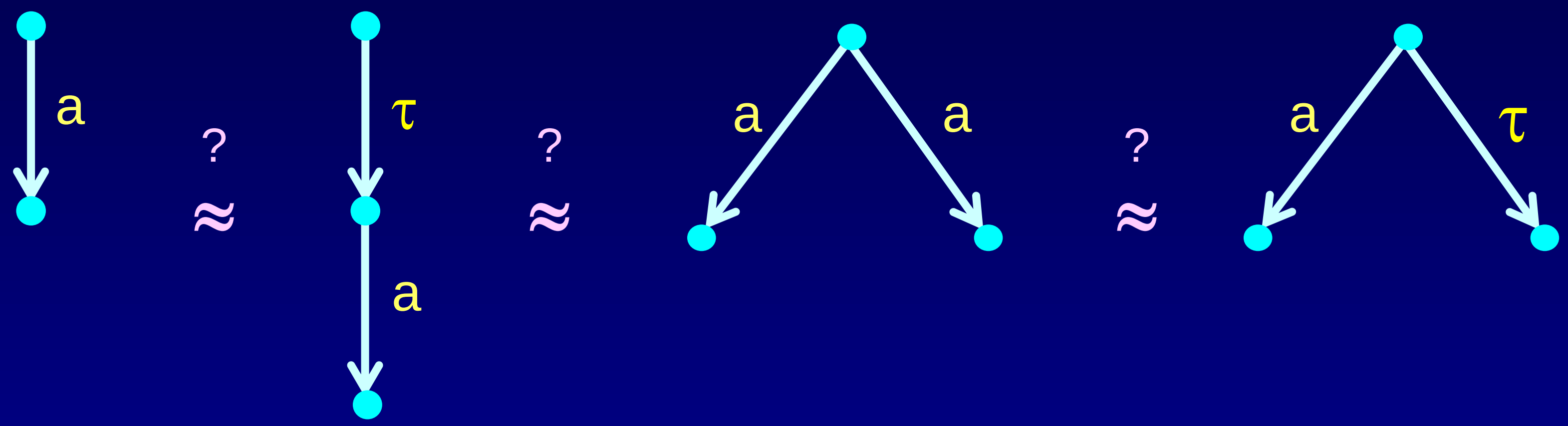
Equivalences on

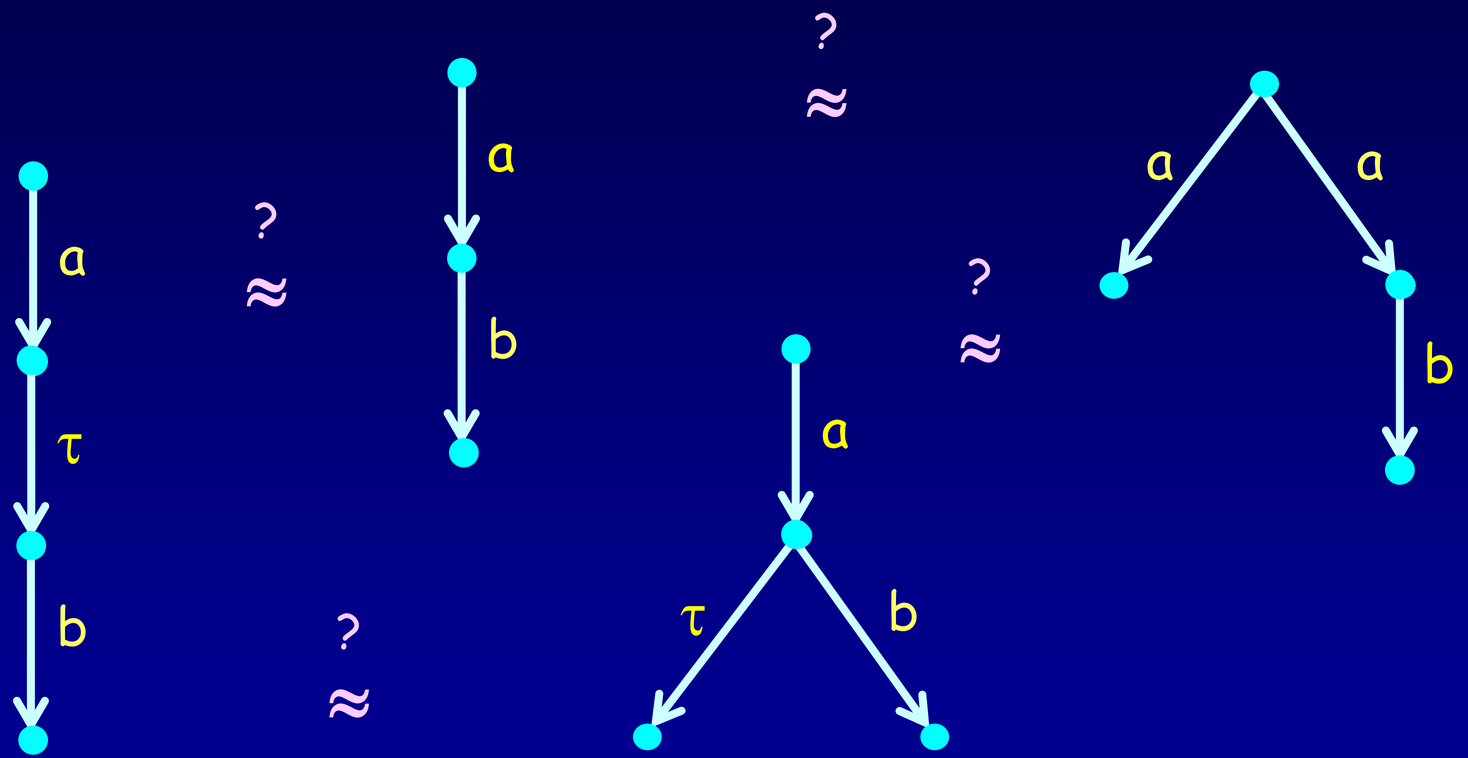
Labelled Transition Systems

