the tomato falling one second after it was dropped?
Solution: Here we want the instantaneous velocity, the velocity at an instant in time. Unfortunately, the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after one second is simply the average velocity during the entire fall: -40 ft/s. However, the tomato fell slowly at the beginning and rapidly near the end so this estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at $t = 1$ by calculating the average velocities over a short time interval near $t = 1$. The average velocity between $t = 0.5$ and $t = 1$ is $\dfrac{-12 \text{ feet}}{0.5 \text{ s}} = -24$ ft/s, and the average velocity between $t = 1$ and $t = 1.5$ is $\dfrac{-20 \text{ feet}}{0.5 \text{ s}} = -40$ ft/s, so we can be reasonably sure that the instantaneous velocity is between -24 ft/s and -40 ft/s.

Another approximation would be by calculating the average velocity between $t = 0.5$ and $t = 1.5$, that is, the shortest time interval we have with $t = 1$ in the middle: $\dfrac{64 - 96 \text{ feet}}{1.5 - 0.5 \text{ s}} = -32$ ft/s.

Now, in general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. Using just the table in Figure 2.10, we cannot find any shorter time interval. However, with a model or a function describing the height of the tomato, we could get a better approximation. We will plot the points and fit a curve to it and work from there.

Notice that the data, plotted in Figure 2.11, is not linear and has the same concavity (concave down), so a quadratic model may be appropriate. In fact, we know from physics that in the absence of air resistance, that a quadratic model would fit this kind of data perfectly. From LibreOffice, we can obtain the model

$$h(t) = 100 - 16t^2 \text{ feet after } t \text{ seconds}$$

for the height of the tomato. We plot this and the data together.

Now we can approximate the velocity of the tomato at $t = 1$ seconds by going off the model. We often use the symbol Δ (delta) to indicate a finite difference between two values of a variable. In this context, Δt is a difference in time and Δh is a difference in the function h, i.e., a difference in height. With this notation, the average velocity over some time interval is $\frac{\Delta h}{\Delta t}$. In this example,

$$\Delta h = h(1 + \Delta t) - h(1) \ .$$

From the data in Table 2.1, the average velocities are approaching -32 feet per second, which is in fact the (instantaneous) velocity of the tomato one second after being dropped. ■

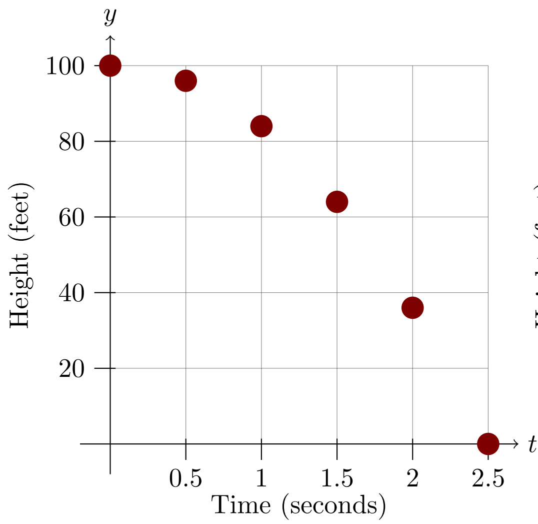
Example: Marginal Cost

Another example of the application of secant and tangent lines is **marginal cost**.

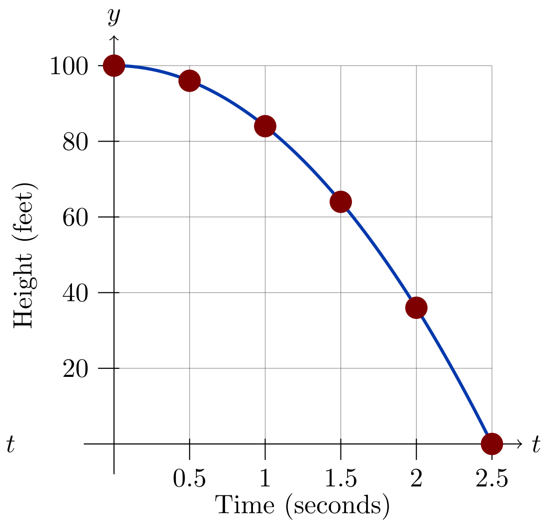
Definition 2.2.1. The **marginal cost** at q items, $MC(q)$, is the cost of producing the next item:

$$MC(q) = C(q + 1) - C(q) \ .$$

In many cases, though, it's easier to approximate this difference using tangent lines, as the instantaneous rate of change of $C(n)$ at $n = q$. Some sources define the marginal cost this way. In this course, we will use both of these definitions as if they were interchangeable.



(a) Data of the height of the tomato.



(b) Modeling the height of the tomato.

Figure 2.11

Δt (seconds)	$\Delta h = h(1+\Delta t) - h(1)$ (feet)	$\frac{\Delta h}{\Delta t}$ (ft/s)
1	$h(2) - h(1) = 36 - 84 = -48$	$\dfrac{-48}{1} = -48$
0.5	$h(1.5) - h(1) = 64 - 84 = -20$	$\dfrac{-20}{0.5} = -40$
0.1	$h(1.1) - h(1) = 80.64 - 84 = -3.36$	$\dfrac{-3.36}{0.1} = -33.6$
0.01	$h(1.01) - h(1) = 83.6784 - 84 = -0.3216$	$\dfrac{-0.3216}{0.01} = -32.16$
0.001	$h(1.001) - h(1) = 83.967984 - 84 = -0.032016$	$\dfrac{-0.032016}{0.001} = -32.016$

Table 2.1: Average velocities approaching the instantaneous velocity at $t = 1$ seconds.

In other words, the marginal cost of n items is the slope of the secant line of $y = C(n)$ through the points $(n, C(n))$ and $(n+1, C(n+1))$, but it can be approximated by the slope of the tangent line of $y = C(n)$ through the point $(n, C(n))$.

Example 2.2.7. Table 2.2 shows the total cost ($C(n)$) of producing n items.

Items (n)	Cost $C(n)$
0	\$20,000
100	\$35,000
200	\$45,000
300	\$53,000

Table 2.2: Cost to produce n items.

When 200 items are made, estimate the marginal cost.

Solution: Since we don't have $C(201)$, We need to estimate the value of $MC(200)$ by estimating the slope of the tangent line at $n = 200$ using secant lines. The secant line from $n = 100$ to $n = 200$ has a slope of:

$$MC(200) \approx \frac{\$45,000 - \$35,000}{200 - 100 \text{ items}} = \$100 \text{ per item} \ .$$

The secant line from $n = 200$ to $n = 300$ has a slope of:

$$MC(200) \approx \frac{\$53,000 - \$45,000}{300 - 200 \text{ items}} = \$80 \text{ per item} \ .$$

We could estimate the tangent slope by averaging these secant slopes, giving us an estimate of \$90 per item.

Alternatively, we could estimate $MC(2000$ by finding the slope of the secant line between $n = 100$ and $n = 300$:

$$MC(200) \approx \frac{\$53,000 - \$35,000}{300 - 100 \text{ items}} = \$90 \text{ per item} \ .$$

This tells us that after 200 items have been made, it will cost about \$90 to make one more item. ■

2.2.4 Exercises

1. What is the slope of the line through the points $(3, 9)$ and (x, y) for $y = x^2$ and:
   (a) $x = 2.97$?
   (b) $x = 3.001$?
   (c) $x = 3 + h$?
   (d) What happens to this last slope when h is very small (close to 0)?
   (e) Sketch the graph of $y = x^2$ for x near 3.
2. What is the slope of the line through the points $(-2, 4)$ and (x, y) for $y = x^2$ and:
   (a) $x = -1.98$?
   (b) $x = -2.03$?
   (c) $x = -2 + h$?
   (d) What happens to this last slope when h is very small (close to 0)?
   (e) Sketch the graph of $y = x^2$ for x near -2.
3. What is the slope of the line through the points $(2, 4)$ and (x, y) for $y = x^2 + x - 2$ and
   (a) $x = 1.99$?
   (b) $x = 2.004$?
   (c) $x = 2 + h$?
   (d) What happens to this last slope when h is very small (close to 0)?
   (e) Sketch the graph of $y = x^2 + x - 2$ for x near 2.
4. What is the slope of the line through the points $(-1, -2)$ and (x, y) for $y = x^2 + x - 2$ and:
   (a) $x = -0.98$?
   (b) $x = -1.03$?
   (c) $x = -1 + h$?
   (d) What happens to this last slope when h is very small (close to 0)?
   (e) Sketch the graph of $y = x^2 + x - 2$ for x near -1.

5. The graph in Figure 2.12 shows the temperature during a day in Ames.

   (a) What was the average rate of change in temperature from 9 am to 1 pm?

   (b) Sketch a tangent line on the curve and estimate how fast the temperature was changing at 10 am.

   (c) Sketch a tangent line on the curve and estimate how fast the temperature was changing at 7 pm.

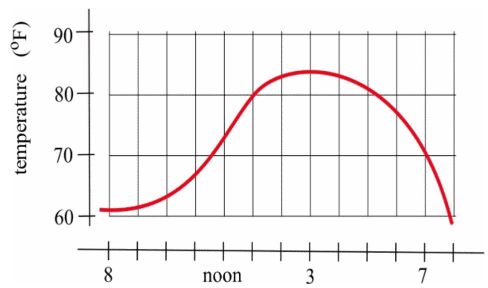


Figure 2.12: Temperature in Ames.

6. The graph in Figure 2.13 shows the distance of a car from a measuring position located on the edge of a straight road.

   (a) What was the average velocity of the car from $t = 0$ to $t = 30$ seconds?

   (b) What was the average velocity of the car from $t = 10$ to $t = 30$ seconds?

   (c) Sketch a tangent line on the curve and estimate the velocity of the car at $t = 10$, at $t = 20$, and at $t = 30$ seconds.

   (d) What does the horizontal part of the graph between $t = 15$ and $t = 20$ seconds mean?

   (e) What does the negative velocity at $t = 25$ represent?

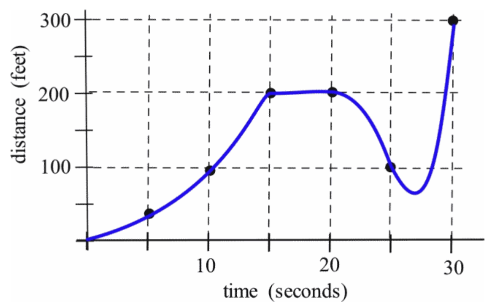


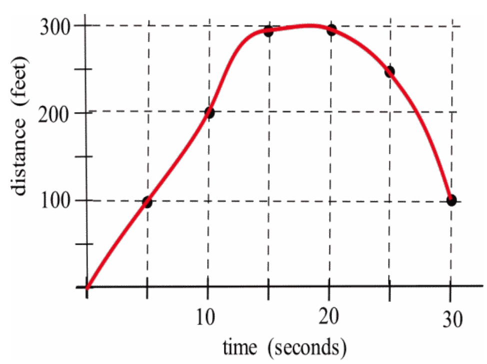
Figure 2.13: Net distance a car has traveled.

7. The graph in Figure 2.14 shows the distance of a car from a measuring position located on the edge of a straight road.

   (a) What was the average velocity of the car from $t = 0$ to $t = 20$ seconds?

   (b) What was the average velocity from $t = 10$ to $t = 30$ seconds?

   (c) About how fast was the car traveling at $t = 10$ seconds? at $t = 20$ seconds? at $t = 30$ seconds?



8. The graph in Figure 2.15 shows the composite developmental skill level of chess masters at different ages as determined by their performance against other chess masters. (From *Rating Systems for Human Abilities*, by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)

   (a) At what age is the "typical" chess master playing the best chess?

   (b) At approximately what age is the chess master's skill level increasing most rapidly?

   (c) Describe the development of the "typical" chess master's skill in words.

   (d) Sketch graphs which you think would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician, and a professional in your major field.

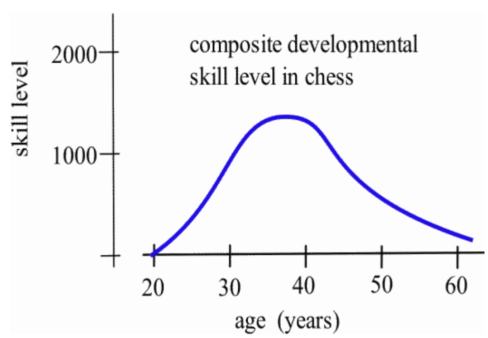


Figure 2.15: Developmental skill level of chess masters versus age.

2.3 The Derivative

2.3.1 Derivative Concepts and Notation

Definition 2.3.1. We can view the **derivative** of a function, $f(x)$ in several different equivalent ways. Here are a three of them. The derivative of $f(x)$ at a point $(a, f(a))$ is:

- the instantaneous rate of change of $f(x)$ at $x = a$,
- the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$, and
- the slope of the curve $y = f(x)$ at the point $(a, f(a))$.

We've been acting as if derivatives exist everywhere for every function. This is true for most of the functions that you will run into in this textbook. But there are some common places where the derivative doesn't exist.

Definition 2.3.2. A function, $f(x)$, is called **differentiable** at the point $(a, f(a))$ if its derivative exists at $x = a$.

Remember that the derivative is the slope of the tangent line to the curve. That's what to think about. Where can a slope not exist? If the tangent line is vertical, the derivative will not exist.

Example 2.3.1. Show that $f(x) = \sqrt[3]{x} = x^{1/3}$ is not differentiable at $x = 0$.
Solution: From the graph in Figure 2.16, we can see that the tangent line to $y = \sqrt[3]{x}$ at $x = 0$ is vertical with undefined slope, which is why the derivative does not exist at $x = 0$.

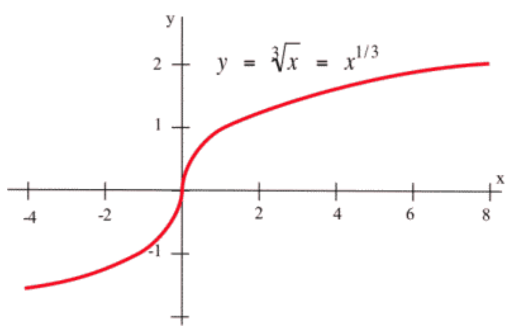


Figure 2.16: $f(x) = \sqrt[3]{x}$

■

Where can a tangent line not exist?

If there is a sharp corner (cusp) in the graph, the derivative will not exist at that point because there is no well-defined tangent line (a teetering tangent, if you will).

If there is a discontinuity in the graph (a jump, a break, a hole in the graph, or a vertical asymptote), the tangent line will be different on either side and the derivative will not exist at that point.

Example 2.3.2. Show that $f(x) = |x|$ is not differentiable at $x = 0$.
Solution: On the left side of the graph in Figure 2.17, the slope of the line is -1. On the right side of the graph, the slope is 1. There is no well-defined tangent line at the sharp corner at $x = 0$, so the function $|x|$ is not differentiable at that point.

■

Definition 2.3.3. When we find the derivative of a function, or take the derivative of a function, we say that we **differentiate** the function.

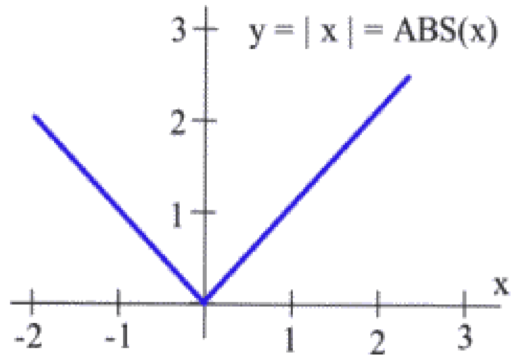


Figure 2.17: $f(x) = |x|$

Notation for the Derivative Calculus was invented by Leibniz and Newton, but it was developed by many more throughout the subsequent centuries. Many developed their own notation for the derivative of a function. The notation developed by Leibniz and Joseph-Louis Lagrange are the most convenient and most used.

The derivative of $y = f(x)$, with respect to x, is written in many different ways:

- $f'(x)$: "f prime of x",
- y': "y prime,"
- $\dfrac{dy}{dx}$: "$d\ y\ d\ x$,"
- $\dfrac{df}{dx}$: "$d\ f\ d\ x$," or
- $\dfrac{d}{dx}f(x)$: "$d\ d\ x$ of f of x."

The notation that resembles a fraction is called **Leibniz notation**. It displays not only the name of the function (f or y), but also the name of the variable (in this case, x). It looks like a fraction because the derivative is a slope. In fact, this is simply $\dfrac{\Delta y}{\Delta x}$ written in Roman letters instead of Greek letters. The nuance here is that Δv represents a finite difference of some variable v, whereas dv represents an **infinitesimal** difference of a variable v. An infinitesimal difference would be a difference that is infinitely small, but not 0. That may sound like an oxymoron, but until about 1850, calculus was developed using infinitesimal differences, also called **differentials**.

The best way to understand differentials is with an analogy: $1 : \infty :: dx : 1$. Suppose you had a bin with an infinite number of balls and you throw one ball in. You still throw in a ball, but the total quantity of balls in the bin is unchanged. In the same way, if you have a line that is one inch long and you add on a line of length dx inches, then the line is still one inch long, despite "lengthening" the line by dx.

Looking Ahead We will figure out ways to compute exact values of derivatives from many kinds of functions in Sections 2.5-2.10. If the function is given to you as a table or graph, you will still need to approximate the derivative of the function this way.

This is the foundation for the rest of this chapter. It's remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative $f'(x)$) will lead to so many important ideas and applications.

2.3.2 The Derivative as a Function

We now know how to find (or at least approximate) the derivative of a function for any x-value. This means that we can think of the derivative as a function too. That is the origin of the term "derivative." The derivative of a function is a function that is derived from that function. The input of the derivative is the same as the function, but the output is the value of the derivative at that x value and the units reflect the rate of change concept.

Example 2.3.3. Below is the graph of a function $y = f(x)$. We can use the information in the graph to fill in a table showing values of $f'(x)$:

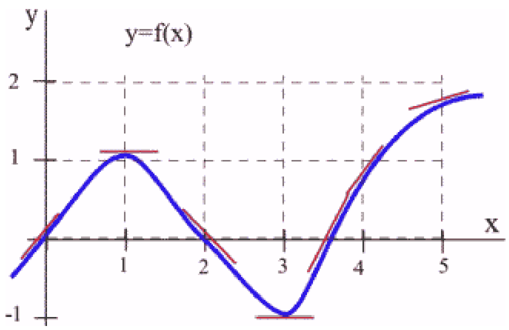


Figure 2.18: $y = f(x)$

Solution: At various values of x, draw your best guess at the tangent line and measure its slope. You might have to extend your lines so you can read some points. In general, your estimate of the slope will be better if you choose points that are easy to read and far away from each other. Here are estimates for a few values of x (parts of the tangent lines used are shown above in the graph): Based on this, we can estimate

x	$y = f(x)$	$f'(x)$
0	0	1
1	1	0
2	0	-1
3	-1	0
3.5	0	2

Table 2.3: Table of inputs, outputs, and derivatives of $f(x)$.

the values of $f'(x)$ at some non-integer values of x too: $f'(0.5) \approx 0.5$ and $f'(1.3) \approx -0.3$. We can even think

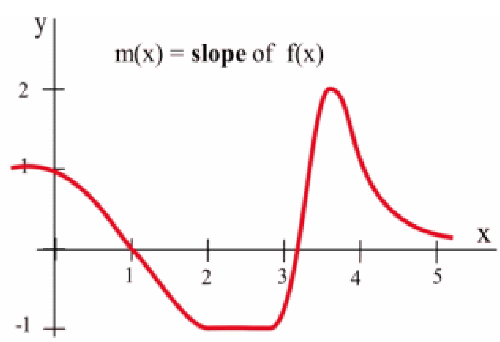


Figure 2.19: Slope of $y = f(x)$

about entire intervals. For example, if $0 < x < 1$, then $f(x)$ is increasing; all the slopes are positive, and so $f'(x) > 0$.

The values of $f'(x)$ definitely depend on the values of x, and $f'(x)$ is a function of x. We can use the results in the table to help sketch the graph of $f'(x)$.

■

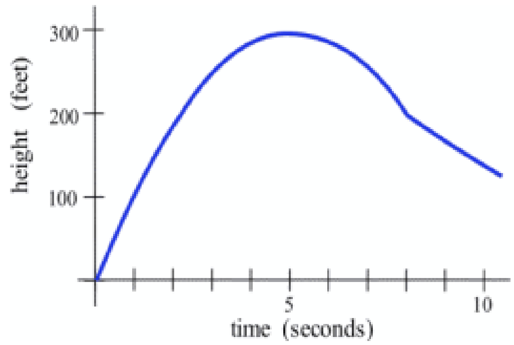


Figure 2.20: $y = h(t)$

Example 2.3.4. Figure 2.20 shows the graph of the height, $h(t)$, of a rocket t seconds after liftoff.

Sketch the graph of the velocity of the rocket at time t. (Velocity is the derivative of the height function, so it is the slope of the tangent to the graph of position or height.)

Solution: We can estimate the slope of the function at several points. The graph in Figure 2.21 shows the velocity of the rocket. This is $v(t) = h'(t)$.

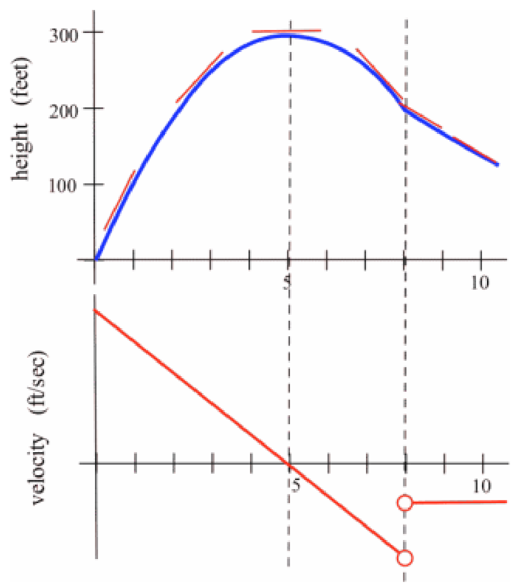


Figure 2.21: $y = h(x)$ and $y = h'(x)$

■

2.3.3 Interpreting the Derivative

So far we have emphasized the derivative as the slope of the line tangent to a graph. That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative that are commonly used.

General Rate of Change: $f'(x)$ is the rate of change of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are people per year, a rate of change in population.

Graphical Slope: $f'(x)$ is the slope of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$.

Physical **Velocity:** If $f(x)$ is the position of an object at time x, then $f'(x)$ is the **velocity** of the object at time x. If the units for x are hours and $f(x)$ is distance measured in miles, then the units for $f'(x) = \frac{df}{dx}$ are miles per hour, which is a measure of velocity.

Acceleration: If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the **acceleration** of the object at time x. If the units are for x are hours and $f(x)$ has the units miles per hour, then the units for the acceleration $f'(x) = \frac{df}{dx}$ are miles per hour per hour, or miles per hour^2.

Business **Marginal Cost**, **Marginal Revenue**, and **Marginal Profit**: We introduced marginal cost in Section 2.2.3 and marginal revenue and profit are defined in a similar way. In business contexts, the word "marginal" usually means the derivative or rate of change of some function.

Therefore, **marginal revenue**, $MR(n)$ is the additional revenue from selling one more item once we have already sold n item. Just as with marginal cost, we will use both this definition and the derivative definition.

$$MR(n) = R(n+1) - R(n) \approx R'(n) \ .$$

Marginal profit is the additional profit from selling one more item once we have already sold n items.

$$MP(n) = P(n+1) - P(n) \approx P'(n) \ .$$

If $f(x)$ is the cost to produce x bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are dollars per bicycle, the cost per bicycle.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.

2.3.4 Graphical Interpretations of the Basic Business Math Terms

Figure 2.22 shows graphs of $TR = R(q)$ and $TC = C(q)$ for producing and selling a certain item. The horizontal axis is the number of items, in thousands. The vertical axis is the number of dollars, also in thousands.

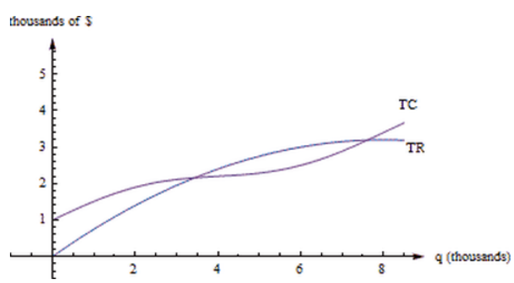


Figure 2.22: Total Revenue (TR) and Total Cost (TC) versus q items.

First, notice how to find the fixed cost and variable cost from the graph here. The fixed cost, FC, is the y-intercept of the TC graph. ($FC = C(0)$.) The graph of variable costs would have the same shape as the graph of TC, shifted down.

Marginal cost for q items is $MC(q) = C(q+1) - C(q)$, but that's impossible to read on this graph. How could you distinguish between $C(4022)$ and $C(4023)$? On this graph, that interval is too small to see, and our best guess at the secant line is actually the tangent line to the TC curve at that point. (This is the reason we want to have the derivative definition handy.)

$MC(q)$ is the slope of the tangent line to the TC curve at the point $(q, C(q))$.

$MR(q)$ is the slope of the tangent line to the TR curve at the point $(q, R(q))$.

Profit is the distance between the TR and TC curves. If you experiment with a clear ruler, you'll see that the biggest profit occurs exactly when the tangent lines to the TR and TC curves are parallel. This is the rule that profit is maximized when $MR = MC$ which we'll explore in Section 3.6.

Example 2.3.5. Suppose the demand for widgets is $D(p) = \frac{1}{p}$, where D is the quantity of widgets, in thousands, at a price of p dollars. Approximate and interpret the derivative of D at $p = \$3$.
Solution: We can approximate $D'(3)$ by looking at a table, Table 2.4, of secant line slopes through the points $(3, D(3)) = \left(3, \frac{1}{3}\right)$ and $(3 + \Delta p, D(3 + \Delta p))$, where Δp gets closer to 0.

Δp	$D(3+\Delta p)$	$\dfrac{D(3+\Delta p)-D(3)}{\Delta p}$
1	$\dfrac{1}{3+1} = 0.25$	$\dfrac{D(4)-D(3)}{1} \approx -0.083333$
0.1	$\dfrac{1}{3+0.1} \approx 0.322581$	$\dfrac{D(3.1)-D(3)}{0.1} \approx -0.107527$
0.01	$\dfrac{1}{3+0.01} \approx 0.332226$	$\dfrac{D(3.01)-D(3)}{0.01} \approx -0.110742$
0.001	$\dfrac{1}{3+0.001} \approx 0.333222$	$\dfrac{D(3.001)-D(3)}{0.001} \approx -0.111074$
0.0001	$\dfrac{1}{3+0.0001} \approx 0.333322$	$\dfrac{D(3.0001)-D(3)}{0.0001} \approx -0.111107$

Table 2.4: Average rate of change of demand approaching $D'(3)$.

Based on this, the quantities in the last column appear to be approaching $-0.1111\ldots$, which is $\frac{-1}{9}$. Later, we will show that that is indeed true. Since $D(p)$ has units "thousands of widgets" and the units for p is "dollars," the units for $D'(3)$ will be thousands of widgets per dollar. Therefore, $D'(3) \approx -0.111$ thousand widgets per dollar, or -111 widgets per dollar. This shows how the demand for widgets will change as the price increases.

Specifically, $D'(3) \approx -111$ widgets per dollar tells us that when the price of a widget is \$3, then the demand for widgets will decrease by about 111 widgets for every dollar the price increases. ■

In the next section, we will lay the theoretical foundation to establish a rigorous and precise definition of the derivative of a function.

2.4 Limits of Functions

2.4.1 Limits

In Section 2.2, we saw that as the interval over which we calculated got smaller, the secant slopes approached the tangent slope. The **limit** gives us better language with which to discuss the idea of "approaches." The limit of a function describes the behavior (i.e., output) of the function when the variable is near, but not necessarily equal to, a specified number. (See Figure 2.23.)

Definition 2.4.1 (Limit)**.** If the values of $f(x)$ get closer and closer, as close as we want, to one single number L, as we take values of x very close to (but not equal to) a number c, then we say, "the **limit** of $f(x)$ as x approaches c is L" and we write

$$\lim_{x \to c} f(x) = L \ .$$

The symbol "$\to$" means "approaches" or, less formally, "gets very close to."

This definition of the limit isn't stated as formally as it could be, but it is sufficient for our purposes in this textbook.

Note:

$f(c)$ is a single number that describes the behavior (value) of $f(x)$ at the point $x = c$.

$\lim\limits_{x \to c} f(x)$ is a single number that describes the behavior of $f(x)$ near, but NOT at, the point $x = c$.

If we have a graph of $f(x)$ near $x = c$, then it is usually easy to determine $\lim_{x \to c} f(x)$.

Example 2.4.1. Use the graph of $y = f(x)$ in Figure 2.24 to determine the following limits.

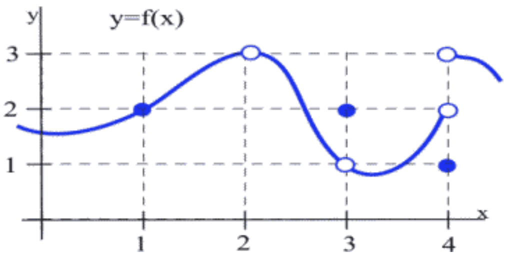


Figure 2.24: $y = f(x)$

(a) $\lim_{x \to 1} f(x)$

Solution: When x is very close to 1, the values of $f(x)$ are very close to $y = 2$. In this example, it happens that $f(1) = 2$, but that is irrelevant for the limit. The only thing that matters is what happens for x close to 1 but $x \neq 1$. ∎

(b) $\lim_{x \to 2} f(x)$

Solution: $f(2)$ is undefined, but we only care about the behavior of $f(x)$ for x close to 2 but not equal to 2. When x is close to 2, the values of $f(x)$ are close to 3. If we restrict x close enough to 2, the values of y will be as close to 3 as we want, so $\lim_{x \to 2} f(x) = 3$. ∎

(c) $\lim_{x \to 3} f(x)$

Solution: When x is close to 3 (or "as x approaches the value 3"), the values of $f(x)$ are close to 1 (or "approach the value 1"), so $\lim_{x \to 3} f(x) = 1$. For this limit it is completely irrelevant that $f(3) = 2$. We only care about what happens to $f(x)$ for x close to and not equal to 3. ∎

(d) $\lim_{x \to 4} f(x)$

Solution: This one is harder and we need to be careful. When x is close to 4 and slightly less than 4 (x is just to the left of 4 on the x-axis), then the values of $f(x)$ are close to 2. But if x is close to 4 and slightly larger than 4 then the values of $f(x)$ are close to 3. If we only know that x is very close to 4, then we cannot say whether $y = f(x)$ will be close to 2 or close to 3. It depends on whether x is on the right or the left side of 4. In this situation, the $f(x)$ values are not close to a single number so we say $f(x)$ does not exist. It is irrelevant that $f(4) = 1$. The limit, as x approaches 4, would still be undefined if $f(4)$ was 3, 2, or anything else. ∎

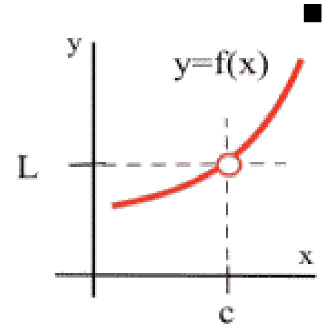


Figure 2.23: The limit of $f(x)$ as $x \to c$ is L.

We can also explore limits using tables and using algebra.

Example 2.4.2. Find $\lim_{x \to 1} f(x)$, where $f(x) = \dfrac{2x^2 - x - 1}{x - 1}$.

Solution: You might try to evaluate at $x = 1$, but $f(x)$ is not defined at $x = 1$. It is tempting, but incorrect, to conclude that this function does not have a limit as x approaches 1.

Using tables: Trying some "test" values for x which get closer and closer to 1 from both the left and the right, we get the following table. Table 2.5 suggests that $\lim_{x \to 1} f(x) = 3$. This is not a proof, but is very convincing. A proof would require algebraic techniques.

Using algebra: We can prove this result by noting that

$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x+1)(x-1)}{x-1} = 2x + 1 \ ,$$

x	$f(x)$	x	$f(x)$
0.9	2.82	1.1	3.2
0.9998	2.9996	1.003	3.006
0.999994	2.999988	1.0001	3.0002
0.9999999	2.9999998	1.000007	3.000014
$\to 1$	$\to 3$	$\to 1$	$\to 3$

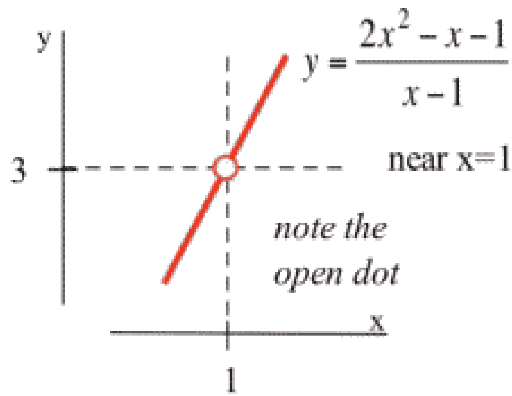
Table 2.5: Finding the limit of $f(x)$ as $x \to 1$

as long as $x \neq 1$. (If $x \neq 1$, then $x - 1 \neq 0$, so it is valid to divide the numerator and denominator by the factor $x - 1$.) The "$x \to 1$" part of the limit means that x is close to 1 but not equal to 1, so our division step is valid and

$$\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x\to 1} 2x + 1 = 3 \ ,$$

which is our answer.

Using a graph: We can graph $y = f(x) = \dfrac{2x^2 - x - 1}{x - 1}$ for x close to 1. Notice that whenever x is close



to 1, the values of $y = f(x)$ are close to 3. Since f is not defined at $x = 1$, the graph has a hole above $x = 1$, but we only care about what $f(x)$ is doing for x close to but not equal to 1. Therefore,

$$\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x\to 1} 2x + 1 = 3 \ .$$

■

2.4.2 One Sided Limits

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.

Definition 2.4.2 (Left and Right Limits)**.** The **left limit** of $f(x)$ as x approaches c is L if the values of $f(x)$ get as close to L as we want when x is very close to and left of c (i.e., $x < c$). We write:

$$\lim_{x\to c^-} f(x) = L \ .$$

The**right limit** of $f(x)$ as x approaches c is L if the values of $f(x)$ get as close to L as we want when x is very close to and right of c (i.e., $x > c$). We write

$$\lim_{x \to c^+} f(x) = L \ .$$

Example 2.4.3. Evaluate the one sided limits of the function $f(x)$ graphed in Figure 2.25 at $x = 0$ and $x = 1$.

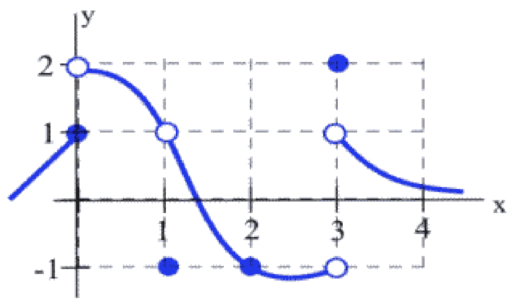


Figure 2.25: $y = f(x)$

Solution: As x approaches 0 from the left, the value of the function is getting closer to 1, so $\lim_{x \to 0^-} f(x) = 1$. As x approaches 0 from the right, the value of the function is getting closer to 2, so $\lim_{x \to 0^+} f(x) = 2$. Notice that since the limit from the left and limit from the right are different, the general limit, $\lim_{x \to 0} f(x)$, does not exist.

At x approaches 1 from either direction, the value of the function is approaching 1, so

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1 \ .$$

■

2.4.3 Continuity

A function that is "friendly" and doesn't have any breaks or jumps in it is called **continuous**. More formally,

Definition 2.4.3 (Continuity at a Point)**.** A function $f(x)$ is **continuous** at $x = a$ if and only if:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

The graph in Figure 2.26 illustrates some of the different ways a function can behave at and near a point, and the table contains some numerical information about the function and its behavior.

a	$f(a)$	$\lim_{x \to a} f(x)$
1	2	2
2	1	2
3	2	Does not exist (DNE)
4	Undefined	2

Based on the information in the table, we can conclude that $f(x)$ is continuous at 1 since $\lim_{x \to 1} f(x) = 2 = f(1)$. We can also conclude from the information in the table that $f(x)$ is not continuous at 2, 3, or 4, because $\lim_{x \to 2} f(x) \neq f(2)$, $\lim_{x \to 3} f(x) \neq f(3)$, and $\lim_{x \to 4} f(x) \neq f(4)$.

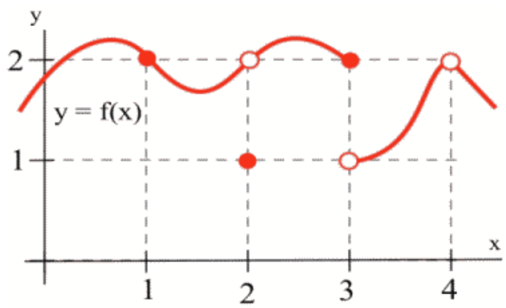


Figure 2.26: $y = f(x)$

The behaviors at $x = 2$ and $x = 4$ exhibit a hole in the graph, sometimes called a **removable discontinuity**, since the graph could be made continuous by changing the value of a single point. The behavior at $x = 3$ is called a **jump discontinuity** or **step discontinuity**, since the graph jumps between two values. Jump discontinuities and vertial asymptotes are both **nonremovable discontinuities** because they cannot be fixed by changing the value of a single point.

So which functions are continuous? It turns out pretty much every function you've studied is continuous where it is defined: polynomial, radical, rational, exponential, and logarithmic functions are all continuous where they are defined. Moreover, any sum, difference, or product of continuous functions is also continuous.

This is helpful, because the definition of continuity says that for a continuous function, $\lim\limits_{x\to a} f(x) = f(a)$. That means for a continuous function, we can find the limit by direct substitution (evaluating the function) if the function is continuous at a.

Example 2.4.4. Evaluate using continuity, if possible:

(a) $\lim\limits_{x\to 2} x^3 - 4x$

Solution: The given function is a polynomial, so it is defined for all values of x. Therefore, we can find the limit by direct substitution:

$$\lim_{x\to 2} x^3 - 4x = 2^3 - 4(2) = 0 \ .$$

■

(b) $\lim\limits_{x\to 2} \dfrac{x-4}{x+3}$

Solution: The given function is rational. It is not defined at $x = -3$, but we are taking the limit as x approaches 2, and the function is defined at that point, so we can use direct substitution:

$$\lim_{x\to 2} \frac{x-4}{x+3} = \frac{2-4}{2+3} = -\frac{2}{5} \ .$$

■

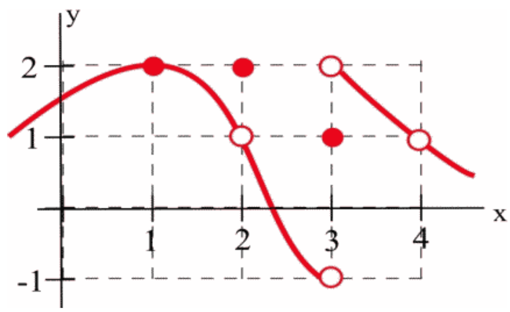
(c) $\lim\limits_{x\to 2} \dfrac{x-4}{x-2}$

Solution: This function is not defined at $x = 2$, so is not continuous at $x = 2$. We cannot use direct substitution. There is also no factor of $x - 2$ in the numerator, so the $x - 2$ in the denominator cannot be canceled out to remove the discontinuity. Therefore, the limit does not exist. ■

2.4.4 Exercises

1. Use the graph to determine the following limits.

   (a) $\lim\limits_{x\to 1} f(x)$

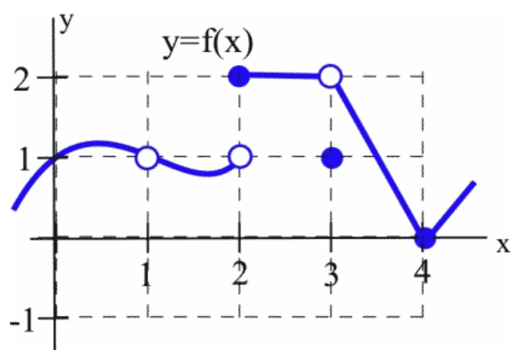


(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to 3} f(x)$

(d) $\lim_{x \to 4} f(x)$

2. Use the graph to determine the following limits.



(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to 3} f(x)$

(d) $\lim_{x \to 4} f(x)$

3. Evaluate the following.

(a) $\lim_{x \to 1} \dfrac{x^2 + 3x + 3}{x - 2}$

(b) $\lim_{x \to 2} \dfrac{x^2 + 3x + 3}{x - 2}$

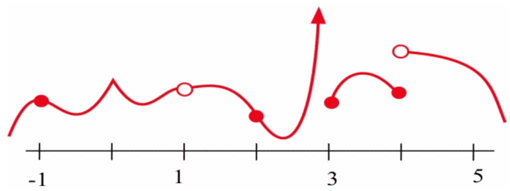
4. Evaluate the following.

