

# Open Applied Calculus

Eric Landquist

June 26, 2019

The textbook is adapted from *Modeling, Functions, and Graphs: Algebra for College Students* by Katherine Yoshiwara, *Business Calculus* by Shana Calaway, and *Applied Calculus* by Shana Calaway, Dale Hoffman, and David Lippman. The latter two texts were adapted from *Precalculus: An Investigation of Functions* by David Lippman and Melonie Rasmussen and *Contemporary Calculus*, by Dale Hoffman.  
Copyright © 2019 Eric Landquist CC-BY-SA 4.0

This text is licensed under a Creative Commons Attribution-Share Alike 4.0 United States License.

To view a copy of this license, visit <http://creativecommons.org/licenses/by-sa/4.0/us/> or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA. You are free:

- **to Share** to copy, distribute, display, and perform the work
- **to Remix** to make derivative works

Under the following conditions:

- **Attribution.** You must attribute the work in the manner specified by the author or licensor (but not in any way that suggests that they endorse you or your use of the work).
- **Share Alike.** If you alter, transform, or build upon this work, you may distribute the resulting work only under the same, similar or a compatible license.

With the understanding that:

- **Waiver.** Any of the above conditions can be waived if you get permission from the copyright holder.
- **Other Rights.** In no way are any of the following rights affected by the license:
  - Your fair dealing or fair use rights;
  - Apart from the remix rights granted under this license, the author's moral rights;
  - Rights other persons may have either in the work itself or in how the work is used, such as publicity or privacy rights.
  - Notice For any reuse or distribution, you must make clear to others the license terms of this work. The best way to do this is with a link to this web page: <http://creativecommons.org/licenses/by-sa/4.0/us/>

**Acknowledgments.** I gratefully acknowledge support for the initial development of this book with a grant from the Pennsylvania State System of Higher Education (PASSHE) Faculty Professional Development Committee.

*First release, August 2019*

# Contents

<b>0</b>	<b>What is Calculus?</b>	<b>5</b>
0.1	Two Problems . . . . .	5
0.2	Problem Solving . . . . .	5
<b>1</b>	<b>Models, Graphs, and Functions</b>	<b>9</b>
1.1	What is a Function? . . . . .	9
1.1.1	Function Concepts . . . . .	9
1.1.2	Representing Functions . . . . .	9
1.1.3	Evaluating Functions. . . . .	12
1.1.4	Cost, Revenue, and Profit . . . . .	13
1.1.5	Domain and Range. . . . .	15
1.1.6	Exercises . . . . .	16
1.2	Models . . . . .	19
1.3	Linear Functions . . . . .	20
1.4	Polynomial Functions . . . . .	20
1.5	Exponential Functions . . . . .	20
1.6	Logistic Functions . . . . .	20
1.7	Logarithmic Functions . . . . .	20
1.8	Trigonometric Functions . . . . .	20
1.9	Combining Functions . . . . .	20
<b>2</b>	<b>The Derivative: Rates of Change of a Function</b>	<b>21</b>
2.1	Slopes and Average Rates of Change . . . . .	21
2.2	Tangent Lines and Instantaneous Rates of Change . . . . .	21
2.3	Basic Applications of Slope . . . . .	21
2.4	The Derivative . . . . .	21
2.5	Limits of Functions . . . . .	21
2.6	Finding Derivatives Algebraically . . . . .	21
2.7	The Power Rule . . . . .	21
2.8	The Product and Quotient Rules . . . . .	21
2.9	The Chain Rule . . . . .	21
2.10	Higher Order Derivatives . . . . .	21
<b>3</b>	<b>Applications of Derivatives</b>	<b>23</b>
3.1	Linear Approximations . . . . .	23
3.2	Elasticity of Demand . . . . .	23
3.3	Concavity and Inflection Points . . . . .	23
3.4	Maximum and Minimum Function Values . . . . .	23
3.5	Curve Sketching . . . . .	23
3.6	Applied Optimization . . . . .	23

<b>4</b>	<b>The Integral: Accumulation of a Rate Function</b>	<b>25</b>
4.1	Accumulation . . . . .	25
4.2	Approximating Area . . . . .	25
4.3	Integrals . . . . .	25
4.4	The Fundamental Theorem of Calculus . . . . .	25
4.5	Integral Formulas . . . . .	25
4.6	Integration by Substitution . . . . .	25
4.7	Applications of Integrals . . . . .	25
<b>5</b>	<b>Functions of Many Variables</b>	<b>27</b>
5.1	Examples of Functions of Multiple Variables . . . . .	27
5.2	Multivariable Functions . . . . .	27
5.3	Partial Derivatives . . . . .	27
5.4	Optimization with Multivariable Functions . . . . .	27
5.5	Integrals with Multivariable Functions . . . . .	27
	<b>References</b>	<b>29</b>
	<b>Index</b>	<b>29</b>

# Chapter 0

## What is Calculus?

A good speaker will often begin a talk with an outline of what will be discussed in the talk. In the same way, it is incumbent upon the author of a textbook to give the reader the big picture of the subject of the textbook. That is the purpose of this brief chapter.

**Calculus is the study of change.** Over time, the population of a city may grow. The price of a stock will fluctuate: sometimes increasing and sometimes decreasing. The cumulative profits of a company may grow or wane. These situations represent changing quantities. We may want to dig deeper and find out how quickly the city is growing or the rate at which the stock price or the company's profits are growing. Those are questions that calculus attempts to answer.