Observable Behaviour



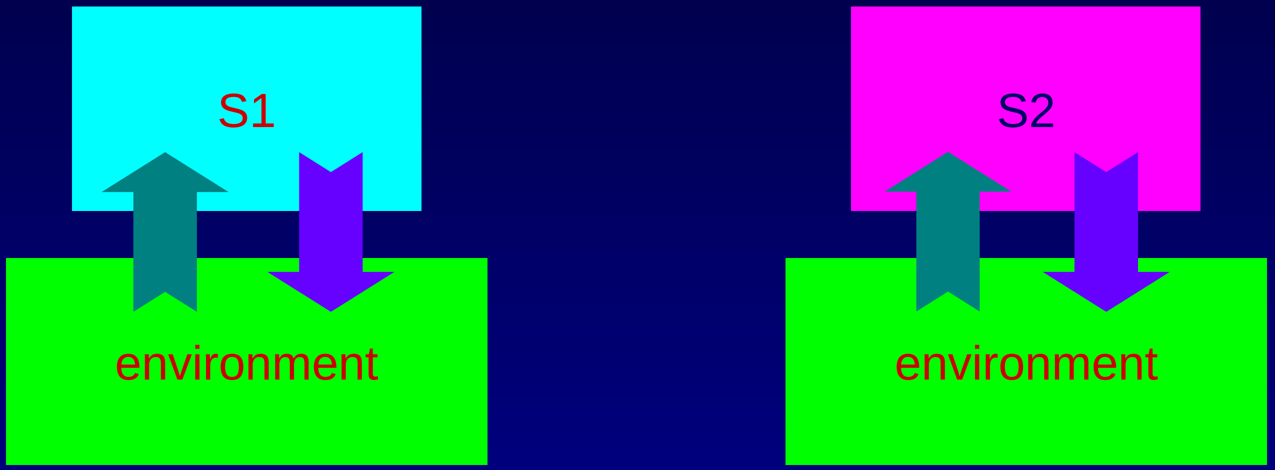
“ Some transition systems are more equal than others “