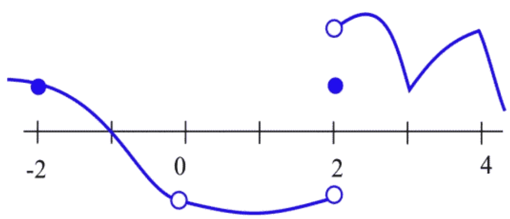
(a) $\lim_{x \to 0} \dfrac{x + 7}{x^2 + 9x + 14}$

(b) $\lim_{x \to 3} \dfrac{x + 7}{x^2 + 9x + 14}$

(c) $\lim_{x \to -4} \dfrac{x + 7}{x^2 + 9x + 14}$

(d) $\lim_{x \to -7} \dfrac{x + 7}{x^2 + 9x + 14}$





5. At which points is the function shown discontinuous?

6. At which points is the function shown discontinuous?

7. Find at least one point at which each function is not continuous and state which of the three conditions in the definition of continuity is violated at that point.

   (a) $\dfrac{x-5}{x+3}$

   (b) $\dfrac{x^2+x-6}{x-2}$

   (c) $\dfrac{x}{x}$

   (d) $\dfrac{\pi}{x^2-6x+9}$

   (e) $\ln(x^2)$

2.5 Finding Derivatives Algebraically

2.5.1 Formal Limit Definition of the Derivative

Definition 2.5.1 (Limit Definition of the Derivative)**.** The **derivative** of a function $f(x)$, at any point $(x, f(x))$ where $f(x)$ is differentiable is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad .$$

Example 2.5.1. Find the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x = 3$.
Solution: The slope of the tangent line is the value of the derivative $f'(3)$. $f(3) = \frac{1}{3}$ and $f(3+h) = \frac{1}{3+h}$, so using the limit definition of the derivative, we have:

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h} \quad .$$

We can simplify by giving the fractions a common denominator:

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} &= \lim_{h\to 0}\frac{\frac{1}{3+h}\cdot\frac{3}{3}-\frac{1}{3}\cdot\frac{3+h}{3+h}}{h}\\
&= \lim_{h\to 0}\frac{\frac{3}{9+3h}-\frac{3+h}{9+3h}}{h}\\
&= \lim_{h\to 0}\frac{\frac{3-(3+h)}{9+3h}}{h}\\
&= \lim_{h\to 0}\frac{\frac{3-3-h}{9+3h}}{h}\\
&= \lim_{h\to 0}\frac{\frac{-h}{9+3h}}{h}\\
&= \lim_{h\to 0}\frac{-h}{9+3h}\cdot\frac{1}{h}\\
&= \lim_{h\to 0}\frac{-1}{9+3h}\ .
\end{aligned}$$

Evaluating this using direct substitution, we have:

$$\lim_{h\to 0}\frac{-1}{9+3h}=\frac{-1}{9+3(0)}=-\frac{1}{9}\ .$$

Thus, the slope of the tangent line to $f(x)=\frac{1}{x}$ at $x=3$ is $-\frac{1}{9}$. ■

Example 2.5.2. Find $\frac{d}{dx}(2x^2-4x-1)$.
Solution: Setting up the derivative using a limit, we have:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\ .$$

We will start by simplifying $f(x+h)$ by expanding:

$$\begin{aligned}
f(x+h) &= 2(x+h)^2-4(x+h)-1\\
&= 2(x^2+2xh+h^2)-4(x+h)-1\\
&= 2x^2+4xh+2h^2-4x-4h-1
\end{aligned}$$

Now finding the limit:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(2x^2+4xh+2h^2-4x-4h-1)-(2x^2-4x-1)}{h}\\
&= \lim_{h\to 0}\frac{2x^2+4xh+2h^2-4x-4h-1-2x^2+4x+1}{h} && \text{(Substitute in the formulas.)}\\
&= \lim_{h\to 0}\frac{4xh+2h^2-4h}{h} && \text{(Now simplify.)}\\
&= \lim_{h\to 0}\frac{h(4x+2h-4)}{h} && \text{(Factor out the } h\text{, then cancel.)}\\
&= \lim_{h\to 0}(4x+2h-4)
\end{aligned}$$

We can find the limit of this expression by direct substitution:

$$f'(x)=\lim_{h\to 0}(4x+2h-4)=4x-4\ .$$

Notice that the derivative depends on x, and that this formula will tell us the slope of the tangent line to $y = f(x)$ at any value x. For example, if we wanted to know the tangent slope of $y = f(x)$ at $x = 3$, we would simply evaluate: $f'(3) = 4 \cdot 3 - 4 = 12 - 4 = 8$. ■

A formula for the derivative function is very powerful, but as you can see, calculating the derivative using the limit definition is very time consuming. In the next section, we will identify some patterns that will allow us to start building a set of rules for finding derivatives without needing the limit definition.

2.5.2 Exercises

For each function $f(x)$ in Problems 1 - 6:

(a) Compute $f'(x)$,

(b) Compute $f'(2)$, and

(c) Find the equation of the line tangent to the graph of $f(x)$ at the point $(2, f(2))$.

1. $f(x) = 3x - 7$
2. $f(x) = 2 - 7x$
3. $f(x) = ax + b$, where a and b are constants
4. $f(x) = x^2 + 3x$
5. $f(x) = 8 - 3x^2$
6. $f(x) = ax^2 + bx + c$, where a, b, and c are constants
7. Match the graphs of the three functions in Figure 2.27 with the graphs of their derivatives.

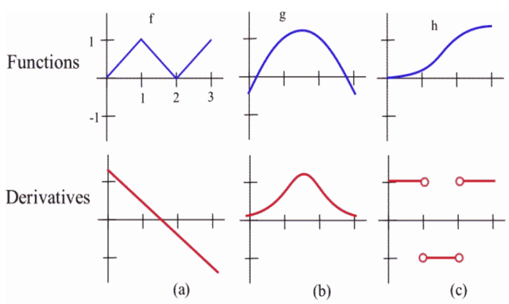


Figure 2.27: Functions and Derivateives

2.6 Sum, Difference, and Constant Multiple Rules

In the next few sections, we'll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula.

2.6.1 Sum and Difference

Given two functions, $f(x)$ and $g(x)$, what can be said about the derivative of their sum? Think about it. What should we expect for the slope of the curve $y = f(x) + g(x)$ or the curve $y = f(x) - g(x)$? We can prove the following result using the limit definition of the derivative, but it should make intuitive sense that the slope of a sum is the sum of the slopes.

Theorem 2.6.1 (Sum and Difference Rule)**.** If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x) \ .$$

2.6.2 Constant Multiples

Given a single function, $f(x)$, we have $2f(x) = f(x) + f(x)$ and $3f(x) = f(x) + f(x) + f(x)$, so it would make sense that the slope of $y = k \cdot f(x)$ is k times the slope of $y = f(x)$. This too, can be proved using the limit definition of the derivative. We often say that when taking the derivative of a multiple of a function, that "constants come along for the ride."

Theorem 2.6.2 (Constant Multiple Rule)**.** If $f(x)$ is differentiable and k is a real number, then

$$\frac{d}{dx}(k \cdot f(x)) = k \cdot f'(x) \ .$$

2.6.3 Exercises

1. Prove Theorem 2.6.1 using the limit definition of the derivative, Definition 2.5.1.
2. Prove Theorem 2.6.2 using the limit definition of the derivative, Definition 2.5.1.

2.7 The Power Rule

This section is a very algebraic section and you should get lots of practice. When you tell someone you have studied calculus, this is the one skill they will expect you to have.

2.7.1 Main Results

These are the simplest rules – rules for the basic functions. We won't prove each of these rules; we'll just use them. But first, let's look at a few examples so that we can see they make sense.

Example 2.7.1. Find the derivative of $f(x) = 135$.
Solution: Think about this one graphically. The graph of $f(x)$ is a horizontal line, so its slope is zero:

$$f'(x) = 0 \ .$$

■

This obviously generalizes.

Theorem 2.7.1. The derivative of a constant is zero.

$$\frac{d}{dx}c = 0$$

Example 2.7.2. Find the derivative of $f(x) = 5x + 2$.
Solution: This is a linear function, so its graph is its own tangent line! The slope of the tangent line, the derivative, is the slope of the line:

$$f'(x) = 5$$

■

Theorem 2.7.2. The derivative of a linear function is its slope.

$$\frac{d}{dx}(mx + b) = m$$

Example 2.7.3. Find the derivative of $f(x) = x^2$.
Solution: Recall the formal definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \ .$$

Using our function $f(x) = x^2$, we use the distributive property to show that $f(x+h) = (x+h)^2 = x^2+2xh+h^2$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x \ . \end{aligned}$$

From all that, we find that $f'(x) = 2x$. ■

Luckily, there is a handy rule we use to skip using the limit for all power functions.

Theorem 2.7.3 (Power Rule)**.** For all real n, we have

$$\frac{d}{dx}x^n = nx^{n-1} \ .$$

2.7.2 Polynomials and Power Functions

Combining Theorem 2.7.3 with the Sum and Difference Rule (Theorem 2.6.1) and the Constant Multiple Rule (Theorem 2.6.2, we can easily find the derivative of any polynomial.

Example 2.7.4. Find the derivative of $g(x) = 4x^3$.
Solution: Using the Power Rule, we know that if $f(x) = x^3$, then $f'(x) = 3x^2$. Notice that $g(x)$ is 4 times the function $f(x)$. So we have:

$$g'(x) = \frac{d}{dx}4x^3 = 4\frac{d}{dx}x^3 = 4\cdot 3x^2 = 12x^2 \ .$$

■

Example 2.7.5. Find the derivative of $p(x) = 17x^{10} + 13x^8 - 1.8x + 1003$.
Solution:

$$\begin{aligned} \frac{d}{dx}\left(17x^{10}+13x^8-1.8x+1003\right) &= \frac{d}{dx}\left(17x^{10}\right) + \frac{d}{dx}\left(13x^8\right) - \frac{d}{dx}(1.8x) + \frac{d}{dx}(1003) \\ &= 17\frac{d}{dx}\left(x^{10}\right) + 13\frac{d}{dx}\left(x^8\right) - 1.8\frac{d}{dx}(x) + \frac{d}{dx}(1003) \\ &= 17\left(10x^9\right) + 13\left(8x^7\right) - 1.8\,(1) + 0 \\ &= 170x^9 + 104x^7 - 1.8 \end{aligned}$$

■

You don't have to show every single step. Do be careful when you're first working with the rules, but pretty soon you'll be able to just write down the derivative directly:

Example 2.7.6. Find $\frac{d}{dx}(17x^2 - 33x + 12)$.
Solution: Writing out the rules, we'd write

$$\frac{d}{dx}(17x^2-33x+12) = 17(2x) - 33(1) + 0 = 34x - 33 \ .$$

Once you're familiar with the rules, you can, in your head, multiply the 2 times the 17 and the 33 times 1, and just write

$$\frac{d}{dx}(17x^2 - 33x + 12) = 34x - 33 \ .$$

■

The Power Rule works even if the power is negative or a fraction. In order to apply it, first translate all roots and basic rational expressions using exponents.

Example 2.7.7. Find the derivative of $y = 3\sqrt{t} - \frac{4}{t^4}$.
Solution: The first step is to rewrite the functions using exponents:

$$y = 3\sqrt{t} - \frac{4}{t^4} = 3t^{1/2} - 4t^{-4} \ .$$

Now you can take the derivative.

$$\begin{aligned} y' = \frac{d}{dt}\left(3t^{1/2} - 4t^{-4}\right) &= 3\left(\frac{1}{2}t^{-1/2}\right) - 4\left(-4t^{-5}\right) \\ &= \frac{3}{2}t^{-1/2} + 16t^{-5} \ . \end{aligned}$$

If there is a reason to, you can rewrite the derivative using radicals and positive exponents:

$$y' = \frac{3}{2}t^{-1/2} + 16t^{-5} = \frac{3}{2\sqrt{t}} + \frac{16}{t^5}$$

■

Be careful when finding the derivatives with negative exponents.

We can immediately apply these rules to solve the problem we started the chapter with – finding a tangent line.

Example 2.7.8. Find the equation of the line tangent to $g(t) = 10 - t^2$ when $t = 2$.
Solution: The slope of the tangent line is the value of the derivative. We can compute $g'(t) = -2t$. To find the slope of the tangent line when $t = 2$, evaluate the derivative at that point. The slope of the tangent line is $g'(2) = -2 \cdot 2 = -4$.

To find the equation of the tangent line, we also need a point on the tangent line. Since the tangent line touches the original function at $t = 2$, we can find the point by evaluating the original function: $g(2) = 10 - 2^2 = 10 - 4 = 6$. The tangent line must pass through the point $(2, 6)$.

Using the point-slope equation of a line, the tangent line will have equation $y - 6 = -4(t - 2)$. Simplifying to slope-intercept form, the equation is $y = -4t + 14$.

Graphing $y = g(t)$ and $y = -4t + 14$, we can verify this line is indeed tangent to the curve.

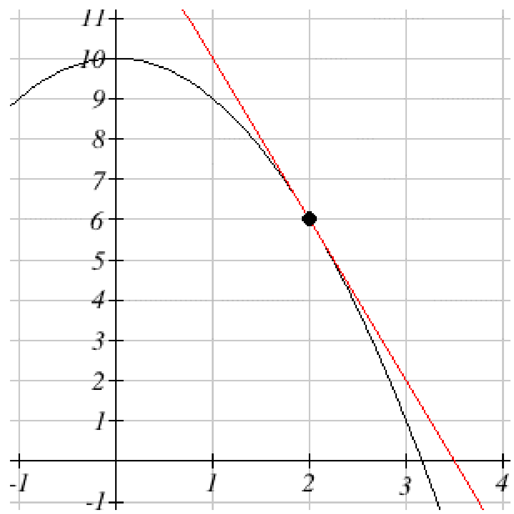
■

We can also use these rules to help us find the derivatives we need to interpret the behavior of a function.

Example 2.7.9. In a memory experiment, a researcher asks the subject to memorize as many words from a list as possible in 10 seconds. Recall is tested, then the subject is given 10 more seconds to study, and so on. Suppose the number of words remembered after t seconds of studying could be modeled by $W(t) = 4t^{2/5}$. Find and interpret $W'(20)$.
Solution: $W'(t) = 4 \cdot \frac{2}{5}t^{-3/5}\frac{\text{words}}{\text{second}} = \frac{8}{5}t^{-3/5}\frac{\text{words}}{\text{second}}$, so $W'(20) = \frac{8}{5} \cdot 20^{-3/5}\frac{\text{words}}{\text{second}} \approx 0.2652$ words per second.

Since $W(t)$ is measured in words and t is in seconds, W' has units "words per second." $W'(20) \approx 0.2652$ words per second means that after 20 seconds of studying, the subject is learning about 0.27 more words for each additional second of studying. ■

Example 2.7.10. The cost to produce x items is $C(x) = \sqrt{x}$ hundred dollars.



(a) What is the cost to produce 100 items? 101 items? What is cost of the 101st item?

Solution: The cost to produce 100 items is $C(100) = \sqrt{100}$ hundred dollars = \$1000 and $C(101) =$ \$1004.99, so it costs \$4.99 for that 101st item. By the definition of marginal cost, the marginal cost is \$4.99. ■

(b) Calculate $C'(x)$ and evaluate $C'(100)$. How does $C'(100)$ compare with the last answer in Part (a)?

Solution: Let's rewrite $C(x) = \sqrt{x} = x^{1/2}$ hundred dollars, the cost for x items. $C'(x) = \dfrac{1}{2}x^{-1/2}$ hundred dollars per item, so $C'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}$ hundred dollars per item = \$5.00 per item. Note how close these answers are! This shows (again) why it's OK that we use both definitions for marginal cost. ■

Example 2.7.11. The demand, D, for a product at a price of p dollars is given by $D(p) = 200 - 0.2p^2$. Find the marginal revenue when the price is \$10.

Solution: First we need to form a revenue equation. Since revenue is price times quantity, $R(p) = p \cdot D(p)$, and the demand equation shows the quantity of product that can be sold at price p. So we have:

$$R(p) = \left(p\,\frac{\text{dollars}}{\text{item}}\right) \cdot (D(p) \text{ items}) = p \cdot (200 - 0.2p^2) \text{ dollars} = 200p - 0.2p^3 \text{ dollars.}$$

Now we can find marginal revenue by finding the derivative:

$$R'(p) = 200 \cdot 1 - 0.2 \cdot 3p^2 = 200 - 0.6p^2 \text{ dollars per dollar.}$$

At a price of \$10, $R'(10) = 200 - 0.6 \cdot 10^2 = 200 - 0.6 \cdot 100 = 200 - 60 =$ \$140 per dollar.

Notice the units for $R'(p)$ are dollars (of revenue) per dollar (of price), so $R'(10) = 140$ means that if the price per item is \$10, then the revenue would increase by \$140 for each dollar that the price per item is increased. ■

2.7.3 Exercises

2.8 Exponential and Logarithmic Functions

Theorem 2.8.1. Exponential Functions

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = \ln(a)a^x$$

Logarithmic Functions

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)} \quad , b > 0, b \neq 1$$

2.9 The Product and Quotient Rules

The basic rules will let us tackle simple functions. But what happens if we need the derivative of a combination of these functions?

Example 2.9.1. Find the derivative of $h(x) = (4x^3 - 11)(x + 3)$.
Solution: This function is not a simple sum or difference of polynomials. It's a product of polynomials. We can simply multiply it out to find its derivative:

$$\begin{aligned} h(x) &= \left(4x^3 - 11\right)(x+3) \\ &= 4x^4 - 11x + 12x^3 - 33 \\ h'(x) &= 16x^3 - 11 + 36x^2 \end{aligned}$$

■

Now suppose we wanted to find the derivative of

$$f(x) = \left(4x^5 + x^3 - 1.5x^2 - 11\right)\left(x^7 - 7.25x^5 + 120x + 3\right)$$

) This function is not a simple sum or difference of polynomials. It's a product of polynomials. We could 'simply' multiply it out to find its derivative as before – who wants to volunteer? Nobody?

We'll need a rule for finding the derivative of a product so we don't have to multiply everything out.

It would be great if we can just take the derivatives of the factors and multiply them, but unfortunately that won't give the right answer. To see that, consider finding derivative of $h(x) = (4x^3 - 11)(x + 3)$. We already worked out the derivative, it is $h'(x) = 16x^3 - 11 + 36x^2$. What if we try differentiating the factors and multiplying them? We'd get $h'(x) = (12x^2)(1) = 12x^2$, which is radically different from the correct answer.

In other words, the derivative of a product is not the product of the derivatives.

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out. They also let us deal with products where the factors are not polynomials. We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.

2.9.1 Derivative Rules: Product and Quotient Rules

Theorem 2.9.1 (Product Rule)**.** If $f(x)$ and $g(x)$ are differentiable, then $\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$.

The derivative of the first factor times the second left alone, plus the first left alone times the derivative of the second.

The Product Rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up:

$$\frac{d}{dx}f(x)\cdot g(x)\cdot h(x) = f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot g'(x)\cdot h(x) + f(x)\cdot g(x)\cdot h'(x)$$

Theorem 2.9.2 (Quotient Rule)**.**

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$$

An easy way to remember this is with a chant.

"Low dee high
minus high dee low
over low squared
it's the way to go."

The numerator of the result resembles the Product Rule, but there is a minus instead of a plus; the minus sign goes with the $g'(x)$. The denominator is simply the square of the original denominator – no derivatives there.

Example 2.9.2. Find the derivative of $h(x) = (4x^3 - 11)(x+3)$
Solution: This is the same function we found the derivative of in Example 2.9.1 , but let's use the Product Rule and check to see if we get the same answer. For this first example, we will provide a lot more detail and steps than one usually actually shows when working a problem like this.

Notice we can think of $h(x)$ as the product of two functions $f(x) = 4x^3 - 11$ and $g(x) = x + 3$. Finding the derivative of each of these, $f'(x) = 12x^2$ and $g'(x) = 1$. Using the Product Rule, we have:

$$\begin{aligned} h'(x) &= (f')(g) + (f)(g') \\ &= \left(12x^2\right)(x+3) + \left(4x^3 - 11\right)(1) \end{aligned}$$

To check if this is equivalent to the answer we found in Example 2.9.1 we could simplify:

$$\begin{aligned} h'(x) &= \left(12x^2\right)(x+3) + \left(4x^3 - 11\right)(1) \\ &= 12x^3 + 36x^2 + 4x^3 - 11 \\ &= 16x^3 + 36x^2 - 11 \end{aligned}$$

From this, we can see the answers are equivalent. ■

Example 2.9.3. Find the derivative of $F(t) = e^t \ln(t)$.
Solution: This is a product, so we need to use the Product Rule.

$$F'(t) = \left(e^t\right)(\ln(t)) + \left(e^t\right)\left(\frac{1}{t}\right) = e^t \ln(t) + \frac{e^t}{t}$$

■

Notice that this was one we could not have done by multiplying out.

Example 2.9.4. Find the derivative of $y = \dfrac{x^4 + 4^x}{3 + 16x^3}$.
Solution: This is a quotient, so we need to use the Quotient Rule.

$$y' = \frac{\left(4x^3 + \ln(4)\cdot 4^x\right)\left(3 + 16x^3\right) - \left(x^4 + 4^x\right)\left(48x^2\right)}{\left(3 + 16x^3\right)^2}$$

■

Now for goodness' sake don't try to simplify that! Remember that simple depends on what you will do next; in this case, we were asked to find the derivative, and we've done that. I expect you to do any basic simplifications, such as multiplying constants together or doing obvious cancellations or combining of terms, but otherwise please STOP unless there is a reason to simplify further.

Example 2.9.5. Suppose a large tank contains 8 kg of a chemical dissolved in 50 liters of water. If a tap is opened and water is added to the tank at a rate of 5 liters per minute, at what rate is the concentration of chemical in the tank changing after 4 minutes?
Solution: First we need to set up a model for the concentration of chemical. The concentration would be measured as kg of chemical per liter of water, kg/L. The number of kg of chemical stays constant at 8 kg, but the quantity of water in the tank is increasing at a constant rate of 5 L/min. The total volume of water in the tank after t minutes is $50 + 5t$, so the concentration after t minutes is

$$c(t) = \frac{8}{50+5t} \quad .$$

To find the rate at which the concentration is changing, we need the derivative:

$$\begin{aligned} c'(t) &= \frac{\frac{d}{dt}(8)\cdot(50+5t) - (8)\cdot\frac{d}{dt}(50+5t)}{(50+5t)^2} \\ &= \frac{(0)\cdot(50+5t)-(8)(5)}{(50+5t)^2} \\ &= -\frac{40}{(50+5t)^2} \quad . \end{aligned}$$

At $t = 4$ minutes, we have

$$c'(4) = -\frac{40}{(50+5(4))^2} \approx -0.00816 \quad .$$

Note that the units here are kg per liter, per minute, or $\frac{\text{kg/L}}{\text{min}}$. In other words, this tells us that after 4 minutes, the concentration of chemical is decreasing at a rate of 0.00816 kg/L per minute. ■

2.9.2 More Graphical Interpretations of the Basic Business Math Terms

Returning to our discussion of business and economics topics, in addition to total cost and marginal cost, we often also want to talk about average cost or average revenue.

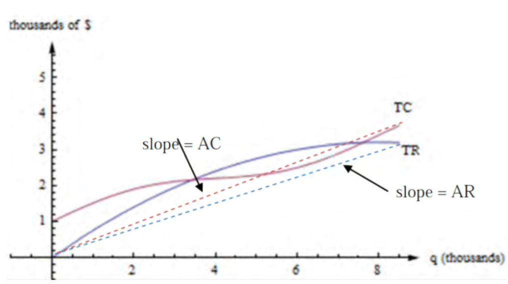
Recall from Section 1.1 (Definition 1.1.5 that the average cost for q items is the total cost divided by q, or $A(q) = \frac{C(q)}{q}$. You can also talk about the average fixed cost, $\frac{FC(q)}{q}$, or the average variable cost, $\frac{VC(q)}{q}$.

Also recall that the Average Revenue (AR) for q items is the total revenue divided by q, or $AR(q) = \frac{R(q)}{q}$. But $\frac{R(q)}{q} = \frac{q\cdot D(q)}{q} = D(q)$, so $AR(q) = D(q)$, which just the price since $p = D(q)$.

We already know that we can find average rates of change by finding slopes of secant lines. AC, AR, MC, and MR are all rates of change, and we can find them with slopes too.

$AC(q)$ is the slope of a diagonal line, from $(0, 0)$ to $(q, C(q))$.

$AR(q)$ is the slope of the line from $(0, 0)$ to $(q, R(q))$, which is the also equal to the price at $(q, R(q))$ by the calulations in the previous paragraph.



And just as we found marginal Total Cost, we can also find marginal Average Cost.

Example 2.9.6. The cost, in thousands of dollars, for producing x thousand cellphone cases is given by $C(x) = 22 + x - 0.004x^2$. Find

(a) the Fixed Costs, **Solution:** The fixed costs are the costs when no items are produced: $C(0) = 22$ thousand dollars. ■

(b) the Average Cost when 5 thousand, 10 thousand, or 20 thousand cases are produced, **Solution:** The average cost function is total cost divided by number of items, so

$$AC(x) = \frac{C(x)}{x} = \frac{22 + x - 0.004x^2}{x}.$$

Note the units are thousands of dollars per thousands of items, which simplifies to just dollars per item.

At a production of 5 thousand items: $AC(5) = \dfrac{22 + 5 - 0.004(5)^2}{5} = 5.38$ dollars per item.

At a production of 10 thousand items: $AC(10) = \dfrac{22 + 5 - 0.004(10)^2}{10} = 3.16$ dollars per item.

At a production of 20 thousand items: $AC(20) = \dfrac{22 + 5 - 0.004(20)^2}{20} = 2.02$ dollars per item.

Notice that while the total cost increases with production, the average cost per item decreases, because the initial fixed costs are being distributed across more items. ■

(c) the Marginal Average Cost when 5 thousand cases are produced.

Solution: For the marginal average cost, we need to find the derivative of the average cost function. We can either calculate this using the Quotient Rule, or we could use algebra to simplify the equation first (this is the easier option – remember, simplifying before differentiating is almost always easier):

$$\begin{aligned} AC(x) &= \frac{22 + x - 0.004x^2}{x} \\ &= \frac{22}{x} + \frac{x}{x} - \frac{0.004x^2}{x} \\ &= \frac{22}{x} + 1 - 0.004x \\ &= 22 \end{aligned}$$

(Note: we haven't differentiated yet, only simplified.)

Taking the derivative,

$$AC'(x) = -22x^{-2} - 0.004 = -\frac{22}{x^2} - 0.004 \ .$$

When 5 thousand items are produced,

$$AC'(5) = -\frac{22}{5^2} - 0.004 = -0.884 \ .$$

Since the units on $AC(x)$ are dollars per item, and the units on x are thousands of items, the units on $AC'(x)$ are dollars per item per thousands of items. This tells us that when 5 thousand items are produced, the average cost per item is decreasing at a rate of \$0.884 per additional thousand items produced. ■

2.10 The Chain Rule

The big idea of the **Chain Rule** is that it allows us to find the derivative of a composition of functions.

Example 2.10.1. Find the derivative of $y = (4x^3 + 15x)^2$.
Solution: This is not a simple polynomial, so we can't use the basic building block rules yet. It is a product, so we could write it as $y = (4x^3 + 15x)^2 = (4x^3 + 15x)(4x^3 + 15x)$ and use the Product Rule. Or we could multiply it out and simply differentiate the resulting polynomial. I'll do it the second way:

$$\begin{aligned} y &= \left(4x^3 + 15x\right)^2 \\ &= 16x^6 + 120x^4 + 225x^2 \\ y' &= 96x^5 + 480x^3 + 450x \ . \end{aligned}$$

■

Now suppose we want to find the derivative of $y = \left(4x^3 + 15x\right)^{20}$. We could write it as a product with 20 factors and use the Product Rule, or we could multiply it out. But I don't want to do that. Do you? I didn't think so.

We need an easier way, a rule that will handle a composition like this. The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out. It will also handle compositions where it wouldn't be possible to multiply it out.

The Chain Rule is a common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.

Theorem 2.10.1 (Chain Rule)**.** Let $f(u)$ and $g(x)$ be differentiable functions, with $y = f(u)$ and $u = g(x)$.

Chain Rule (Leibniz notation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Notice that the du's seem to cancel. This is one advantage of the Leibniz notation – it can remind you of how the Chain Rule chains together.

Chain Rule (Lagrange notation)

$$f'(x) = f'(u) \cdot g'(x) = f'\left(g(x)\right) \cdot g'(x)$$

Chain Rule (in words)

The derivative of a composition is the derivative of the outside (with the inside staying the same) times the derivative of the inside.

I recite the version in words each time I take a derivative, especially if the function is complicated.

Example 2.10.2. Find the derivative of $y = \left(4x^3 + 15x\right)^2$.
Solution: This is the same one we did before by multiplying out. This time, let's use the Chain Rule: The inside function is what appears inside the parentheses: $4x^3 + 15x$. The outside function is the first thing we find as we come in from the outside – it's the square function, u^2.

The derivative of this outside function is $2u$. Now using the Chain Rule, the derivative of our original function is $2u$ times the derivative of the inside, which is

$$\frac{du}{dx} = \frac{d}{dx}(4x^3 + 15x) = 12x^2 + 15 \ .$$

So,

$$y' = 2\left(4x^3 + 15x\right)\left(12x^2 + 15\right) \ .$$

■

If you multiply this out, you get the same answer we got before. Hurray! Algebra works!

Example 2.10.3. Find the derivative of $y = (4x^3 + 15x)^{20}$.
Solution: Now we have a way to handle this one. It's the derivative of the outside times the derivative of the inside.

The outside function is u^{20}, which has derivative $u' = 20 \cdot u^{19}$, so

$$y' = 20\left(4x^3 + 15x\right)^{19}\left(12x^2 + 15\right) \ .$$

■

Example 2.10.4. Differentiate $y = e^{x^2+5}$.
Solution: This isn't a simple exponential function; it's a composition. Typical calculator or computer syntax can help you see what the "inside" function is here. On a TI calculator, for example, when you push the e^x key, it opens up parentheses: `e^(`. This tells you that the "inside" of the exponential function is the exponent. Here, the inside is the exponent $u = x^2 + 5$. Now we can use the Chain Rule: We want the derivative of the outside times the derivative of the inside. The outside is e^u, so its derivative is $\frac{d}{du}e^u = e^u$. The derivative of the inside function is $\frac{d}{du}(x^2 + 5) = 2x$. So

$$\frac{d}{dx}\left(e^{x^2+5}\right) = \left(e^{x^2+5}\right) \cdot (2x).$$

■

Example 2.10.5. The table gives values for f , f' , g, and g' at a number of points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and 0.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(g \circ f)(x)$
−1	2	3	1	0		
0	−1	1	3	2		
1	1	0	−1	3		
2	3	−1	0	1		
3	0	2	2	−1		

Solution:

$$(f \circ g)(-1) = f(g(-1)) = f(3) = 0$$

$$(f \circ g)(0) = f(g(0)) = f(1) = 1$$

$$(f \circ g)'(-1) = f'(g(-1)) \cdot g'(-1) = f'(3) \cdot (0) = (2)(0) = 0$$

$$(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot (2) = (-1)(2) = -2$$

■

Example 2.10.6. If 2400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in t years is $p(t) = 2400 \cdot 2^{t/3}$.

(a) How many people are expected to have the disease in 2 years?

Solution: In 2 years, $p(2) = 2400 \cdot 2^{2/3} \approx 3,810$ people will have the disease. ■

(b) When are 50,000 people expected to have the disease?

Solution: We know $p(t) = 50{,}000$, for some t, so we need to solve $50{,}000 = 2400 \cdot 2^{t/3}$ for t. We could start by isolating the exponential by dividing both sides by 2400:

$$\begin{aligned}
\frac{50000}{2400} &= 2^{t/3} \\
\ln\left(\frac{50000}{2400}\right) &= \ln\left(2^{t/3}\right) && \text{(Taking the natural log of both sides.)} \\
\ln\left(\frac{500}{24}\right) &= \frac{t}{3}\ln(2) && \text{(Using the exponent property for logs.)} \\
t &= \frac{3\ln\left(\frac{500}{24}\right)}{\ln(2)} \approx 13.14 \text{ years} && \text{(Solving for } t\text{.)}
\end{aligned}$$

We expect 50,000 people to have the disease about 13.14 years from now. ■

(c) How fast is the number of people with the disease expected to grow now and 2 years from now?

Solution: This is asking for $p'(t)$ when $t = 0$ and 2 years. Using the Chain Rule,

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left(2400 \cdot 2^{t/3}\right)\\ &= 2400 \cdot 2^{t/3} \cdot \ln(2) \cdot \frac{1}{3}\\ &\approx 554.5 \cdot 2^{t/3}\end{aligned}$$

So, at $t = 0$, the rate of growth of the disease is approximately $554.5 \cdot 2^0 \approx 554.5$ people per year. In two years, the rate of growth will be approximately $554.5 \cdot 2^{2/3} \approx 880$ people per year. ■

2.10.1 Derivatives of Complicated Functions

You're now ready to take the derivative of some mighty complicated functions. But how do you tell what rule applies first? Work your way in from the outside – what do you encounter first? That's the first rule you need. Use the Product, Quotient, and Chain Rules to peel off the layers, one at a time, until you're all the way inside.

Example 2.10.7. Find $\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right)$.

Solution: Coming in from the outside, we see that this is a product of two (complicated) functions. So we'll need the Product Rule first. we'll fill in the pieces we know, and then we can figure the rest as separate steps and substitute in at the end:

$$\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right) = \left(\frac{d}{dx}\left(e^{3x}\right)\right) \cdot \ln(5x+7) + e^{3x} \cdot \left(\frac{d}{dx}\left(\ln(5x+7)\right)\right)$$

Now as separate steps, we'll find

$$\frac{d}{dx}\left(e^{3x}\right) = 3e^{3x} \qquad \text{(using the Chain Rule)}$$

and

$$\frac{d}{dx}\left(\ln(5x+7)\right) = \frac{1}{5x+7} \cdot 5 \qquad \text{(also using the Chain Rule).}$$

Finally, to substitute these in their places:

$$\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right) = \left(3e^{3x}\right) \cdot \ln(5x+7) + e^{3x} \cdot \left(\frac{1}{5x+7} \cdot 5\right)$$

(We can stop here – we don't need to try to simplify any further.) ■

Example 2.10.8. Differentiate $z = \left(\frac{3t^3}{e^t(t-1)}\right)^4$.

Solution: Don't panic! As we come in from the outside, what's the first thing we encounter? It's that fourth power. That tells us that this is a composition, a (complicated) function raised to the fourth power.

Step One: Use the Chain Rule. The derivative of the outside times the derivative of the inside:

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4 = 4\left(\frac{3t^3}{e^t(t-1)}\right)^3 \cdot \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$$

Now we're one step inside, and we can concentrate on just the $\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$ part. Now, as you come in from the outside, the first thing you encounter is a quotient – this is the quotient of two (complicated) functions.

Step Two: Use the Quotient Rule. The derivative of the numerator is straightforward, so we can just calculate it. The derivative of the denominator is a bit trickier, so we'll leave it for now:

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right) = \frac{\left(9t^2\right)\left(e^t(t-1)\right) - \left(3t^3\right)\left(\frac{d}{dt}\left(e^t(t-1)\right)\right)}{\left(e^t(t-1)\right)^2}.$$

Now we've gone one more step inside, and we can concentrate on just the $\frac{d}{dt}\left(e^t(t-1)\right)$ part, which involves a product.

Step Three: Use the Product Rule:

$$\frac{d}{dt}\left(e^t(t-1)\right) = \left(e^t\right)(t-1) + \left(e^t\right)(1)$$

And now we're all the way in – no more derivatives to take!

Step Four: Now it's just a question of substituting back – be careful now!

$$\frac{d}{dt}\left(e^t(t-1)\right) = \left(e^t\right)(t-1) + \left(e^t\right)(1)$$

so

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right) = \frac{\left(9t^2\right)\left(e^t(t-1)\right) - \left(3t^3\right)\left(\left(e^t\right)(t-1) + \left(e^t\right)(1)\right)}{\left(e^t(t-1)\right)^2}$$

so

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4 = 4\left(\frac{3t^3}{e^t(t-1)}\right)^3 \cdot \left(\frac{\left(9t^2\right)\left(e^t(t-1)\right) - \left(3t^3\right)\left(\left(e^t\right)(t-1) + \left(e^t\right)(1)\right)}{\left(e^t(t-1)\right)^2}\right)$$

Phew! ∎

2.11 Trigonometric Functions

2.11.1 Derivative Formulas

Theorem 2.11.1.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

From this, we can figure out the derivatives of the four other trigonometric functions.

Theorem 2.11.2.

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Proof. Here, we'll use the fact that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and use the Quotient Rule, since we are finding the derivative of a quotient (fraction).

$$
\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\
&= \frac{\cos(x)\cdot\frac{d}{dx}\sin(x) - \sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}
$$

□

2.11.2 Examples

2.11.3 Exercises

2.12 Higher Order Derivatives

2.12.1 Second Derivative

Definition 2.12.1 (Second Derivative)**.** Let $y = f(x)$. The **second derivative** of $f(x)$ is the derivative of its derivative: $\frac{d}{dx}y' = \frac{d}{dx}f'(x)$.

Using Lagrange notation, this is $f''(x)$ or y''. You can read this aloud as "f double prime of x" or "y double prime."

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as "the second derivative of y (or f)."

In Section 3.3, we will discuss geometric interpretations of the second derivative and how it is applied in that context. For now, we will do some examples and highlight a number of important applications related to its conceptual understanding as the rate of change of the derivative of a function.

Example 2.12.1. Find the second derivative of $f(x) = 3x^7$.
Solution: First, we need to find the first derivative:

$$f'(x) = 21x^6 \quad .$$

Then we take the derivative of that function:

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}21x^6 = 126x^5 \quad .$$

■

Definition 2.12.2. If $f(x)$ represents the position of a particle at time x, then $v(x) = f'(x)$ represents the **velocity** (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ represents the **acceleration** (the rate of change of the velocity) of the particle.

You are probably familiar with acceleration from driving or riding in a car. The speedometer tells you your velocity (speed). When you leave from a stop and press down on the accelerator, you are accelerating – increasing your speed.

Example 2.12.2. The height (feet) of a particle after t seconds is $f(t) = t^3 - 4t^2 + 8t$ feet. Find the height, velocity and acceleration of the particle when $t = 0, 1$, and 2 seconds.

Solution: $f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity after t seconds is $v(t) = f'(t) = 3t^2 - 8t + 8$ feet per second so $v(0) = 8$ ft/s, $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = f''(t) = 6t - 8$ feet per second per second, or $6t - 8$ feet per second squared so $a(0) = -8$ ft/s^2, $a(1) = -2$ ft/s^2 and $a(2) = 4$ ft/s^2.

At time 0 and 1, the velocity is positive but the acceleration is negative, so the particle is going up, but its velocity is decreasing at those points; the particle was slowing down. At time 2, the velocity and acceleration are positive, so the particle is speeding up. ■

2.12.2 Third and Higher Derivatives

2.12.3 Exercises

Chapter 3

Applications of Derivatives

3.1 Linear Approximations

3.1.1 Tangent Line Approximation

Back when we first thought about the derivative, we used the slope of secant lines over tiny intervals to approximate the derivative:

$$f'(a) \approx \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a} \ .$$

Now that we have other ways to find derivatives, we can exploit this approximation to go the other way. Solve the expression above for $f(x)$, and you'll get the tangent line approximation.

Definition 3.1.1 (Tangent Line Approximation (TLA) or Linear Approximation)**.** To approximate the value of $f(x)$ using TLA, find some a where a and x are close, and you know the exact values of both $f(a)$ and $f'(a)$. Then:

$$f(x) \approx f(a) + f'(a)(x - a) \ .$$

Another way to look at the same formula:

$$\Delta y \approx f'(a)\Delta x \ .$$

How close is close? It depends on the shape of the graph of $f(x)$. In general, the closer the better.

Example 3.1.1. Suppose we know that $g(20) = 5$ and $g'(20) = 1.4$. Use this information to approximate $g(23)$ and $g(18)$.
Solution: Using the tangent line approximation:

$$g(23) \approx 5 + (1.4)(23 - 20) = 9.2$$

and

$$g(18) \approx 5 + (1.4)(18 - 20) = 2.2 \ .$$

■

Note that we don't know if these approximations are close – but they're the best we can do with the limited information we have to start with. Note also that 18 and 23 are sort of close to 20, so we can hope these approximations are pretty good. We would feel more confident using this information to approximate $g(20.003)$. We would feel very unsure using this information to approximate $g(55)$.

3.2 Elasticity of Demand

3.2.1 Elasticity

We know that demand functions are decreasing, so when the price increases, the quantity demanded goes down. But what about revenue = price × quantity? When the price increases will revenue go down because the demand dropped so much? Or will revenue increase because demand didn't drop very much?

Elasticity of demand is a measure of how demand reacts to price changes. It's normalized − that means the particular prices and quantities don't matter, and everything is treated as a percent change. The formula for elasticity of demand involves a derivative, which is why we're discussing it here.