Calculus is the study of two seemingly different questions that we will show are actually inverses of each other: the rate of change of a quantity and the accumulation of a quantity. Geometrically, these questions will be visualized as the slope of a curve and the area under a curve.

### 0.1 Two Problems

Calculus was first developed in the late 1600s independently by Sir Isaac Newton and Gottfried Wilhelm Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic and social phenomena and to describe and solve problems in those areas. Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields.

Calculus seeks to solve two main problems.

1. What is the rate of change of a function? Geometrically, what is the slope of a curve?
2. What is the accumulation of a function? Geometrically, what is the area under a curve?

In this book, we will unpack these two problems and show how they are useful to business, economics, finance, the social sciences, and the life sciences. We will take a data-driven and a problem-solving approach to this course, working with real and realistic data.

### 0.2 Problem Solving

Have you ever looked at a math problem and said to yourself, "I don't know where to start!" No matter how you answered, this section will offer some tips.

One often thinks of the study of a mathematical subject as mere rote memorization of formulas and step-by-step procedures. On the contrary, the approach that mathematicians take to the subject is much more creative. We see mathematics as a way to apply abstract concepts to solve problems. Often there is

just one correct answer. Often, however, the “correct” or “best” answer may be impossible to find, so we are instead interested in a solution that would be considered “good,” “plausible,” or “reasonable.” That is the approach that will drive much of the content of this textbook. In short, one of our objectives is to retrain the way you view mathematics: it’s not about knowing or not knowing the answer, it’s about figuring out a solution. This textbook is not a book to tell you what to think, but how to think about quantitative problem solving.

In light of this, we introduce a well-known problem-solving process and thoughts on how to implement it in practice. In his famous book *How to Solve It*, mathematician George Pólya described his problem-solving process in four steps, which we have modified to follow the acronym “C.O.P.E.”

1. **Comprehend** (Understand) the problem.

- **What is the problem asking?**
- It is impossible to solve a problem without this step, but it is often skipped.
- Read and understand the instructions if there are any.
- Read the problem over and over until you understand.
- Look up words and symbols you don’t understand.
- Ask yourself and answer the following questions:
  - Is the problem assuming that I know something that I don’t?
  - What knowledge gaps must I fill in order to really understand the problem?
  - Is there enough information given to solve the problem?
  - Do I need to look up information for the problem?
  - Is there any irrelevant information that can be ignored?
  - Do I need to make any assumptions? If so, are these assumptions reasonable?
- It may help to draw a picture or diagram.
- It may help to simplify the problem, i.e., work with a simpler “toy” problem.
- Study examples similar to the problem.
- It may help to compare the problem to a similar one.
- Be patient with yourself. Sufficient understanding may take time.
- What is a range of plausible solutions?
- Often there is more than one level of comprehension to a problem or concept. A deeper understanding of a problem or concept results in a better or more advanced solution to the problem.
- You may obtain a deeper understanding of the problem by pursuing the next step: “Observe.”

2. **Observe** the information and applicable tools and devise a plan.

- **What information is given in the problem and what tools do you have that apply?**
- “Look” around. What (mathematical or other) tools, results, theorems, algorithms, etc. apply in this situation, based on the information given?
- If you studied examples similar to the problem, do the same tools, results, theorems, or algorithms apply in this case? Why or why not?
- Here are some ideas to work with the information.
  - Define variables to describe components of the problem and solution. Consider units.
  - Express the information we have mathematically.
  - Draw a picture or diagram. Label the picture with variables, numbers, etc.
  - Make a chart or list or plot some of the data or information.
  - Experiment. Play with the problem and the math until something works.
  - Guess and check: Trial and error helps to develop intuition.

- Be systematic. Use an organized method to check all cases.
- Look for patterns.
- Work backwards.
- Look for counterexamples.
- Divide and conquer. Split the problem up into pieces and solve the individual pieces.

3. **Proceed** with the plan and get a solution.

- **It often helps to work out a solution on scratch paper first.**
- **Make your solution easy to follow for yourself and others.**
- Check your work at each step.
- If you get stuck or the solution doesn't work, go back to the planning stage and try something else. Thomas Edison tested thousands of designs before developing a practical working lightbulb.
- Think about the problem during otherwise unproductive times such as when waiting in line, walking, driving, or falling asleep.
- When stuck, take a break and let your brain work on the problem subconsciously.
- Be patient. It may take a while for the solution to "click."

4. **Evaluate** (in the sense of "Reflect on") your solution and check your work.

- (1) **Does the solution answer the question or solve the problem?**
- (2) **Does the solution make sense?**
- Skipping this step often results in "stupid" mistakes that are easy to catch.
- Are there any obvious errors or contradictions?
- Does the solution fit within a range of plausible solutions?
- Is there another, independent approach to validate the solution or confirm that it is plausible?
- Do the units make sense?
- Check your work. This is why organized solutions are important.
- Proofread any written work, ideally at least a day after you wrote it.

5. **Extend** (Generalize) the solution or problem and reflect.

- Is there another, possibly easier, way to solve the problem?
- Can you generalize the problem?
- Would you do something different to improve the solution?





# Chapter 1

## Models, Graphs, and Functions

### 1.1 What is a Function?

#### 1.1.1 Function Concepts

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask “If I know one quantity, can I then determine the other?” This establishes the idea of an input quantity, or **independent variable**, and a corresponding output quantity, or **dependent variable**. From this, we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

**Definition 1.1.1.** A **function** is a rule for a relationship between an **input** (or **independent**) quantity and an **output** (or **dependent**) quantity in which each input value uniquely determines one output value. We say “the output is a function of the input.”