Definition 3.2.1 (Elasticity of Demand)**.** Given a demand function $q = D(p)$, the **elasticity of demand** is

$$E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right|$$

We expect $D(p)$ to be a decreasing function in most cases, so $D'(p)$ is usually negative. That's why we have the absolute value: so that E will always be positive. This means we can also write E as

$$E = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p \cdot D'(p)}{D(p)} \quad \text{if } D'(p) < 0 \ .$$

- If $E < 1$, we say demand is **inelastic**. In this case, raising prices increases revenue.
- If $E > 1$, we say demand is **elastic**. In this case, raising prices decreases revenue.
- If $E = 1$, we say demand is **unitary**. $E = 1$ when the revenue function is a maximum.

3.2.2 Interpretation of elasticity

If the price increases by 1%, the demand will decrease by E%.

Example 3.2.1. A company sells q ribbon winders per year at \$$p$ per ribbon winder. The demand function for ribbon winders is given by $p = 300 - 0.02q$. Find the elasticity of demand when the price is \$70 each. Will an increase in price lead to an increase in revenue?

Solution: First, we need to solve the demand equation so it gives q in terms of p. Then we can find $\frac{dq}{dp}$.

$$\begin{aligned} p &= 300 - 0.02q \\ p + 0.02q &= 300 \\ 0.02q &= 300 - p \\ q = D(p) &= 15000 - 50p \ . \end{aligned}$$

Then $\frac{dq}{dp} = -50$. Now We need to find q when $p = 70$:

$$q = D(70) = 15000 - 50 \cdot 70 = 15000 - 3500 = 11500 \ .$$

Now compute

$$E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right| = \left|\frac{70}{11500} \cdot (-50)\right| \approx 0.3$$

$E < 1$, so demand is inelastic. Increasing the price by 1% would only cause a 0.3% drop in demand. Increasing the price would lead to an increase in revenue, so it seems that the company should increase its price. ■

The demand for necessary products, such as food, tends to be inelastic. Even if the price goes up, people still have to buy about the same amount of food, and revenue will not go down. The demand for products that people can do without, or put off buying, such as cars, tends to be elastic. If the price goes up, people will just not buy cars right now, and revenue will drop.

Example 3.2.2. A company finds the demand q, in thousands, for their kites to be $q = 400 - p^2$ at a price of p dollars. Find the elasticity of demand when the price is \$5 and when the price is \$15. Then find the price that will maximize revenue.

Solution: Calculating the derivative, $\frac{dq}{dp} = -2p$. The elasticity equation as a function of p will be:

$$E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right| = \left|\frac{p}{400 - p^2} \cdot (-2p)\right| = \left|\frac{-2p^2}{400 - p^2}\right|$$

Evaluating this to find the elasticity at \$5 and at \$15:

$$E = \left|\frac{-2(5)^2}{400 - (5)^2}\right| \approx 0.133$$

So the demand is inelastic when the price is \$5.

At a price of \$5, a 1% increase in price would decrease demand by only 0.133%. Revenue could be raised by increasing prices.

$$E = \left|\frac{-2(15)^2}{400 - (15)^2}\right| \approx 2.571$$

So the demand is elastic when the price is \$15.

At a price of \$15, a 1% increase in price would decrease demand by about 2.571%. Revenue could be raised by decreasing prices.

To maximize the revenue, we could solve for p when $E = 1$:

$$\begin{aligned} \left|\frac{-2p^2}{400 - p^2}\right| &= 1 \\ 2p^2 &= 400 - p^2 \\ 3p^2 &= 400 \\ p &= \sqrt{\frac{400}{3}} \approx 11.55\ . \end{aligned}$$

A price of \$11.55 will maximize the revenue. ∎

3.3 Concavity and Inflection Points

3.3.1 Second Derivative and Concavity

Definition 3.3.1. Graphically, a function is **concave up** if its graph is curved with the opening upward (Figure 3.1(a)). Similarly, a function is **concave down** if its graph opens downward (Figure 3.1(b)).

Concave up: looks like a cup.

Concave down: looks like a frown.

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is concave up on that interval. We can say that $f(x)$ is increasing (or decreasing) at an increasing rate.

If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is concave down on that interval. We can say that $f(x)$ is increasing (or decreasing) at a decreasing rate.

Figure 3.2 shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.

For example, suppose an epidemic has started, and you, as a member of Congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In Figure 3.3 below, $f(x)$ is the number of people who have the disease at time x, and two different situations are shown. In both Figure 3.3(a) and Figure 3.3(b), the number of people with the disease, $f(now)$, and the rate at which new people are getting sick, $f'(now)$, are the same. The

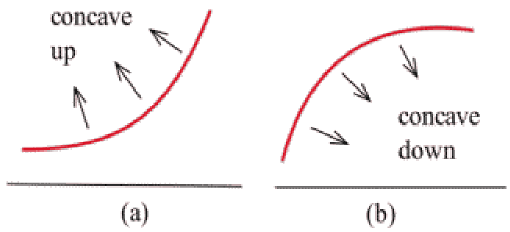


Figure 3.1: Concave up and down.

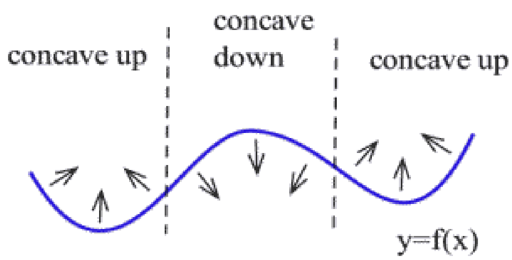


Figure 3.2: Concave up and down.

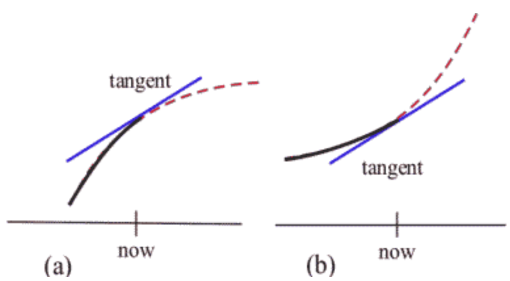


Figure 3.3: Epidemic models

difference in the two situations is the concavity of $f(x)$, and that difference in concavity might have a big effect on your decision.

In Figure 3.3(a), $f(x)$ is concave down at "now," the slopes are decreasing, and it looks as if the epidemic is tailing off. We can say "$f(x)$ is increasing at a decreasing rate." It appears that the current methods are starting to bring the epidemic under control.

In Figure 3.3(b), $f(x)$ is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the derivative is increasing or decreasing.

The derivative of a function $f(x)$ is a function that gives information about the slope of $y = f(x)$. The derivative tells us if the original function is increasing or decreasing.

Because $f'(x)$ is a function, we can take its derivative. This second derivative also gives us information about our original function, $f(x)$. The second derivative gives us a mathematical way to tell how the graph of a function is curved. The second derivative tells us if the original function is concave up or down.

3.3.2 Inflection Points

Definition 3.3.2 (Inflection Point)**.** An **inflection point** of a curve is a point on the graph where the concavity of the curve changes, from concave up to down or from concave down to up.

Example 3.3.1. Which of the labeled points in the graph in Figure 3.4 are inflection points?
Solution: The concavity changes at points b and g. At points a and h, the graph is concave up on both sides, so the concavity does not change. At points c and f, the graph is concave down on both sides. At

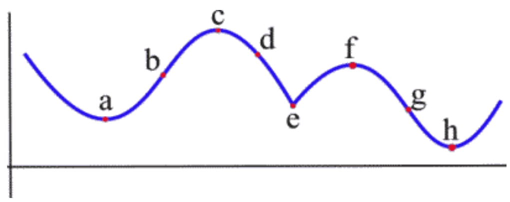


Figure 3.4: Sample curve

point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change. Therefore, only points b and g are inflection points on the curve. ∎

Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

Theorem 3.3.1. If $(a, f(a))$ is an inflection point of the curve $y = f(x)$, then either $f''(a) = 0$ or $f''(a)$ is undefined.

So to find the inflection points of a function we only need to check the points where $f''(x)$ is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined at a point on the curve $y = f(x)$. We still need to check that $f''(x)$ changes sign at that point. The functions in the next example illustrate what can happen.

Example 3.3.2. Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$. For which of these functions is the point $(0, 0)$ an inflection point?

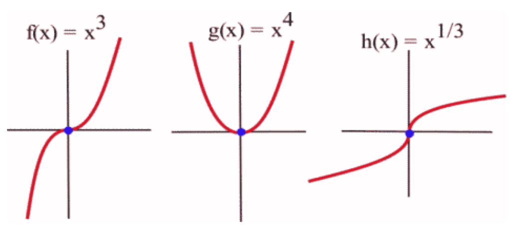


Figure 3.5: Examples and nonexamples of inflection points

Solution: Graphically, it is clear that the concavity of $y = f(x) = x^3$ and $y = h(x) = x^1/3$ changes at the point $(0, 0)$, so $(0, 0)$ is an inflection point of the curves $y = f(x)$ and $y = h(x)$. The curve $y = f(x)$ goes from concave down to concave up and the curve $y = h(x)$ goes from concave up to concave down. The function $g(x) = x^4$ is concave up everywhere (except at the point $(0, 0)$), so $(0, 0)$ is not an inflection point of the curve $y = g(x)$.

We can also compute the second derivatives and check the sign change.

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. The only point at which $f''(x) = 0$ or is undefined ($f'(x)$ is not differentiable) is at $x = 0$. If $x < 0$, then $f''(x) < 0$ so $y = f(x)$ is concave down. If $x > 0$, then $f''(x) > 0$, so $y = f(x)$ is concave up. At $x = 0$, the concavity changes so the point $(0, f(0)) = (0, 0)$ is an inflection point of the curve $y = f(x) = x^3$.

If $g(x) = x^4$, then $g'(x) = 4x^3$ and $g''(x) = 12x^2$. The only point at which $g''(x) = 0$ or is undefined is at $x = 0$. If $x < 0$, then $g''(x) > 0$ so $y = g(x)$ is concave up. If $x > 0$, then $g''(x) > 0$, so $y = g(x)$ is also concave up. At $x = 0$, the concavity does not change, so the point $(0, g(0)) = (0, 0)$ is not an inflection point of the curve $y = g(x) = x^4$. Keep this example in mind!

If $h(x) = x^{1/3}$, then $h'(x) = \frac{1}{3}x^{-2/3}$ and $h''(x) = -\frac{2}{9}x^{-5/3}$. $h''(x)$ is not defined at $x = 0$, but if $x < 0$, then $h''(x) > 0$ and if $x > 0$, then $h''(x) < 0$, so $y = h(x)$ changes concavity at the point $(0, h(0)) = (0, 0)$ and $(0, 0)$ is an inflection point of the curve $y = h(x) = x^{1/3}$. ∎

Example 3.3.3. Sketch the graph of a function, $f(x)$ with $f(2) = 3$, $f'(2) = 1$, and an inflection point at the point $(2, 3)$.

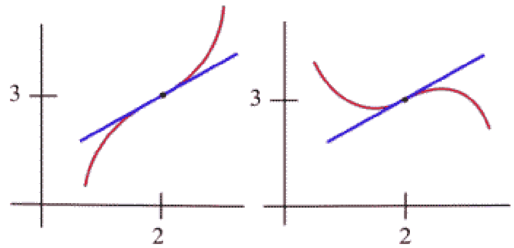


Figure 3.6: Two possible solutions to Example 3.3.3.

Solution: Two possible solutions are shown in Figure 3.6.

■

3.4 Maximum and Minimum Function Values

"Nothing takes place in the world whose meaning is not that of some maximum or minimum."
(Leonhard Euler)

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Without calculus, we only know how to find the optimum points in a few specific examples (for example, we know how to find the vertex of a parabola). But what if we need to optimize an unfamiliar function?

The best way we have without calculus is to examine the graph of the function, perhaps using technology. But our view depends on the viewing window we choose – we might miss something important. In addition, we'll probably only get an approximation this way. (In some cases, that will be good enough.)

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven't missed anything important.

3.4.1 Local Maxima and Minima

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will develop and use.

Definition 3.4.1 (Local Extrema)**.** Suppose that $f(x)$ is a function defined over some interval containing $x = a$.

$f(a)$ is a **local maximum** of $f(x)$ if $f(a) \geq f(x)$ for all x near a.

$f(a)$ is a **local minimum** of $f(x)$ if $f(a) \leq f(x)$ for all x near a.

$f(a)$ is a **local extremum** of $f(x)$ if $f(a)$ is a local maximum or a local minimum.

We say that $f(x)$ is **maximized** or **minimized** at $x = a$, respectively.

Note 3.4.1 (Latin grammar)**.** The plurals of maximum, minimum, and extremum are **maxima**, **minima**, and **extrema**, respectively. these words are Latin words, so we use the Latin plurals.

Note 3.4.2 (Relative and Local)**.** The terms **relative** and **local** are used interchangably. A relative extremum is a local extremum.

Definition 3.4.2 (Optimization)**.** The process of finding extrema of a function is called **optimization**.

Definition 3.4.3 ((Global Extrema)**.** Suppose that $f(x)$ is a function defined over some interval containing $x = a$.

$f(a)$ is a **global maximum** of $f(x)$ if $f(a) \geq f(x)$ for all x in the domain of $f(x)$.

$f(a)$ is a **global minimum** of $f(x)$ if $f(a) \leq f(x)$ for all x in the domain of $f(x)$.

$f(a)$ is a **global extremum** of $f(x)$ if $f(a)$ is a global maximum or a global minimum.

Note 3.4.3 (Absolute and Global)**.** The terms **absolute** and **global** are used interchangably. An absolute extremum is a global extremum.

The local and global extrema of the function in Figure 3.7 are labeled. You should notice that every global extremum is also a local extremum, but there are local extrema that are not global extrema.

Figure 3.7: Local and global extrema

If $h(x)$ is the height of the earth above sea level at the location x, then the global maximum of $h(x)$ is the elevation at the summit of Mt. Everest, 29,031.7 feet. The local maximum of $h(x)$ for the United States is the elevation at the summit of Denali (formerly Mt. McKinley), 20,310 feet. The local minimum of $h(x)$ for the United States is the lowest point of Death Valley, -282 feet.

Example 3.4.1. Table 3.1 shows the annual calculus enrollments at a large university.

(a) Which years had local maximum or minimum calculus enrollments?

(b) What were the global maximum and minimum enrollments in calculus?

Year:	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Enrollment:	1257	1324	1378	1336	1389	1450	1523	1582	1567	1545	1571

Table 3.1: Calculus enrollments at a large university from 2000-2010

Solution:

(a) There were local maxima in 2002 and 2007 since $1378 > 1324, 1336$ and $1582 > 1523, 1567$. There were local minima in 2003 and 2009 since $1336 < 1378, 1389$ and $1545 < 1567, 1571$.

(b) The global maximum was 1582 students in 2007. The global minimum was 1257 students in 2000.

Note: We choose not to think of the 1257 students in 2000 or the 1571 students in 2010 as local extrema, but some books would include the endpoints. We may have global extrema at an endpoint of the domain. ■

3.4.2 Finding Extrema of a Function

What must the tangent line look like at a local maximum or minimum? Look at the graph in Figure 3.8. At all the extreme points, either the tangent line is horizontal, like at the points a, c, e, and g, or the tangent line doesn't exist, like at the cusp at the point f. It follows that at these extrema, either the derivative of the function is 0 or is undefined. This gives us the method to find extreme values of a function.

Definition 3.4.4. A **critical number** of a function $f(x)$ is a value $x = a$ in the domain of $f(x)$ where either $f'(a) = 0$ or $f'(a)$ is undefined.

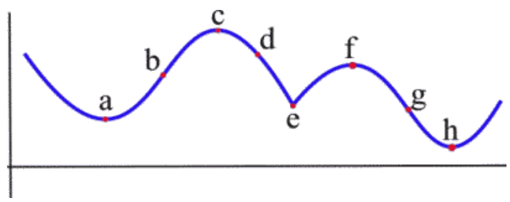


Figure 3.8: Local and global extrema

A **critical point** of a function $f(x)$ is a point $(a, f(a))$ where a is a critical number of $f(x)$.

Theorem 3.4.1. A local maximum or local minimum of a function can only occur at a critical point of the function.

Example 3.4.2. Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$.
Solution: A critical number of $f(x)$ can occur only where $f'(x) = 0$ or where $f'(x)$ does not exist.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$

So $f'(x) = 0$ at $x = 1$ and $x = 3$ (and no other values of x). There are no places where $f'(x)$ is undefined. Therefore, the critical numbers of $f(x)$ are $x = 1$ and $x = 3$, so the critical points of $f(x)$ are $(1, f(1)) = (1, 6)$ and $(3, f(3)) = (3, 2)$. ■

We haven't discussed yet how to tell whether either of these points is actually a local extremum of $f(x)$, and if so, which kind of extremem it is. We can be certain, though, that no other point is a local extremum. The graph of $f(x) = x^3 - 6x^2 + 9x + 2$ in Figure 3.9 shows that the point $f(1) = 6$ is a local maximum and $f(3) = 2$ is a local minimum of $f(x)$. This function does not have a global maximum or minimum.

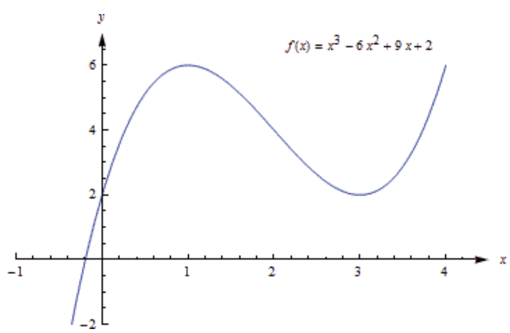


Figure 3.9: $f(x) = x^3 - 6x^2 + 9x + 2$

Example 3.4.3. Find all local extrema of $f(x) = x^3$.
Solution: $f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$. The only place where $f'(x) = 0$ is at $x = 0$, so the only candidate is the critical point $(0, 0)$. Notice that if $x > 0$, then $f(x) = x^3 > 0 = f(0)$, so $f(0)$ is not a local maximum. Similarly, if $x < 0$ then $f(x) = x^3 < 0 = f(0)$ so $f(0)$ is not a local minimum.

Therefore, $f(x) = x^3$ does not have a local extremum.

Figure 3.10 plots this function. We see, graphically, that this function does not have an extremum. The critical point is a **saddle point**. ■

Remember this example! It is not enough to find the critical points – we can only say that $f(x)$ *might* have a local extremum at any critical point.

3.4.3 First and Second Derivative Tests

Is that critical point a Maximum, Minimum, or Neither? Once we have found the critical points of $f(x)$, we still have the problem of determining whether these points are maxima, minima, or neither.

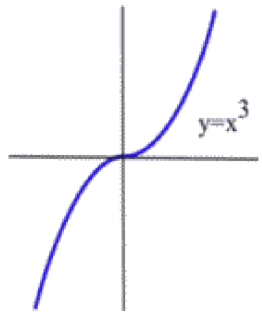


Figure 3.10: $f(x) = x^3$

All of the graphs in Figure 3.11 have a critical point of $(2, 3)$. It is clear from the graphs that the point $(2, 3)$ is a local maximum in (a) and (d), $(2, 3)$ is a local minimum in (b) and (e), and $(2, 3)$ is not a local extremum in (c) and (f).

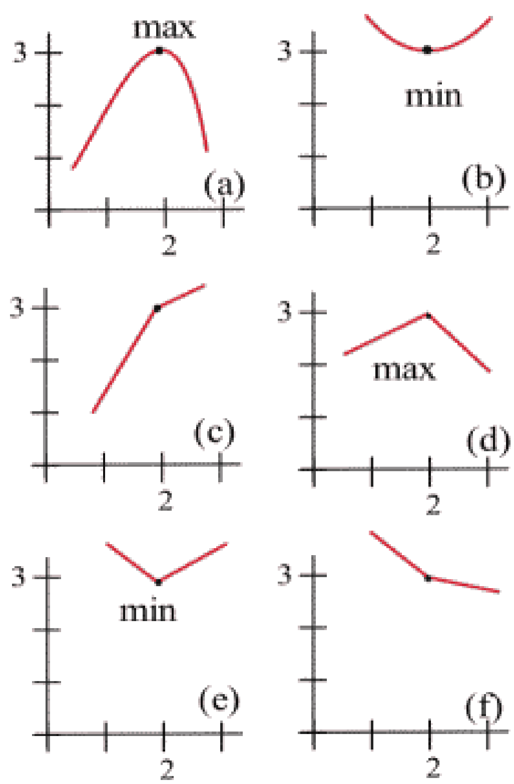


Figure 3.11: Six examples

The critical numbers only give the possible locations of extrema, and some critical numbers are not the locations of extrema. The critical numbers are the candidates for the locations of maxima and minima.

$f'(x)$ and Extrema of $f(x)$ Four possible shapes of graphs are shown in Figure 3.12. In each graph, the point marked by an arrow is a critical point, where $f'(x) = 0$. What happens to the derivative near the critical point?

At a local maximum, such as in the graph on the left, the function increases on the left of the local maximum, then decreases on the right. The derivative is first positive, then negative at a local maximum. At a local minimum, the function decreases to the left and increases to the right, so the derivative is first

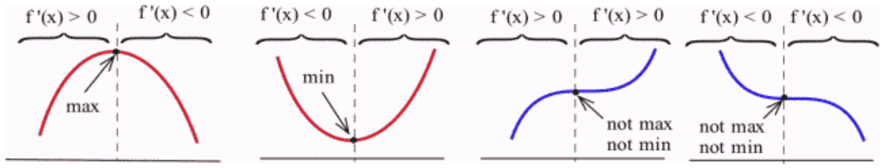


Figure 3.12: Six examples

negative, then positive. When there isn't a local extreme, the function continues to increase (or decrease) right past the critical point – the derivative doesn't change sign. This is the argument and picture behind the first test we have to identify the kind of point a critical point is.

Theorem 3.4.2 (The First Derivative Test for Extrema)**.** Suppose that $f(x)$ is a function defined in a neighborhood around a critical value $x = c$, so that $(c, f(c))$ is a critical point. Suppose we examine the sign of $f'(x)$ to the left and to the right of $x = c$.

- If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum** of $f(x)$.
- If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum** of $f(x)$.
- If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is neither a local maximum nor a local minimum.

Example 3.4.4. Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$ and classify them as local maximum, local minimum, or neither.
Solution: We already found the critical points of $f(x)$ in Example 3.4.2: $(1, 6)$ and $(3, 2)$.

Now we can use the first derivative test to classify each. We have $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. The factored form is easiest to work with here, so let's use that.

At the point $(1, 6)$, we could choose a number slightly less than 1 to plug into $f'(x)$ – perhaps use $x = 0$ or $x = 0.9$. Then we examine its sign. We don't care about the numerical value. We just need to know if $f'(x) < 0$ or $f'(x) > 0$. Since $f'(x)$ is factored, we don't have to actually plug anything in.

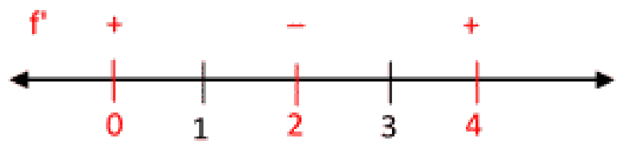
- If $x < 1$, then $x - 1 < 0$ and $x - 3 < 0$, so $f'(x) = 3(x - 1)(x - 3) = (+)(-)(-) > 0$; a positive times a negative times a negative is positive.
- If $1 < x < 3$, that is, between the critical values, then we can do the same thing. We evaluate $f'(x)$ at a number between 1 and 3, perhaps $x = 2$. Or we can make a quick sign argument like what we did above: for x a little more than 1, $f'(x) = 3(x - 1)(x - 3) = (+)(+)(-) < 0$.
- If $x > 3$, then $x - 1 > 0$ and $x - 3 > 0$, so $f'(x) = 3(x - 1)(x - 3) = (+)(+)(+) > 0$;

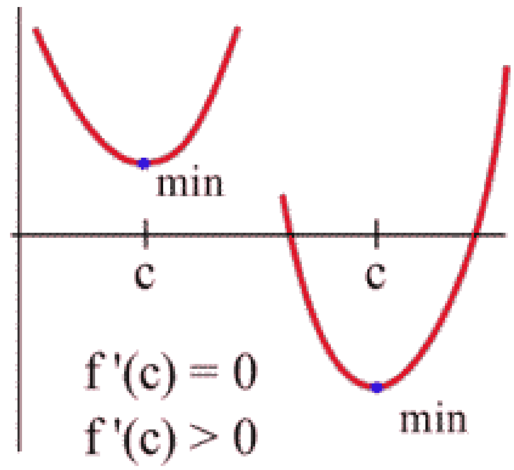
So $f'(x)$ changes signs from positive to negative around $x = 1$, which means there is a local maximum at the point $(1, 6)$; 6 is the local maximum. Also, $f'(x)$ changes signs from negative to positive around $x = 3$, which means there is a local minimum at the point $(3, 2)$; 2 is the local minimum.

As another approach, we could draw a number line, and mark the critical numbers. We already know

that $f'(1) = f'(3) = 0$. On each interval between these values, the derivative will stay the same sign. To determine the sign, we could pick a test value in each interval, and evaluate the derivative at those points (or use the sign approach used above). ■

$f''(x)$ **and Extrema of** $f(x)$ The concavity of a function can also help us determine whether a critical point is a maximum, minimum, or neither. For example, if a point is at the bottom of a concave up function, then the point is a minimum.





Theorem 3.4.3 (The Second Derivative Test for Extrema)**.** Suppose $(c, f(c))$ is a critical point of $f(x)$, $f'(c) = 0$, and $f''(c)$ exists.

- If $f''(c) < 0$, then $y = f(x)$ is **concave down** and $f(c)$ is a **local maximum** of $f(x)$.
- If $f''(c) > 0$, then $y = f(x)$ is **concave up** and $f(c)$ is a **local minimum** of $f(x)$.
- If $f''(c) = 0$ then the test is **inconclusive**; $f(x)$ may have a local maximum, a minimum, or neither at $x = c$.

The cartoon faces in Figure 3.13 can help you remember the Second Derivative Test.

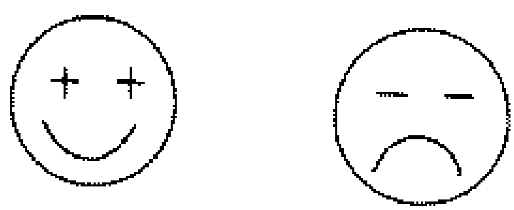


Figure 3.13: Visualizing the Second Derivative Test

Example 3.4.5. $f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers $x = 1$ and 4. Use the Second Derivative Test for extrema to determine whether $f(1)$ and $f(4)$ are maxima, minima, or neither.
Solution: We need to find the second derivative:

$$\begin{aligned}
f(x) &= 2x^3 - 15x^2 + 24x - 7 \\
f'(x) &= 6x^2 - 30x + 24 \\
f''(x) &= 12x - 30
\end{aligned}$$

Then we just need to evaluate $f''(x)$ at each critical number.

$x = 1$: $f''(1) = 12(1) - 30 < 0$, so $f(1) = 4$ is a local maximum of $f(x)$.

$x = 4$: $f''(4) = 12(4) - 30 > 0$, so $f(4) = -23$ is a local minimum of $f(x)$.

■

Many students like the Second Derivative Test. The Second Derivative Test is often easier to use than the First Derivative Test. You only have to find the sign of one number for each critical number rather than two. And if your function is a polynomial, its second derivative will probably be a simpler function than the derivative.

However, if you needed a product rule, quotient rule, or chain rule to find the first derivative, finding the second derivative can be a lot of work. Also, even if the second derivative is easy, the Second Derivative Test doesn't always give an answer. The First Derivative Test will always give you an answer.

Use whichever test you want to. But remember – you have to do some test to be sure that your critical point actually is a local maximum or minimum.

3.4.4 Global Maxima and Minima

In applications, we often want to find the global extrema; knowing that a critical point is a local extreme is not enough.

For example, if we want to make the greatest profit. we want to make the absolutely greatest profit of all. How do we find global maximum and minimum of a function?

There are just a few additional things to think about.

Endpoint Extrema The local extrema of a function occur at critical points – these are points in the function that we can find by thinking about the shape (and using the derivative to help us). But if we're looking at a function on a closed interval, the endpoints could be extrema. These endpoint extrema are not related to the shape of the function; they have to do with the interval, the window through which we're viewing the function.

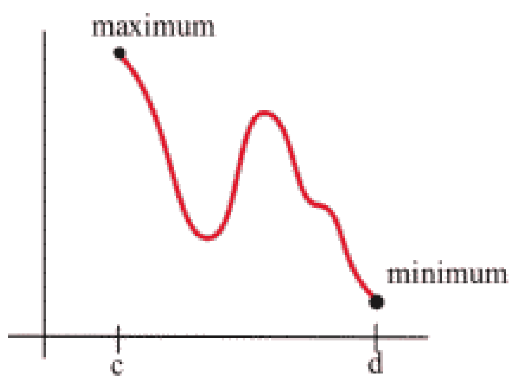


Figure 3.14: Global Extrema at Domain Endpoints

In Figure 3.14, it appears that there are three critical points – one local minimum, one local maximum, and one saddle point. But the global maximum, the highest point of all, is at the left endpoint. The global minimum, the lowest point of all, is at the right endpoint.

How do we decide if an endpoint is a global maximum or minimum? It's easier than you expected – simply plug in the endpoints, along with all the critical numbers, and compare y-values.

Theorem 3.4.4. If $f(x)$ is a continuous function on a closed domain: $a \leq x \leq b$, then $f(x)$ has a **global maximum** and a **global minimum**. Global extrema will occur at:

- $x = a$ (the left endpoint of the domain),
- $x = b$ (the right endpoint of the domain), or
- a critical value of $f(x)$.

Example 3.4.6. Find the global maximum and minimum of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \le x \le 6$.
Solution: $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$. We need to find critical points and we need to check the endpoints, $x = -2$ and $x = 6$.

$f'(x) = 3(x+1)(x-3) = 0$ when $x = -1$ and $x = 3$.

Now we simply compare the values of $f(x)$ at these four values of x.

x	$f(x)$
-2	3
-1	10
3	-22
6	59

The global minimum of $f(x)$ on $[-2, 6]$ is -22, when $x = 3$, and the global maximum of $f(x)$ on $[-2, 6]$ is 59, when $x = 6$. ∎

If there's only one critical point If the function has only one critical point and it's a local maximum (or minimum), then it must be the global maximum (or minimum). To see this, think about the geometry. Look at the graph on the left in Figure 3.15 – there is a local maximum, and the graph goes down on either side of the critical point. Suppose there was some other point that was higher – then the graph would have to turn around. But that turning point would have shown up as another critical point. If there's only one critical point, then the graph can never turn back around.

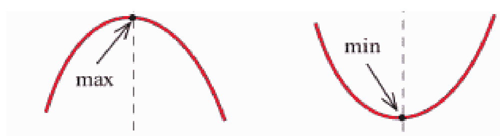


Figure 3.15: Global Extrema at Critical Points

When in doubt, graph it and look. If you are trying to find a global maximum or minimum on an open interval (or the whole real line), and there is more than one critical point, then you need to look at the graph to decide whether there is a global maximum or minimum. Be sure that all your critical points show in your graph, and that you graph beyond them – that will tell you what you want to know.

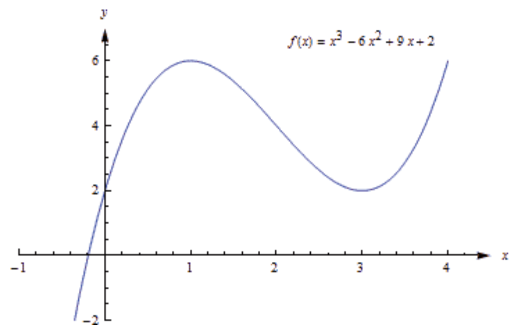
Example 3.4.7. Find the global maximum and minimum of $f(x) = x^3 - 6x^2 + 9x + 2$.
Solution: We have previously found that $f(1) = 6$ is a local maximum and $f(3) = 2$ is a local minimum. This is not a closed interval, and there are two critical points, so we must turn to the graph of the function to find global extrema.

The graph of $f(x)$ shows that points to the left of $x = 4$ have y-values greater than 6, so $f(1) = 6$ is not a global maximum. Likewise, if x is negative, then y is less than 2, so 2 is not a global minimum. There are no endpoints, so we've exhausted all the possibilities. This function does not have a global maximum or minimum.

∎

Theorem 3.4.5 (Finding Global Extrema)**.** The only places where a function can have a global extreme are critical points or endpoints of the domain.

- If the function has only one critical point, and it's a local extremum, then it is also a global extremum.
- If the domain has endpoints, find the global extrema by comparing y-values at all the critical points and at the endpoints.
- When in doubt, graph the function to be sure. (However, unless the problem explicitly tells you otherwise, it is not enough to just use the graph to get your answer.)



3.5 Curve Sketching

This section examines some of the interplay between the shape of the graph of $f(x)$ and the behavior of $f'(x)$. If we have a graph of $f(x)$, we will see what we can conclude about the values of $f'(x)$. If we know values of $f'(x)$, we will see what we can conclude about the graph of $f(x)$. We will also utilize the information from $f''(x)$ that we learned in the last section.

3.5.1 First Derivative Information

Definition 3.5.1. Suppose that a function $f(x)$ is defined on the open interval (a, b).

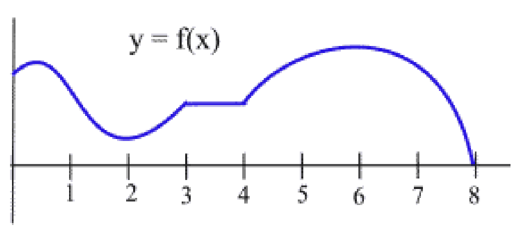
The function $f(x)$ is **increasing** on (a, b) if $a < x_1 < x_2 < b$ implies $f(x_1) < f(x_2)$.

The function $f(x)$ is **decreasing** on (a, b) if $a < x_1 < x_2 < b$ implies $f(x_1) > f(x_2)$.

Graphically, $f(x)$ is increasing (decreasing) if, as we move from left to right along the graph of $f(x)$, the height of the graph increases (decreases).

These same ideas make sense if we consider $h(t)$ to be the height (in feet) of a rocket t seconds after liftoff. We naturally say that the rocket is rising or that its height is increasing if the height $h(t)$ increases over a period of time, as t increases.

Example 3.5.1. List the intervals on which the function shown is increasing or decreasing.



Solution: $f(x)$ is increasing on the intervals $[0, 0.5)$, $(2, 3)$ and $(4, 6)$. Notice that we don't include critical values in the intervals. A function is neither increasing nor decreasing at a critical point.

$f(x)$ is decreasing on $(0.5, 2)$ and $(6, 8)$.

On the interval $[3, 4]$, $f(x)$ is neither increasing nor decreasing – it is constant. ■

Theorem 3.5.1 (First Derivative Information about Shape (Part 1))**.** Let $f(x)$ be a function which is differentiable on an interval (a, b).

- If $f(x)$ is increasing on (a, b), then $f'(x) \geq 0$ for all x in (a, b).
- If $f(x)$ is decreasing on (a, b), then $f'(x) \leq 0$ for all x in (a, b).
- If $f(x)$ is constant on (a, b), then $f'(x) = 0$ for all x in (a, b).

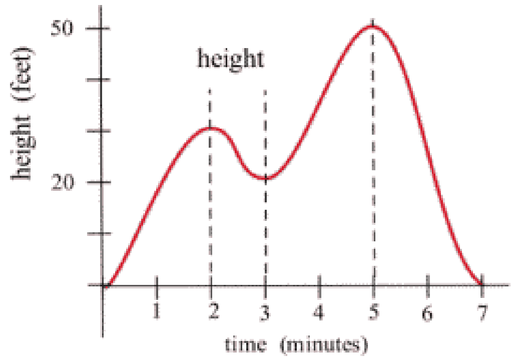


Figure 3.16: Elevation of a Helicopter

Example 3.5.2. The graph in Figure 3.16 shows the elevation of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, $\dfrac{dh}{dt}$.
Solution: Notice that $h(t)$ has a local maximum when $t = 2$ and $t = 5$, so $v(2) = 0$ and $v(5) = 0$, where $v(t) = h'(t)$ is the upward velocity of the helicopter. Similarly, $h(t)$ has a local minimum when $t = 3$, so $v(3) = 0$.

When $h(x)$ is increasing, $v(t) > 0$. When $h(t)$ is decreasing, $v(t) < 0$.

Using this information, we can sketch a graph of $v(t) = \dfrac{dh}{dt}$.

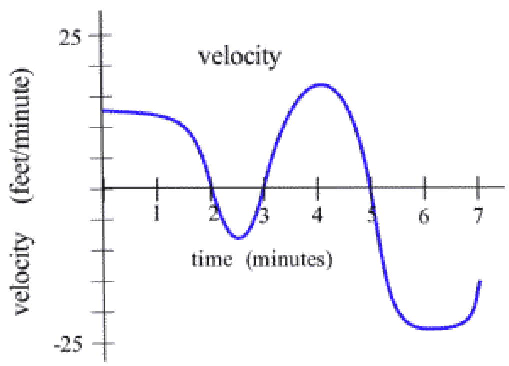


Figure 3.17: Vertical Velocity of the Helicopter

■

The next theorem is almost the converse of the First Shape Theorem (Theorem 3.5.1) and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of $f'(x)$, then we can draw some conclusions about the shape of the graph of $f(x)$.

Theorem 3.5.2 (First Derivative Information about Shape (Part 2))**.** Let $f(x)$ be a function which is differentiable on an interval I.

- If $f'(x) > 0$ for all x in the interval I, then $f(x)$ is **increasing** on I.
- If $f'(x) < 0$ for all x in the interval I, then $f(x)$ is **decreasing** on I.
- If $f'(x) = 0$ for all x in the interval I, then $f(x)$ is **constant** on I.

Example 3.5.3. Use information about the values of $f'(x)$ to help graph $f(x) = x^3 - 6x^2 + 9x + 1$.
Solution: $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, so $f'(x) = 0$ only when $x = 1$ or $x = 3$. $f'(x)$ is a polynomial so it is always defined.

The only critical numbers for $f(x)$ are $x = 1$ and $x = 3$, and they divide the real number line into three intervals: $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. On each of these intervals, the function is either always increasing or always decreasing.

If $x < 1$, then $f'(x) = 3(-)(-) > 0$, so $f(x)$ is increasing.

If $1 < x < 3$, then $f'(x) = 3(+)(-) < 0$, so $f(x)$ is decreasing.

If $x > 3$, then $f'(x) = 3(+)(+) > 0$, so $f(x)$ is increasing.

Even though we don't know the value of $f(x)$ anywhere yet, we do know a lot about the shape of the graph of $f(x)$: as we move from left to right along the x-axis, the graph of $f(x)$ increases until $x = 1$, then the graph decreases until $x = 3$, and then the graph increases again. The graph of $f(x)$ makes "turns" when $x = 1$ and $x = 3$; it has a local maximum at $x = 1$, and a local minimum at $x = 3$. Figure 3.18 sketches this information.

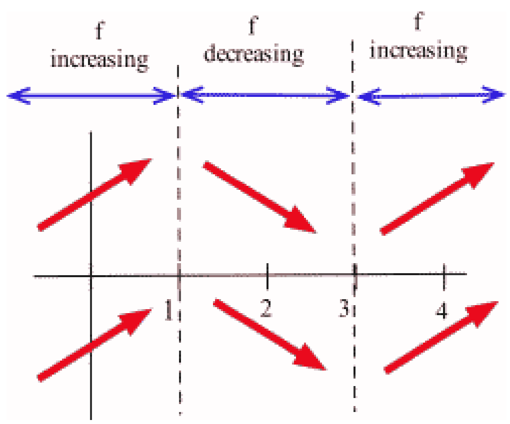


Figure 3.18: Sketching where $f(x)$ increases and decreases.

To plot the graph of $f(x)$, we still need to evaluate $f(x)$ at a few values of x, but only at a very few values. $f(1) = 5$, and $(1, 5)$ is a local maximum of $f(x)$. $f(3) = 1$, and $(3, 1)$ is a local minimum of $f(x)$. The resulting graph of $f(x)$ is shown in Figure 3.19.

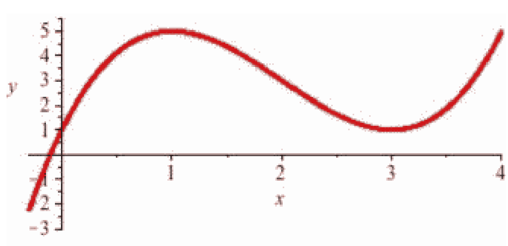


Figure 3.19: Sketch of $y = f(x)$

■

3.5.2 Second Derivative Information

Until now, we've only used first derivative information, but we could also use information from the second derivative to provide more information about the shape of the function.

Theorem 3.5.3 (Second Derivative Information about Shape)**.** Suppose $f(x)$ is continuous and both $f'(x)$ and $f''(x)$ exist on the interval (a, b).

- If $f(x)$ is concave up on (a, b), then $f''(x) \geq 0$ for all x in (a, b).

- If $f(x)$ is concave down on (a,b), then $f''(x) \leq 0$ for all x in (a,b).

The converse of both of these are also true.

- If $f''(x) \geq 0$ for all x in (a,b), then $f(x)$ is **concave up** on (a,b).
- If $f''(x) \leq 0$ for all x in (a,b), then $f(x)$ is **concave down** on (a,b).

Example 3.5.4. Use information about the values of $f''(x)$ to help determine the intervals on which the graph of the function $f(x) = x^3 - 6x^2 + 9x + 1$ is concave up and concave down.
Solution: For concavity, we need the second derivative: $f'(x) = 3x^2 - 12x + 9$, so $f''(x) = 6x - 12$.

To find possible inflection points, set the second derivative equal to zero. $6x - 12 = 0$, so $x = 2$. This divides the real number line into two intervals: $(-\infty, 2)$ and $(2, \infty)$.

For $x < 2$, the second derivative is negative (for example, $f''(0) = 6(0) - 12 = -12$), so $y = f(x)$ is concave down. For $x > 2$, the second derivative is positive, so $y = f(x)$ is concave up.

We could have incorporated this concavity information when sketching the graph for the previous example, and indeed we can see the concavity reflected in the graph shown. ■

Example 3.5.5. Use information about the values of $f'(x)$ and $f''(x)$ to help graph $f(x) = x^{2/3}$.
Solution: $f'(x) = \dfrac{2}{3}x^{-1/3}$. This is undefined at $x = 0$.

$f''(x) = \dfrac{-2}{9}x^{-4/3}$. This is also undefined at $x = 0$.

This creates two intervals: $x < 0$, and $x > 0$.

On the interval $x < 0$, we could test out a value like $x = -1$:

$$f'(-1) = \frac{2}{3}(-1)^{-1/3} = -\frac{2}{3} < 0$$

and

$$f''(-1) = -\frac{2}{9}(-1)^{-4/3} = -\frac{2}{9} < 0 \ .$$

$f'(-1)$ is negative and $f''(-1)$ is negative, so we can conclude that $f(x)$ is decreasing and concave down on the interval $(-\infty, 0)$.

On the interval $x > 0$, we could test out a value like $x = 1$:

$$f'(1) = \frac{2}{3}(1)^{-1/3} = \frac{2}{3} > 0$$

and

$$f''(1) = -\frac{2}{9}(1)^{-4/3} = -\frac{2}{9} < 0 \ .$$

$f'(1)$ is positive and $f''(1)$ is negative, so we can conclude that $f(x)$ is increasing and concave down on the interval $(0, \infty)$.

We can also calculate that $f(0) = 0^{2/3} = 0$, giving us a base point for the graph. Using this information, we can conclude the graph must look like this.

■

3.5.3 Sketching without an Equation

Of course, graphing calculators and computers are great at graphing functions. Calculus provides a way to illuminate what may be hidden or out of view when we graph using technology. More importantly, calculus gives us a way to look at the derivatives of functions for which there is no equation given. We already saw the idea of this back in Section 2.3, where we sketched the derivative of two graphs by estimating slopes on the curves.

We can summarize all the derivative information about shape in a table.

Theorem 3.5.4. When $f'(x) = 0$, the graph of $f(x)$ may have a local maximum or minimum.

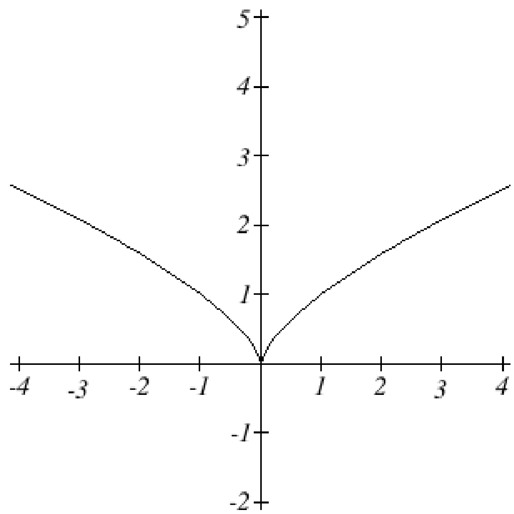


Figure 3.20: Sketch of $y = x^{2/3}$

$f(x)$	Increasing	Decreasing	Concave Up	Concave Down
$f'(x)$	$+$	$-$	Increasing	Decreasing
$f''(x)$			$+$	$-$

Table 3.2: Summary of Derivative Information about the Graph

When $f''(x) = 0$, the graph of $f(x)$ may have an inflection point.

Example 3.5.6. A company's bank balance, B, in millions of dollars, t weeks after releasing a new product is shown in Figure 3.21. Sketch a graph of the marginal balance – the rate at which the bank balance was changing over time.

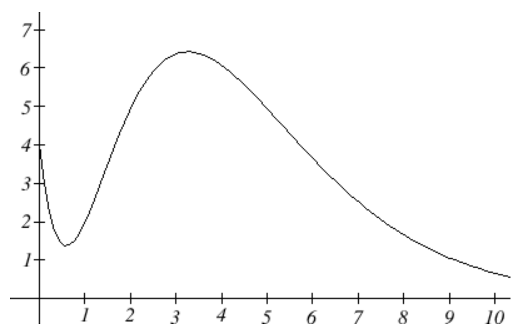


Figure 3.21: Company Balance after t Years

Solution: Notice that since the tangent line of $y = B(t)$ will be horizontal at about $t = 0.6$ and $t = 3.2$; the derivative will be 0 at those points.

We can then identify intervals on which the original function is increasing or decreasing. This will tell us when the derivative function, $B'(t)$, is positive or negative. There appear to be inflection points at about $t = 1.5$ and $t = 5.5$. At these points, the derivative will be changing from increasing to decreasing or vice versa, so the derivative will have a local maximum or minimum at those points.

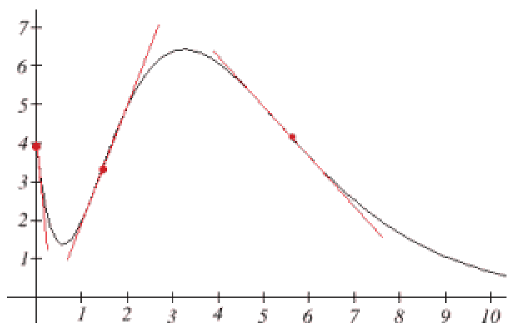
Let's look at the intervals of concavity.

If we wanted a more accurate sketch of the derivative function, we could also estimate the derivative at a few points by sketching a collection of tangent lines.

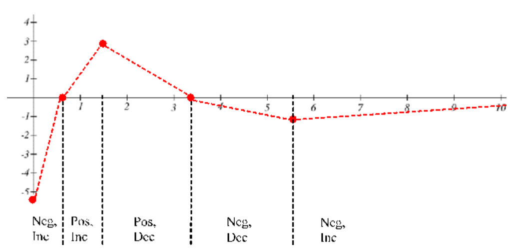
Interval	$B(t)$	$B'(t)$
$0 < t < 0.6$	Decreasing	Negative
$0.6 < t < 3.2$	Increasing	Positive
$t > 3.2$	Decreasing	Negative

Interval	$y = B(t)$	$B'(t)$
$0 < t < 1.5$	Concave Up	Increasing
$1.5 < t < 5.5$	Concave Down	Decreasing
$t > 5.5$	Concave Up	Increasing

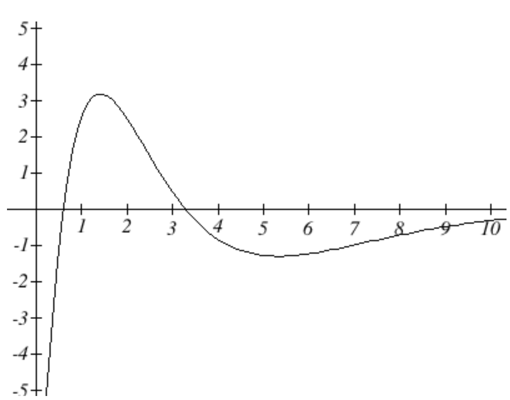
t	$B'(t)$
0	-10
1.5	3
6	-1



Now we can sketch the derivative. We know a few values for the derivative function, and on each interval we know the shape we need. We can use this to create a rough idea of what the graph should look like.



Smoothing this out gives us a good estimate for the graph of the derivative.



■

3.6 Applied Optimization

We have used derivatives to help find the maxima and minima of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask your help in maximizing or minimizing something. What is the largest volume package which the post office will take? What is the quickest way to get from here to there? What is the least expensive way to accomplish some task? In this section, we'll discuss how to find these extreme values using calculus.

3.6.1 Applications of Extrema

Example 3.6.1. The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$7 per running foot. The fourth side will be built of cement blocks, at a cost of \$14 per running foot. Find the dimensions of the least costly such enclosure.

The process of finding maxima or minima is called **optimization**. The function we're optimizing is called the **objective function** (or objective equation). The objective function can be recognized by its proximity to superlatives (greatest, least, highest, farthest, most, etc.). Look at the garden store example; the cost function is the objective function.

In many cases, there are two (or more) variables in the problem. In the garden store example again, the length and width of the enclosure are both unknown. If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable. Equations that relate the variables in this way are called **constraint equations** or simply **constraints**. The constraint equations are always equations, so they will have equals signs. For the garden store, the fixed area relates the length and width of the enclosure. This will give us our constraint equation.

Applied Optimization Problem Technique

1. Translate the English statement of the problem line by line into a picture (if that applies) and into math. This is often the hardest step!
2. Identify the objective function. Look for words indicating a largest or smallest value.
   (a) If you seem to have two or more variables, find the constraint equation. Think about the English meaning of the word constraint, and remember that the constraint equation will have an equals sign.
   (b) Solve the constraint equation for one variable and substitute into the objective function. Now you have an equation of one variable.
3. Use calculus to find the optimum values. (Take derivative, find critical points, test. Don't forget to check the endpoints!)
4. Look back at the question to make sure you answered what was asked. Translate your number answer back into English.

Example 3.6.2. The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$7 per running foot. The fourth side will be built of cement blocks, at a cost of \$14 per running foot. Find the dimensions of the least costly such enclosure.
Solution: First, translate line by line into math and a picture:

- The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment.

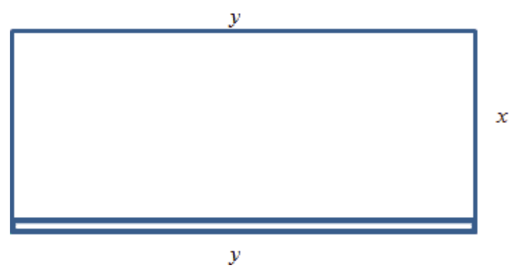
  We will give variable names to the unknown quantities. We're given a rectangular area of 600 ft^2. Since area is length times width, let x be the width of the enclosure and let y its length, the length of the side made of blocks. Then area is $A(x, y) = xy = 600$. This is a constraint. Since length and width must be positive, we have two more constraints: $x \geq 0$ and $y \geq 0$.

- Three sides of the enclosure will be built of redwood fencing, at a cost of \$7 per running foot. The fourth side will be built of cement blocks, at a cost of \$14 per running foot.

  A length of $2x + y$ costs \$7 per foot, and the length of y costs \$14 per foot, so cost is $C(x, y) = 7(2x + y) + 14y = 14x + 21y$.

- Find the dimensions of the least costly such enclosure.

  Find x and y so that $C(x, y)$ is minimized.



The objective function is the cost function, and we want to minimize it. As it stands, though, it has two variables, so we need to use the constraint to eliminate one variable. The constraint equation is the fixed area $A = xy = 600$. Solve A for x to get $x = \dfrac{600}{y}$, and then substitute into $C(x, y)$:

$$C(y) = 14 \cdot \frac{600}{y} + 21y = \frac{8400}{y} + 21y \ .$$

Now we have a function of just one variable, so we can find the minimum using calculus.

$$C'(y) = -\frac{8400}{y^2} + 21$$

$C'(y)$ is undefined for $y = 0$, so we'll solve $C' = 0$ for y.

$$\begin{aligned}
-\frac{8400}{y^2} + 21 &= 0 \\
21 &= \frac{8400}{y^2} \\
21y^2 &= 8400 \\
y^2 &= 400 \\
y &= \pm 20
\end{aligned}$$

So cost is optimized when $y = 20$ or $y = -20$, but of the three critical numbers, only $y = 20$ makes sense (since it is within the constraints).

Test $y = 20$ (here we chose the second derivative test):

$$C''(20) = \frac{16800}{y^3} > 0 \ ,$$

so this is a local minimum.

Since this is the only critical point in the domain, this must be the global minimum. Going back to our constraint function, we can find that when $y = 20$, $x = 30$. The dimensions of the enclosure that minimize the cost are 20 feet by 30 feet. ■

When trying to maximize their revenue, businesses also face the constraint of consumer demand. While a business would love to see lots of products at a very high price, typically demand decreases as the price of goods increases. In simple cases, we can construct that demand curve to allow us to maximize revenue.

Example 3.6.3. A concert promoter has found that if she sells tickets for \$50 each, she can sell 1200 tickets, but for each \$5 she raises the price, 50 less people attend. What price should she sell the tickets at to maximize her revenue?

Solution: We are trying to maximize revenue, and we know that $R = pq$, where p is the price per ticket, and q is the quantity of tickets sold.

The problem provides information about the demand relationship between price and quantity – as price increases, demand decreases. We need to find a formula for this relationship. To investigate, let's calculate what will happen to attendance if we raise the price in Table 3.3.

Price, p	50	55	60	65
Quantity, q	1200	1150	1100	1050

Table 3.3: Demand for tickets as a function of price.

You might recognize this as a linear relationship. We can find the slope for the relationship by using two points:

$$m = \frac{1150 - 1200}{55 - 50} = -\frac{50}{5} = -10 \ .$$

You may notice that the second step in that calculation corresponds directly to the statement of the problem: the attendance drops 50 people for every \$5 the price increases.

Using the point-slope form of the line, we can write the equation relating price and quantity:

$$q - 1200 = -10(p - 50) \ .$$

Simplifying to slope-intercept form gives the demand equation

$$q = 1700 - 10p \ .$$

Substituting this into our revenue equation, we get an equation for revenue involving only one variable:

$$R = pq = p(1700 - 10p) = 1700p - 10p^2 \ .$$

Now, we can find the maximum of this function by finding critical numbers. $R'(p) = 1700-20p$, so $R'(p) = 0$ when $p = 85$.

Using the second derivative test, $R''(p) = -20 < 0$ (for any value of p), so the critical number is a local maximum. Since it is the only critical number, we can also conclude that $R(85)$ is the global maximum.

The promoter will be able to maximize revenue by charging \$85 per ticket. At this price, she will sell $q(85) = 1700 - 10(85) = 850$ tickets, generating $R(85) = \$85 \cdot 850 = \$72,250$ in revenue. ■

3.6.2 When Marginal Revenue Equals Marginal Cost

You may have heard before that profit is maximized when marginal cost and marginal revenue are equal. Now you can see why people say that! (Even though it's not completely true.)

Example 3.6.4. Suppose we want to maximize profit.

Solution: Now we know what to do – find the profit function, find its critical points, test them, etc.

But remember that profit is revenue minus cost: $P(p) = R(p) - C(p)$. So $P'(p) = R'(p) - C'(p)$. That is, the derivative of the profit function is $MR - MC$.

Now let's find the critical points – those will be where $P'(p) = 0$ or is undefined. $P'(p) = 0$ when $R'(p) - C'(p) = 0$, or where $R'(p) = C'(p)$.

In summary, profit has a critical point when marginal revenue and marginal cost are equal. ■

In all the cases we'll see in this class, profit will be very well behaved, and we won't have to worry about looking for critical points where P' is undefined. But remember that not all critical points are local maxima! Any place where $R' = C'$ could represent a local maximum, a local minimum, or neither.

Example 3.6.5. A company sells q ribbon winders per year at \$$p$ per ribbon winder. The demand function for ribbon winders is given by: $p = 300 - 0.02q$. The ribbon winders cost \$30 each to manufacture, plus there are fixed costs of \$9000 per year. Find the quantity where profit is maximized.

Solution: We want to maximize profit, but there isn't a formula for profit given. So let's make one. We can find a function for $R = pq$ using the demand function for p.

$$R(q) = (300 - 0.02q)q = 300q - 0.02q^2 \ .$$

We can also find a function for cost, using the variable cost of \$30 per ribbon winder, plus the fixed cost:

$$C(q) = 9000 + 30q \ .$$

Putting them together, we get a function for profit:

$$P(q) = R(q) - C(q) = (300q - 0.02q^2) - (9000 + 30q) = -0.02q^2 + 270q - 9000 \ .$$

Now we have two choices. We can find the critical points of profit by taking the derivative of $P(q)$ directly, or we can find $R'(p)$ and $C'(p)$ and set them equal. (Naturally, we'll get the same answer either way.)

Let's use $R'(p) = C'(p)$ this time.

$$\begin{aligned} R'(p) &= 300 - 0.04q \\ C'(p) &= 30 \\ 300 - 0.04q &= 30 \\ 270 &= 0.04q \\ q &= 6750 \end{aligned}$$

The only critical point is at $q = 6750$. Now we need to be sure this is a local max and not a local min. In this case, we'll look to the graph of $P(q)$ – it's a downward opening parabola, so this must be a local maximum. And since it's the only critical point, it must also be the global maximum.

Profit is maximized when they sell 6750 ribbon winders. ■

3.6.3 When Average Cost Equals Marginal Cost

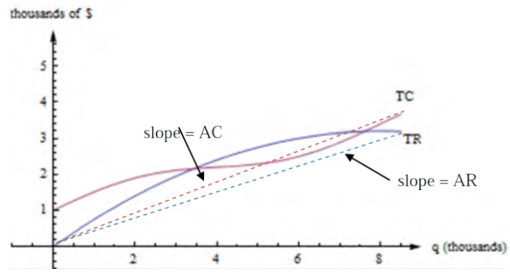
Average cost is minimized when average cost equals marginal cost is another saying that isn't quite true; in this case, the correct statement is:

Theorem 3.6.1. Average Cost has critical points when average cost and marginal cost are equal.

Let's look at a geometric argument here. Remember that the average cost is the slope of the diagonal line, the line from the origin to the point on the total cost curve. If you move your clear plastic ruler around, you'll see (and feel) that the slope of the diagonal line is smallest when the diagonal line just touches the cost curve – when the diagonal line is actually a tangent line – when the average cost is equal to the marginal cost.

3.6.4 Exercises

1. A farmer wants to build a pen with two dividers in order to separate elephants, donkeys, and penguins. If 600 ft of fence is available and one side of the pen is bounded by a river and needs no fence since all the animals just happen to have an irrational fear of water, then what is the maximum area that can be enclosed?



2. A carpenter wants to build a rectangular box with square sides in which to put round things. The material for the bottom costs \$20 per ft^2, material for the sides costs \$10 per ft^2, and the material for the top costs \$50 per ft^2. If the volume of the box must be 5 ft^3, then find the dimensions that will minimize the cost (and find the minimum cost).

3. A knight sees a damsel in distress 3 miles downstream on the opposite side of a straight raging river 0.5 miles wide. The knight can swim at 4 mi/hr and run at 7 mi/hr. At what point on the opposite side should the knight swim in order to reach the distressed damsel as soon as possible?

4. A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that will minimize the amount of material needed.

5. A coffee shop has \$1000 in fixed daily costs and daily costs of \$60 per seat. The daily revenue per seat is \$90 if there are 100 or fewer seats. However, if the seating capacity is more than 100 places, the daily profit per seat will be decreased by \$1 for each additional seat over 100. If the fire marshal will only allow up to 150 seats, what should the seating capacity be in order to maximize the coffee shop's daily profit?

6. The total cost in dollars for Alicia to make q oven mitts is given by

$$C(q) = 64 + 1.5q + 0.01q^2 \ .$$

   (a) What is the fixed cost?
   (b) Find a function that gives the marginal cost.
   (c) Find a function that gives the average cost.
   (d) Find the quantity that minimizes the average cost.
   (e) Confirm that the average cost and marginal cost are equal at your answer to part (d).

7. A 5 in. $\times$ 8 in. piece of paper has a square cut out of each corner (same size from each) and is then folded to make an open-top box. Find the size of the square that will maximize the volume.

8. Find the area of the largest rectangle that can be inscribed inside an isosceles triangle with side lengths $\sqrt{2}$, $\sqrt{2}$, and 2.

9. A right circular cylinder is inscribed in a sphere of radius 6 in. Find the largest possible volume of such a cylinder.

Chapter 4

Functions of Many Variables

If you've ever hiked, you have probably seen a topographical map. Figure 4.1 shows part of a topographic map of Stowe, Vermont.

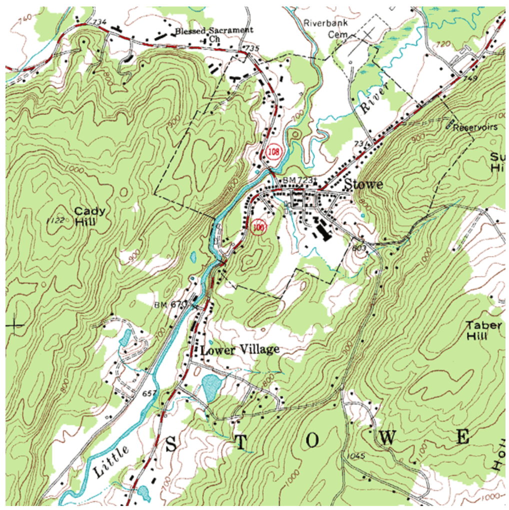


Figure 4.1: Topographical map of Stowe, VT: USGS, http://en.wikipedia.org/wiki/File:Topographic_map_example.png.

Points with the same elevation are connected with curves, so you can read not only your east-west and your north-south location, but also your elevation. You may have also seen weather maps that use the same principle – points with the same temperature are connected with curves (isotherms), or points with the same atmospheric pressure are connected with curves (isobars). These maps let you read not only a place's location but also its temperature or atmospheric pressure.

In this chapter, we will use that same idea to make graphs of functions of two variables.

4.1 Multivariable Functions

4.1.1 Introduction

Real life is rarely as simple as one input / one output. Many relationships depend on lots of variables. Here are some examples.

- If I put a deposit into an interest-bearing account and let it sit, the amount I have at the end of three years depends on P (how much my initial deposit is), r (the annual interest rate), and n (the number of compoundings per year).

- The air resistance on a wing in a wind tunnel depends on the shape of the wing, the speed of the wind, the wing's orientation (pitch, yaw, and roll), plus a myriad of other things that I can't begin to describe.

- The amount of your television cable bill depends on which basic rate structure you have chosen and how many pay-per-view movies you ordered.

Since the real world is so complicated, we want to extend our calculus ideas to functions of several variables.

4.1.2 Functions of Two Variables

If $x_1, x_2, x_3, \ldots, x_n$ are real numbers, then $(x_1, x_2, x_3, \ldots, x_n)$ is called an **n-tuple**. This is an extension of ordered pairs and triples. A function of n variables is a function whose domain is some set of n-tuples and whose range is some set of real numbers.

For much of what we do here, everything will work the same whether we were working with 2, 3, or 47 variables. Because we're trying to keep things a little bit simpler, we'll concentrate on functions of two variables.

Definition 4.1.1 (A Function of Two Variables)**.** A **function of two variables** is a function, that is, to each input is associated exactly one output.

- The inputs are ordered pairs, (x, y).

- The outputs are real numbers (each output is a single real number).

- The domain of a function is the set of all possible inputs (ordered pairs);

- the range is the set of all possible outputs (real numbers).

- The function can be written $z = f(x, y)$.

Functions of two variables can be described numerically (a table), graphically, algebraically (a formula), or in English.

We will often now call the familiar $y = f(x)$ a **function of one variable**.

Example 4.1.1. The cost of renting a car depends on how many days you keep it and how far you drive. Represent this using a function.
Solution: Let d be the number of days you rent the car, and m be the number of miles you drive. Then the cost of the car rental $C(d, m)$ is a function of two variables. ■

Example 4.1.2. The demand for hot dog buns depends on the price for the hot dog buns and also on the price for hot dogs. Represent this as a function.
Solution: The demand $q_B = f(p_B, p_D)$ is a function of two variables. (The demand for hot dogs also depends on the price of both dogs and buns). ■

4.1.3 Formulas and Tables

Just as in the case of functions of one variable, we can display a function of two variables in a table. The two inputs are shown in the margin (top row, left column), and the outputs are shown in the interior cells.

Example 4.1.3. Table 4.1 shows the cost $C(d, m)$ in dollars to rent a car for d days and drive it m miles.

(a) What is the cost to rent a car for 3 days and drive it 200 miles?

Solution: According to the table, renting the car for three days (row with $d = 3$) and driving it 200 miles (column with $m = 200$) will cost \$150. ■

d	m			
	100	200	300	400
1	55	70	85	100
2	95	110	125	140
3	135	150	165	180

Table 4.1: Cost, in dollars, to rent a car for d days and drive m miles.

(b) What is $C(100, 4)$? What is $C(4, 100)$?

Solution: Careful now – the input is an ordered pair, so in $C(100, 4)$, the 100 has to be a value of d and the 4 has to be a value of m, so $C(100, 4)$ would be the cost of renting a car for 100 days and driving it 4 miles. That cost is not in the table. (And that would be a pretty silly way to rent a car.) On the other hand, $C(4, 100)$ is the cost of renting for 4 days and driving 100 miles. The table says that would cost \$175. ■

(c) Suppose we rent the car for three days. Is C an increasing function with d fixed at 3?

Solution: If we know that d is fixed at 3, we're looking at $C(3, m)$. This is now a function of one variable: just m. We can see the table that displays values of this function by focusing our attention on just the row where $d = 3$. Now we can see that if we rent for 3 days, the cost appears to be an

d	m			
	100	200	300	400
3	135	150	165	180

increasing function of the number of miles we drive, which shouldn't be surprising. ■

The idea of fixing one variable and watching what happens to the function as the other varies will come up again and again.

It's hard to display a function of more than two variables in a table. But it's convenient to work with formulas for functions of two variables, or as many variables as you like.

Example 4.1.4. The cost $C(d, m)$ in dollars to rent a car for d days and drive it m miles is given by the formula

$$C(d, m) = 40d + 0.15m \ .$$

(a) What is the cost of renting a car for 3 days and driving it 200 miles?

Solution: $C(3, 200) = 40(3) + 0.15(200) = \$150$. This is the same value we got from the table. The formula will give us the same answers for any of the table values. ■

(b) What is $C(100, 4)$? What is $C(4, 100)$?

Solution: $C(100, 4)$ makes perfect sense to the formula (even if it doesn't make sense for actually renting a car). So now we can get an answer. To rent the car for 100 days and drive it for 4 miles should cost \$4000.60. $C(4, 100) = \$175$, as before. ■

(c) Suppose we rent the car for three days. Is C an increasing function with d fixed at 3?

Solution: If we fix $d = 3$, then $C(d, m)$ becomes $C(3, m) = 40(3) + 0.15m = 120 + 0.15m$. Yes, this is an increasing function of m – we can tell because it's linear and its slope is $0.15 > 0$. ■

Reality check – the formula that gives the cost for the rental car makes sense for all values of d and m. But that's not how the real cost works – you can't rent the car for a negative number of days or drive a negative number of miles. (That is, there are domain restrictions.) In addition, most car rental agreements don't compute a charge for fractions of days; they round up to the next whole number of days.

Example 4.1.5. Let $f(x, y, z, w) = 35x^2w - \frac{1}{z} + yz^2$. Evaluate $f(0, 1, 2, 3)$.
Solution: Remember that this is an ordered 4-tuple; make sure the numbers get substituted into the correct places.

$$f(0, 1, 2, 3) = 35(0)^2(3) - \frac{1}{2} + (1)(2)^2 = 3.5$$

■

4.1.4 Graphs

The graph of a function of two variables is a surface in three-dimensional space. Let's start by looking at the 3-dimensional rectangular coordinate system, how to locate points in three dimensions, and distance between points in three dimensions.

In the 2-dimensional rectangular coordinate system we have two coordinate axes that meet at right angles at the origin, and it takes two numbers, an ordered pair (x, y), to specify the rectangular coordinate location of a point in the plane (two dimensions).

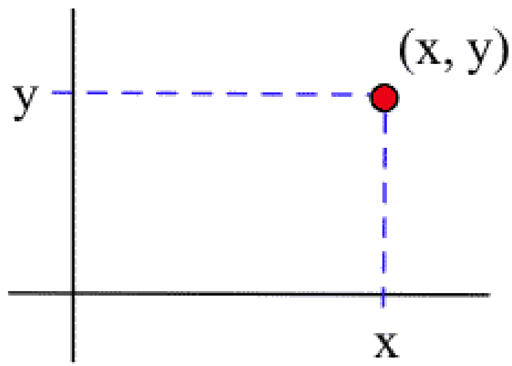


Figure 4.2: A point in the xy-plane.

Each ordered pair (x, y) specifies the location of exactly one point, and the location of each point is given by exactly one ordered pair (x, y). The x and y values are the coordinates of the point (x, y).

The situation in three dimensions is very similar. In the 3-dimensional rectangular coordinate system we have three coordinate axes that meet at right angles, and three numbers, an ordered triple (x, y, z), are needed to specify the location of a point.

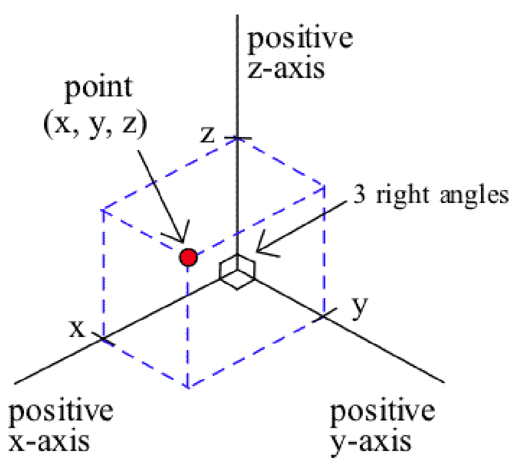


Figure 4.3: A point in 3-dimensional space.

Each ordered triple (x, y, z) specifies the location of exactly one point, and the location of each point is given by exactly one ordered triple (x, y, z). The x, y, and z values are the coordinates of the point (x, y, z).

Figure 4.4 shows the location of the point $(4, 2, 3)$.

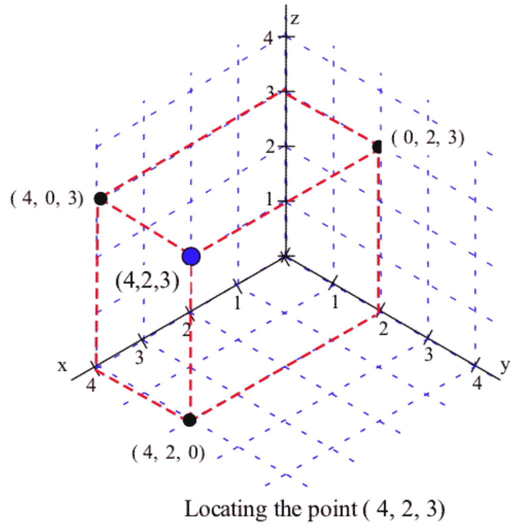


Figure 4.4: Locating the point $(4, 2, 3)$

Typically we use a **right-hand orientation**. To see what this means, imagine your right hand in front hand in front of you with the palm toward your face, your thumb pointing up, you index finger straight out, and your next finger toward your face (and the two bottom fingers bent into the palm). Then, in the right hand coordinate system, your thumb points along the positive z-axis, your index finger along the positive x-axis, and the other finger along the positive y-axis. See Figure 4.5

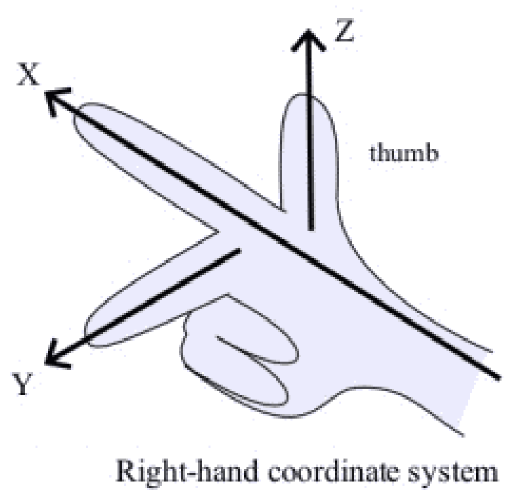


Figure 4.5: The right-hand rule for 3-dimensional coordinates

Other orientations of the axes are possible and valid (with appropriate labeling), but the right-hand system is the most common orientation and is the one we will generally use. If another orientation is used, then the axes will be explicitly labeled.

Each ordered triple (x, y, z) specifies the location of a single point, and this location point can be plotted by locating the point $(x, y, 0)$ on the xy-plane and then going up z units (the red path in Figure 4.6).

We could also get to the same (x, y, z) point in other ways. For instance, we could start by finding the point $(x, 0, z)$ on the xz-plane and then going y units parallel to the y-axis, or by finding $(0, y, z)$ on the yz-plane and then going x units parallel to the x-axis (the blue path in Figure 4.6).

Example 4.1.6. Plot the locations of the points

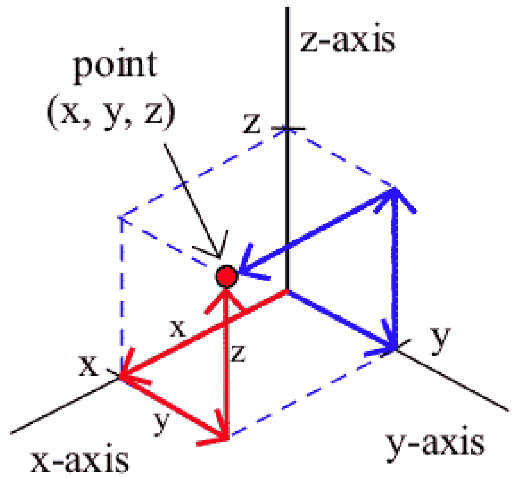


Figure 4.6: Plotting a point in 3-dimensional space

- $P = (0, 3, 4)$,
- $Q = (2, 0, 4)$,
- $R = (1, 4, 0)$,
- $S = (3, 2, 1)$, and
- $T = (-1, 2, 1)$.

Solution: The points are shown in Figure **??**.

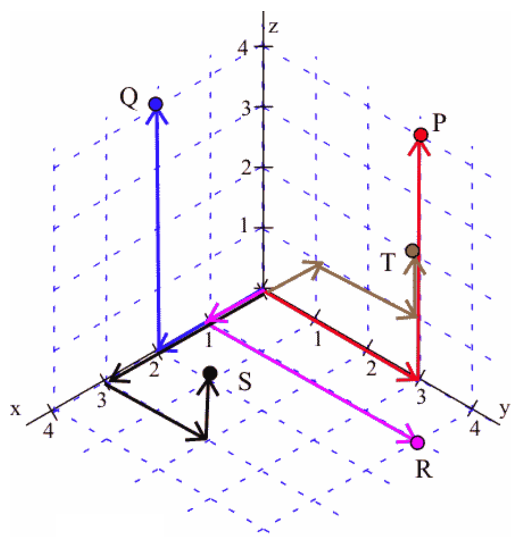


Figure 4.7: Points P, Q, R, S, and T

■

Once we can locate points, we can begin to consider the graphs of various collections of points. By the graph of $z = 2$ we mean the collection of all points (x, y, z) which have the form $(x, y, 2)$. Since no condition is imposed on the x and y variables, they take all possible values. The graph of $z = 2$ is a plane parallel to the xy-plane and 2 units above the xy-plane. Similarly, the graph of $y = 3$ is a plane parallel to the xz-plane, and $x = 4$ is a plane parallel to the yz-plane. (Note: The planes have been drawn as rectangles in Figures 4.8 and 4.9, but they actually extend infinitely far.)

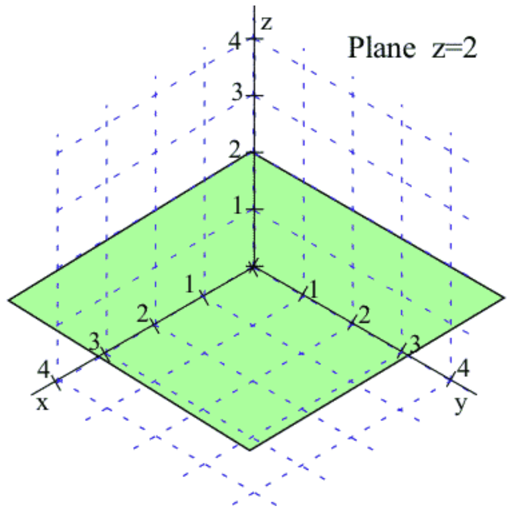


Figure 4.8: Plane $z = 2$

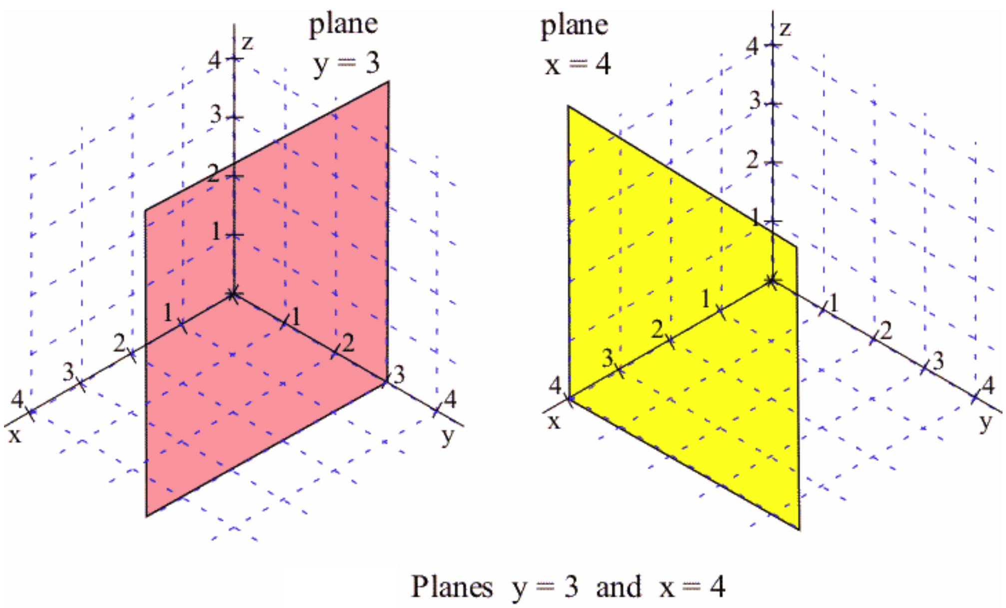


Figure 4.9: Planes $y = 3$ and $x = 4$

Distance Between Points In two dimensions we can think of the distance between points as the length of the hypotenuse of a right triangle, and that leads to the Pythagorean formula:

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} \ .$$

In three dimensions we can also think of the distance between points as the length of the hypotenuse of a right triangle.

In this situation the calculations may appear more complicated, but they are straightforward and the final formula is what we hope it would be given the 2-dimensional formula:

$$\begin{aligned} \text{distance}^2 &= \text{base}^2 + \text{height}^2 \\ &= \left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right)^2 + (\Delta z)^2 \\ &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \ , \end{aligned}$$

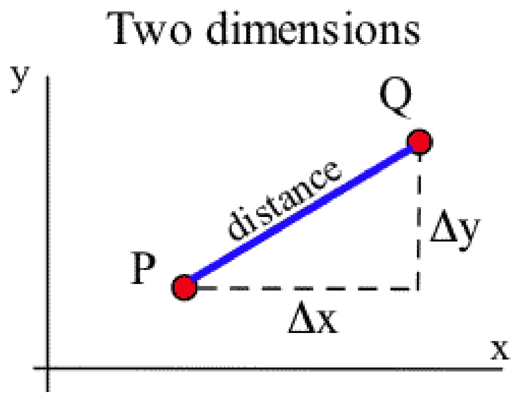


Figure 4.10: Distance in two dimensions

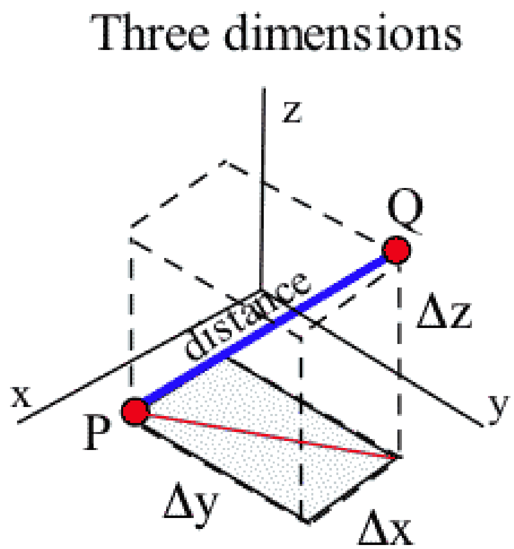


Figure 4.11: Distance in three dimensions

so

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\ .$$

Theorem 4.1.1 (Distance in 3-Dimensions)**.** If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are points in space, then the distance between P and Q is

$$\begin{aligned} \text{distance} &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}$$

The 3-dimensional pattern is very similar to the 2-dimensional pattern with the additional piece $\Delta z)^2$.