**Example 1.1.1.** In the height and age example above, is height a function of age? Is age a function of height?

**Solution:** In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. You cannot have two different heights at any one instant in time.

However, age is not a function of height, since one height input might correspond with more than one output age. Once you have stopped growing, your height essentially remains constant for the rest of your life.

#### 1.1.2 Representing Functions

Functions can be represented in many ways:

1. Words,
2. Tables of values,
3. Graphs, or
4. Formulas.

Example 1.1.1 represented a function in words, but it will be convenient to streamline the discussion of a function. To that end, we introduce notation of functions.

**Function Notation.** To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write “height is a function of age”, we could use the descriptive variable  $h$  to represent height and we could use the descriptive variable  $a$  to represent age.

If we name the function  $f$  we could write “height is a function of age” as “ $h$  is  $f$  of  $a$ ,” or more simply:

$$h = f(a) .$$

We could instead name the function  $h$  and write  $h(a)$ , which is read “ $h$  of  $a$ .”

We can use any variable to name the function; the notation  $h(a)$  shows us that  $h$  depends on  $a$ . The value “ $a$ ” must be put into the function “ $h$ ” to get a result.

**Note 1.1.1.** Be careful! The parentheses indicate that age is the input into the function. Do not confuse these parentheses with multiplication!

**Example 1.1.2.** A function  $N = f(y)$  gives the number of police officers,  $N$ , in a town in year  $y$ . What does  $f(2005) = 300$  tell us?

**Solution:** When we read  $f(2005) = 300$ , we see the input quantity is 2005, which is a value for the input quantity of the function: the year ( $y$ ). The output value is 300, the number of police officers ( $N$ ), a value for the output quantity. Remember  $N = f(y)$ . This tells us that in the year 2005 there were 300 police officers in the town.

**Tables as Functions.** A table lists the input and corresponding output values of a function.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1.1 below represents the age of a child in years and his corresponding height. This represents just some of the data available for the age and height of the child.

<b>(Input:)</b> $a$ , age (years)	4	5	6	7	8	9	10
<b>(Output:)</b> $h$ , height (inches)	40	42	44	47	50	52	54

Table 1.1: Tabulating height as a function of age for a child.

From this, we can create equations such as  $h(6) = 44$ , meaning that when the child was 6 years old, he was 44 inches tall.

**Example 1.1.3.** Which of these tables define a function (if any)?

Table A.		Table B.		Table C.	
Input	Output	Input	Output	Input	Output
2	1	-3	5	1	0
5	3	0	1	5	2
8	6	4	5	5	4

**Solution:** Tables A and B define functions. In both tables, each input corresponds to exactly one output. Table C does not define a function since the input value of 5 corresponds with two different output values: 2 and 4.

**Graphs as Functions** A function can often be represented as a **graph**, a set of **points** plotted on **coordinate axes**: the **horizontal axis** and the **vertical axis**. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical axis.

Points on the graph are represented by ordered pairs of the form  $(a, b)$ . Beginning at the **origin** (the point  $(0, 0)$ ), you move  $a$  units to the left and  $b$  units up to arrive at the point  $(a, b)$ . If  $a < 0$ , then movement is to the left and if  $b < 0$ , then movement is down.

The horizontal and vertical axes are typically called the  $x$ -axis and the  $y$ -axis, but these axes can be labeled with any variable name, not just  $x$  and  $y$ . We say  $y$  is a function of  $x$ , or  $y = f(x)$  when the function is named  $f$ . The point  $(a, b)$  lies on the graph of the function  $f$  if and only if  $f(a) = b$ .

As an example, Figure 1.1 is a plot of the data from Table 1.1.

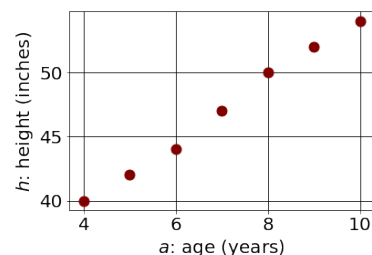
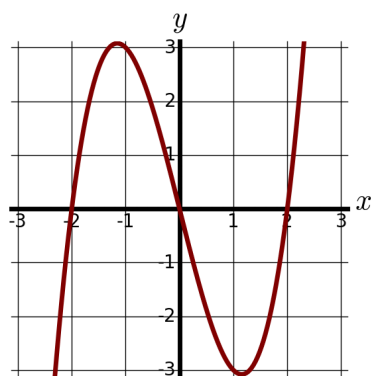
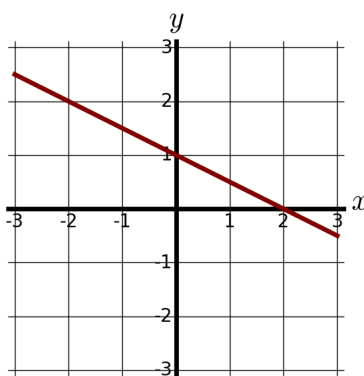


Figure 1.1: A plot of height data.

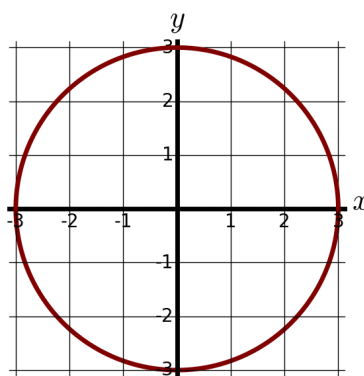
**Example 1.1.4.** Which of these graphs defines a function  $y = f(x)$ ?



Graph A.



Graph B.



Graph C.

**Solution:** Looking at the graphs above, Graphs A and B define a function  $y = f(x)$ , since for each input value along the horizontal ( $x$ ) axis, there is exactly one corresponding output value, determined by the  $y$ -value of the graph. Graph C does not define a function  $y = f(x)$  since some input values, such as  $x = 2$ , correspond with more than one output value.

The **vertical line test** is an easy way to determine whether a graph defines a function or not. Imagine drawing vertical lines through the graph. Since a function has exactly one output for every input, if there is a vertical line that would cross the graph more than once, then the graph does not define a function.

Figure 1.2 illustrates how Graph C from Example 1.1.4 fails the vertical line test. The vertical line  $x = 2$  (in gold) intersects Graph C (in maroon) at two points; there are two outputs for the input  $x = 2$ .

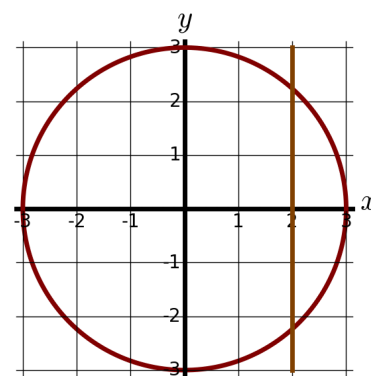


Figure 1.2: A graph failing the vertical line test.

**Formulas as Functions** When possible, it is very convenient to define relationships between quantities using a formula. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

**Example 1.1.5.** Express the relationship  $2N + 6p = 12$  as a function  $p = f(n)$  if possible.

**Solution:** To express the relationship in this form, we need to be able to write the relationship where  $p$  is a function of  $N$ , which means writing it as  $p = \text{something involving } N$ , or  $p$  in terms of  $N$ . We proceed using algebra.

$$2N + 6p = 12$$

$$6p = 12 - 2N$$

$$p = \frac{12 - 2N}{6}$$

$$p = \frac{12}{6} - \frac{2N}{6}$$

$$p = 2 - \frac{1}{3}N$$

Subtract  $2N$  from both sides.

Divide both sides by 6 and simplify.

We can now express  $p$  as a function of  $N$ :

$$p = f(N) = 2 - \frac{1}{3}N .$$

It is important to note that not every relationship can be expressed as a function with a formula. Consider the examples of the boy's height as a function of age in Table 1.1 or a company's stock price as a function of time. Neither situation allows one to define the relationship using a formula precisely, yet we still may want to analyze aspects of these relationships using tools of calculus. The rest of the chapter will introduce us to some tools to help us with that.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed.

### 1.1.3 Evaluating Functions.

The fundamental use of a function is to **evaluate** the function: “plugging in” some number into the function, or more precisely, to determine the corresponding output for a given input. In other words, we substitute or replace the input variable of the function with the input value. Evaluating will always produce one result, since each input of a function corresponds to exactly one output. We can evaluate a function from a table, graph, or formula.

Another related use is to determine the input or inputs of a function, given an output of the function. This is called **solving** an equation and could produce more than one solution, since different inputs can produce the same output.

**Note 1.1.2.** The concepts of evaluating and solving often get confused. When we use the word *solve*, we will be *solving a problem* or *solving an equation for an unknown quantity or variable*. It does not make sense to solve a function, since a function is merely a mathematical expression and not an equation.

**Example 1.1.6.** Let  $Q = g(n)$  and use the table shown.

$Q = g(n)$ as a table.					
$n$	1	2	3	4	5
$Q$	8	6	7	6	8

- (a) Evaluate  $g(3)$ .

**Solution:** Evaluating  $g(3)$ , read “ $g$  of 3,” means that we need to determine the output value,  $Q$ , of the function  $g$  given the input value of  $n = 3$ . Looking at the table, we see the output corresponding to  $n = 3$  is  $Q = 7$ , allowing us to conclude  $g(3) = 7$ .

- (b) Solve  $g(n) = 6$  for  $n$ .

**Solution:** Solving  $g(n) = 6$  means we need to determine what input values,  $n$ , produce an output value of 6. Looking at the table we see there are two solutions:  $n = 2$  and  $n = 4$ .

When we evaluate  $g(n)$  at 2, our output is  $Q = 6$ .

When we evaluate  $g(n)$  at 4, our output is also  $Q = 6$ .

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

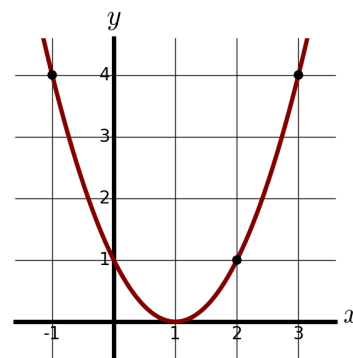
**Example 1.1.7.** Consider the graph of a function  $f(x)$  to the right.

- (a) Evaluate  $f(2)$ .

**Solution:** To evaluate  $f(2)$ , we find the input of  $x = 2$  on the horizontal ( $x$ ) axis. Moving up to the graph gives the point  $(2, 1)$ , giving an output of  $y = 1$ . Therefore,  $f(2) = 1$ .

- (b) Solve  $f(x) = 4$  for  $x$ .

**Solution:** To solve  $f(x) = 4$ , we find the value 4 on the vertical ( $y$ ) axis because if  $f(x) = 4$  then 4 is the output. Moving horizontally across the graph gives two points with the output of 4:  $(-1, 4)$  and  $(3, 4)$ . These give the two solutions to  $f(x) = 4$ :  $x = -1$  or  $x = 3$ . This means  $f(-1) = 4$  and  $f(3) = 4$ , or when the input is  $-1$  or  $3$ , the output is  $4$ .