Example 4.1.7. Find the distances between points $A = (1, 2, 3)$ and $B = (7, 5, -3)$.
Solution:

$$\text{Dist}(A, B) = \sqrt{6^2 + 3^2 + (-6)^2} = \sqrt{36 + 9 + 36} = \sqrt{81} = 9\ .$$

■

In two dimensions, the set of points at a fixed distance from a given point is a circle, and we used the distance formula to determine equations describing circles: the circle with center $(2, 3)$ and radius 5 is given by $(x - 2)^2 + (y - 3)^2 = 5^2$ or $x^2 + y^2 - 4x - 6y = 12$.

The same ideas work for spheres in three dimensions.

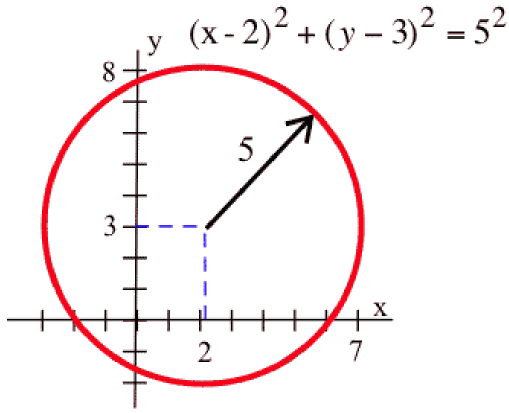


Figure 4.12: $(x-2)^2 + (y-3)^2 = 25$

Definition 4.1.2 (Sphere)**.** The set of points (x, y, z) at a fixed distance r from a point (a, b, c) is a **sphere** with center (a, b, c) and radius r.

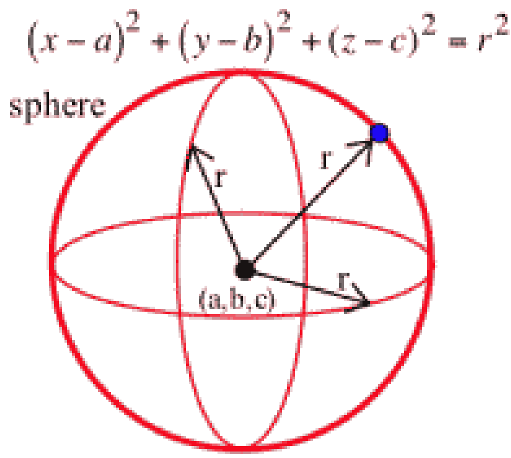


Figure 4.13: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

The sphere is given by the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \ .$$

Example 4.1.8. Write the equations of a sphere with center $(2, -3, 4)$ and radius 3.
Solution: The equation is

$$(x-2)^2 + (y+3)^2 + (z-4)^2 = 3^2 = 9.$$

■

Now suppose that we want to graph a surface. We can think of each input (x, y) as a location on the plane, and plot the point $f(x, y)$ units above that point. Graphing that can be challenging. We have a few options:

- Use a computer program (such as GeoGebra or SageMath) to draw beautiful perspective drawings. If such a program is available, then this is usually the most accurate option.
- Try to draw a perspective drawing by hand. This is very challenging, and usually not worth the effort.
- Use level curves to draw contour diagrams (or contour maps), which is the approach we'll focus on here. A contour diagram is like a topographical map – points with the same elevation (outputs) are

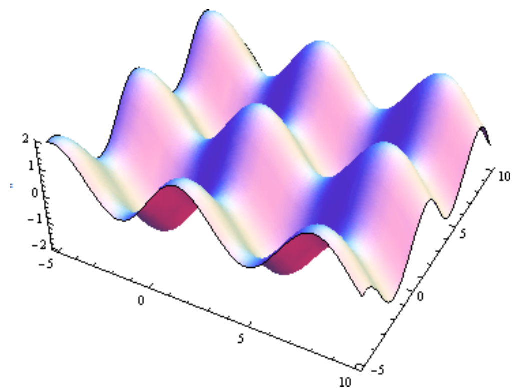


Figure 4.14: Surface

connected with curves. Each particular output is called a level, and these curves are called **level curves** or **contours**. The closer the curves are to each other, the steeper that section of the surface. Topographical maps give hikers information about elevation, steep and shallow grades, peaks and valleys. Contour diagrams give us the same kind of information about a function.

Figure 4.15 is a contour diagram of the same surface shown in Figure 4.14. The level curves are graphs in the xy-plane of curves $f(x, y) = c$ for various constants c.

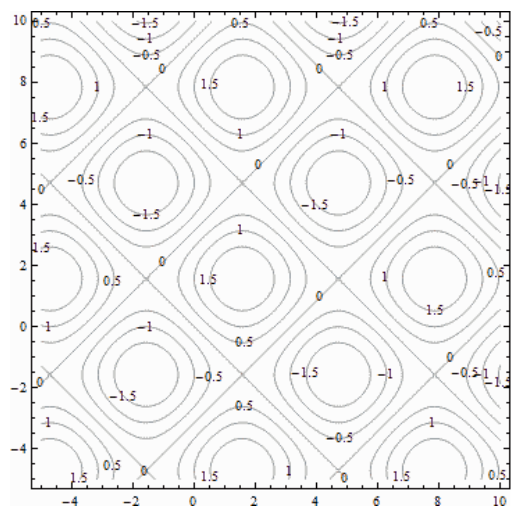


Figure 4.15: Contour map

Each of the squares in Figure 4.15 corresponds to one of the bumps on the surface. If the contours are positive, as highlighted in Figure 4.16, the bump is above the xy-plane. If the contours are negative, the bump extends below the xy-plane.

Everywhere on the crisscrossed pattern of diagonal lines, the height of the surface is 0, so the surface is on the xy-plane. This is a feature that we wouldn't necessarily have seen when we looked at the perspective drawing. Contour maps can help us see features of the surface that the 3-dimensional graph doesn't show.

To better understand contour diagrams, suppose we had a table of elevation data. We could graph this by plotting the height at each point and connecting the dots with smooth curves, which would result in the something like the graph shown in Figure 4.18

If we slice the surface in Figure 4.18 with the plane $z = 8$, the points where the plane cuts the surface are those points where the elevation of the surface is 8 units above the xy-plane. Figure 4.19 shows the surface being sliced by the planes $z = 8$ and $z = 4$. Slicing the surface at different elevations and sketching the curves where the plane intersects the surface results in the graph in Figure 4.20.

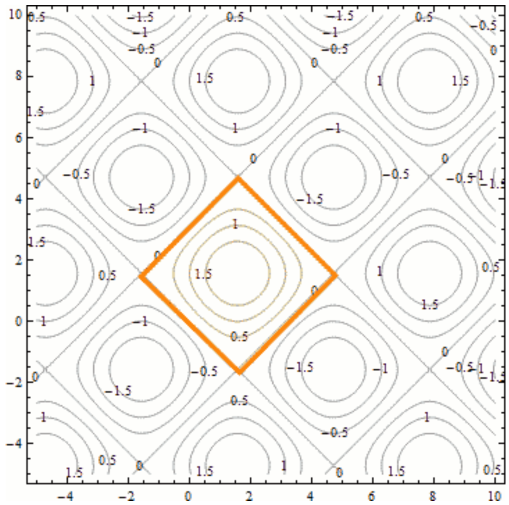


Figure 4.16: Contour map

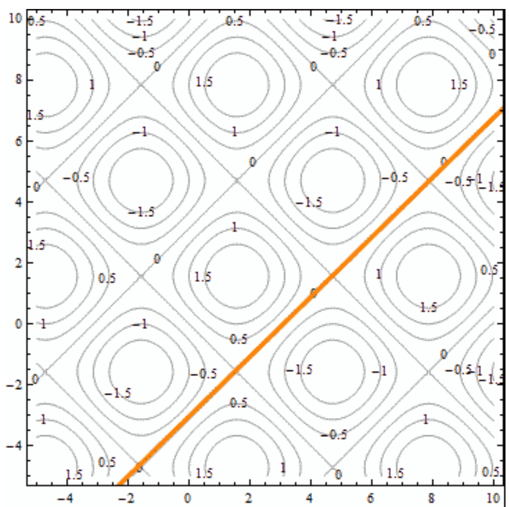


Figure 4.17: Contour map

x \ y	0	1	2	3	4	5	6
0	1.6	1.8	2.0	2.1	2.2	2.1	2.0
1	2.4	2.9	3.3	3.7	5.9	3.7	3.3
2	3.3	4.3	5.6	6.7	7.1	6.7	5.6
3	3.8	5.3	7.1	9.1	9.9	9.1	7.1
4	3.3	4.3	5.6	6.7	7.1	6.7	5.6
5	2.4	2.9	3.3	3.7	3.8	3.7	3.3
6	1.6	1.8	2.0	2.1	2.2	2.1	2.0

Elevations arranged in a table

If we move all of those curves to the xy-plane (or, equivalently, view them from directly overhead), the result is a 2-dimensional graph of the level curves of the original surface. This is the contour diagram in Figure 4.21.

Example 4.1.9. Create a contour diagram for our car rental example with cost function $C(d, m) = 40d + 0.15m$. Draw curves for when the cost is 0, 100, 200, 300, and 400.

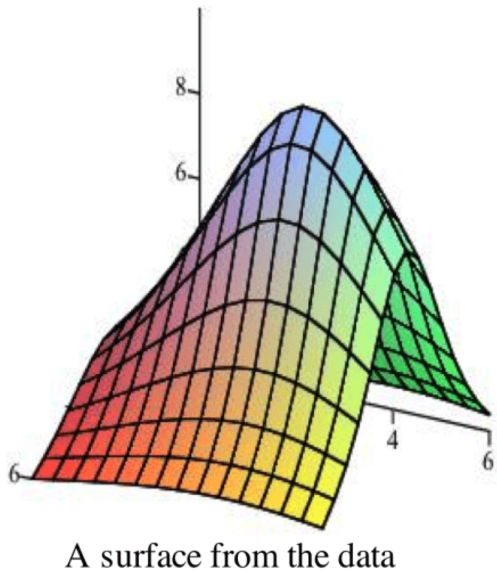


Figure 4.18: Surface from the data

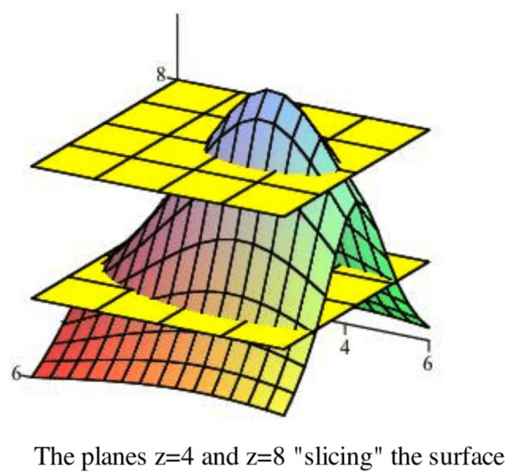


Figure 4.19: Slices of the surface

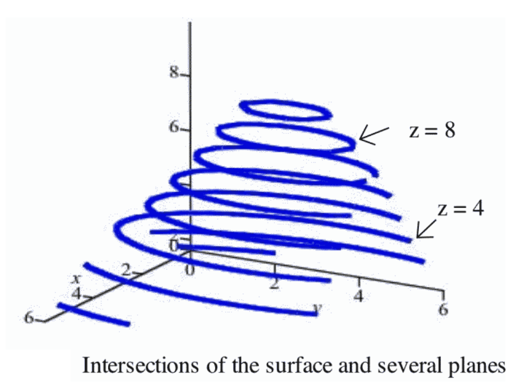


Figure 4.20: Slices of the surface

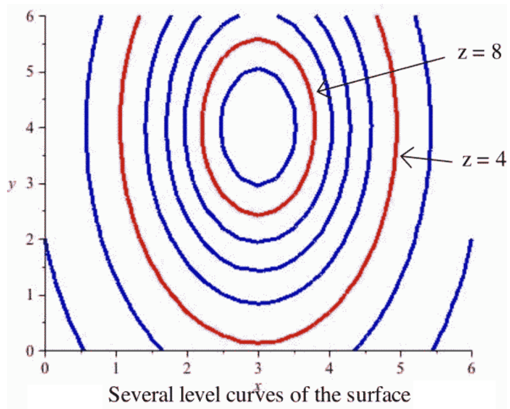


Figure 4.21: Level curves of the surface

Solution: We'll set $C(d,m) = 40d + 0.15m = c$ for $c = 0$, 100, 200, 300, and 400 and draw the curves in the dm-plane.

The first coordinate of the ordered pair is d, so the d-axis will be horizontal; the m-axis will be vertical. Remember that the domain for this function is really just where $d \geq 0$ and $m \geq 0$, so we will only draw the curves in the first quadrant.

When $c = 0$:

$$\begin{aligned} C(d,m) = 40d + 0.15m &= 0 \\ 0.15m &= -40d \\ m &= \frac{40}{0.15}d \approx -267d \end{aligned}$$

This is the equation of a line, with slope about -267, passing through the origin. Because of the domain restrictions, the curve we will draw for this level is simply the origin. Putting this back into the car rental context, the only point where we pay \$0 for renting the car is when we rent the car for 0 days and drive it 0 miles – that is, if we don't rent it at all.

When $c = 100$:

$$\begin{aligned} C(d,m) = 40d + 0.15m &= 100 \\ 0.15m &= -40d + 100 \\ m &= -\frac{40}{0.15}d + \frac{100}{0.15} \approx -267d + 667 \end{aligned}$$

This is the equation of a line, with slope about -267, and d-intercept of about 667. This section of this line that lies in the first quadrant is shown with 100 labeling it.

Putting this into context, any point on that line represents a (d,m) combination of days and miles that will make the cost exactly \$100. So, for example – if we rent the car for 0 days and drive it 667 miles, it will cost us \$100. If we rent the car for 2.5 days and don't drive it at all, it will cost us \$100.

We continue for $c = 200, 300$, and 400 and sketch the curves in the plane, resulting in the contour diagram shown in Figure 4.22.

■

Example 4.1.10. The contour diagram for the cost $C(d,m)$ in dollars for renting a car for d days and driving it m miles is shown in the previous example. Use the diagram to answer the following questions.

(a) What is the cost of renting a car for 3 days and driving it 200 miles?

Solution: The point $(3,200)$ is between contours on this graph, so we can't get an exact answer for $C(3,200)$. (But it's typical for a graph that we would have to estimate). It looks to me as if $(3,200)$ is halfway between the 100 and the 200 contours, so let's estimate that $C(3,200)$ is about \$150.

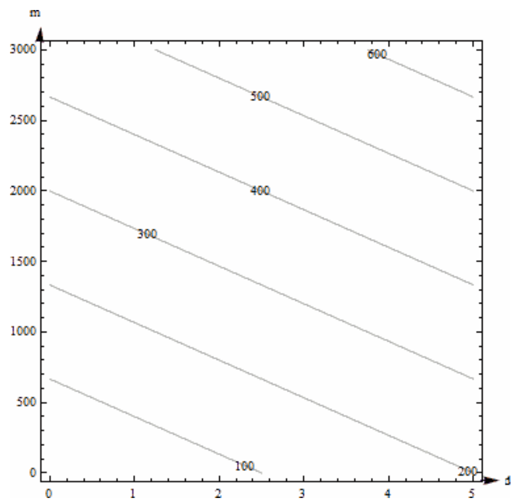


Figure 4.22: Level curves of $C(d, m) = 40d + 0.15m$

Estimates from the graph are necessarily very rough. The graph only shows a little information (in this way, a contour diagram is like a table), so we have to extrapolate in between. But for most graphs, we don't actually know what happens between the contours. All we know for sure is that the output at $(3, 200)$ is between the two levels we see. For this car rental example, we also know a formula, and my table showed this particular input, so we have other ways to get a better answer. ■

(b) What is $C(100, 4)$? What is $C(4, 100)$?

Solution: We can't find $(100, 4)$ on this diagram, so we can't make an estimate of $C(100, 4)$ from this graph. $(4, 100)$ lies between the contours for 100 and 200. It looks closer to 200, so let's estimate that $C(4, 100)$ is about \$180. ■

(c) Suppose we rent the car for 3 days. Is C an increasing function of miles?

Solution: If we fix $d = 3$, we get a vertical line. What happens as m increases on this vertical line? As m increases, the function values shown on the contours increase, so C appears to be an increasing function of miles. ■

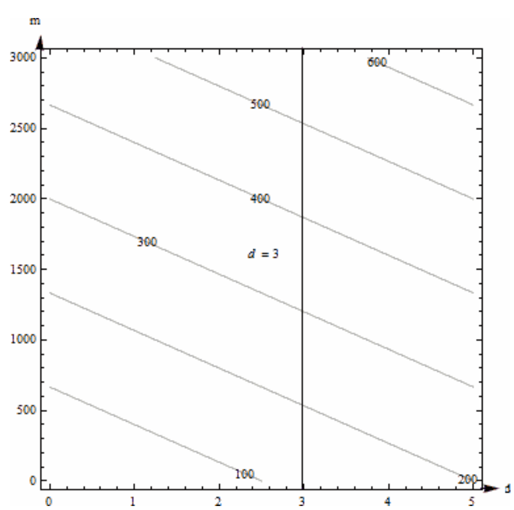


Figure 4.23: Level curves of $C(d, m) = 40d + 0.15m$ with $d = 3$

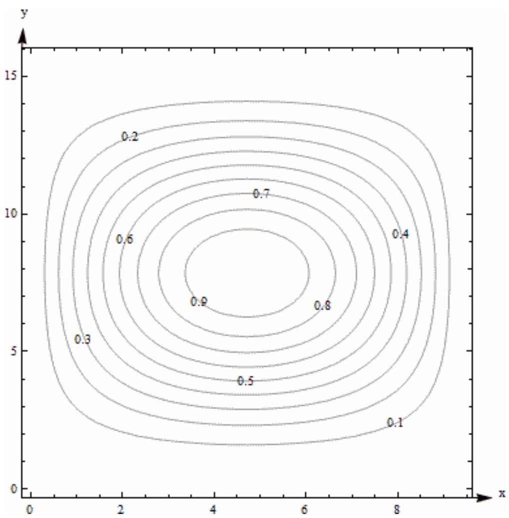


Figure 4.24: Level curves of $z = g(x, y)$

Example 4.1.11. Here is a contour diagram for a function $g(x, y)$.
Use Figure 4.24 to answer the following questions:

(a) What is $g(3, 5)$? **Solution:** $g(3, 5)$ is 0.6. We can tell because the point is right on one of the contours, as illustrated in Figure 4.25.

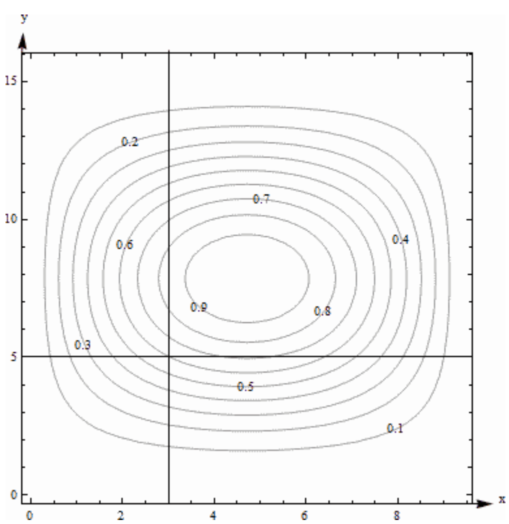


Figure 4.25: Level curves of $z = g(x, y)$ with $x = 3$ and $y = 5$

■

(b) What is the highest point shown on the diagram? What is the lowest point shown? **Solution:** The highest contour shown is 0.9, and there would be a contour for 1.0 if the surface had ever got that high. However, the height seems to be increasing as we move in toward the center, so it appears that g gets to nearly 1 in the center. The lowest contour is 0.1. But again, we will guess that the height continues to decrease, so it appears that g is nearly 0 around the outside. ■

(c) If you start at $(3, 5)$ and head in the positive x direction, do you go uphill or downhill first? **Solution:** Starting at the point $(3, 5, 0.6)$ on the surface and traveling to the right along the horizontal line shown in the previous part, we would cross the contour for 0.7 next. So the function increases first (we go uphill), and then decreases again. ■

Again, remember that we don't really know what happens between the contours. All we can do is estimate from the information in the graph.

Example 4.1.12. Here is a contour diagram for a function $F(x, y)$.

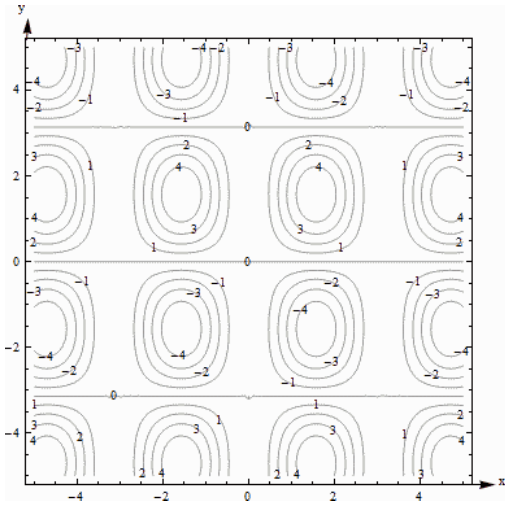


Figure 4.26: Level curves of $z = F(x, y)$

(a) Describe the shape of the surface. **Solution:** The surface is bumpy, with regularly spaced oval bumps. Notice that some of the bumps go up (positive contours), but others go down. Between the bumps, there are horizontal lines that are completely level, with an elevation of 0. ■

(b) Suppose you travel along the surface in the positive y-direction, starting on the surface at the point above (or below) the point $(x, y) = (-1, 1)$. Describe your journey. **Solution:** It looks as if $F(-1, 1)$ is about 3. As we head in the positive y-direction along the line shown in Figure 4.27, we first go uphill, nearly to 4, then we start going downhill. As we keep going north, we keep descending, going into the dip, until nearly -4. We're starting to go uphill again just as we leave the graph.

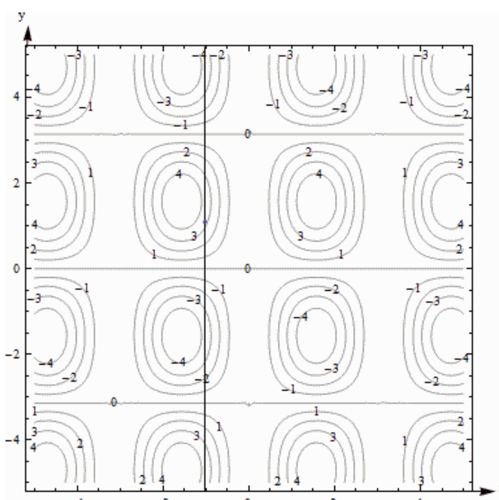


Figure 4.27: Level curves of $z = F(x, y)$ with $x = -1$

■

What happens if you have a function of more than two variables? Its graph will be a **hyper-surface**. For example, the graph of a function of four variables will be a hyper-surface in 5-dimensional space. This is very difficult (impossible for most of us) to visualize. Even the contours are hard to visualize – instead of curves in the plane, they're hyper-surfaces in 4-dimensional space. So if you have more than two variables, the graph isn't usually very useful.

4.1.5 Complementary Goods and Substitute Goods

The demand for some pairs of goods have a relationship, where the quantity demanded for one product depends somehow on the prices for both

Definition 4.1.3 (Complementary Goods)**.** Two goods are **complementary** if an increase in the price of either decreases the demand for both.

Example 4.1.13. Here are two examples of complementary goods.

- The demand for cars depends on both the price for cars and the price of gasoline.
- The demand for hot dog buns depends on both the price for the buns and the price for the hot dogs.

Definition 4.1.4 (Substitute Goods)**.** Two goods are **substitutes** if an increase in the price of one increases the demand for the other.

Example 4.1.14. Here is an example of substitute goods.

- The demand for Brand A depends on its price and also on the price of its main competitor Brand B. If the Brand B raises its price, consumers will switch brands (substitute) and demand for Brand A will increase.

Think about brands of soft drinks, detergent, or paper towels. A traditional example is coffee and tea: the idea is that consumers are simply looking for a hot drink and they'll buy whatever is cheaper. But this has always seemed fishy to me – I've never met any coffee- or tea-drinkers who would happily switch.

These demand functions are functions of two variables.

Example 4.1.15. The demand functions for two products are given below. p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for Products 1 and 2:

$$q_1 = 200 - 3p_1 - p_2$$
$$q_2 = 150 - p_1 - 2p_2$$

Are these two products complementary goods or substitute goods? What is the quantity demanded for each when the price for Product 1 is \$20 per item and the price for Product 2 is \$30 per item?

Solution: These products are complementary: an increase in either price decreases both demands. You can see that because the coefficients are both negative in each demand function.

When $p_1 = 20$ and $p_2 = 30$, we have

$$q_1 = 200 - 3(20) - (30) = 110$$
$$q_2 = 150 - (20) - 2(30) = 70$$

So 110 units are demanded for Product 1 and 70 units are demanded for Product 2 when the price for Product 1 is \$20 per item and the price for Product 2 is \$30 per item. ■

4.1.6 Cobb-Douglas Production Function

Production functions are used to model the total output of a firm for a variety of inputs (doesn't this sound like a function of several variables?). One example is a **Cobb-Douglas Production function**:

$$P = AL^{\alpha}K^{\beta} \ ,$$

where P is the total production, A is a constant, α and β are constants between 0 and 1, L is the labor force, and K is the capital expenditure. (The units must be massaged well.)

You can read more about Cobb-Douglas Production functions at http://en.wikipedia.org/wiki/Cobb-Douglas. You can read about other kinds of production functions at http://en.wikipedia.org/wiki/Production_function.

4.2 Partial Derivatives

Now that you have some familiarity with functions of two variables, it's time to start applying calculus to help us solve problems with them. In Chapter 2, we learned about the derivative for functions of two variables. Derivatives told us about the shape of the function, and let us find local max and min – we want to be able to do the same thing with a function of two variables.

First let's think. Imagine a surface, the graph of a function of two variables. Imagine that the surface is smooth and has some hills and some valleys. Concentrate on one point on your surface. What do we want the derivative to tell us? It ought to tell us how quickly the height of the surface changes as we move... Wait, which direction do we want to move? This is the reason that derivatives are more complicated for functions of several variables – there are so many (in fact, infinitely many) directions we could move from any point.

It turns out that our idea of fixing one variable and watching what happens to the function as the other changes is the key to extending the idea of derivatives to more than one variable.

Definition 4.2.1 (Partial Derivatives)**.** Suppose that $z = f(x, y)$ is a function of two variables.

- The **partial derivative** of f **with respect to** x is the derivative of the function $f(x, y)$ where we think of x as the only variable and act as if y is a constant.
- The partial derivative of f with respect to y is the derivative of the function $f(x, y)$ where we think of y as the only variable and act as if x is a constant.

The "with respect to x" or "with respect to y" part is really important – you have to know and tell which variable you are thinking of as THE variable.

Geometrically, the partial derivative with respect to x gives the slope of the curve as you travel along a cross-section, a curve on the surface parallel to the x-axis. The partial derivative with respect to y gives the slope of the cross-section parallel to the y-axis.

Notation for the Partial Derivative The partial derivative of $z = f(x, y)$ with respect to x is written as

$$f_x(x, y)$$

or simply

$$f_x \quad \text{or} \quad z_x \ .$$

The **Leibniz notation** is

$$\frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial z}{\partial x} \ .$$

We use an adaptation of the $\dfrac{\partial z}{\partial x}$ notation to mean find the partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial f}{\partial x} \ .$$

Estimate a partial derivative from a table or contour diagram The partial derivative with respect to x can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval in the x-direction (holding y constant). The tinier the interval, the closer this is to the true partial derivative.

Compute a partial derivative from a formula If $f(x, y)$ is given as a formula, you can find the partial derivative with respect to x algebraically by taking the ordinary derivative thinking of x as the only variable (and treating y as an ordinary number).

Of course, everything here works the same way if we're trying to find the partial derivative with respect to y – just think of y as your only variable and act as if x is constant.

The idea of a partial derivative works perfectly well for a function of several variables: you focus on one variable to be THE variable and act as if all the other variables are constants.

Example 4.2.1. Figure 4.28 shows a contour diagram for a function $g(x, y)$.

Use the diagram to answer the following questions:

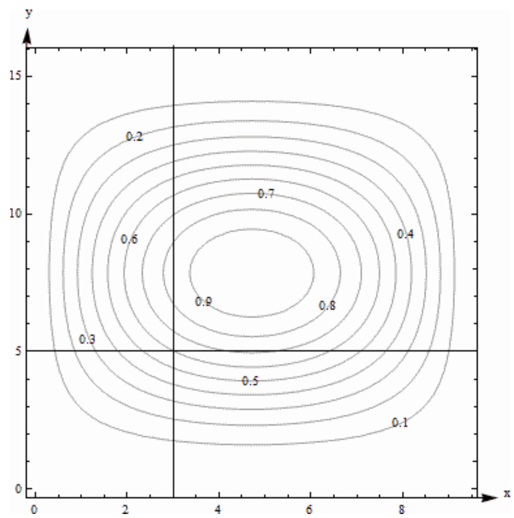


Figure 4.28: $z = g(x, y)$

(a) Estimate $g_x(3,5)$ and $g_y(3,5)$.

Solution: $g_x(3,5)$ means we're thinking of x as the only variable, so we'll hold y fixed at $y = 5$. That means we'll be looking along the horizontal line $y = 5$. To estimate g_x, we need two function values. $(3,5)$ lies on the contour line, so we know that $g(3,5) = 0.6$. The next point as we move to the right is $g(4.2,5) = 0.7$.

Now we can find the average rate of change:

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{change in output}}{\text{change in input}}\\ &= \frac{\Delta g}{\Delta x}\\ &= \frac{0.7-0.6}{4.2-3}\\ &= \frac{1}{12} \approx 0.083\end{aligned}$$

We can do the same thing by going to the next point we can read to the left, which is $g(2.4,5) = 0.5$. Then the average rate of change is

$$\frac{\Delta g}{\Delta x} = \frac{0.5-0.6}{2.4-3} = \frac{1}{6} \approx 0.167 \ .$$

Either of these would be a fine estimate of $g_x(3,5)$ given the information we have, or we could take their average. We can estimate that $g_x(3,5) \approx 0.125$.

Estimate $g_y(3,5)$ the same way, but moving on the vertical line. Using the next point up, we get the average rate of change is

$$\frac{\Delta g}{\Delta y} = \frac{0.7-0.6}{5.8-5} = \frac{1}{8} = 0.125 \ .$$

Using the next point down, we get

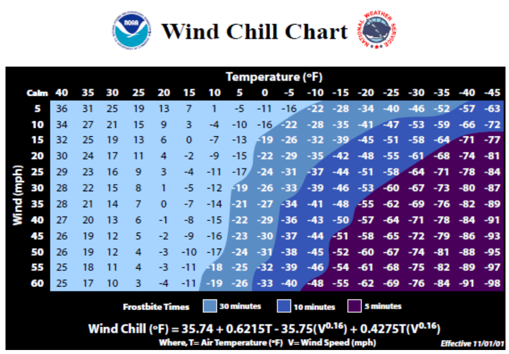
$$\frac{\Delta g}{\Delta y} = \frac{0.5-0.6}{4.5-5} = \frac{1}{5} = 0.2 \ .$$

Taking their average, we estimate $g_y(3,5) \approx 0.1625$. ■

(b) Where on this diagram is g_x greatest? Where is g_y greatest?

Solution: g_x means x is our only variable, and we're thinking of y as a constant. So we're thinking about moving across the diagram on horizontal lines. g_x will be greatest when the contour lines are closest together, i.e., when the surface is steepest – then the denominator in $\dfrac{\Delta g}{\Delta x}$ will be small, so $\dfrac{\Delta g}{\Delta x}$ will be big. Scanning the graph in Figure 4.28, we can see that the contour lines are closest together when we head to the left or to the right from about $(0.5, 8)$ and $(9, 8)$. So g_x is greatest at about $(0.5, 8)$ and $(9, 8)$. For g_y, we want to look at vertical lines. g_y is greatest at about $(5, 3.8)$ and $(5, 12)$. ■

Example 4.2.2. Cold temperatures feel colder when the wind is blowing. Windchill, W, is the perceived temperature, and it depends on both the actual temperature, T, and the wind speed, V, so $W = W(T, V)$ is a function of two variables! You can read more about windchill at http://www.nws.noaa.gov/om/windchill/. The table in Figure 4.29 shows the perceived temperature for various temperatures and windspeeds.



Wind Chill Chart

Wind (mph) \ Temperature (°F): Calm	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45
5	36	31	25	19	13	7	1	-5	-11	-16	-22	-28	-34	-40	-46	-52	-57	-63
10	34	27	21	15	9	3	-4	-10	-16	-22	-28	-35	-41	-47	-53	-59	-66	-72
15	32	25	19	13	6	0	-7	-13	-19	-26	-32	-39	-45	-51	-58	-64	-71	-77
20	30	24	17	11	4	-2	-9	-15	-22	-29	-35	-42	-48	-55	-61	-68	-74	-81
25	29	23	16	9	3	-4	-11	-17	-24	-31	-37	-44	-51	-58	-64	-71	-78	-84
30	28	22	15	8	1	-5	-12	-19	-26	-33	-39	-46	-53	-60	-67	-73	-80	-87
35	28	21	14	7	0	-7	-14	-21	-27	-34	-41	-48	-55	-62	-69	-76	-82	-89
40	27	20	13	6	-1	-8	-15	-22	-29	-36	-43	-50	-57	-64	-71	-78	-84	-91
45	26	19	12	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-72	-79	-86	-93
50	26	19	12	4	-3	-10	-17	-24	-31	-38	-45	-52	-60	-67	-74	-81	-88	-95
55	25	18	11	4	-3	-11	-18	-25	-32	-39	-46	-54	-61	-68	-75	-82	-89	-97
60	25	17	10	3	-4	-11	-19	-26	-33	-40	-48	-55	-62	-69	-76	-84	-91	-98

Frostbite Times: 30 minutes, 10 minutes, 5 minutes

Wind Chill (°F) = 35.74 + 0.6215T - 35.75($V^{0.16}$) + 0.4275T($V^{0.16}$)

Where, T= Air Temperature (°F) V= Wind Speed (mph) *Effective 11/01/01*

Figure 4.29: Windchill table, National Weather Service

Note that they also include the formula, but for this example we'll use the information in the table.

1. What is the perceived temperature when the actual temperature is 25°F and the wind is blowing at 15 miles per hour?

   Solution: Reading the table, we see that the perceived temperature is $W(25, 15) = 13$°F. ■

2. Suppose the actual temperature is 25°F. Use information from Table 4.29 to describe how the perceived temperature would change if the wind speed increased from 15 miles per hour?

   Solution: This is a question about a partial derivative. We're holding the temperature fixed at $T = 25$°F, and asking what happens as wind speed (V) increases from 15 miles per hour. We're thinking of V as the only variable, so we want to estimate $W_V(25, 15)$. We'll find the average rate of change by looking in the column where $T = 25$ and letting V increase, and use that to approximate the partial derivative.

   $$W_V \approx \frac{\Delta W}{\Delta V} = \frac{11 - 13}{20 - 15} = -0.4 \ .$$

   What are the units? W is measured in °F and V is measured in mph, so the units here are °F/mph. And that lets us describe what happens. The perceived temperature would decrease by about 0.4°F for each mph increase in wind speed. ■

Example 4.2.3. Find f_x and f_y at the points $(0, 0)$ and $(1, 1)$ if $f(x, y) = x^2 - 4xy + 4y^2$.
Solution: To find f_x, take the ordinary derivative of f with respect to x, acting as if y is constant:

$$f_x(x, y) = 2x - 4y \ .$$

Note that the derivative of the term $4y^2$, with respect to x, is 0 because it's a constant (as far as x is concerned).

Similarly,

$$f_y(x, y) = -4x + 8y \ .$$

Now we can evaluate these at the points:

$f_x(0, 0) = 0$ and $f_y(0, 0) = 0$; this tells us that the cross sections parallel to the x and y-axes are both flat at $(0, 0)$.

$f_x(1, 1) = -2$ and $f_y(1, 1) = 4$; this tells us that above the point $(1, 1)$, the surface decreases if we move to more positive x values and increases if we move to more positive y values. ∎

Example 4.2.4. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ if $f(x, y) = \frac{e^{x+y}}{y^3+y} + y\ln(y)$.

Solution: $\dfrac{\partial f}{\partial x}$ means x is our only variable, we're thinking of y as a constant. Then we'll just find the ordinary derivative. From the point of view of x, this is an exponential function, divided by a constant, with a constant added. The constant pulls out in front, the derivative of the exponential function is the same thing, and we need to use the chain rule, so we multiply by the derivative of that exponent (which is just 1):

$$\frac{\partial f}{\partial x} = \frac{1}{y^3 + y} \cdot e^{x+y} \ .$$

$\dfrac{\partial f}{\partial y}$ means that we're thinking of y as the variable, acting as if x is constant. From y's point of view, f is a quotient plus a product, so we'll need the quotient rule and the product rule:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{()() - ()()}{()^2} + ()() + ()() \\ &= \frac{\left(e^{x+y}(1)\right)\left(y^3 + y\right) - \left(e^{x+y}\right)\left(3y^2 + 1\right)}{(y^3 + y)^2} + (1)(\ln(y)) + (y)\left(\frac{1}{y}\right)\end{aligned}$$

∎

Example 4.2.5. Find f_z if $f(x, y, z, w) = 35x^2w - \frac{1}{z} + yz^2$.

Solution: f_z means we act as if z is our only variable, so we'll act as if all the other variables (x, y, and w) are constants and take the ordinary derivative:

$$f_z(x, y, z, w) = \frac{1}{z^2} + 2yz \ .$$

∎

4.2.1 Using Partial Derivatives to Estimate Function Values

We can use the partial derivatives to estimate values of a function. The geometry is similar to the tangent line approximation in one variable. Recall the one-variable case: if x is close enough to a known point a, then

$$f(x) \approx f(a) + f'(a)(x - a) \ .$$

In two variables, we do the same thing in both directions at once:

Theorem 4.2.1 (Approximating Function Values with Partial Derivatives)**.** To approximate the value of $f(x, y)$ where the point (x, y) is near a point (a, b), especially when the exact values of $f(a, b)$, $f_x(a, b)$, and $f_y(a, b)$ exist and are easy to compute, then:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \ .$$

Notice that the total change in f is being approximated by adding the approximate changes coming from the x and y directions. Another way to look at the same formula is :

$$\Delta f \approx f_x \Delta x + f_y \Delta y \ .$$

How close is close? It depends on the shape of the graph of f. In general, the closer the better.

Example 4.2.6. Use partial derivatives to estimate the value of $f(x,y) = x^2 - 4xy + 4y^2$ at the point $(0.9, 1.1)$.

Solution: Note that the point $(0.9, 1.1)$ is close to an easy point, $(1,1)$. In fact, we already worked out the partial derivatives at $(1,1)$: $f_x(x,y) = 2x - 4y$ so $f_x(1,1) = -2$, and $f_y(x,y) = -4x + 8y$ so $f_y(1,1) = 4$. We also know that $f(1,1) = 1$.

So,

$$f(0.9, 1.1) \approx 1 - 2(-0.1) + 4(0.1) = 1.6 \ .$$

Note that in this example it would have been possible to simply compute the exact answer:

$$f(0.9, 1.1) = (0.9)2 - 4(0.9)(1.1) + 4(1.1)2 = 1.69 \ .$$

Our estimate is not perfect, but it's pretty close. ■

Example 4.2.7. Figure 4.30 shows a contour diagram for a function $g(x,y)$. Use partial derivatives to estimate the value of $g(3.2, 4.7)$.

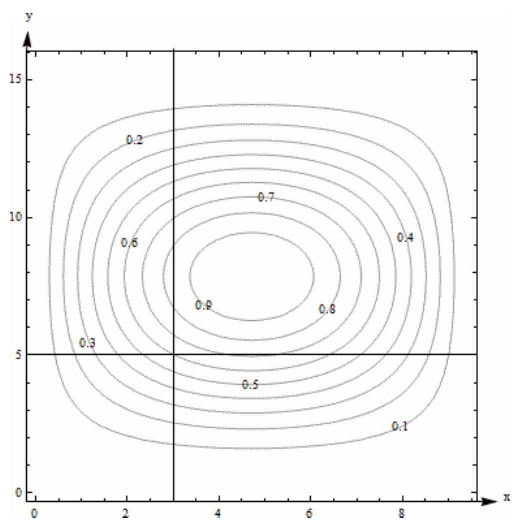


Figure 4.30: $z = g(x,y)$

Solution: This is the same diagram from before, so we already estimated the value of the function and the partial derivatives at the nearby point $(3,5)$. $g(3,5)$ is 0.6, our estimate of $g_x(3,5) \approx 0.125$, and our estimate of $g_y(3,5) \approx 0.1625$. So

$$g(3.2, 4.7) \approx 0.6 + (0.125)(0.2) + (0.1625)(-0.3) = 0.57625 \ .$$

Note that in this example we have no way to know how close our estimate is to the actual value. ■

4.3 Optimization with Multivariable Functions

The partial derivatives tell us something about where a surface has local maxima and minima. Remember that even in the one-variable cases, there were critical points which were neither maxima nor minima – this is also true for functions of many variables. In fact, as you might expect, the situation is even more complicated.