**Example 1.1.8.** Let  $k(t) = t^3 + 2$ .

- (a) Evaluate  $k(2)$ .

**Solution:** To evaluate  $k(2)$ , we replace  $t$  with 2 in the expression  $t^3 + 2$ , then simplify.

$$\begin{aligned} k(2) &= 2^3 + 2 \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

So  $k(2) = 10$ .

- (b) Solve  $k(t) = 1$  for  $t$ .

**Solution:** To solve  $k(t) = 1$ , we set the formula for  $k(t)$  equal to 1, and solve for the input value that will produce that output.

$\begin{aligned} k(t) &= 1 \\ t^3 + 2 &= 1 \\ t^3 &= -1 \\ t &= \sqrt[3]{-1} = -1 \end{aligned}$	<p>Substitute the original formula <math>k(t) = t^3 + 2</math>.</p> <p>Subtract 2 from each side.</p> <p>Evaluate the cube root of each side.</p>
--	---

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

To check our work, we want to know if  $k(t) = 1$  is a true statement when  $t = -1$ .

$$\begin{aligned} k(-1) &= (-1)^3 + 2 \\ &= -1 + 2 \\ &= 1, \end{aligned}$$

which was the desired result.

### 1.1.4 Cost, Revenue, and Profit

Suppose that your club wants to raise funds by selling T-shirts. The screen printing shop that will make the shirts will charge your club \$50 to cover overhead costs and \$5 per shirt for the shirts themselves. You decide to charge \$15 per shirt. Some questions naturally arise. How many shirts need to be sold to **break even**? How much **profit** can be expected?

We can describe situations such as this with functions. The **total cost** to produce these shirts combines **fixed costs** and **variable costs**. The fixed costs are also called **overhead costs** and do not depend on the number of T-shirts made, while the variable costs are the per item cost. If  $n$  is the number of T-shirts the screen printing shop will make,  $F(n)$  is the fixed costs and  $V(n)$  is the variable costs, then the **cost function** to produce  $n$  T-shirts is

$$\begin{aligned}\text{total cost} &= \text{fixed costs} + \text{variable costs} \\ C(n) &= F(n) + V(n) \\ &= \$50 + (\$5 \text{ per shirt})(n \text{ shirts}) \\ &= 50 + 5n \text{ dollars}\end{aligned}$$

If you sell each shirt for \$15 and you sell  $n$  shirts, then your **revenue** from selling  $n$  shirts will be  $15n$  dollars. This is your **revenue function**:  $R(n) = 15n$  dollars.

Finally, the **profit** that your club will earn from selling  $n$  shirts is the revenue minus the total costs. If  $P(n)$  is the profit from selling  $n$  items, then the **profit function** is

$$P(n) = R(n) - C(n) .$$

In this example, we have

$$\begin{aligned}P(n) &= R(n) - C(n) \\ &= 15n - (50 + 5n) \text{ dollars} \\ &= 15n - 50 - 5n \text{ dollars} \\ &= 10n - 50 \text{ dollars}\end{aligned}$$

**Example 1.1.9.** Consider the T-shirt scenario above.

- (a) What is the profit from selling 30 shirts?

**Solution:**  $P(30) = 10 \cdot 30 - 50 = 300 - 50 = \$250$ .

If you sell 30 T-shirts, then you profit \$250.

- (b) What is the profit from selling 3 shirts?

**Solution:**  $P(3) = 10 \cdot 3 - 50 = 30 - 50 = \$(-20)$ .

If you only sell 3 T-shirts, then you lose \$20; your profit is negative.

- (c) What is the profit from selling 0 shirts?

**Solution:**  $P(0) = 10 \cdot 0 - 50 = 0 - 50 = \$(-50)$ .

If you don't sell any T-shirts, then you lose \$50; your profit is negative.

The **break-even point** in this context is the minimum number of T-shirts that must be sold in order to have a profit of at least \$0. How do we find this? If our profit is \$0, then we turn that sentence into a mathematical equation and solve for the number of shirts. The profit from selling  $n$  shirts is  $P(n)$ , "is" is "=", and 0 is 0.

$$\begin{aligned}P(n) &= 0 \\ 10n - 50 &= 0 \\ 10n &= 50 \\ n &= \frac{50}{10} = 5\end{aligned}$$

So you need to sell at least five T-shirts in order to break even.

**Note 1.1.3.** Note the distinction between  $P(0)$  and  $P(n) = 0$ .  $P(0)$  is the profit from selling 0 items, while  $P(n) = 0$  is an equation whose solution tells you the number of items that must be sold in order to have no profit.

**Definition 1.1.2.** In summary, the **total cost** to produce  $n$  items,  $C(n)$ , is the combination of **fixed costs** ( $F(n)$ ) and **variable costs** ( $V(n)$ ). The fixed costs are also called **overhead costs** and is constant regardless of the number of items made, while the variable costs depend on the number of items being made. So

$$C(n) = F(n) + V(n) .$$

The **revenue** function,  $R(n)$  gives the amount of money brought in from selling  $n$  items. The difference between revenue and total costs is **profit** ( $P(n)$ ) from selling  $n$  items:

$$P(n) = R(n) - C(n) .$$

The **break-even point** is the fewest number of items,  $n$ , that must be sold in order for  $P(n) \geq 0$ , in other words, the fewest number of items to guarantee that you won't have negative profit and be losing money.