4.3.1 Second Derivatives

When you find a partial derivative of a function of two variables, you get another function of two variables – you can take its partial derivatives, too. We've done this before, in the one-variable setting. In the one-variable setting, the second derivative gave information about how the graph was curved. In the two-variable setting, the second partial derivatives give some information about how the surface is curved, as you travel on cross-sections – but that's not very complete information about the entire surface.

Imagine that you have a surface that's ruffled around a point, like what happens near a button on an overstuffed sofa, or a pinched piece of fabric, or the wrinkly skin near your thumb when you make a fist. Right at that point, every direction you move, something different will happen – it might increase, decrease, curve up, curve down... A simple phrase like concave up or concave down can't describe all the things that can happen on a surface.

Surprisingly enough, though, there is still a second derivative test that can help you decide if a point is a local maximum or minimum or neither, so we still do want to find second derivatives.

Definition 4.3.1 (Second Partial Derivatives)**.** Suppose $f(x, y)$ is a function of two variables. Then it has four **second partial derivatives**:

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = (f_x)_x \quad f_{xy} = \frac{\partial}{\partial y}(f_x) = (f_x)_y$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = (f_y)_x \quad f_{yy} = \frac{\partial}{\partial y}(f_y) = (f_y)_y \ .$$

f_{xy} and f_{yx} are called the **mixed (second) partial derivatives** of f.

Leibniz notation for the second partial derivatives is a bit confusing, and we won't use it as often.

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \quad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that the order of the variables for the mixed partials goes from right to left in the Leibniz notation instead of left to right.

Example 4.3.1. Find all four partial derivatives of $f(x, y) = x^2 - 4xy + 4y^2$.
Solution: We have to start by finding the (first) partial derivatives:

$$\begin{aligned} f_x(x, y) &= -4x + 8y \\ f_y(x, y) &= 2x - 4y \end{aligned}$$

Now we're ready to take the second partial derivatives:

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial}{\partial x}(2x - 4y) = 2 \\ f_{xy}(x, y) &= \frac{\partial}{\partial y}(2x - 4y) = -4 \\ f_{yx}(x, y) &= \frac{\partial}{\partial x}(-4x + 8y) = -4 \\ f_{yy}(x, y) &= \frac{\partial}{\partial y}(-4x + 8y) = 8 \end{aligned}$$

■

You might have noticed that the two mixed partial derivatives were equal in this last example. It turns out that it's not a coincidence – it's a theorem!

Theorem 4.3.1 (Mixed Partial Derivative Theorem)**.** If f, f_x, f_y, f_{xy}, and f_{yx} are all continuous (no breaks in their graphs), then

$$f_{xy} = f_{yx} \quad .$$

In fact, as long as f and all its appropriate partial derivatives are continuous, the mixed partials are equal even if they are of higher order, and even if the function has more than two variables.

This theorem means that the confusing Leibniz notation for second derivatives is not a big problem – in almost every situation the mixed partials are equal, so the order in which we compute them doesn't matter.

Example 4.3.2. Find $\dfrac{\partial^2 f}{\partial x \partial y}$ for $f(x, y) = \dfrac{e^{x+y}}{y^3 + y} + y\ln(y)$.

Solution: We already found the first partial derivatives in Example 4.2.4:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{e^{x+y}}{y^3+y} \\
\frac{\partial f}{\partial y} &= \frac{(e^{x+y(1)})(y^3+y) - (e^{x+y})(3y^2+1)}{(y^3+y)^2} + (1)(\ln(y)) + (y)\frac{1}{y}
\end{aligned}$$

Now we need to find the mixed partial derivative. Theorem 4.3.1 says that $\dfrac{\partial f^2}{\partial x \partial y} = \dfrac{\partial f^2}{\partial y \partial x}$, so it doesn't matter whether we find the partial derivative of $\dfrac{\partial f}{\partial x}$ with respect to y or the partial derivative of $\dfrac{\partial f}{\partial y}$ with respect to x. Which would you rather do?

It looks like it will be easier to compute the mixed partial by finding the partial derivative of $\dfrac{\partial f}{\partial x} = \dfrac{e^{x+y}}{y^3+y}$ with respect to y. It still looks messy, but it looks less messy:

$$\frac{\partial f^2}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{e^{x+y}}{y^3+y}\right) = \frac{e^{x+y}(y^3+y) - e^{x+y}(3y^2+1)}{(y^3+y)^2} \quad .$$

If we had decided to do this the other way, we'd end up in the same place. Eventually. ∎

4.3.2 Local Maxima, Local Minima, and Saddle Points

Let's briefly review optimization problems in one variable.

A **local maximum** is a point on a curve that is higher than all the nearby points. A **local minimum** is lower than all the nearby points. We know that local maximum or minimum can only occur at critical points, where the derivative is zero or undefined. But we also know that not every critical point is a maximum or minimum, so we also need to test them, with the First Derivative or Second Derivative Test.

The situation with a function of two variables is much the same. Just as in the one-variable case, the first step is to find critical points, places where both the partial derivatives are either zero or undefined

Definition 4.3.2 (Local Maximum and Minimum)**.** Let $f(x, y)$ be a function of two variables that exists at a point (a, b).

- f has a **local maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for all points (x, y) near (a, b).
- f has a **local minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for all points (x, y) near (a, b).

A **critical point** of a function $f(x, y)$ is a point (x, y) (or $(x, y, f(x, y))$) where both the following are true:

1. $f_x = 0$ or is undefined, and
2. $f_y = 0$ or is undefined.

Just as in the one-variable case, a local maximum or minimum of f can only occur at a critical point.

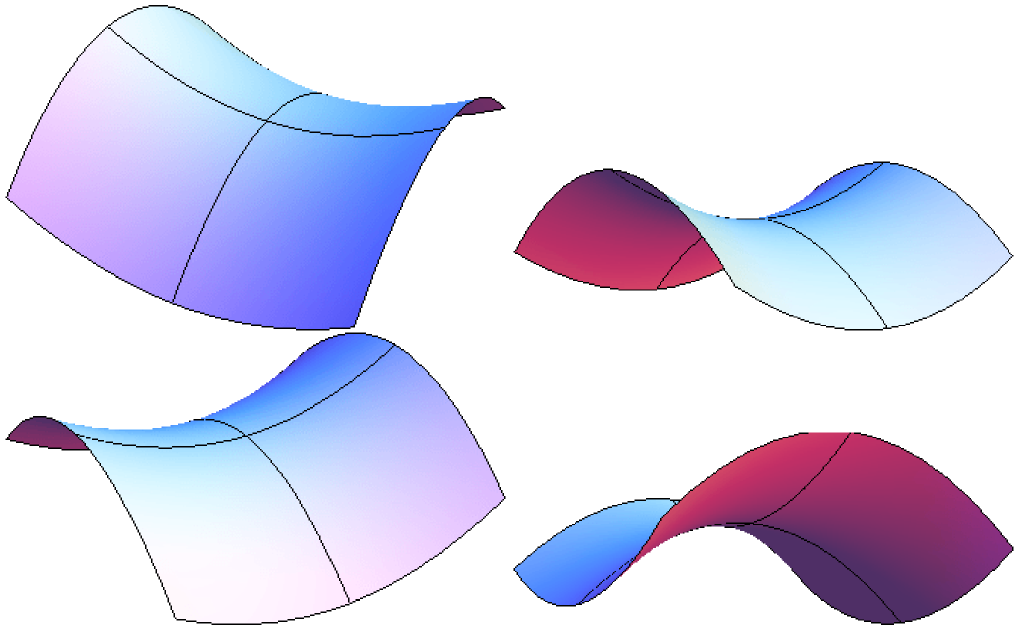


Figure 4.31: Saddle point of $z = f(x, y) = 5x^2 - 3y^2 + 10$

Just as in the one-variable setting, not every critical point is a local maximum or minimum. For a function of two variables, the critical point could be a local maximum, local minimum, or a saddle point.

A point on a surface is a local maximum if it's higher than all the points nearby; a point is a local minimum if it's lower than all the points nearby.

A saddle point is a point on a surface that is a minimum along some paths and a maximum along some others. It's called this because it's shaped a bit like a saddle you might use to ride a horse. You can see a saddle point by making a fist – between the knuckles of your index and middle fingers, you can see a place that is a minimum as you go across your knuckles, but a maximum as you go along your hand toward your fingers.

Figure 4.31 shows a saddle point of the surface $z = f(x, y) = 5x^2 - 3y^2 + 10$ from a few different angles. The saddle point is above the origin and the lines show what the surface looks like above the x and y-axes. Notice how the point above the origin, where the lines cross, is a local minimum in one direction, but a local maximum in the other direction.

4.3.3 Second Derivative Test

Just as in the one-variable case, we'll need a way to test a critical point to see whether it is a local maximum or minimum. There is a second derivative test for functions of two variables that can help, but, just as in the one-variable case, it isn't always conclusive.

Theorem 4.3.2 (The Second Derivative Test for Functions of Two Variables)**.** Find all critical points of $f(x, y)$. Compute

$$D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}) \ ,$$

and evaluate $D = D(x, y)$ at each critical point.

(a) If $D > 0$, then f has a local maximum or minimum at the critical point. To see which, look at the sign of f_{xx}:

- If $f_{xx} > 0$, then f has a local minimum at the critical point.
- If $f_{xx} < 0$, then f has a local maximum at the critical point.

(b) If $D < 0$ then f has a saddle point at the critical point.

(c) If $D = 0$, there could be a local maximum, local minimum or neither (i.e., the test in inconclusive).

Example 4.3.3. Find all local maxima, minima, and saddle points for the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8 \ .$$

Solution: First we find the partial derivatives: $f_x = 3x^2 + 6x$ and $f_y = 3y^2 - 6y$.

Critical points are the places where both of these are zero (neither is ever undefined): $f_x = 3x^2 + 6x = 3x(x + 2) = 0$ when $x = 0$ or when $x = -2$. $f_y = 3y^2 - 6y = 3y(y - 2) = 0$ when $y = 0$ or when $y = 2$.

Putting these together, we get four critical points: $(0, 0)$, $(-2, 0)$, $(0, 2)$, and $(-2, 2)$.

Now to classify them, we'll use Theorem 4.3.2, the Second Derivative Test. We'll need all the second partial derivatives:

$$f_{xx} = 6x + 6 \ , \quad f_{yy} = 6y - 6 \ , \quad f_{xy} = f_{yx} = 0 \ .$$

Then

$$D(x, y) = (6x + 6)(6y - 6) - (0)(0) = (6x + 6)(6y - 6) \ .$$

Now look at each critical point in turn:

- At $(0, 0)$: $D(0, 0) = (6(0) + 6)(6(0) - 6) = (6)(-6) = -36 < 0$, so there is a saddle point at the point $(0, 0)$.
- At $(-2, 0)$: $D(-2, 0) = (6(-2) + 6)(6(0) - 6) = (-6)(-6) = 36 > 0$ and $f_{xx}(-2, 0) = 6(-2) + 6 = -6 < 0$, so there is a local maximum at the point $(-2, 0)$.
- At $(0, 2)$: $D(0, 2) = (6(0) + 6)(6(2) - 6) = (6)(6) = 36 > 0$ and $f_{xx}(0, 2) = 6(0) + 6 = 6 > 0$, so there is a local minimum at the point $(0, 2)$.
- At $(-2, 2)$: $D(-2, 2) = (6(-2) + 6)(6(2) - 6) = (-6)(6) = -36 < 0$, so there is another saddle point at the point $(-2, 2)$.

■

Example 4.3.4. Find all local maxima, minima, and saddle points for the function

$$z = 9x^3 + \frac{y^3}{3} - 4xy \ .$$

Solution: We'll need all the partial derivatives and second partial derivatives, so let's compute them all first:

$$z_x = 27x^2 - 4y \ , \quad z_y = y^2 - 4x \ ,$$
$$z_{xx} = 54x \ , \quad z_{xy} = z_{yx} = -4 \ , \quad z_{yy} = 2y \ .$$

Now to find the critical points, we need both z_x and z_y to be zero (neither is ever undefined), so we need to solve this set of equations simultaneously:

$$\begin{aligned} z_x &= 27x^2 - 4y = 0 \\ z_y &= y^2 - 4x = 0 \end{aligned}$$

Perhaps it's been a while since you solved systems of equations. One solution method is the substitution method – solve one equation for one variable and substitute into the other equation:

$$\begin{cases} 27x^2 - 4y &= 0 \\ y^2 - 4x &= 0 \end{cases} \to \text{ Solve } y^2 - 4x = 0 \text{ for } x = \frac{y^2}{4} \ ,$$

then substitute into the other equation:

$$\begin{aligned} 27\left(\frac{y^2}{4}\right)^2 - 4y &= 0 \\ \frac{27}{16}y^4 - y &= 0 \end{aligned}$$

Now we have just one equation in one variable to solve. Factoring out a y gives

$$y\left(\frac{27}{16}y^3 - 1\right) = 0 \ ,$$

so $y = 0$ or $\frac{27}{16}y^3 - 1 = 0$, giving $y = \sqrt[3]{\frac{1}{27/16}} = \frac{\sqrt[3]{4}}{3}$.

Plugging back in to the equation $x = \frac{y^2}{4}$ to find x gives us the two critical points: $(0,0)$ and $\left(\frac{4}{9}, \frac{4}{3}\right)$.

Now to test them. First compute

$$\begin{aligned} D(x,y) &= (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}) \\ &= (54x)(2y) - (-4)(-4) \\ &= 108xy - 16 \end{aligned}$$

Then evaluate D at the two critical points:

- At $(0,0)$: $D(0,0) = -16 < 0$, so there is a saddle point at $(0,0)$.
- At $\left(49, \frac{\sqrt[3]{4}}{3}\right)$: $D\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right) = 16\left(\sqrt[3]{4} - 1\right) > 0$, and $f_{xx}\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right) > 0$, so there is a local minimum at the point $\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right)$.

■

4.3.4 Applied Optimization

Example 4.3.5. A company makes two products. The demand equations for the two products are given below. p_1, p_2, q_1, and q_2 are the prices and quantities for Products 1 and 2.

$$\begin{aligned} q_1 &= 200 - 3p_1 - p_2 \\ q_2 &= 150 - p_1 - 2p_2 \end{aligned}$$

Find the price the company should charge for each product in order to maximize total revenue. What is that maximum revenue?

Solution: Revenue is still price×quantity. If we're selling two products, the total revenue will be the sum of the revenues from the two products:

$$\begin{aligned} R(p_1, p_2) &= p_1q_1 + p_2q_2 \\ &= p_1(200 - 3p_1 - p_2) + p_2(150 - p_1 - 2p_2) \\ &= 200p_1 - 3p_1^2 - 2p_1p_2 + 150p_2 - 2p_2^2 \ . \end{aligned}$$

This is a function of two variables, the two prices, and we need to optimize it (just as in the previous examples). First we find critical points. The notation here gets a bit hard to look at, but hang in there – this is the same stuff we've done before.

$$R_{p_1} = 200 - 6p_1 - 2p_2 \quad \text{and} \quad R_{p_2} = 150 - 2p_1 - 4p_2.$$

Solving these simultaneously gives the one critical point $(p_1, p_2) = (25, 25)$. To confirm that this gives maximum revenue, we need to use the Second Derivative Test. Find all the second derivatives:

$$R_{p_1p_2} = -6, R_{p_2p_2} = -4, \quad \text{and} \quad R_{p_1p_2} = R_{p_2p_1} = -2 \ .$$

So $D(25,25) = (-6)(-4) - (-2)(-2) > 0$ and $R_{p_1p_2}(25,25) < 0$, so this really is a local maximum.

Thus, to maximize revenue the company should charge \$25 per unit for both products. This will yield a maximum revenue of \$4375. ■

Chapter 5

The Integral: Accumulation of a Rate Function

The previous few chapters dealt with differential calculus, which revolved around the concept of the rate of change of a function. We started with the simple geometric idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter deals with integral calculus and starts with the simple geometric idea of area and relates it to the accumulation of a rate function. This idea will be developed into another combination of theory, techniques, and applications.

5.1 Approximating Area

5.1.1 PreCalculus Idea – The Area of a Rectangle

If you look on the inside cover of nearly any traditional math book, you will find a bunch of area and volume formulas – the area of a square, the area of a trapezoid, the volume of a right circular cone, and so on. Some of these formulas are pretty complicated. But you still won't find a formula for the area of a jigsaw puzzle piece or the volume of an egg. There are lots of things for which there is no formula. Yet we might still want to find their areas.

One reason areas are so useful is that they can represent quantities other than simple geometric shapes. If the units for each side of the rectangle are meters, then the area will have the units meters×meters = square meters = m^2. But if the units of the base of a rectangle are hours and the units of the height are miles/hour, then the units of the area of the rectangle are hours×miles/hour = miles, a measure of distance. Similarly, if the base units are centimeters and the height units are grams, then the area units are gram-centimeters, a measure of work.

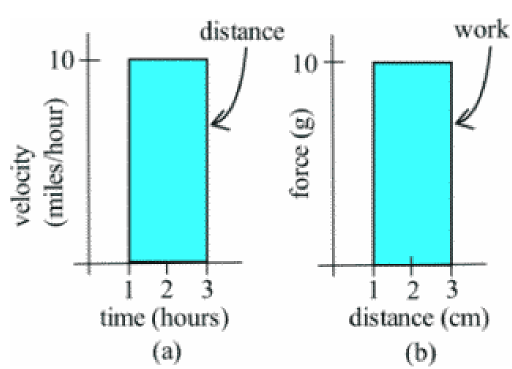


Figure 5.1: Modeling distance traveled and work with rectangles

The basic shape we will use is the rectangle; the area of a rectangle is base×height. You should also know the area formulas for triangles: $A = \frac{1}{2}bh$, and for circles: $A = \pi r^2$.

5.1.2 Distance from Velocity

Example 5.1.1. Suppose a car travels on a straight road at a constant speed of 40 miles per hour for two hours. See the graph of its velocity in Figure 5.2. How far has it gone?

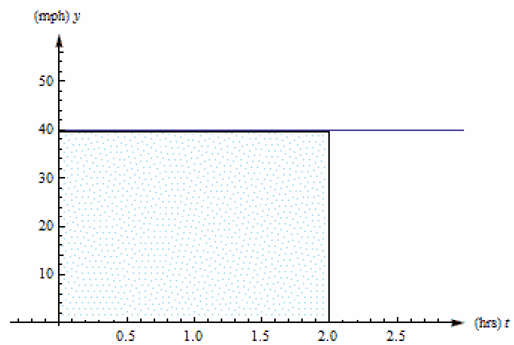


Figure 5.2: Car velocity as a function of time.

Solution: We all remember distance = rate×time, so this one is easy. The car has gone (40 miles/hour)×(2 hours) = 80 miles. ■

Example 5.1.2. Now suppose that a car travels so that its speed increases steadily from 0 to 40 miles per hour, for two hours. (Just be grateful you weren't stuck behind this car on the highway.) See the graph of its velocity in below. How far has this car gone?

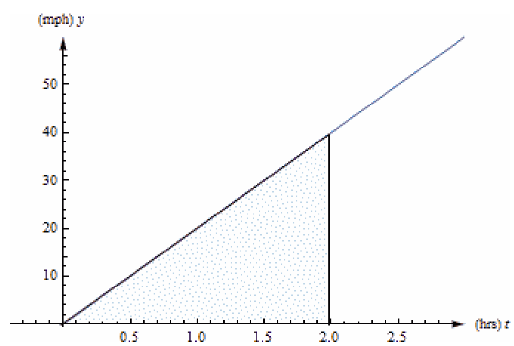


Figure 5.3: Car velocity as a function of time.

Solution: The trouble with our old reliable distance = rate×time relationship is that it only works if the rate is constant. If the rate is changing, there isn't a good way to use this formula.

But look at the graph from the last example again. Notice that distance = rate×time also describes the area between the velocity graph and the t-axis, between $t = 0$ and $t = 2$ hours. The rate is the height of the rectangle, the time is the length of the rectangle, and the distance is the area of the rectangle. This is the way we can extend our simple formula to handle more complicated velocities. And this is the way we can answer the second example.

The distance the car travels is the area between its velocity graph, the t-axis, $t = 0$ and $t = 2$. This region is a triangle, so its area is

$$\frac{1}{2}bh = \frac{1}{2}(2 \text{ hours})(40 \text{ miles per hour}) = 40 \text{ miles} \ .$$

So the car travels 40 miles during its annoying trip. ■

In our distance/velocity examples, the function represented a rate of travel (miles per hour), and the area represented the total distance traveled. This principle works more generally.

For functions representing other rates such as the production of a factory (bicycles per day), the flow of water in a river (gallons per minute), traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.

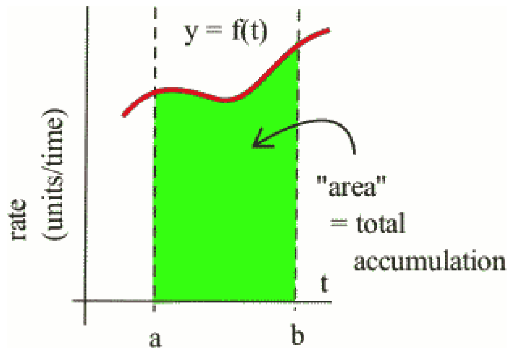


Figure 5.4: Total accumulation

Example 5.1.3. The graph in Figure 5.5 shows the flow rate (cubic feet per second) of water in the Skykomish river at the town of Goldbar in Washington state.

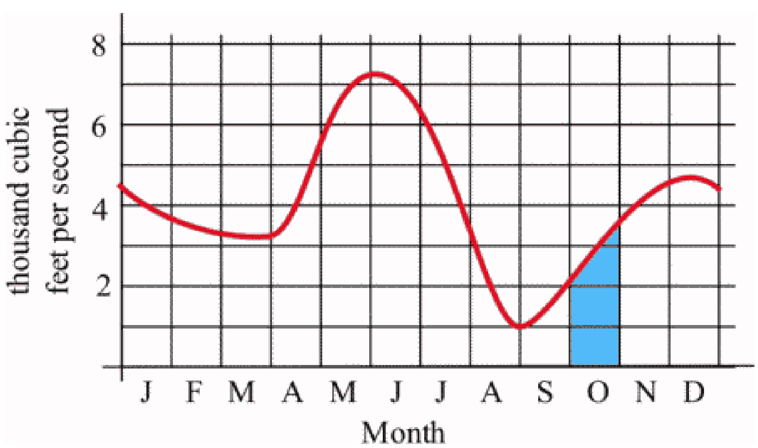


Figure 5.5: Flow rate of the Skykomish River at Goldbar, WA

Solution: The area of the shaded region represents the total volume (cubic feet or ft^3) of water flowing past the town during the month of October. We can approximate this area to approximate the total water by thinking of the shaded region as a rectangle with a triangle on top.

$$\begin{aligned}
\text{Total water} &= \text{total area} \\
&\approx \text{area of rectangle} + \text{area of the ``triangle''} \\
&\approx \left(2000\frac{\mathrm{ft}^3}{\mathrm{s}}\right)(30 \text{ days}) + 12\left(1500\frac{\mathrm{ft}^3}{\mathrm{s}}\right)(30 \text{ days}) \\
&= \left(2570\frac{\mathrm{ft}^3}{\mathrm{s}}\right)(30 \text{ days})
\end{aligned}$$

Note that we need to convert the units to make sense of our result:

$$\begin{aligned} \text{Total water} &\approx \left(2570\frac{\text{ft}^3}{\text{s}}\right)(30 \text{ days}) \\ &= \left(2570\frac{\text{ft}^3}{\text{s}}\right)(2592000 \text{ s}) \\ &\approx 7.128 \cdot 10^9 \text{ ft}^3 \end{aligned}$$

About 7 billion cubic feet of water flowed past Goldbar in October. ■

5.1.3 Approximating with Rectangles

How do we approximate the area if the rate curve is, well, curvy? We could use rectangles and triangles, like we did in the last example. But it turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.

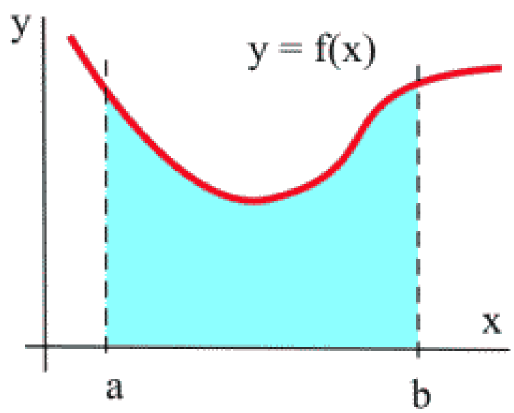


Figure 5.6: Area

Suppose we want to calculate the area between the graph of a positive function $f(x)$ and the x-axis on the interval $[a, b]$ (graphed in Figure 5.6). The **Riemann Sum method** builds several rectangles with bases on the interval $[a, b]$ and sides that reach up to the graph of $f(x)$, as in Figure 5.7. Then the areas of the rectangles can be calculated and added together to get a number called a **Riemann Sum** of f on $[a, b]$. The area of the region formed by the rectangles is an approximation of the area we want.

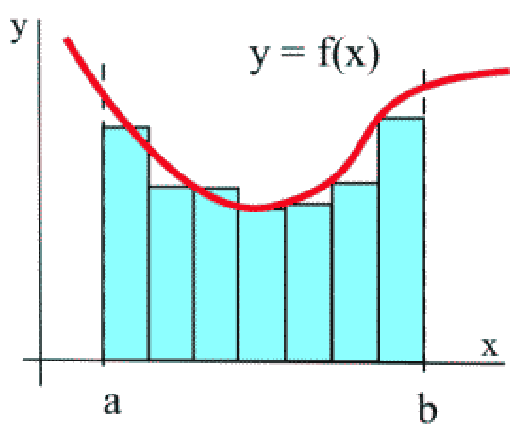


Figure 5.7: Area Approximation with rectangles

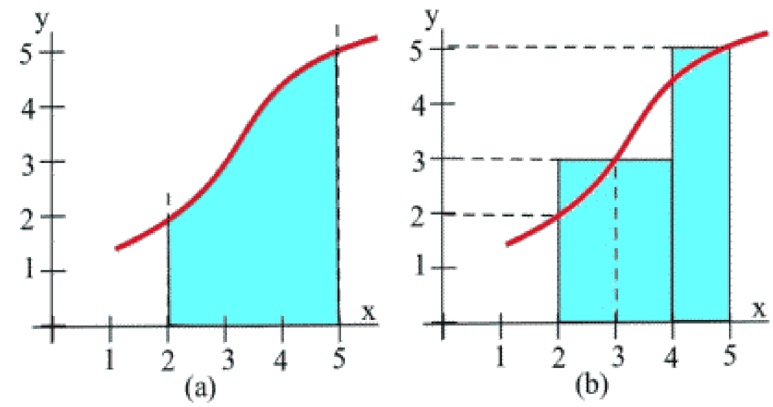


Figure 5.8: Area under $y = f(x)$

Example 5.1.4. Approximate the area between the graph of f and the x-axis on the interval $[2, 5]$ in Figure 5.8 by adding the areas of the rectangles in the graph on the right in Figure 5.8.
Solution: The total area of the rectangles is

$$(2)(3) + (1)(5) = 11 \ .$$

■

Example 5.1.5. Let A be the region bounded by the graph of $f(x) = \frac{1}{x}$, the x-axis, and vertical lines at $x = 1$ and $x = 5$. We can't find the area exactly (with what we know now), but we can approximate it using rectangles.
Solution: When we make our rectangles, we have a lot of choices. We could pick any (non-overlapping) rectangles whose bottoms lie within the interval on the x-axis, and whose tops intersect with the curve somewhere. But it is easiest to choose rectangles that (a) have all the same width and (b) take their heights from the function at one edge. Below are graphs showing two ways to use four rectangles to approximate this area. In the first graph, we used left-endpoints; the height of each rectangle comes from the function value at its left edge. In the second graph on the next page, we used right-hand endpoints.

Left-endpoint approximation: The area is approximately the sum of the areas of the rectangles. Each rectangle gets its height from the function $f(x) = \frac{1}{x}$ and each rectangle has a width of 1.

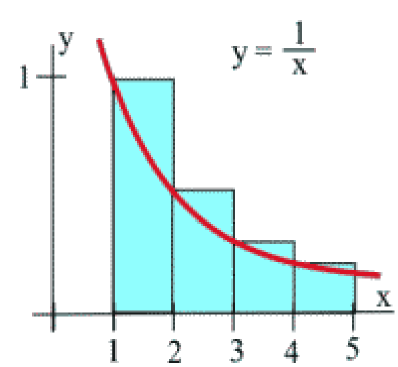


Figure 5.9: Left-endpoint approximation for $y = \frac{1}{x}$.

You can find the area of each rectangle using area = height×width. So the total area of the rectangles, the left-hand estimate of the area under the curve, is

$$f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.08 \ .$$

Notice that because this function is decreasing, all the left endpoint rectangles stick out above the region we want – using left-hand endpoints will overestimate the area.

Right-endpoint approximation: The right-hand estimate of the area is

$$f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{77}{60} \approx 1.28 \ .$$

All the right-hand rectangles lie completely under the curve, so this estimate will be an underestimate.

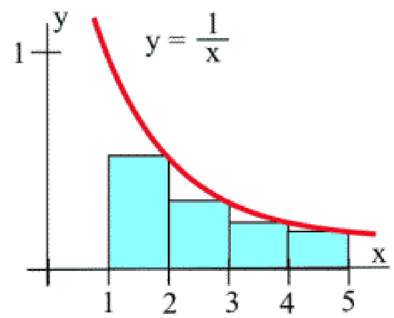


Figure 5.10: Right-endpoint approximation for $y = \frac{1}{x}$.

We can see that the true area is actually in between these two estimates. So we could take their average:

$$\text{Average} = \frac{\frac{25}{12}+\frac{77}{60}}{2} = \frac{101}{60} \approx 1.68 \ .$$

In general, the average of the left-hand and right-hand estimates will be closer to the real area than either individual estimate.

Our estimate of the area under the curve is about 1.68. (The actual area is about 1.61.)

■

If we wanted a more accurate solution, we could use even more and even narrower rectangles. But there is a limit to how much work we want to do by hand. In practice, choose a manageable number of rectangles. We will have better methods to get more accurate answers before long.

These sums of areas of rectangles are called **Riemann sums**. You may see a shorthand notation used when people talk about sums. We won't use it much in this book, but you should know what it means.

Definition 5.1.1 (Riemann Sum)**.** A Riemann sum of a function $f(x)$ over an interval $[a,b]$ is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval $[a,b]$ into n subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width Δx. The height of each rectangle comes from the function evaluated at some point in its sub interval. Then the Riemann sum is

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x$$

or, factoring out the Δx,

$$\Delta x(f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)) \ .$$

Definition 5.1.2 (Sigma Notation)**.** The upper-case Greek letter Sigma Σ is used to stand for sum. **Sigma notation** is a way to compactly represent a long, infinite, or arbitrarily long sum of many similar terms, such as a Riemann sum.

Using the Sigma notation, the Riemann sum can be written

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x \ .$$

This is read aloud as "the sum as i goes from 1 to n of f of x sub i times Δx." The i is a **counter** or **index**, like you might have seen in a programming class. The index increases by 1 each term and must always be an integer.

5.2 Integrals

5.2.1 Definition of the Definite Integral

Because the area under the curve is so important, it has a special vocabulary and notation.

Definition 5.2.1 (The Definite Integral)**.** The **definite integral** of a positive function $f(x)$ over an interval $[a, b]$ is the area between $y = f(x)$, the x-axis, $x = a$, and $x = b$.

The definite integral of a positive function $f(x)$ from a to b is the area under the curve between a and b.

If $f(t)$ represents a positive rate (in y-units per t-units), then the definite integral of $f(t)$ from a to b is the total y-units that **accumulate** between $t = a$ and $t = b$.

Notation for the Definite Integral. The definite integral of $f(x)$ from a to b is written

$$\int_a^b f(x)\, dx \quad .$$

The $\int$ symbol is called the **integral sign**; it is an elongated letter S, standing for sum. (The $\int$ corresponds to the Σ from the Riemann sum).

The dx on the end must be included! The dx tells what the variable is – in this example, the variable is x. (The dx corresponds to the Δx from the Riemann sum).

The function f is called the **integrand**.

The a and b are called the **limits of integration**.

Verb forms. We **integrate**, or find the definite integral of a function. This process is called **integration**.

Formal Algebraic Definition

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad .$$

Practical Definition The definite integral can be approximated with a Riemann sum (dividing the area into rectangles where the height of each rectangle comes from the function, computing the area of each rectangle, and adding them up). The more rectangles we use, the narrower the rectangles are, the better our approximation will be.

Looking Ahead We will have methods for computing exact values of some definite integrals from formulas soon. In many cases, including when the function is given to you as a table or graph, you will still need to approximate the definite integral with rectangles.

Example 5.2.1. Figure 5.11 shows $y = r(t)$, the number of telephone calls made per hour on a Tuesday. Approximately how many calls were made between 9 pm and 11 pm? Express this as a definite integral and approximate with a Riemann sum.

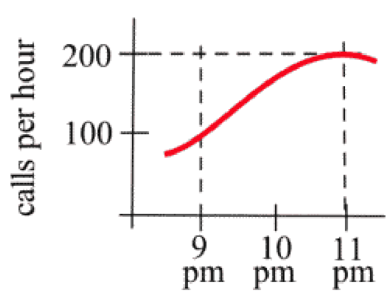


Figure 5.11: Calls per hour between 9 pm and 11 pm.

Solution: We know that the accumulated calls will be the area under this rate graph over that two-hour period, the definite integral of this rate from $t = 9$ to $t = 11$.

The total number of calls will be $\int_9^{11} r(t)\,dt$.

The top of the area in Figure 5.12 is a curve, so we can't get an exact answer. But we can approximate the area using rectangles. Let's choose to use four rectangles and left-endpoints.

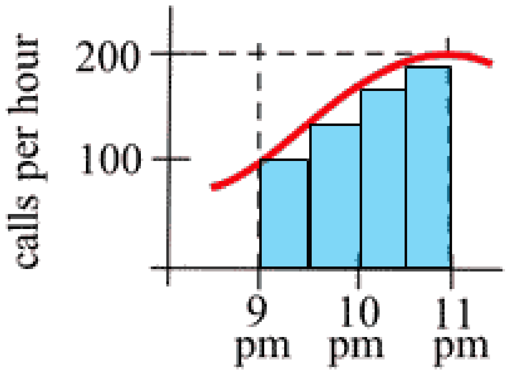


Figure 5.12: Approximating the number of calls between 9 pm and 11 pm.

$$\int_9^{11} r(t)\,dt \approx 0.5(100+150+180+195) = 312.5 \ .$$

The units are (calls per hour)×hours = calls. Our estimate is that about 312 calls were made between 9 pm and 11 pm. Is this an under-estimate or an over-estimate? ■

Example 5.2.2. Describe the area between the graph of $f(x) = \dfrac{1}{x}$, the x-axis, and the vertical lines at $x = 1$ and $x = 5$ as a definite integral.

Solution: This is the same area we estimated to be about 1.68 before. Now we can use the notation of the definite integral to describe it. Our estimate of $\int_1^5 \frac{1}{x}\,dx$ was 1.68. The true value of $\int_1^5 \frac{1}{x}\,dx$ is about 1.61. ■

Example 5.2.3. Using the idea of area, determine the value of $\int_1^3 (1+x)\,dx$.

Solution: $\int_1^3 (1+x)\,dx$ represents the area between the graph of $f(x) = 1+x$, the x-axis, and the vertical lines at 1 and 3.

Since this area can be broken into a rectangle and a triangle, we can find the area exactly. The area equals $4 + \frac{1}{2}(2)(2) = 6$. ■

Example 5.2.4. Table 5.1 shows rates of population growth for Berrytown for several years. Use this table to estimate the total population growth from 1970 to 2000:

Year, t	1970	1980	1990	2000
Rate of population growth $R(t)$ in thousands of people per year	1.5	1.9	2.2	2.4

Table 5.1: Population growth of Berrytown from 1970 to 2000.

Solution: The definite integral of this rate will give the total change in population over the thirty-year period. We only have a few pieces of information, so we can only estimate. Even though we haven't made a graph, we're still approximating the area under the rate curve, using rectangles. How wide are the rectangles? We have information every 10 years, so the rectangles have a width of 10 years. How many rectangles? Be careful here – this is a thirty-year span, so there are three rectangles.

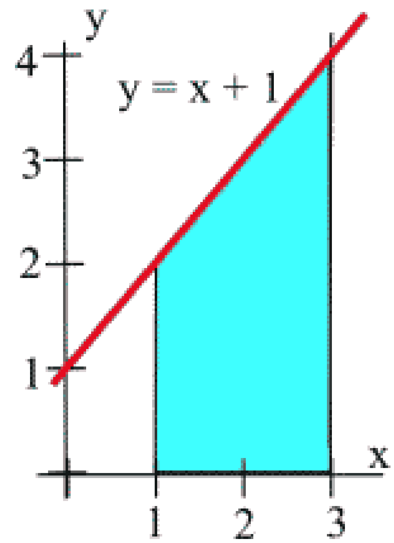


Figure 5.13: Calls per hour between 9 pm and 11 pm.

- Using left-hand endpoints: $(1.5)(10) + (1.9)(10) + (2.2)(10) = 56$.
- Using right-hand endpoints: $(1.9)(10) + (2.2)(10) + (2.4)(10) = 65$.

Taking the average of these two:

$$\frac{56 + 65}{2} = 60.5 \ .$$

Our best estimate of the total population growth from 1970 to 2000 is 60.5 thousand people. ■

5.2.2 Signed Area

You may have noticed that until this point, we've insisted that the integrand (the function we're integrating) be positive. That's because we've been talking about area, which is always positive.

If the height (from the function) is a negative number, then multiplying it by the width doesn't give us actual area, it gives us the area with a negative sign.

But it turns out to be useful to think about the possibility of negative area. We'll expand our idea of a definite integral now to include integrands that might not always be positive. The heights of the rectangles, the values from the function, now might not always be positive.

Definition 5.2.2 (The Definite Integral and Signed Area)**.** The **definite integral** of a function $f(x)$ over an interval $[a, b]$ is the **signed area** between the curve $y = f(x)$, the x-axis, $x = a$ and $x = b$.

The **definite integral** of a function $f(x)$ from a to b is the **signed area** under the curve between a and b.

If the function is positive, the signed area is positive, as before (and we can call it area.)

If the function dips below the x-axis, the areas of the regions below the x-axis here will be negative. In this case, we cannot call it simply "area." These negative areas take away from the definite integral.

$$\int_a^b f(x)\,dx = (\text{Area above } x\text{-axis}) - (\text{Area below } x\text{-axis})$$

If $f(t)$ represents a positive rate (in y-units per t-units), then the definite integral of $f(t)$ from a to b is the total y-units that accumulate between $t = a$ and $t = b$.

If $f(t)$ represents any rate (in y-units per t-units), then the definite integral of $f(t)$ from a to b is the net y-units that accumulate between $t = a$ and $t = b$.

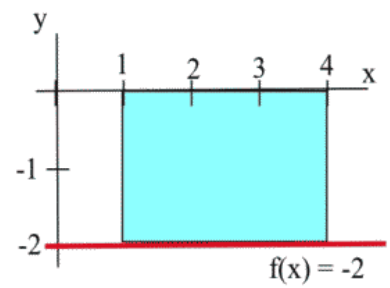


Figure 5.14: A region with negative signed area.

Example 5.2.5. Find the definite integral of $f(x) = -2$ over the interval $[1, 4]$.

Solution: $\int_1^4 -2\,dx$ is the signed area of the region shown in Figure 5.14. The region lies below the x-axis, so the area, 6, comes in with a negative sign. So the definite integral is $\int_1^4 -2\,dx = -6$. ■

Negative rates indicate that the amount is decreasing. For example, if $f(t)$ is the velocity of a car in the positive direction along a straight line at time t (miles/hour), then negative values of f indicate that the car is traveling in the negative direction, backwards. The definite integral of f is the net change in position of the car during the time interval. If the velocity is positive, positive distance accumulates. If the velocity is negative, distance in the negative direction accumulates.

This is true of any rate. For example, if $f(t)$ is the rate of population change (in people/year) for a town, then negative values of f would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would be the change in the population, a decrease, during the time interval.

Example 5.2.6. In 1980 there were 12,000 ducks nesting around a lake, and the rate of population change (in ducks per year) is shown in Figure 5.15. Write a definite integral to represent the total change in the duck population from 1980 to 1990, and estimate the population in 1990.

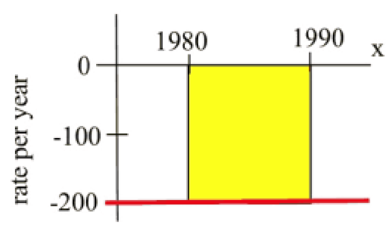


Figure 5.15: Rate of change in duck population from 1980 to 1990.

Solution: The change in population is:

$$\begin{aligned}\int_{1980}^{1990} f(t)\,dt &= -(\text{area between } f \text{ and axis})\\ &\approx -(200\text{ducks/year}) \cdot (10 \text{ years})\\ &= -2000 \text{ ducks.}\end{aligned}$$

Then (1990 duck population) = (1980 population) + (change from 1980 to 1990) = (12000)+(-2000) = 10000 ducks. ■

Example 5.2.7. A bug starts at the location $x = 12$ on the x-axis at 1 pm walks along the axis with the velocity $v(x)$ shown in the graph in Figure 5.16. How far does the bug travel between 1 pm and 3 pm, and where is the bug at 3 pm?