### 1.1.5 Domain and Range.

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of the models we create. In our most recent example, it wouldn't make sense to sell  $-4$  T-shirts or  $4.4$  T-shirts. We also wouldn't expect to sell a billion T-shirts for a university club fund raiser. When using a function to describe a real-world scenario, we need to set common sense boundaries for the input and output. Here is another example.

Table 1.2 shows a relationship between the circumference,  $c$ , and height,  $h$ , of a tree as it grows. In this table, we would consider height as a function of circumference:  $h = h(c)$ .

<b>Circumference:</b> $c$ (feet)	1.7	2.5	5.5	8.2	13.7
<b>Height:</b> $h$ (feet)	24.5	31.0	45.2	54.6	92.1

Table 1.2: Height of a tree as a function of its circumference.

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of  $-3$  feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the **domain** and **range** of the function.

**Definition 1.1.3.** The **domain** of a function is the set of possible input values to the function.

The **range** of a function is the set of possible output values of the function.

**Example 1.1.10.** Using Table 1.2 above, determine a reasonable domain and range of the function  $h$ .

**Solution:** We can combine the data provided with additional research and our own reason to determine an appropriate domain and range of the function  $h = h(c)$ . For the domain, it doesn't make sense for the circumference (input) to be negative, so  $c \geq 0$ . For a maximum circumference, we could make an educated guess at a reasonable value, or look up that the maximum recorded circumference is about 119 feet<sup>1</sup>. With this information, we would say a reasonable domain is  $0 \leq c \leq 119$  feet.

Similarly for the range, if we only consider the tree when the sprout has broken through the ground, it doesn't make sense to have negative heights. The maximum recorded height of a tree could be looked up to be 379 feet, so a reasonable range is  $0 \leq h \leq 379$  feet.

**Interval Notation.** A convenient alternative to the notation using inequalities is **interval notation**, in which intervals of values are referred to by the starting and ending values. Parentheses “ $()$ ” are used for “strictly less than,” and square brackets “[ $]$ ” are used for “less than or equal to.” Since infinity,  $\infty$ , is not a number, neither  $-\infty$  nor  $\infty$  are included in the domain and range of a function, so we always use curved parentheses with

<sup>1</sup><http://en.wikipedia.org/wiki/Tree>, retrieved July 19, 2010

Inequality	Interval Notation
$5 \leq x \leq 10$	$[5, 10]$
$5 < x \leq 10$	$(5, 10]$
$5 \leq x < 10$	$[5, 10)$
$5 < x < 10$	$(5, 10)$
$x < 10$	$(-\infty, 10)$
$5 \leq x$	$[5, \infty)$
All real numbers	$(-\infty, \infty)$

Table 1.3: Interval Notation

$\pm\infty$ . Table 1.3 shows how inequalities correspond to interval notation for an arbitrary variable  $x$ .

To combine two intervals together, we can use the word “or”. In interval notation, we use the union symbol,  $\cup$ , to combine two unconnected intervals together.

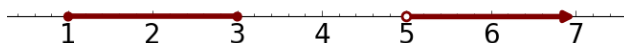


Figure 1.3: A union of intervals.

**Example 1.1.11.** Describe the intervals of values shown in Figure 1.3 using inequalities and using interval notation.

**Solution:** To describe the values,  $x$ , that lie in the intervals shown above we would say, “ $x$  is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

As an inequality it is:  $1 \leq x \leq 3$  or  $x > 5$ .

In interval notation:  $[1, 3] \cup (5, \infty)$ .

**Example 1.1.12.** Find the domain of each function.

(a)  $f(x) = 2\sqrt{x+4}$

**Solution:** Since we cannot evaluate the square root of a negative number, we need the inside of the square root to be non-negative.  $x+4 \geq 0$  when  $x \geq -4$ . (Subtract 4 from both sides of the inequality.) Therefore, the domain of  $f(x)$  is  $[-4, \infty)$ .

(b)  $g(x) = \frac{3}{6-3x}$

**Solution:** We cannot divide by zero, so we need the denominator to be non-zero. Solving  $6-3x=0$  for  $x$ , we have  $x=2$ , so we must exclude 2 from the domain. Therefore, the domain of  $g(x)$  is  $(-\infty, 2) \cup (2, \infty)$ .