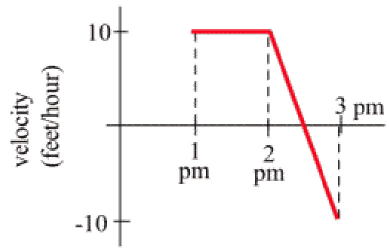


Figure 5.16: Velocity of a bug from 1 pm to 3 pm.

Solution: Note that the velocity is positive from 1 until 2:30, then becomes negative. So the bug moves in the positive direction from 1 until 2:30, then turns around and moves back toward where it started. The area under the velocity curve from 1 to 2:30 shows the total distance traveled by the bug in the positive direction; the bug moved 12.5 feet in the positive direction. The area between the velocity curve and the x-axis, between 2:30 and 3, shows the total distance traveled by the bug in the negative direction, back toward home; the bug traveled 2.5 feet in the negative direction. The definite integral of the velocity curve, $\int_1^3 v(t)\,dt$, shows the net change in distance:

$$\int_1^3 v(t)\,dt = 12.5 - 2.5 = 10 \ .$$

The bug ended up 10 feet farther in the positive direction than he started. At 3 pm, the bug is at $x = 22$. ■

Example 5.2.8. Use the graph in Figure 5.17 to calculate $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$, and $\int_0^5 f(x)\,dx$.

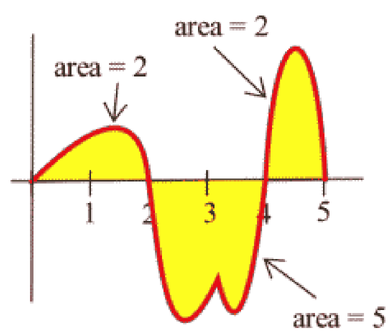


Figure 5.17: Graph for Example 5.2.8.

Solution: Using the given areas, $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = -5$, $\int_4^5 f(x)\,dx = 2$, and $\int_0^5 f(x)\,dx =$ (area above) $-$ (area below) $= (2+2) - (5) = -1$. ■

5.3 The Fundamental Theorem of Calculus

This section contains the most important and most used theorem of calculus, the Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz in the late 1600s, it establishes the connection between derivatives and integrals, provides a way of easily calculating many integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure.

Theorem 5.3.1 (The Fundamental Theorem of Calculus)**.** Let $f(x)$ be a function whose derivative is continuous on an interval $[a, b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a) \ .$$

This is actually not new for us; we've been using this relationship for some time; we just haven't written it this way. This says what we've said before: the definite integral of a rate from a to b is the net y-units, the change in y, that accumulate between $x = a$ and $x = b$. Here we've just made it plain that that the rate is a derivative.

Thinking about the relationship this way gives us the key to finding exact answers for some definite integrals. If the integrand is the derivative of some function f, then maybe we could simply find f and subtract. That would be easier than approximating with rectangles. Going backwards through the differentiation process will help us evaluate definite integrals.

Example 5.3.1. Find $f(x)$ if $f'(x) = 2x$.
Solution: Oooh, I know this one. It's $f(x) = x^2 + 3$. Oh, wait, you were thinking something else? Yes, I guess you're right – $f(x) = x^2$ works too. So does $f(x) = x^2 - \pi$, and $f(x) = x^2 + 104589.2$. In fact, there are lots of answers. ■

In fact, there are infinitely many functions that all have the same derivative. And that makes sense – the derivative tells us about the shape of the function, but it doesn't tell about the location. We could shift the graph up or down and the shape wouldn't be affected, so the derivative would be the same.

This leads to one of the trickier definitions – pay careful attention to the articles (the versus an), because they're important.

Definition 5.3.1 (Antiderivatives)**.** An **antiderivative** of a function $f(x)$ is any function $F(x)$ where $F'(x) = f(x)$.

The antiderivative of a function $f(x)$ is a whole family of functions, written $F(x)+C$, where $F'(x) = f(x)$ and C represents any constant (i.e., a real number).

The antiderivative of $f(x)$ is also called the **indefinite integral** of $f(x)$.

Notation for the Antiderivative The antiderivative of f is written

$$\int f(x)\,dx$$

This notation resembles the definite integral, because the Fundamental Theorem of Calculus says antiderivatives and definite integrals are intimately related. But in this notation, there are no limits of integration.

The $\int$ symbol is still called an **integral sign**; the dx on the end still must be included; you can still think of $\int$ and dx as left and right parentheses. The function f is still called the **integrand**.

Verb Forms We **antidifferentiate**, or **integrate**, or **find the indefinite integral** of a function. This process is called **antidifferentiation** or **integration**.

There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an infinite number of antiderivatives and every one of them has the form $F(x) + C$.

Example 5.3.2. Find an antiderivative of $2x$.
Solution: We can choose any function we like as long as its derivative is $2x$, so we can pick, say, $F(x) = x^2 - 5.2$. ■

Example 5.3.3. Find the antiderivative of $2x$.
Solution: Now we need to write the entire family of functions whose derivatives are $2x$. We can use the $\int$ notation:

$$\int 2x\,dx = x^2 + C \ .$$

(Don't forget the $+C$!) ■

Example 5.3.4. Find $\displaystyle\int e^x \, dx$.

Solution: This is likely one you remember: e^x is its own derivative, so it is also its own antiderivative. The integral sign tells us that we need to include the entire family of functions, so we need that $+C$ on the end:

$$\int e^x \, dx = e^x + C \ .$$

■

5.3.1 Antiderivatives Graphically or Numerically

Another way to think about the Fundamental Theorem of Calculus is to solve the expression for $F(b)$:

Theorem 5.3.2 (The Fundamental Theorem of Calculus (restated))**.**

$$\int_a^b F'(x) \, dx = F(b) - F(a) \ .$$

The definite integral of a derivative from a to b gives the net change in the original function.

$$F(b) = F(a) + \int_a^b F'(x) \, dx$$

The amount we end up is the amount we start with plus the net change in the function.

This lets us get values for the antiderivative – as long as we have a starting point, and we know something about the area.

Example 5.3.5. Suppose $F(t)$ has the derivative $f(t)$ shown in Figure 5.18, and suppose that we know $F(0) = 5$. Find values for $F(1)$, $F(2)$, $F(3)$, and $F(4)$.

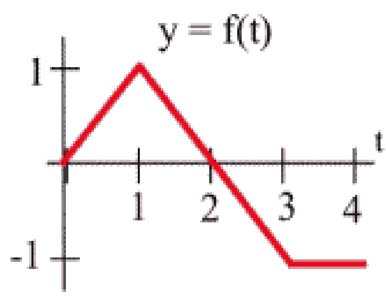


Figure 5.18: Graph for Example 5.3.5.

Solution: Using the second way to think about the Fundamental Theorem of Calculus, $F(b) = F(a) + \displaystyle\int_a^b F'(x) \, dx$, we can see that $F(1) = F(0) + \int_0^1 f(x) \, dx$. We know the value of $F(0)$, and we can easily find $\displaystyle\int_0^1 f(x) \, dx$ from the graph – it's just the area of a triangle. So

$$F(1) = 5 + 0.5 = 5.5$$

$$F(2) = F(0) + \int_0^2 f(x) \, dx$$
$$= 5 + 1 = 6$$

Note that we can start from any place of which we know the value; for example, now that we know $F(2)$, we can use that to find

$$\begin{aligned}
F(3) &= F(2) + \int_2^3 f(x)\,dx \\
&= 6 - 0.5 = 5.5 \\
F(4) &= F(3) + \int_3^4 f(x)\,dx \\
&= 5.5 - 1 = 4.5
\end{aligned}$$

■

Example 5.3.6. The graph of $F'(t) = f(t)$ is shown in Figure 5.19 below. Where does $F(t)$ have maximum and minimum values on the interval $[0, 4]$?

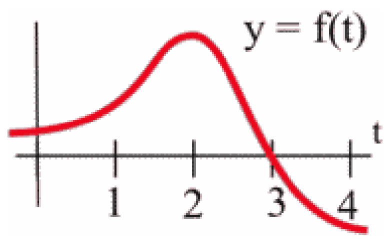


Figure 5.19: Graph for Example 5.3.6.

Solution: Since $F(b) = F(a) + \int_a^b F'(x)\,dx$, we know that F is increasing as long as the area accumulating under $y = F'(t) = f(t)$ is positive (until $t = 3$), and then decreases when the curve dips below the t-axis so that negative area starts accumulating. The area between $t = 3$ and $t = 4$ is much smaller than the positive area that accumulates between 0 and 3, so we know that $F(4)$ must be larger than $F(0)$. The maximum value is when $t = 3$; the minimum value is when $t = 0$. ■

Note that this is a different way to look at a problem we already knew how to solve – in Chapter 2, we would have found critical points of F, where $f = 0$: there's only one, when $t = 3$. $f(t) = F'(t)$ goes from positive to negative there, so F has a local max at that point. It's the only critical point, so it must be a global maximum. Then we would look at the values of F at the endpoints to find which was the global minimum.

We can also attempt to sketch a function based on the graph of the derivative.

Example 5.3.7. The graph in Figure 5.20 shows $f'(x)$ – the rate of change of $f(x)$. Use it to sketch a graph of $f(x)$ that satisfies $f(0) = 0$.

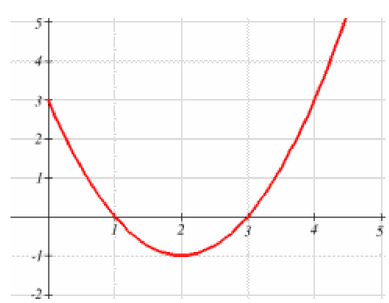


Figure 5.20: Graph for Example 5.3.7.

Solution: Recall from Section 3.5 the relationships between the function graph and the derivative graph:

$f(x)$	Increasing	Decreasing	Concave Up	Concave Down
$f'(x)$	$+$	$-$	Increasing	Decreasing

Table 5.2: The relationship between a function and its derivative.

In the graph shown in Figure 5.20, we can see the derivative is positive on the interval $(0, 1)$ and $(3, \infty)$, so the graph of f should be increasing on those intervals. Likewise, f should be decreasing on the interval $(1, 3)$.

In the graph, f' is decreasing on the interval $(0, 2)$, so f should be concave down on that interval. Likewise, f should be concave up on the interval $(2, \infty)$.

The derivative itself is not enough information to know where the function f starts, since there are a family of antiderivatives, but in this case we are given a specific point at which to start.

To start the sketch, we might note first the shapes we need:

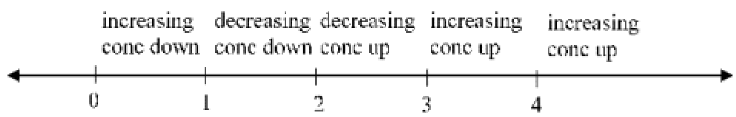


Figure 5.21: Geometric characteristics of $y = f(x)$.

then sketch the basic shapes:

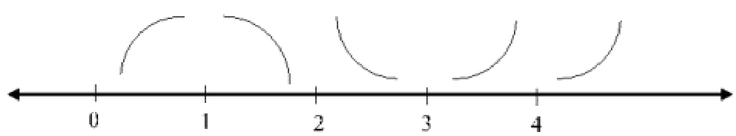


Figure 5.22: Basic shapes of $y = f(x)$.

Now we can attempt to sketch the graph, starting at the point $(0, 0)$. Notice we are very roughly sketching this, as we don't have much information to work with. We can tell, though, from the graph that the area from $x = 0$ to $x = 1$ is about the same as the area from $x = 1$ to $x = 3$, so we would expect the net area from $x = 0$ to $x = 3$ to be close to 0.

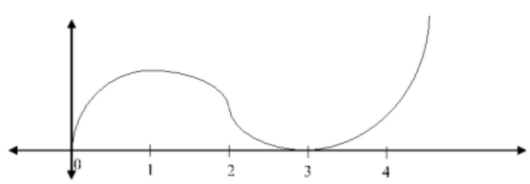


Figure 5.23: Basic sketch of $y = f(x)$.

It turns out this graph isn't horribly bad. Smoothing it out would give a graph closer to the actual antiderivative graph, shown in Figure 5.24.

■

5.3.2 Derivative of the Integral

There is another important connection between the integral and derivative.

Theorem 5.3.3 (The Fundamental Theorem of Calculus (Part 2))**.** If

$$A(x) = \int_a^x f(t)\,dt$$

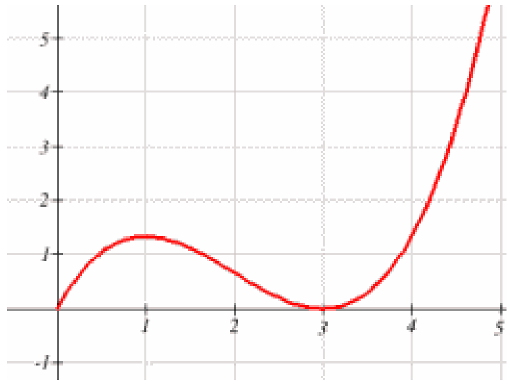


Figure 5.24: Actual sketch of $y = f(x)$.

then

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x) \ .$$

I.e., the derivative of the accumulation function is the original function.

Example 5.3.8. Let $F(x) = \displaystyle\int_0^x f(t)\,dt$, where $f(x)$ is graphed in Figure 5.25. Estimate $F'(3)$.

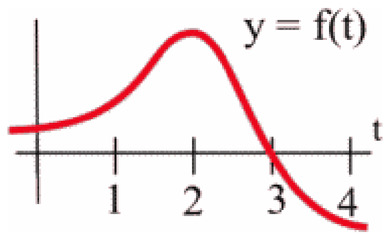


Figure 5.25: Graph for Example 5.3.8.

Solution: The function $F(x)$ measures the area from $t = 0$ to some $t = x$. To estimate $F'(3)$, we want to estimate how much the area is increasing when $t = 3$. Since the value of the function $f(t)$ is 0 at $t = 3$, the area will not be increasing or decreasing, so we can estimate $F'(3) = 0$.

Directly using the Fundamental Theorem of Calculus part 2,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

so

$$F'(x) = f(3) = 0 \ .$$

■

5.4 Integral Formulas

Now we can put the ideas of areas and antiderivatives together to get a way of evaluating definite integrals that is exact and often easy. To evaluate a definite integral $\displaystyle\int_a^b f(t)\,dt$, we can find any antiderivative $F(t)$ of $f(t)$ and evaluate $F(b) - F(a)$. The problem of finding the exact value of a definite integral reduces to finding some (any) antiderivative $f(x)$ of the integrand and then evaluating $F(b) - F(a)$. Even finding one antiderivative can be difficult, and we will stick to functions that have easy antiderivatives.

5.4.1 Building Blocks

Antidifferentiation is going backwards through the derivative process. So the easiest antiderivative rules are simply backwards versions of the easiest derivative rules. Recall the following from Chapter 2.

Theorem 5.4.1 (Derivative Rules: Building Blocks)**.** In what follows, $f(x)$ and $g(x)$ are differentiable functions of x.
Constant Multiple Rule

$$\frac{d}{dx}kf = kf'$$

Sum and Difference Rule

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

Special cases:

$$\frac{d}{dx}k = 0 \quad \text{(Because } k = kx^0.\text{)}$$

$$\frac{d}{dx}x = 1 \quad \text{(Because } x = x^1.\text{)}$$

Exponential Functions

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = \ln(a)a^x$$

Natural Logarithm

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Thinking about these basic rules was how we came up with the antiderivatives of $2x$ and e^x before.

The corresponding rules for antiderivatives are next – each of the antiderivative rules is simply rewriting the derivative rule. All of these antiderivatives can be verified by differentiating.

There is one surprise – the antiderivative of $\frac{1}{x}$ is actually not simply $\ln(x)$, it's $\ln(|x|)$. This is a good thing – the antiderivative has a domain that matches the domain of $\frac{1}{x}$, which is bigger than the domain of $\ln(x)$, so we don't have to worry about whether our x's are positive or negative. But we must be careful to include those absolute values – otherwise, we could end up with domain problems.

Theorem 5.4.2 (Antiderivative Rules: Building Blocks)**.** In what follows, $f(x)$ and $g(x)$ are differentiable functions of x, and k, n, and C are constants.
Constant Multiple Rule

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$$

Sum and Difference Rule

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{provided that } n \neq -1$$

Special case:

$$\int k\,dx = kx + C \quad \text{(Because } k = kx^0.\text{)}$$

(The other special case ($n = -1$) is covered next.)

Natural Logarithm

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(|x|) + C$$

Exponential Functions

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$$

Example 5.4.1. Find the antiderivative of $y = 3x^7 - 15\sqrt{x} + \dfrac{14}{x^2}$.

Solution:

$$\begin{aligned}\int \left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right) dx &= \int \left(3x^7 - 15x^{1/2} + 14x^{-2}\right) dx \\ &= 3\frac{x^8}{8} - 15\frac{x^{3/2}}{3/2} + 14\frac{x^{-1}}{-1} + C \\ &= \frac{3}{8}x^8 - 10x^{3/2} - 14x^{-1} + C\end{aligned}$$

■

Example 5.4.2. Find $\displaystyle\int \left(e^x + 12 - \frac{16}{x}\right) dx$.

Solution:

$$\int \left(e^x + 12 - \frac{16}{x}\right) dx = e^x + 12x - 16\ln(|x|) + C$$

■

Example 5.4.3. Find $F(x)$ so that $F'(x) = e^x$ and $F(0) = 10$.

Solution: This time we are looking for a particular antiderivative; we need to find exactly the right constant. Let's start by finding the antiderivative:

$$\int e^x\,dx = e^x + C \ .$$

So we know that $F(x) = e^x + C$, for some real number C. Now we just need to find which one. To do that, we'll use the other piece of information (the initial condition):

$$\begin{aligned}F(x) &= e^x + C \\ F(0) &= e^0 + C = 1 + C = 10 \\ C &= 9\end{aligned}$$

The particular constant we need is 9; thus, $F(x) = e^x + 9$. ■

The reason we are looking at antiderivatives right now is so we can evaluate definite integrals exactly. Recall the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \ .$$

If we can find an antiderivative for the integrand, we can use that to evaluate the definite integral. The evaluation $F(b) - F(a)$ is represented as $F(x)|_a^b$.

Example 5.4.4. Evaluate $\displaystyle\int_1^3 x\,dx$ in two ways.

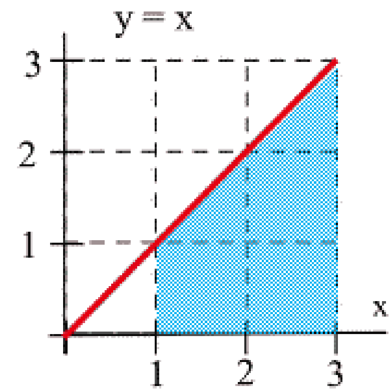


Figure 5.26: Graph for Example 5.4.4.

(a) By sketching the graph of $y = x$ and geometrically finding the area.

Solution: The graph of $y = x$ is shown in Figure 5.26, and the shaded region corresponding to the integral has area 4.

■

(b) By finding an antiderivative of $F(x)$ of the integrand and evaluating $F(3) - F(1)$.

Solution: One antiderivative of x is $F(x) = \frac{1}{2}x^2$, and

$$\begin{aligned}\int_1^3 x\,dx &= \frac{1}{2}x^2\Big|_1^3 \\ &= \left(\frac{1}{2}(3)^2\right) - \left(\frac{1}{2}(1)^2\right) \\ &= \frac{9}{2} - \frac{1}{2} \\ &= 4.\end{aligned}$$

Note that this answer agrees with the answer we got geometrically.

If we had used another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$, then

$$\begin{aligned}\int_1^3 x\,dx &= \frac{1}{2}x^2 + 7\Big|_1^3 \\ &= \left(\frac{1}{2}(3)^2 + 7\right) - \left(\frac{1}{2}(1)^2 + 7\right) \\ &= \frac{9}{2} + 7 - \frac{1}{2} - 7 \\ &= 4.\end{aligned}$$

In general, whatever constant we choose gets subtracted away during the evaluation, so we might as well always choose the easiest one, where the constant is 0. ■

Example 5.4.5. Find the area between the graph of $y = 3x^2$ and the horizontal axis for x between 1 and 2.
Solution: This is

$$\int_1^2 3x^2\,dx = x^3\big|_1^2 = 2^3 - 1^3 = 7 \ .$$

■

Example 5.4.6. A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet/second.

(a) How far will the robot travel during its first 4 seconds of movement?

Solution: The distance during the first 4 seconds will be the area under the graph of velocity, from $t = 0$ to $t = 4$.

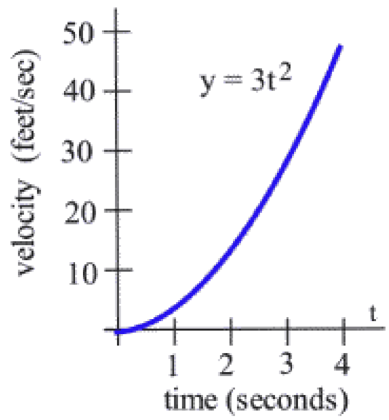


Figure 5.27: Robot velocity after t seconds

That area is the definite integral $\int_0^4 3t^2\,dt$. An antiderivative of $3t^2$ is t^3, so $\int_0^4 3t^2\,dt = t^3\Big|_0^4 = 4^3 - 0^3 = 64$ feet. ■

(b) How far will the robot travel during its next 4 seconds of movement?

Solution: $\int_4^8 3t^2\,dt = t^3\Big|_4^8 = 8^3 - 4^3 = 512 - 64 = 448$ feet. ■

Example 5.4.7. Suppose that t minutes after putting 1000 bacteria on a Petri plate the rate of growth of the population is $6t$ bacteria per minute.

(a) How many new bacteria are added to the population during the first 7 minutes?

Solution: The number of new bacteria is the area under the rate of growth graph in Figure 5.28, and one antiderivative of $6t$ is $3t^2$.

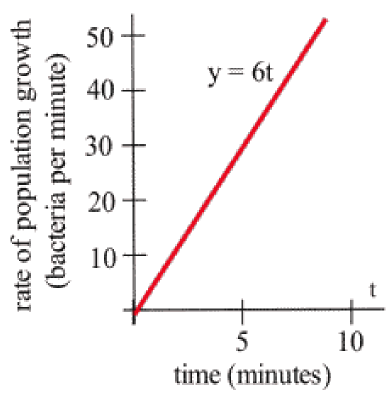


Figure 5.28: Bacteria growth rate after t minutes.

So the number of new bacteria is

$$\int_0^7 6t\,dt = 3t^2\Big|_0^7 = 3(7)^2 - 3(0)^2 = 147\ .$$

■

(b) What is the total population after 7 minutes?

Solution: Therefore, the new population is the old population plus the new bacteria: $1000 + 147 = 1147$ bacteria. ■

Example 5.4.8. A company determines their marginal cost for production, in dollars per item, is $MC(x) = \frac{4}{\sqrt{x}} + 2$ when producing x thousand items. Find the cost of increasing production from 4 thousand items to 5 thousand items.

Solution: Remember that marginal cost is the rate of change of cost, and so the fundamental theorem tells us that

$$\int_a^b MC(x)\,dx = \int_a^b C'(x)\,dx = C(b) - C(a) \ .$$

In other words, the integral of marginal cost will give us a net change in cost. To find the cost of increasing production from 4 thousand items to 5 thousand items, we need to integrate $\int_4^5 MC(x)\,dx$.

We can write the marginal cost as $MC(x) = 4x^{-1/2} + 2$. We can then use the basic rules to find an antiderivative:

$$C(x) = 4\frac{x^{1/2}}{1/2} + 2x = 8\sqrt{x} + 2x \ .$$

Using this, the net change in cost is

$$\begin{aligned} \int_4^5 \left(4x^{-1/2} + 2\right)\,dx &= \left(8\sqrt{x} + 2x\right)\Big|_4^5 \\ &= (8\sqrt{5} - 2(5)) - (8\sqrt{4} + 2(4)) \\ &\approx 3.889 \end{aligned}$$

It will cost about 3.889 thousand dollars to increase production from 4 thousand items to 5 thousand items. (The final answer would be better written as \$3889.) ■

5.5 Integration by Substitution

We don't have many integration rules. For quite a few of the problems we see, the rules won't directly apply; we'll have to do some algebraic manipulation first. In practice, it is much harder to write down the antiderivative of a function than it is to find a derivative. (In fact, it's very easy to write a function that doesn't have any antiderivative you can find with algebra, although proving that it doesn't have an antiderivative is much more difficult.)

The **Substitution Method** (also called u-**Substitution**) is one way of algebraically manipulating an integrand so that the rules apply. This is a way to unwind or undo the Chain Rule for derivatives. When you find the derivative of a function using the Chain Rule, you end up with a product of something like the original function times a derivative. We can reverse this to write an integral:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

so

$$f(g(x)) = \int f'(g(x))g'(x)dx$$

With substitution, we will substitute $u = g(x)$ (hence the name u-substitution). This means $\frac{du}{dx} = g'(x)$, so $du = g'(x)dx$. Making these substitutions, $\int f'(g(x))g'(x)\,dx$ becomes $\int f'(u)\,du$, which will probably be easier to integrate.

Try u-substitution when you see a product in your integral, especially if you recognize one factor as the derivative of some part of the other factor.

5.5.1 The u-Substitution Method for Antiderivatives

The goal is to turn $\int f(g(x))\,dx$ into $\int f(u)\,du$, where $f(u)$ is much less messy than $f(g(x))$.

1. Let u be some part of the integrand. A good first choice is one step inside the messiest bit.
2. Compute $du = \dfrac{du}{dx}dx$.
3. Translate all your x's into u's everywhere in the integral, including the dx. When you're done, you should have a new integral that is entirely in terms of u. If you have any x's left, then that's an indication that the substitution didn't work or isn't complete; you may need to go back to step 1 and try a different choice for u.
4. Integrate the new u-integral, if possible. If you still can't integrate it, go back to step 1 and try a different choice for u.
5. Finally, substitute back x's for u's everywhere in your answer.

Example 5.5.1. Evaluate $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

Solution: This integrand is more complicated than anything in our list of basic integral formulas, so we'll have to try something else. The only tool we have is substitution, so let's try that!

1. Let u be some part of the integrand. A good first choice is one step inside the messiest bit. Let $u = 4 - x^2$.
2. Compute $du = \dfrac{du}{dx}dx$: $du = -2xdx$. There is xdx in the integrand, so that's a good sign; that will be $-12du$.
3. Translate all your x's into u's everywhere in the integral, including the dx:

$$\begin{aligned}\int \frac{x}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{4-x^2}}(x\,dx)\\ &= \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right)\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\frac{1}{2}\int u^{-1/2}\,du\ .\end{aligned}$$

   Alternatively, we could have solved for dx and substituted that and simplified: $dx = \dfrac{du}{-2x}$, so

$$\begin{aligned}\int \frac{x}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{u}}\left(\frac{du}{-2x}\right)\\ &= \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right)\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\frac{1}{2}\int u^{-1/2}\,du\ .\end{aligned}$$

4. Integrate the new u-integral, if possible:

$$-\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\frac{u^{1/2}}{1/2} + C = -u^{1/2} + C\ .$$

5. Finally, substitute back x's for u's everywhere in the answer.
   Undoing our $u = 4 - x^2$ substitution yields

$$-u^{1/2} + C = -\sqrt{4 - x^2} + C \ .$$

   Thus we have found

$$\int \frac{x}{\sqrt{4 - x^2}}\,dx = -\sqrt{4 - x^2} + C \ .$$

How would we check this? By differentiating:

$$\begin{aligned}
\frac{d}{dx}\left(-\sqrt{4 - x^2} + C\right) &= \frac{d}{dx}(-(4 - x^2)^{1/2} + C) \\
&= -\frac{1}{2}(4 - x^2)^{-1/2}(-2x) \\
&= x(4 - x^2)^{-1/2} \\
&= \frac{x}{\sqrt{4 - x^2}} \ .
\end{aligned}$$

This is the integrand that we started with, so the solution makes sense. ■

Example 5.5.2. Evaluate $\displaystyle\int \frac{e^x\,dx}{(e^x + 15)^3}$.

Solution: This integral is not in our list of building blocks. But notice that the derivative of $e^x + 15$ (which we see in the denominator) is just e^x (which we see in the numerator), so substitution will be a good choice for this.

Let $u = e^x + 15$. Then $du = e^x dx$, and this integral becomes $\displaystyle\int \frac{du}{u^3} = \int u^{-3}\,du$.

Luckily, that is on our list of building block formulas: $\displaystyle\int u^{-3}\,du = \frac{u^{-2}}{-2} + C = -\frac{1}{2u^2} + C$.

Finally, translating back:

$$\int \frac{e^x\,dx}{(e^x + 15)^3} = -\frac{1}{2(e^x + 15)^2} + C \ .$$

■

Example 5.5.3. Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2}{x^3 + 5}\,dx$

Solution: This is not a basic integral, but the composition is less obvious. Here, we can treat the denominator as the inside of the function $\dfrac{1}{x}$.

Let $u = x^3 + 5$. Then $du = 3x^2 dx$. Solving for dx, $dx = \dfrac{du}{3x^2}$. Substituting, we have

$$\int \frac{x^2}{x^3 + 5}\,dx = \int \frac{x^2}{u}\frac{du}{3x^2} = \int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\int \frac{1}{u}\,du \ .$$

Using our basic formulas, we have

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln(|u|) + C \ .$$

Undoing the substitution, we conclude that

$$\int \frac{x^2}{x^3 + 5}\,dx = \frac{1}{3}\ln(|x^3 + 5|) + C \ .$$

■

(b) $\displaystyle\int \frac{x^3+5}{x^2}\,dx$

Solution: It is tempting to start this problem the same way we did the last, but if we try it will not work, since the numerator of this fraction is not the derivative of the denominator. Instead, we need to try a different approach. For this problem, we can use some basic algebra:

$$\begin{aligned}\int \frac{x^3+5}{x^2}\,dx &= \int \left(\frac{x^3}{x^2}+\frac{5}{x^2}\right)\,dx \\ &= \int \left(x+5x^{-2}\right)\,dx \ .\end{aligned}$$

We can integrate this using our basic rules, without needing substitution:

$$\begin{aligned}\int \left(x+5x^{-2}\right)\,dx &= \frac{x^2}{2}+5\frac{x^{-1}}{-1}+C \\ &= \frac{1}{2}x^2-\frac{5}{x}+C.\end{aligned}$$

■

5.5.2 Substitution and Definite Integrals

When you use substitution to help evaluate a definite integral, you have a choice for how to handle the limits of integration. You can do either of these, whichever seems better to you. The important thing to remember is that the original limits of integration were values of the original variable (say, x), not values of the new variable (say, u).

1. You can find the antiderivative as a side problem, translating back to x's, and then use the antiderivative with the original limits of integration.

2. Or, you can substitute for the limits of integration at the same time as you're substituting for everything inside the integral, and then skip the translate back into x step.

   If the original integral had endpoints $x=a$ and $x=b$, and we make the substitution $u=g(x)$ and $du=g'(x)dx$, then the new integral will have endpoints $u=g(a)$ and $u=g(b)$ and

$$\int_{x=b}^{x=a} (\text{original integrand})\,dx$$

   becomes

$$\int_{u=g(b)}^{u=g(a)} (\text{new integrand})\,du \ .$$

Method 1 seems more straightforward for most students, but it can involve some messy algebra. Method 2 is often neater and usually involves fewer steps.

Example 5.5.4. Evaluate $\displaystyle\int_0^1 (3x-1)^4\,dx$.

Solution: We'll need substitution to find an antiderivative, so we'll need to handle the limits of integration carefully. Let's solve this example both ways.

1. **Doing the antiderivative as a side problem:**

   Step One – find the antiderivative, using substitution.

   Let $u=3x-1$. Then $du=3dx$ and

$$\int (3x-1)^4\,dx = \int \frac{1}{3}u^4\,du = \frac{1}{3}\cdot\frac{u^5}{5}+C \ .$$

Translating back to x, we have

$$\frac{1}{3} \cdot \frac{u^5}{5} + C = \frac{(3x-1)^5}{15} + C \ .$$

Step Two – evaluate the definite integral.

$$\int_0^1 (3x-1)^4 \, dx = \left. \frac{(3x-1)^5}{15} \right|_0^1 = \frac{(3(1)-1)^5}{15} - \frac{(3(0)-1)^5}{15} = \frac{32}{15} - \frac{-1}{15} = \frac{33}{15} \ .$$

2. **Substituting for the limits of integration:**

   Let $u = 3x - 1$. Then $du = 3dx$. Substituting for the limits of integration, when $x = 0$, $u = -1$, and when $x = 1$, $u = 2$. So,

$$\begin{aligned} \int_{x=0}^{x=1} (3x-1)^4 \, dx &= \int_{u=-1}^{u=2} \frac{1}{3} u^4 \, du \\ &= \left. \frac{u^5}{15} \right|_{u=-1}^{u=2} \\ &= \frac{(2)^5}{15} - \frac{(-1)^5}{15} \\ &= \frac{32}{15} - \frac{-1}{15} \\ &= \frac{33}{15} \ . \end{aligned}$$

■

Example 5.5.5. Evaluate $\displaystyle\int_2^{10} \frac{(\ln(x))^6}{x} \, dx$.

Solution: I can see the derivative of ln(x) in the integrand, so I can tell that substitution is a good choice.

Let $u = \ln(x)$. Then $du = \frac{1}{x} dx$. When $x = 2$, $u = \ln(2)$. When $x = 10$, $u = \ln(10)$. So the new definite integral is

$$\begin{aligned} \int_{x=2}^{x=10} \frac{(\ln(x))^6}{x} \, dx &= \int_{u=\ln(2)}^{u=\ln(10)} u^6 \, du \\ &= \left. \frac{u^7}{7} \right|_{u=\ln(2)}^{u=\ln(10)} \\ &= \frac{1}{7} \left((\ln(10))^7 - (\ln(2))^7 \right) \\ &\approx 49.01 \ . \end{aligned}$$

■

5.6 Integration by Parts

Integration by parts is an integration method which enables us to find antiderivatives of some new functions such as $\ln(x)$ as well as antiderivatives of products of functions such as $x^2 \ln(x)$ and xe^x.

If the function we're trying to integrate can be written as a product of two functions, u, and dv, then integration by parts lets us trade out a complicated integral for hopefully simpler one.

Theorem 5.6.1 (Integration by Parts Formula)**.**

$$\int u \, dv = uv - \int v \, du$$

For definite integrals:

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$$

Example 5.6.1. Integrate $\int xe^x\,dx$.

Solution: To use the Integration by Parts method, we break apart the product into two parts:

$$u = x \quad \text{and} \quad dv = e^x\,dx \ .$$

We now calculate du, the derivative of u, and v, the integral of dv:

$$du = \left(\frac{d}{dx}x\right)dx \quad \text{and} \quad v = \int e^x\,dx = e^x \ .$$

Using the Integration by Parts formula,

$$\int xe^x\,dx = uv - \int v\,du = xe^x - \int e^x\,dx \ .$$

Notice the remaining integral is simpler that the original, and one which we can easily evaluate:

$$xe^x - \int e^x\,dx = xe^x - e^x + C \ .$$

■

In the last example we could have chosen either x or e^x as our u, but had we chosen $u = e^x$, the second integral would have become messier, rather than simpler.

Note 5.6.1 (Rule of Thumb: LIATE)**.** When selecting the u for Integration by Parts, a common rule of thumb is the "LIATE" mnemonic. Choose u in the following order of preference.

L: **L**ogarithmic functions

I: **I**nverse trigonometric Functions

A: **A**lgebraic functions (polynomials, roots, etc.)

T: **T**rigonometic Functions

E: **E**xponential functions

Example 5.6.2. Integrate $\int_1^4 6x^2\ln(x)\,dx$.

Solution: Since this contains a logarithmic expression, we'll use it for our u.

$$u = \ln(x) \quad and \quad dv = 6x^2\,dx$$

We now calculate du and v.

$$du = \frac{1}{x}\,dx \quad and \quad v = \int 6x^2\,dx = 6\frac{x^3}{3} = 2x^3$$

Using the Integration By Parts formula, we have

$$\int_1^4 6x^2\ln(x)\,dx = 2x^3\ln(x)\big|_1^4 - \int_1^4 6x^2\frac{1}{x}\,dx \ .$$

We can simplify the expression in the integral on the right.

$$\int_1^4 6x^2\ln(x)\,dx = 2x^3\ln(x)\big|_1^4 - \int_1^4 6x\,dx$$

The remaining integral is a basic one we can now evaluate.

$$\int_1^4 6x^2 \ln(x)\, dx = 2x^3 \ln(x)\Big|_1^4 - 3x^2\Big|_1^4$$

Finally, we can evaluate the expressions.

$$\begin{aligned}\int_1^4 6x^2 \ln(x)\, dx &= ((2 \cdot 4^3 \ln(4)) - (2 \cdot 1^3 \ln(1))) - ((3 \cdot 4^2) - (3 \cdot 1^2)) \\ &= 128 \ln(4) - 45 \\ &\approx 132.446\end{aligned}$$

■

5.7 Applications of Integrals

5.7.1 Consumer and Producer Surplus

Two of the most fundamental concepts in economics are **supply** and **demand**.

Definition 5.7.1 (Supply and Demand)**.** Suppose some item is bought and sold for a price. The **supply** of the item is the quantity, q, of the item that is available for sale on some market at some price p. The **demand** of the item is the quantity, q, of the itme that buyers are willing to buy at some price p.

To get a feel for how supply and demand change as price and quantities change, consider Figure 5.29, which shows demand and supply curves for a product. Which is which?

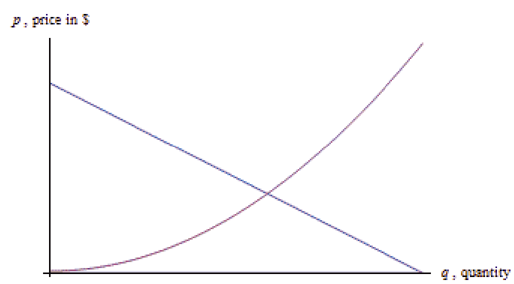


Figure 5.29: Supply and demand curves for q items.

The **demand curve** is decreasing – lower prices are associated with higher quantities demanded, higher prices are associated with lower quantities demanded. Demand curves are often shown as if they were linear, but there's no reason they have to be.

The **supply curve** is increasing – lower prices are associated with lower supply, and higher prices are associated with higher quantities supplied.

The point where the demand and supply curve cross is called the **equilibrium point** (q^*, p^*); p^* is the price at which supply and demand are exactly the same.

Suppose that the price is set at the **equilibrium price**, p^*, so that the quantity demanded equals the quantity supplied. Now think about the folks who are represented on the left of the equilibrium point. The consumers on the left would have been willing to pay a higher price than they ended up having to pay, so the equilibrium price saved them money. On the other hand, the producers represented on the left would have been willing to supply these goods for a lower price – they made more money than they expected to. Both of these groups ended up with extra cash in their pockets!

Graphically, the amount of extra money that ended up in consumers' pockets is the area between the demand curve and the horizontal line $y = p^*$. This is the difference in price, summed up over all the

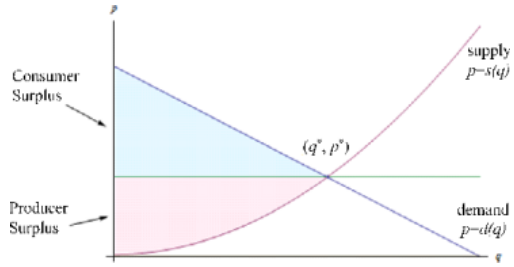


Figure 5.30: Equilibrium point on supply and demand curves.

consumers who spent less than they expected to – a definite integral. Notice that since the area under the horizontal line is a rectangle, we can simplify the area integral:

$$\int_0^{q^*} (d(q) - p^*)\, dq = \int_0^{q^*} d(q)\, dq - \int_0^{q^*} p^*\, dq = \int_0^{q^*} d(q)\, dq - p^* q^* \ .$$

The amount of extra money that ended up in producers' pockets is the area between the supply curve and the horizontal line at p*. This is the difference in price, summed up over all the producers who received more than they expected to. Similar to consumer surplus, this integral can be simplified:

$$\int_0^{q^*} (p^* - s(q))\, dq = \int_0^{q^*} p^*\, dq - \int_0^{q^*} s(q)\, dq = p^* q^* - \int_0^{q^*} s(q)\, dq \ .$$

Definition 5.7.2 (Consumer and Producer Surplus)**.** Given a demand function $p = d(q)$ and a supply function $p = s(q)$, and the equilibrium point (q^*, p^*). The **consumer surplus** is

$$\int_0^{q^*} d(q)\, dq - p^* q^* \ .$$

The **producer surplus** is

$$p^* q^* - \int_0^{q^*} s(q)\, dq.$$

The sum of the consumer surplus and producer surplus is the **total gains from trade**.

What are the units of consumer and producer surplus? The units are (price per item)(quantity of items) = money!