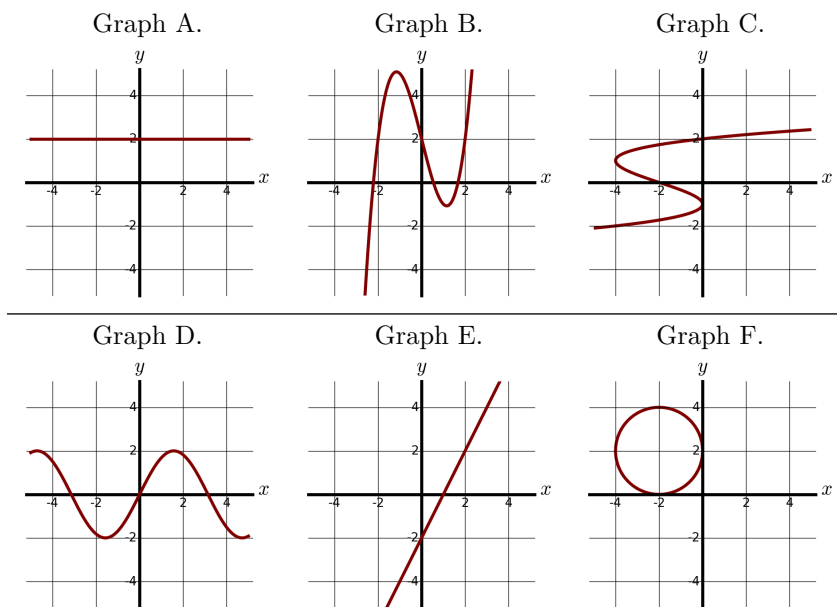
### 1.1.6 Exercises

- The amount of garbage,  $G$ , produced by a city with population  $p$  is given by  $G = f(p)$ .  $G$  is measured in tons per week, and  $p$  is measured in thousands of people.
  - The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function  $f$ .
  - Explain the meaning of the statement  $f(5) = 2$ .
- The number of cubic yards of dirt,  $D$ , needed to cover a garden with area  $a$  square feet is given by  $D = g(a)$ .
  - A garden with area 5000 ft<sup>2</sup> requires 50 cubic yards of dirt. Express this information in terms of the function  $g$ .
  - Explain the meaning of the statement  $g(100) = 1$ .
- Let  $n(t)$  be the number of subscribers to a YouTube channel  $t$  years after 2005. Explain the meaning of each statement.
  - $n(5) = 300$
  - $n(10) = 4000$
- Let  $p(t)$  be the stock price, in dollars, of Valvoline (VTV)  $t$  years after its Initial Public Offering (IPO) on September 23, 2016. Explain the meaning of each statement.

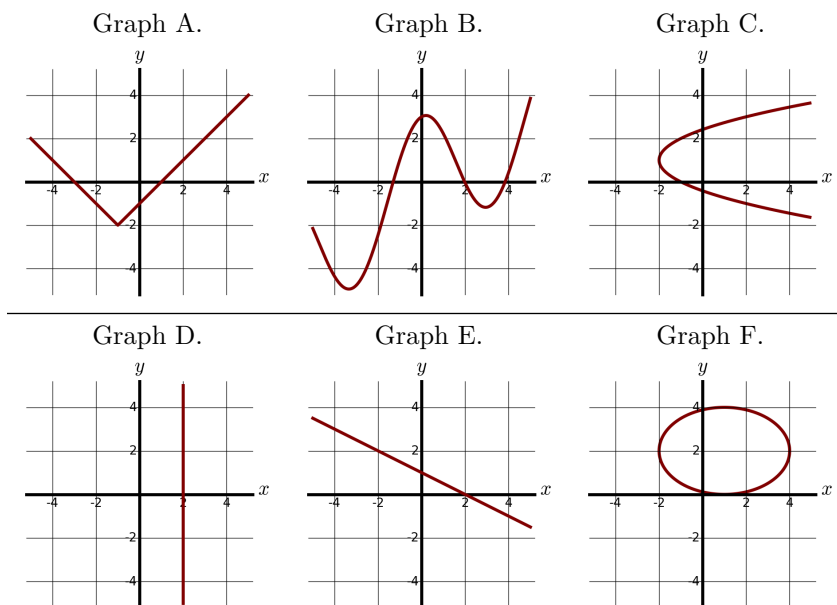


- (a)  $p(0) = 23.10$   
 (b)  $p(1) = 23.27$   
 (c)  $p(2) = 21.47$

5. Select all of the following graphs which represent  $y$  as a function of  $x$ .



6. Select all of the following graphs which represent  $y$  as a function of  $x$ .



7. Select all of the following tables which represent  $y$  as a function of  $x$ .

Table A.				Table B.				Table C.			
$x$	5	10	15	$x$	5	10	15	$x$	5	10	10
$y$	3	8	14	$y$	3	8	8	$y$	3	8	14

8. Select all of the following tables which represent  $y$  as a function of  $x$ .

Table A.				Table B.				Table C.			
$x$	2	6	13	$x$	2	6	6	$x$	2	6	13
$y$	3	10	10	$y$	3	10	14	$y$	3	10	14

9. Select all of the following tables which represent  $y$  as a function of  $x$ .

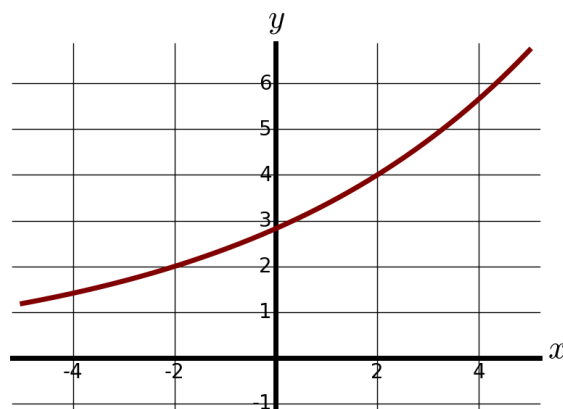
Table A.		Table B.		Table C.		Table D.	
$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$
0	-2	-1	-4	0	-5	-1	-4
3	1	2	3	3	1	1	2
4	6	5	4	3	4	4	2
8	9	8	7	9	8	9	7
3	1	12	11	16	13	12	13

10. Select all of the following tables which represent  $y$  as a function of  $x$ .

Table A.		Table B.		Table C.		Table D.	
$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$
-4	-2	-5	-3	-1	-3	-1	-5
3	2	2	1	1	2	3	1
6	4	2	4	5	4	5	1
9	7	7	9	9	8	8	7
12	16	11	10	1	2	14	12

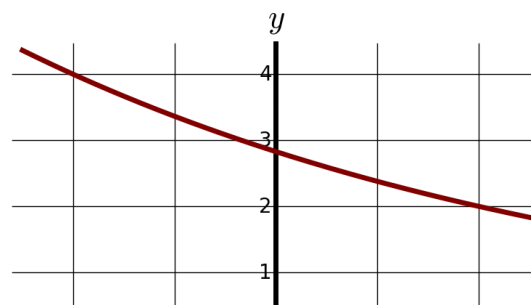
11. Let  $g(x)$  be the function graphed on the right.

- (a) Evaluate  $g(2)$ .  
 (b) Solve  $g(x) = 2$  for  $x$ .



12. Let  $f(x)$  be the function graphed on the right.

- (a) Evaluate  $f(4)$ .  
 (b) Solve  $f(x) = 4$  for  $x$ .



13. Consider the table at right.

(a) Evaluate  $f(3)$ .

(b) Solve  $f(x) = 1$  for  $x$ .

14. Consider the table at right.

(a) Evaluate  $g(8)$ .

(b) Solve  $g(x) = 7$  for  $x$ .

$x$	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

$x$	0	1	2	3	4	5	6	7	8	9
$g(x)$	62	8	7	38	86	73	70	39	75	34

For Exercises 15-22, evaluate:  $f(-2)$ ,  $f(-1)$ ,  $f(0)$ ,  $f(1)$ , and  $f(2)$ .

15.  $f(x) = 4 - 2x$

16.  $f(x) = 8 - 3x$

17.  $f(x) = 8x^2 - 7x + 3$

18.  $f(x) = 6x^2 - 7x + 4$

19.  $f(x) = 3 + \sqrt{x+3}$

20.  $f(x) = 4 - \sqrt[3]{x-2}$

21.  $f(x) = \frac{x-3}{x+1}$

22.  $f(x) = \frac{x-2}{x+2}$

23. Let  $f(t) = 3t + 5$ .

(a) Evaluate  $f(0)$ .

(b) Solve  $f(t) = 0$  for  $t$ .

24. Let  $g(p) = 6 - 2p$ .

(a) Evaluate  $g(0)$ .

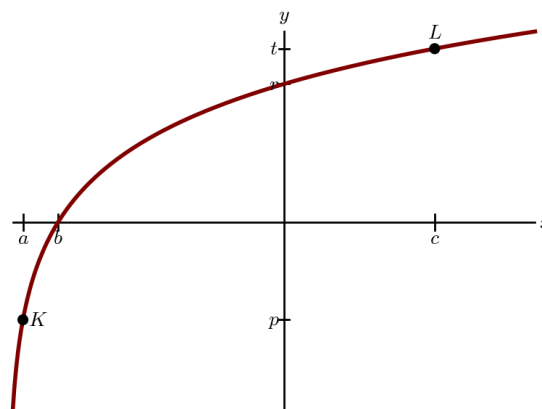
(b) Solve  $g(p) = 0$  for  $t$ .

25. Consider the graph of  $f(x)$  on the right.

(a) Evaluate  $f(c)$ .

(b) Solve  $f(x) = p$  for  $x$ .

(c) Find the coordinates of points  $L$  and  $K$ .



## 1.2 Models

A **model** or a **mathematical model** is a mathematical framework to help describe some phenomenon, specifically how an input or quantity affects or relates to some output. For example, one could construct a model to speculate how the price of an item affects the demand for that item. This model A model is used to simplify reality. Although good models can be used to predict future events, models do not dictate reality. A model has three main components.

1. One or more input variables, with specific descriptions of what these variables represent, including units.
2. One or more functions, with specific descriptions of what the output of the functions represent, including units.
3. Underlying assumptions that were used to simplify the situation.

**1.3 Linear Functions****1.4 Polynomial Functions****1.5 Exponential Functions****1.6 Logistic Functions****1.7 Logarithmic Functions****1.8 Trigonometric Functions****1.9 Combining Functions**

## Chapter 2

# The Derivative: Rates of Change of a Function

- 2.1 Slopes and Average Rates of Change
- 2.2 Tangent Lines and Instantaneous Rates of Change
- 2.3 Basic Applications of Slope
- 2.4 The Derivative
- 2.5 Limits of Functions
- 2.6 Finding Derivatives Algebraically
- 2.7 The Power Rule
- 2.8 The Product and Quotient Rules
- 2.9 The Chain Rule
- 2.10 Higher Order Derivatives



## Chapter 3

# Applications of Derivatives

3.1 Linear Approximations

3.2 Elasticity of Demand

3.3 Concavity and Inflection Points

3.4 Maximum and Minimum Function Values

3.5 Curve Sketching

3.6 Applied Optimization





## Chapter 4

# The Integral: Accumulation of a Rate Function

4.1 Accumulation

4.2 Approximating Area

4.3 Integrals

4.4 The Fundamental Theorem of Calculus

4.5 Integral Formulas

4.6 Integration by Substitution

4.7 Applications of Integrals



## Chapter 5

# Functions of Many Variables

5.1 Examples of Functions of Multiple Variables

5.2 Multivariable Functions

5.3 Partial Derivatives

5.4 Optimization with Multivariable Functions

5.5 Integrals with Multivariable Functions



# Index

Break-even point, 13

Cost

fixed, 12

total, 12

variable, 12

Domain, 13

Function, 9

cost, 12

profit, 13

revenue, 13

Interval notation, 14

Leibniz, Gottfried, 5

model, 18

Newton, Isaac, 5

Pólya, George, 6

Profit, 13

Range, 13

Revenue, 13

Union, 15

Vertical line test, 11