Example 5.7.1. Suppose the demand for a product is given by $p = d(q) = -0.8q + 150$ and the supply for the same product is given by $p = s(q) = 5.2q$. For both functions, q is the quantity and p is the price, in dollars.

(a) Find the equilibrium point.

Solution: The equilibrium point is where the supply and demand functions are equal. Solving $-0.8q + 150 = 5.2q$ gives $q = 25$. ■

(b) Find the consumer surplus at the equilibrium price.

Solution: The consumer surplus is

$$\int_0^{25} (-0.8q + 150)\, dq - (130)(25) = \$250 \ .$$

■

(c) Find the producer surplus at the equilibrium price.

Solution: The producer surplus is

$$(130)(25) - \int_0^{25} 5.2q\,dq = \$1625 \ .$$

■

Example 5.7.2. Tables 5.3 and 5.4 show information about the demand and supply functions for a product. For both functions, q is the quantity and p is the price, in dollars.

Quantity: q (items)	0	100	200	300	400	500	600	700
Price: p (\$)	70	61	53	46	40	35	31	28

Table 5.3: Demand or supply of a product.

Quantity: q (items)	0	100	200	300	400	500	600	700
Price: p (\$)	14	21	28	33	40	47	54	61

Table 5.4: Demand or supply of a product.

(a) Which is which? That is, which table represents demand and which represents supply?

Solution: Table 5.3 shows decreasing price associated with increasing quantity, so that is the demand function. Table 5.4 shows increasing price associated with increasing quantity, so that is the supply function. ■

(b) What is the equilibrium price and quantity?

Solution: For both functions, $q = 400$ is associated with $p = 40$, so the equilibrium price is \$40 and the equilibrium quantity is 400 units. Notice that we were lucky here, because the equilibrium point is actually one of the points shown. In many cases with a table, we would have to estimate. ■

(c) Find the consumer and producer surplus at the equilibrium price.

Solution: The consumer surplus uses the demand function, which comes from the first table. We'll have to approximate the value of the integral using rectangles. There are 4 rectangles, and let's choose to use left endpoints.

The consumer surplus is

$$\int_0^{400} (\textit{demand})\,dq - (40)(400) \approx (100)(70 + 61 + 53 + 46) - (40)(400) = \$7000 \ .$$

So the consumer surplus is about \$7000.

The producer surplus uses the supply function, which comes from the second table. Let's choose to use left endpoints for this integral also.

The producer surplus is

$$(40)(400) - \int_0^{400} (\textit{supply})\,dq \approx (40)(400) - (100)(14 + 21 + 28 + 33) = \$6400 \ ,$$

so the producer surplus is about \$6400. ■

5.7.2 Continuous Income Stream

In precalculus, you learned about compound interest in that really simple situation where you made a single deposit into an interest-bearing account and let it sit undisturbed, earning interest, for some period of time. Recall:

Definition 5.7.3 (Compound Interest Formulas)**.** Let P be the principal (initial investment), r be the annual interest rate expressed as a decimal, and $A(t)$ be the amount in the account after t years.
Compounding n times per year:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Compounding continuously:

$$A(t) = Pe^{rt}$$

If you're using this formula to find what an account will be worth in the future, $t > 0$ and $A(t)$ is called the —bf future value.

If you're using the formula to find what you need to deposit today to have a certain value P sometime in the future, $t < 0$ and $A(t)$ is called the **present value**.

You may also have learned somewhat more complicated annuity formulas to deal with slightly more complicated situations – where you make equal deposits equally spaced in time.

But real life is not usually so neat.

Calculus allows us to handle situations where deposits are flowing continuously into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the present and future value of a continuous income stream. The idea here is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up (a definite integral).

Definition 5.7.4 (Continuous Income Stream)**.** Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) that applies between year 0 and year T.

Then the **present value** of that income stream is given by

$$PV = \int_0^T F(t)e^{-rt}\,dt \quad .$$

The **future value** can be computed by the ordinary compound interest formula

$$FV = PVe^{rt} \quad .$$

This is a useful way to compare two investments – find the present value of each to see which is worth more today.

Example 5.7.3. You have an opportunity to buy a business that will earn \$75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of \$630,000?
Solution: First, please note that we still have to make some simplifying assumptions. We have to assume that the interest rates are going to remain constant for that entire eight years. We also have to assume that the \$75,000 per year is coming in continuously, like a faucet dripping dollars into the business. Neither of these assumptions might be accurate.

But moving on, the present value of the \$630,000 is, well, \$630,000. This is one investment, where we put our \$630,000 in the bank and let it sit there.

To find the present value of the business, we think of it as an income stream. The function F(t) in this case is a constant \$75,000 dollars per year, so $F(t) = 75,000$. The interest rate is 2.8% and the term we're interested in is 8 years, so $r = .028$, and $T = 8$:

$$PV = \int_0^8 75000e^{-0.028t}\,dt \approx 537548.75 \quad .$$

The present value of the business is about \$537,500, which is less than the \$630,000 asking price, so this is not a good deal. ■

While this integral could have been done using substitution, for many of the integrals in this section we don't have the techniques to use antiderivatives or, in some cases, no antiderivative exists. Technology will work quickly, and it will give you an answer that is good enough.

Example 5.7.4. A company is considering purchasing a new machine for its production floor. The machine costs \$65,000. The company estimates that the additional income from the machine will be a constant \$7000 for the first year, then will increase by \$800 each year after that. In order to buy the machine, the company needs to be convinced that it will pay for itself by the end of 8 years with this additional income. Money can earn 1.7% per year, compounded continuously. Should the company buy the machine?
Solution: We'll assume that the income will come in continuously over the 8 years. We'll also assume that interest rates will remain constant over that 8-year time period.

We're interested in the present value of the machine, which we will compare to its \$65,000 price tag. Let t be the time, in years, since the purchase of the machine. The income from the machine is different depending on the time.

From $t = 0$ to $t = 1$ (the first year), the income is constant \$7000 per year. From $t = 1$ to $t = 8$, the income is increasing by \$800 each year; the income flow function $F(t)$ will be $F(t) = 7000 + 800(t - 1) = 6200 + 800t$. To find the present value, we'll have to divide the integral into the two pieces, one for each of the functions:

$$PV = \int_0^1 7000e^{-0.017t}\,dt + \int_1^8 (6200 + 800t)e^{-0.017t}\,dt \approx 70166 \ .$$

The present value is greater than the cost of the machine, so the company should buy the machine. ■

5.7.3 Average Value

We know the average of n numbers $a_1, a_2, \ldots, a_n$ is their sum divided by n. But what if we need to find the average temperature over a day's time – there are too many possible temperatures to add them up! This is a job for the definite integral.

Definition 5.7.5 (Average Value)**.** The average value of a function $f(x)$ on the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx \ .$$

The average value of a positive function $f(x)$ has a nice geometric interpretation. Imagine that the area under the curve $y = f(x)$ (graph (a) in Figure 5.31) is a liquid that can leak through the graph to form a rectangle with the same area (graph (b) in Figure 5.31).

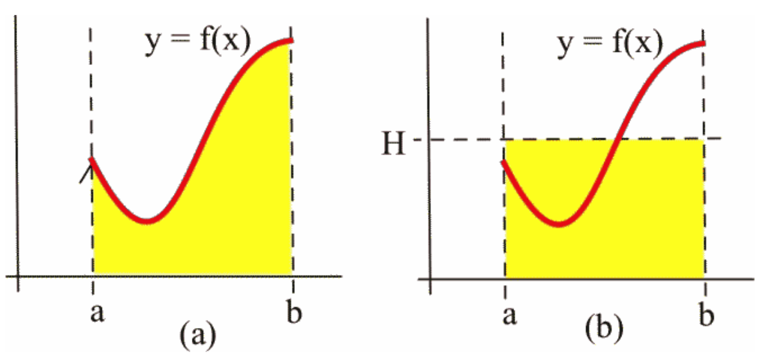


Figure 5.31: Visualizing average value

If the height of the rectangle is H, then the area of the rectangle is $H \cdot (b - a)$. We know the area of the rectangle is the same as the area under $y = f(x)$ between $x = a$ and $x = b$ so $H \cdot (b - a) = \int_a^b f(x)\,dx$. Then

$$H = \frac{1}{b-a}\int_a^b f(x)\,dx \ ,$$

the average value of $f(x)$ on the closed interval $[a, b]$.

The average value of a positive function $f(x)$ is the height H of the rectangle whose area is the same as the area under $y = f(x)$.

Example 5.7.5. During a 9 hour work day, the production rate at time t hours after the start of the shift was given by the function $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

Solution: The average hourly production is $\dfrac{1}{9-0}\displaystyle\int_0^9 \left(5+\sqrt{t}\right)\,dt = 7$ cars per hour. ■

A note about the units – remember that the definite integral has units (cars per hour)·(hours) = cars. But the $\dfrac{1}{b-a}$ in front has units 1hours – the units of the average value are cars per hour, just what we expect an average rate to be.

Note 5.7.1. In general, the average value of a function will have the same units as the integrand.

Function averages, involving means and more complicated averages, are used to smooth data so that underlying patterns are more obvious and to remove high frequency noise from signals. In these situations, the original function $f(x)$ is replaced by some average of $f(x)$. If $f(x)$ is rather jagged time data, then the ten year average of $f(x)$ is the integral $g(x) = \dfrac{1}{10}\displaystyle\int_{x-5}^{x+5} f(t)\,dt$, an average of $f(x)$ over 5 units on each side of x.

For example, Figure 5.32 shows the graphs of a Monthly Average (rather "noisy" data) of surface temperature data, an Annual Average (still rather jagged), and a Five Year Average (a much smoother function).

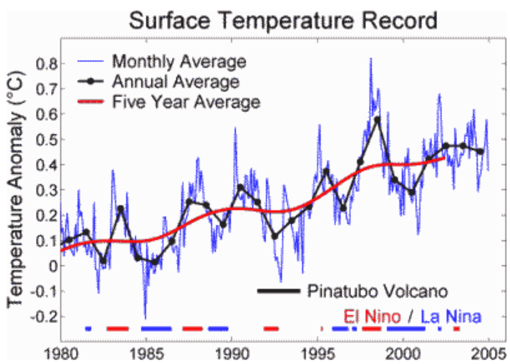


Figure 5.32: Image prepared by Robert A. Rohde,
http://commons.wikimedia.org/wiki/File:Short_Instrumental_Temperature_Record.png.

Typically the average function reveals the pattern much more clearly than the original data. This use of a **moving average** value of noisy data (weather information, stock prices) is a very common, especially data that is a **time series**, data that is a function of time.

Example 5.7.6. The graph in Figure 5.34 shows the volume of water in a reservoir over a 12 hour period. Estimate the average amount of water in the reservoir over this period.

Solution: If $V(t)$ is the volume of the water (in millions of liters) after t hours, then the average amount is $\dfrac{1}{12}\displaystyle\int_0^{12} V(t)\,dt$. In order to find the definite integral, we'll have to estimate. Let's use 6 rectangles and take the heights from their right edges (there's nothing special about using 6 rectangles or right edges – other choices would still give you a valid estimate).

The estimate of the integral is

$$\int_0^{12} V(t)\,dt \approx (18)(2) + (9.7)(2) + (8.2)(2) + (12)(2) + (19.9)(2) + (22)(2) = 179.6\ .$$

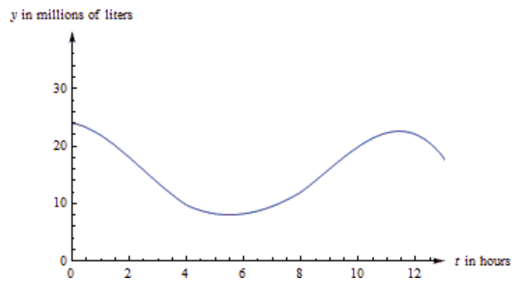


Figure 5.33: Volume of water in a reservoir over a 12-hour span.

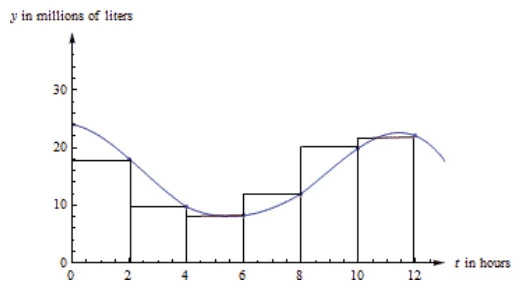


Figure 5.34: Estimating the average volume of water in a reservoir.

The units of this integral are millions of liters. So our estimate of the average volume is $112 \cdot 179.6 \approx 15$ million liters. (The estimate might change a little depending on how we estimate the function values from the graph.)

In Figure 5.35, you can see the same graph as the one in Figure 5.34 with the line $y = 15$ drawn in. The area under the curve and the area under the rectangle are (approximately) the same; this confirms that our solution is plausible and makes sense.

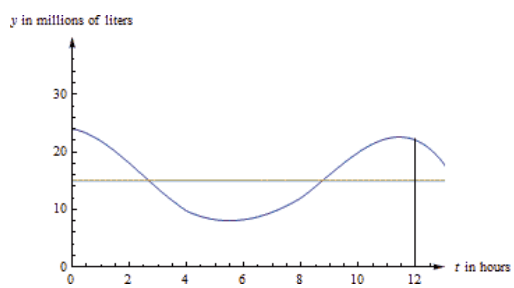


Figure 5.35: Estimating the average volume of water in a reservoir.

In fact, that would be a different way to estimate the average value. We could have estimated the placement of the horizontal line so that the area under the curve and under the line were equal. ■

5.7.4 Area Between Curves

We have already used integrals to find the area between the graph of a function and the horizontal axis. Integrals can also be used to find the area between two graphs.

If $f(x) \geq g(x)$ for all $x \in [a, b]$, then we can approximate the area between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ by partitioning the interval $[a, b]$ and forming a Riemann sum, as shown in Figure 5.36. The height of the ith rectangle is top $-$ bottom, or $f(c_i) - g(c_i)$ so the area of the i-th rectangle is (height)$\cdot$(base) $= (f(c_i) - g(c_i))\Delta x$. Adding up these rectangles gives an approximation of the total area as $\displaystyle\sum_{i=1}^{n}(f(c_i) - g(c_i))\Delta x$, a Riemann sum.

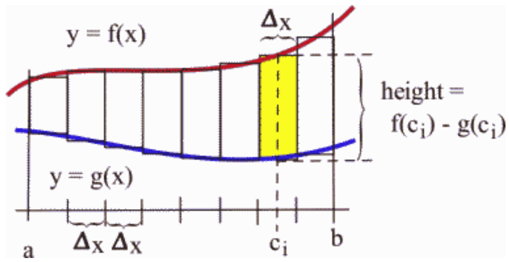


Figure 5.36: Estimating the area between two curves.

The limit of this Riemann sum, as the number of rectangles gets larger and their width gets smaller, is the definite integral $\int_a^b (f(x) - g(x))\,dx$.

Theorem 5.7.1 (Area Between Two Curves)**.** The area between two curves $f(x)$ and $g(x)$, where $f(x) \geq g(x)$, between $x = a$ and $x = b$ is

$$\int_a^b (f(x) - g(x))\,dx \ .$$

The integrand is top $-$ bottom. Make a graph or use test values to be sure which curve is which.

Example 5.7.7. Find the area bounded between the graphs of $f(x) = x$ and $g(x) = 3$ for $1 \leq x \leq 4$.

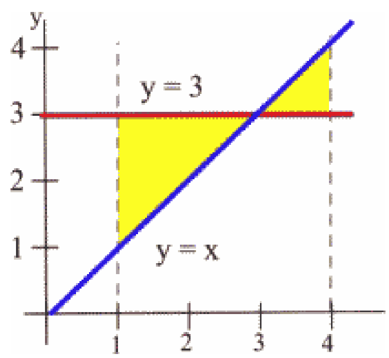


Figure 5.37: Graph for example 5.7.7.

Solution: Always start with a graph so you can see which graph is the top and which is the bottom. In this example, the two curves cross, and they change positions; we'll need to split the area into two pieces. Geometrically, we can see that the area is $2 + 0.5 = 2.5$.

Writing the area as a sum of definite integrals, we get

$$\text{Area} = \int_1^3 (3 - x)\,dx + \int_3^4 (x - 3)\,dx \ .$$

These integrals are easy to evaluate using antiderivatives:

$$\begin{aligned}
\int_1^3 (3 - x)\,dx &= 3x - \frac{x^2}{2}\Big|_1^3 \\
&= \left(9 - \frac{9}{2}\right) - \left(3 - \frac{1}{2}\right) \\
&= 2
\end{aligned}$$

$$\begin{aligned}\int_3^4 (x-3)\,dx &= \frac{x^2}{2} - 3x\Big|_3^4\\ &= \left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right)\\ &= \frac{1}{2}\end{aligned}$$

The sum of these two integrals tells us that the total area between $y = f(x)$ and $y = g(x)$ over the interval $[1, 4]$ is 2.5 square units, which we already knew from the picture. ■

Note that the single integral $\int_1^3 (3-x)\,dx = 1.5$ is not the area we want in the last example. The value of the integral is 1.5, and the value of the area is 2.5. That's because the triangle on the right is below the x-axis. The graph of $y = x$ is above the graph of $y = 3$, so the integrand $3 - x$ is negative; in the definite integral, the area of that triangle comes in with a negative sign.

In this example, it was easy to see exactly where the two curves crossed so we could break the region into the two pieces to figure separately. In other examples, you might need to solve an equation to find where the curves cross.

Example 5.7.8. Two objects, A and B, start from the same location and travel along the same path with velocities $v_A(t) = t + 3$ and $v_B(t) = t^2 - 4t + 3$ meters per second. How far ahead is A after 3 seconds?

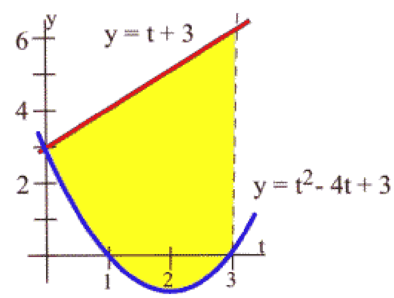


Figure 5.38: Graph for example 5.7.8.

Solution:

Since $v_A(t) \geq v_B(t)$, the area between the graphs of $v_A(t)$ and $v_B(t)$ represents the distance between the objects.

After 3 seconds, the distance apart is

$$\begin{aligned}\int_0^3 (v_A(t) - v_B(t))\,dt &= \int_0^3 ((t+3) - (t^2 - 4t + 3))\,dt\\ &= \int_0^3 (5t - t^2)\,dt\\ &= 5\frac{t^2}{2} - \frac{t^3}{3}\Big|_0^3\\ &= \left(5 \cdot \frac{9}{2} - \frac{27}{3}\right) - (0)\\ &= 13.5\text{meters.}\end{aligned}$$

■

5.7.5 Volume

Just as we can partition an interval and imagine approximating an area with rectangles to find a formula for the area between curves, we can partition an interval and imagine approximating a volume with simple shapes to find a formula for the volume of a solid. While this approach works for a variety of shapes, our focus will be on shapes formed by revolving a curve around the horizontal axis.

We start with an area, the region below a function on the interval $a \leq x \leq b$ (Figure 5.41). We are going to take that region, and rotate it around the x-axis, creating the solid shape shown in Figure 5.42.

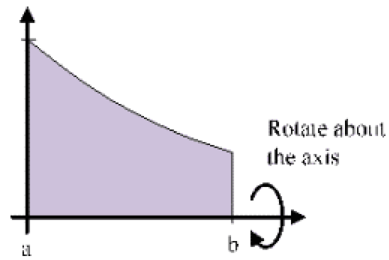


Figure 5.39: Region to rotate around the x-axis.

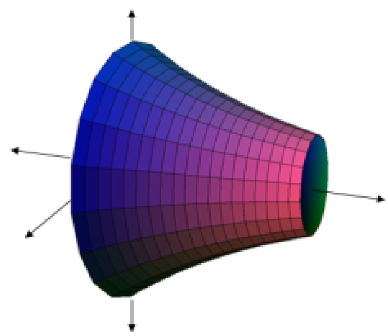


Figure 5.40: Solid formed by rotating the region.

To find the volume of the solid in Figure 5.42, we can start by partitioning the interval $[0, 1]$ and approximating the area with rectangles. As before, the width of each rectangle is Δx and the height is $f(c_i)$.

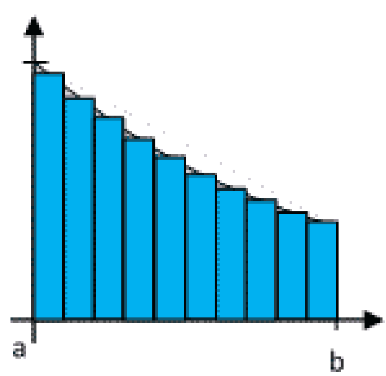


Figure 5.41: Rectangles to approximate the region.

If we took just one of these rectangles and rotated it about the horizontal axis, it would form a cylindrical shape. The radius of that cylinder would be $f(c_i)$, so the volume would be

$$V = \pi r^2 h = \pi (f(c_i))2\Delta x \ .$$

The volume of the whole solid could be approximated by rotating each of the rectangles about the x axis. Adding up the volume of each of the little cylindrical discs gives an approximation of the total volume as

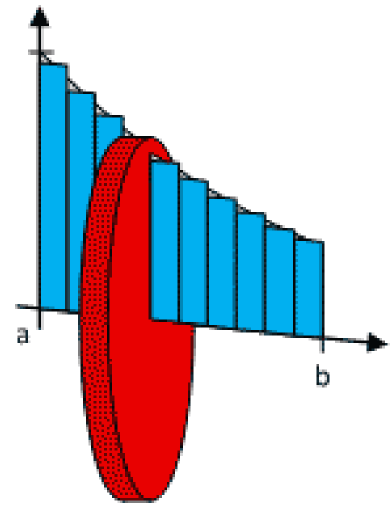


Figure 5.42: Solid formed by rotating a rectangle.

$\sum_{i=1}^{n} \pi(f(c_i))^2 \Delta x$, a Riemann sum.

The limit of this sum as the width of the rectangles becomes small is the definite integral $\int_a^b \pi(f(x))^2\,dx$.

Theorem 5.7.2 (Volume)**.** If $f(x) \geq 0$ over the interval $[ab]$, then the volume of the solid obtained by rotating about the x-axis the area bounded by the curve $y = f(x)$, the x-axis, $x = a$, and $x = b$ is

$$\int_a^b \pi(f(x))^2\,dx \quad .$$

Example 5.7.9. Find the volume of the solid formed by rotating the area under the curve $y = f(x) = e^{-x}$ on the interval $[0, 1]$ about the x-axis.

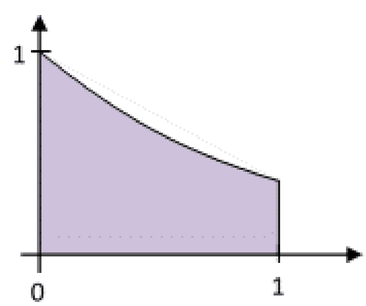


Figure 5.43: e^{-x} over $[0, 1]$.

Solution: This is the region pictured in the earlier example. We substitute in the function and bounds into the formula we derived to set up the definite integral:

$$\text{Volume} = \int_0^1 \pi(e^{-x})^2\,dx \quad .$$

Using exponent rules, the integrand can be simplified. The constant π can be pulled out of the integral:

$$\text{Volume} = \pi \int_0^1 e^{-2x}\,dx \quad .$$

Using the substitution $u = -2x$ and $du = -2\,dx$ we can integrate this function.

$$\begin{aligned}
\pi \int_0^1 e^{-2x}\,dx &= \pi \int_{x=0}^{x=1} -\frac{1}{2} e^u\,du \\
&= -\frac{1}{2}\pi e^u \Big|_{x=0}^{x=1} \\
&= -\frac{1}{2}\pi e^{-2x} \Big|_0^1 \\
&= \left(-\frac{1}{2}\pi e^{-2(1)}\right) - \left(-\frac{1}{2}\pi e^{-2(0)}\right) \\
&\approx 1.358 \text{ cubic units.}
\end{aligned}$$

■

5.8 Differential Equations

A **differential equation** is an equation involving the derivative of a function. They allow us to express with a simple equation the relationship between a quantity and it's rate of change.

Example 5.8.1. A bank pays 2% interest on its certificate of deposit accounts, but charges a \$20 annual fee. Write an equation for the rate of change of the balance, $B'(t)$.
Solution: If the balance $B(t)$ has units of dollars, then $B'(t)$ has units of dollars per year. When we think of what is changing the balance of the account, there are two factors:

1. The interest, which increases the balance, and

2. The fee, which decreases the balance.

Considering the interest, we know each year the balance will increase by 2%, but 2% of what? Each year that will change, since we earn interest on whatever the current balance is. We can represent the amount of increase as 2% of the balance: $0.02B(t)$ dollars/year.

The fee already has the units of dollars/year. Since everything now has the same units, we can put the two together, and create the equation:

$$B'(t) = 0.02B(t) - 20\ .$$

■

The result is an example of a differential equation. Notice this particular equation involves both the derivative and the original function, and so we can't simply find $B(t)$ using basic integration.

Algebraic equations contain constants and variables, and the solutions of an algebraic equation are typically numbers. For example, $x = 3$ and $x = -2$ are solutions of the algebraic equation $x^2 = x + 6$. Differential equations contain derivatives or differentials of functions. Solutions of differential equations are functions. The differential equation $y' = 3x^2$ has infinitely many solutions, and two of those solutions are the functions $y = x^3 + 2$ and $y = x^3 - 4$.

You have already solved lots of differential equations: every time you found an antiderivative of a function $f(x)$, you solved the differential equation $y' = f(x)$ to get a solution y. The differential equation $y' = f(x)$, however, is just the beginning. Other applications generate different differential equations, like in the bank balance example above.

5.8.1 Checking Solutions of Differential Equations

Whether a differential equation is easy or difficult to solve, it is important to be able to check that a possible solution really satisfies the differential equation.

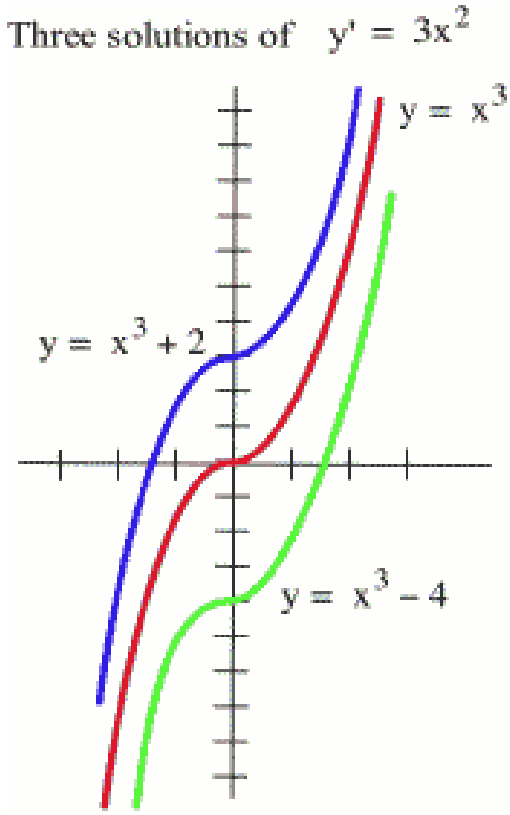


Figure 5.44: Three solutions to $y' = 3x^2$.

A possible solution of an algebraic equation can be checked by putting the solution into the equation to see if the result is true: $x = 3$ is a solution of $5x + 1 = 16$ since $5(3) + 1 = 16$ is true. Similarly, a solution of a differential equation can be checked by substituting the function and the appropriate derivatives into the equation to see if the result is true: $y = x^2$ is a solution of $xy' = 2y$ since $y' = 2x$ and $x(2x) = 2(x^2)$ is true.

Example 5.8.2. Verify the following.

(a) $y = x^2 + 5$ is a solution of $y'' + y = x^2 + 7$.

Solution: $y = x^2 + 5$ so $y' = 2x$ and $y'' = 2$. Substituting these functions for y and y'' into the differential equation $y'' + y = x^2 + 7$, we have

$$y'' + y = (2) + (x^2 + 5) = x^2 + 7 \ ,$$

so $y = x^2 + 5$ is a solution of the differential equation. ■

(b) $y = x + \frac{5}{x}$ is a solution of $y' + \frac{y}{x} = 2$.

Solution: $y = x + \frac{5}{x}$ so $y' = 1 - \frac{5}{x^2}$. Substituting these functions for y and y' in the differential equation $y' + \frac{y}{x} = 2$, we have

$$y' + \frac{y}{x} = \left(1 - \frac{5}{x^2}\right) + \frac{1}{x}\left(x + \frac{5}{x}\right) = 1 - \frac{5}{x^2} + 1 + \frac{5}{x^2} = 2 \ ,$$

the result we wanted to verify.

■

5.8.2 Separable Differential Equations

A differential equation is called **separable** if the variables can be separated algebraically so that all the x's and dx are one one side of the equation, and all the y's and dy are on the other side of the equation. In other words, the differential equation can be written in the form $f(x)\,dx = g(y)\,dy$.

Once separated, separable differential equations can be solved by integrating both sides of the equation.

Example 5.8.3. Find the solution of

$$y' = \frac{6x+1}{2y} \ .$$

Solution: Rewriting y' is a helpful first step:

$$\frac{dy}{dx} = \frac{6x+1}{2y} \ .$$

Now we can multiply both sides by dx and by $2y$ to separate the variables:

$$2y\,dy = (6x+1)\,dx \ .$$

Integrating each side, we have

$$\int 2y\,dy = \int (6x+1)\,dx$$
$$y^2 + C_1 = 3x^2 + x + C_2$$

Notice that we can combine the two constants to create a new, consolidated constant C, so we usually only bother to put a constant on the right side:

$$y^2 = 3x^2 + x + C \ .$$

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As expected, there is a whole family of solutions to this differential equation.

Definition 5.8.1 (Initial Value Problem (IVT))**.** An **initial value problem** is a differential equation that provides additional information about specific values of the function and/or one or more of its derivatives. Often, the specific values will be initial values, that is, when $x = 0$. This allows us to then solve for the constant and find a single solution.

Example 5.8.4. Find the solution of $y' = \dfrac{6x+1}{2y}$ which satisfies $y(2) = 3$.

Solution: In the previous example we found the general solution, $y^2 = 3x^2 + x + C$.

Substituting in the initial condition, $y = 3$ when $x = 2$,

$$3^2 = 3(2)^2 + 2 + C \ ,$$

so $9 = 12 + 2 + C$, giving $C = -5$.

The solution is

$$y^2 = 3x^2 + x - 5 \ .$$

Sometimes it is desirable to solve for y, which would yield $y = \pm\sqrt{3x^2 + x - 5}$, but since the initial condition had a positive y value, we isolate the positive solution:

$$y = \sqrt{3x^2 + x - 5} \ .$$

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Example 5.8.5. A bank pays 2% interest on its certificate of deposit accounts, but charges a \$20 annual fee. If you initially invest \$3,000, how much will you have after 10 years?
Solution: You may recognize this as the example from the beginning of the section, for which we set up the equation

$$B'(t) = 0.02B(t) - 20$$

or, more simply,

$$\frac{dB}{dt} = 0.02B - 20 \ .$$

We can separate this equation by multiplying by dt and dividing by the entire expression on the right:

$$\frac{dB}{0.02B - 20} = dt \ .$$

Integrating the left side of this equation requires substitution. Let $u = 0.02B - 20$, so $du = 0.02\, dB$. Making the substitution,

$$\begin{aligned} \int \frac{dB}{0.02B - 20} &= \int \frac{du/0.02}{u} \\ &= \int \frac{1}{u} \frac{du}{0.02} \\ &= \frac{1}{0.02} \int \frac{1}{u}\, du \\ &= \frac{1}{0.02} \ln|u| + C_1 \\ &= \frac{1}{0.02} \ln|0.02B - 20| + C_1 \end{aligned}$$

Integrating on the right side of the differential equation is easier:

$$\int dt = t + C_2 \ .$$

Together, this gives us the general solution to the differential equation (we're also combining the C's in this step):

$$\frac{1}{0.02} \ln|0.02B - 20| = t + C \ .$$

Now we would like to solve for B. Start by multiplying by 0.02.

$$\begin{aligned} \ln|0.02B - 20| &= 0.02t + 0.02C && \\ \ln|0.02B - 20| &= 0.02t + D && \text{Let } D = 0.02C \\ e^{\ln|0.02B-20|} &= e^{0.02t+D} && \text{Exponentiate both sides} \\ |0.02B - 20| &= e^{0.02t+D} && \text{Use the log rule: } e^{\ln(A)} = A. \\ 0.02B - 20 &= e^{0.02t+D} && \text{The right side is positive.} \\ 0.02B - 20 &= e^{0.02t}e^{D} && \text{Use the rule: } e^{A+B} = e^A e^B. \\ 0.02B - 20 &= ke^{0.02t} && \text{Let } k = e^D \\ 0.02B &= ke^{0.02t} + 20 && \\ B &= \frac{ke^{0.02t} + 20}{0.02} && \\ B &= \frac{ke^{0.02t}}{0.02} + 1000 && \\ B &= Ae^{0.02t} + 1000 && = \text{Let } A = \frac{k}{0.02} \end{aligned}$$

Finally, we can substitute our initial value of $B = 3000$ when $t = 0$ to solve for the constant A:

$$\begin{aligned} 3000 &= Ae^{0.02(0)} + 1000 \\ A &= 2000 \end{aligned}$$

This gives us the equation for the account balance after t years:

$$B(t) = 2000e^{0.02t} + 1000 \ .$$

To find the balance after 10 years, we can evaluate this equation at $t = 10$:

$$B(10) = 2000e^{0.02(10)} + 1000 \approx \$3442.81 \ .$$

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It's worth noting that this answer isn't exactly right. Differential equations assume continuous changes, and it is unlikely interest is compounded continuously or the fee is extracted continuously. However, the answer is probably close to the actual answer, and differential equations provide a relatively simple model of a complicated situation.

5.8.3 Models of Growth

The bank account example demonstrated one basic model of growth: growth proportional to the existing quantity. Bank accounts and populations both tend to grow this way if not constrained. This type of growth is called unlimited growth.

Definition 5.8.2 (Unlimited Growth)**.** If a quantity or population y grows at a rate proportional that quantity's size, it can be modeled with **unlimited growth** (also called **unbounded growth**), which has the differential equation:

$$y' = ry \ ,$$

where r is a constant.

Example 5.8.6. A population grows by 8% each year. If the current population is 5,000, find an equation for the population after t years.
Solution:

$$\begin{aligned} \frac{dy}{dt} &= 0.08y \\ \frac{1}{y}\,dy &= 0.08\,dt \\ \ln|y| &= 0.08t + C \\ e^{\ln|y|} &= e^{0.08t+C} \\ y &= Ae^{0.08t} \end{aligned}$$

Separate the variables.Integrate both sides.Exponentiate both sides.Simplify both sides, using the tricks we used in the bank example. Now substitute in the initial condition: $5000 = Ae^{0.08(0)}$, so $A = 5000$.

The population will grow following the equation

$$y = 5000e^{0.08t} \ .$$

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Notice that the solution to the unlimited growth equation is an exponential equation.

When a product is advertised heavily, sales will tend to grow very quickly, but eventually the market will reach saturation, and sales will slow. In this type of growth, called **limited growth**, **bounded growth**, or **constrained growth**, the population grows at a rate proportional to the distance from the maximum value.

Definition 5.8.3 (Bounded Growth)**.** If a quantity grows at a rate proportional to the distance from the maximum value, M, it can be modeled with **bounded growth**, which has the differential equation

$$y' = k(M - y) \ ,$$

where k is a constant, and M is the maximum size of y.

Example 5.8.7. A new cell phone is introduced. The company estimates they will sell 200 thousand phones. After 1 month they have sold 20 thousand. How many will they have sold after 9 months?
Solution: In this case there is a maximum amount of phones they expect to sell, so $M = 200$ thousand. Modeling the sales, y, in thousands of phones, we can write the differential equation

$$y' = k(200 - y) \ .$$

Since it was a new phone, $y(0) = 0$. We also know the sales after one month, $y(1) = 20$.
Solving the differential equation:

$$\begin{aligned}
\frac{dy}{dt} &= k(200 - y) & \\
\frac{dy}{200 - y} &= k\,dt & \text{Separate the variables.} \\
\int \frac{dy}{200 - y} &= \int k\,dt & \text{Integrate both sides.} \\
-\ln|200 - y| &= kt + C & \text{On the left, use the substitution } u = 200 - y. \\
e^{\ln|200-y|} &= e^{-kt+C} & \text{Multiply both sides by } -1 \text{ and exponentiate both sides.} \\
200 - y &= Be^{-kt} & \text{Simplify.} \\
-y &= Be^{-kt} - 200 & \text{Subtract 200.} \\
y &= Ae - kt + 200 & \text{Divide by } -1, \text{ and simplify.}
\end{aligned}$$

Using the initial condition $y(0) = 0$,

$$0 = Ae^{-k(0)} + 200 \ ,$$

so $0 = A + 200$, giving $A = -200$.
Using the value $y(1) = 20$:

$$\begin{aligned}
20 &= -200e^{-k\cdot 1} + 200 & \\
-180 &= -200e^{-k} & \text{Subtract 200.} \\
\frac{180}{200} &= e^{-k} & \text{Divide by } -200. \\
\ln(0.9) &= \ln\left(e^{-k}\right) = -k & \text{Take the logarithm of both sides.} \\
k &= -\ln(0.9) \approx 0.105 & \text{Divide by } -1.
\end{aligned}$$

As a quick sanity check, this value is positive as we would expect, indicating that the sales are growing over time. We now have the equation for the sales of phones over time:

$$A = -200e^{-0.105t} + 200 \ .$$

Finally, we can evaluate this at $t = 9$ to find the sales after 9 months:

$$A(9) = -200e^{-0.105\cdot 9} + 200 \approx 122.26 \text{ thousand phones.}$$

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Limited growth is also commonly used for learning models, since when learning a new skill, people typically learn quickly at first, then their rate of improvement slows down as they approach mastery.

Earlier we used unlimited growth to model a population, but often a population will be constrained by food, space, and other resources. When a population grows both proportional to its size, and relative to the distance from some maximum, that is called **logistic growth**. This leads to the differential equation $y' = ky(M - y)$, which is accurate but not always convenient to use. We will use a slight modification. Since solving this differential equation requires integration techniques we haven't learned, the solution form is given.

Definition 5.8.4 (Logistic Growth)**.** If a quantity grows at a rate proportional to its size and to the distance from the maximum value, M, it can be modeled with **logistic growth**, which has the differential equation:

$$y' = ry\left(1 - \frac{y}{M}\right) \ .$$

r can be interpreted as the growth rate absent constraints, i.e., how the population would grow if there wasn't a maximum value.

This differential equation has solutions of the form

$$y = \frac{M}{1 + Ae^{-rt}} \ .$$

Example 5.8.8. A colony of 100 rabbits is introduced to a reclaimed forest. After 1 year, the population has grown to 300. It is estimated the forest can sustain 5000 rabbits. The forest service plans to reintroduce wolves to the forest when the rabbit population reaches 3000 rabbits. When will that occur?

Solution: The maximum sustainable population was given as $M = 5000$. Using the solution form

$$y = \frac{M}{1 + Ae^{-rt}}$$

and the initial condition $y(0) = 100$ we can solve for A:

$$\begin{aligned}
100 &= \frac{5000}{1 + Ae^{-r\cdot 0}} \\
100 &= \frac{5000}{1 + A} \quad \text{Simplify.} \\
100(1 + A) &= 5000 \quad \text{Multiply both sides by } 1 + A. \\
1 + A &= 50 \quad \text{Divide by 100.} \\
A &= 49
\end{aligned}$$

Now, using $y(1) = 300$, we can solve for r:

$$\begin{aligned}
300 &= \frac{5000}{1 + 49e^{-r(1)}} \\
300(1 + 49e^{-r}) &= 5000 \\
1 + 49e^{-r} &= \frac{5000}{300} = \frac{50}{3} \\
e^{-r} &= \frac{\frac{50}{3} - 1}{49} \approx 0.3197 \\
-r &\approx \ln(0.3197) \\
r &\approx -\ln(0.3197) \approx 1.1404 \ .
\end{aligned}$$

We now have the equation for the population after t years:

$$y = \frac{5000}{1 + 49e^{-1.1404t}} \ .$$

To answer the original equation, of when the rabbit population will reach 3000, we need to solve for t when $y = 3000$:

$$\begin{aligned}
3000 &= \frac{5000}{1 + 49e^{-1.1404t}} \\
3000\left(1 + 49e^{-1.1404t}\right) &= 5000 \\
1 + 49e^{-1.1404t} &= \frac{5}{3} \\
49e^{-1.1404t} &= \frac{5}{3} - 1 \\
e^{-1.1404t} &= \frac{\frac{5}{3} - 1}{49} \approx 0.01361 \\
t &= \frac{\ln(0.01361)}{-1.1404} \approx 3.77 \text{ years.}
\end{aligned}$$

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Logistic growth is also a good model for unadvertised sales. A new product that is not advertised will have sales increase slowly at first, then grow as word of mouth spreads and people become familiar with the product. Sales will level off as they approach market saturation.

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