Observable Behaviour



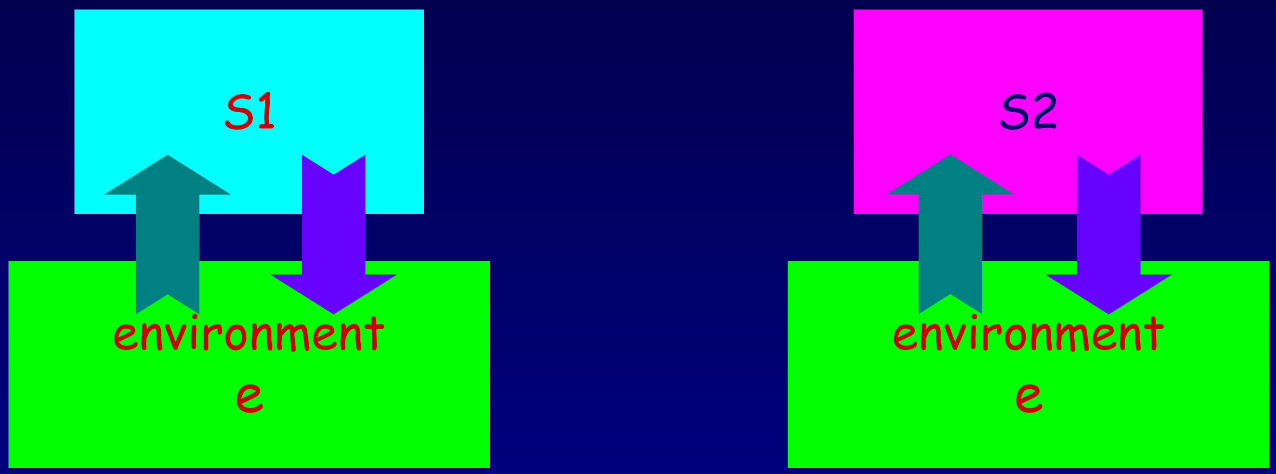
" Some transition systems are more equal than others "

Comparing Transition Systems



- ☞ Suppose an environment interacts with the systems:
  - ♦ the environment tests the system as black box by observing and actively controlling it;
  - ♦ the environment acts as a tester;
- ☞ Two systems are equivalent if they pass the same tests.

Comparing Transition Systems



$$S1 \approx S2 \quad \Leftrightarrow \quad \forall\, e \in E \,.\ \ obs\,(e, S1) = obs\,(e, S2)$$

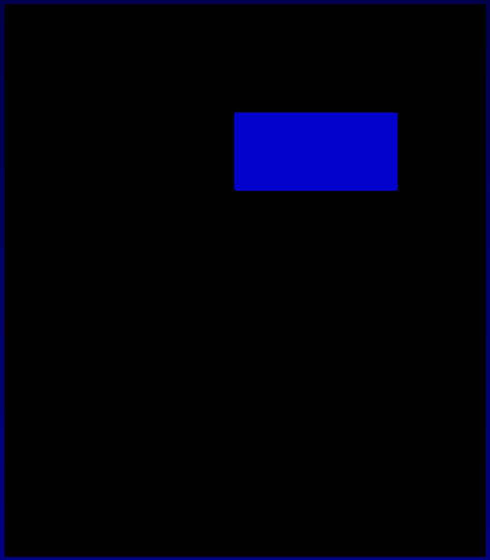
$\downarrow \quad \downarrow$

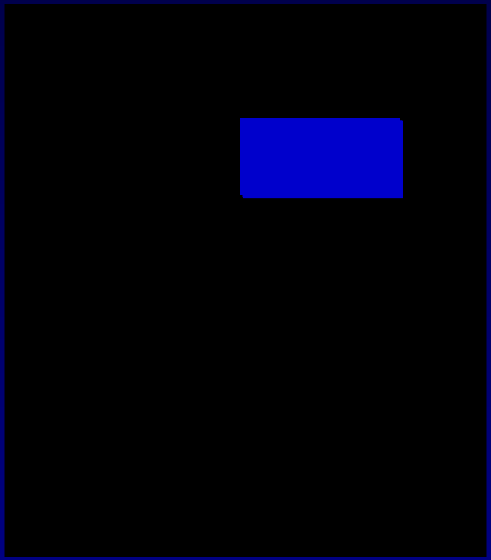
? ?

Equivalence of Transition Systems

a b

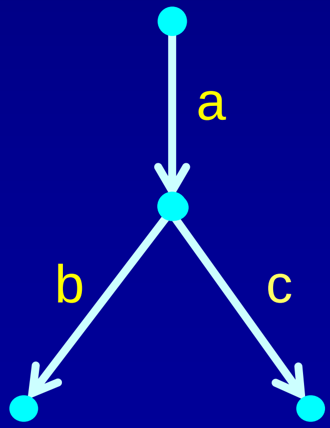
a c



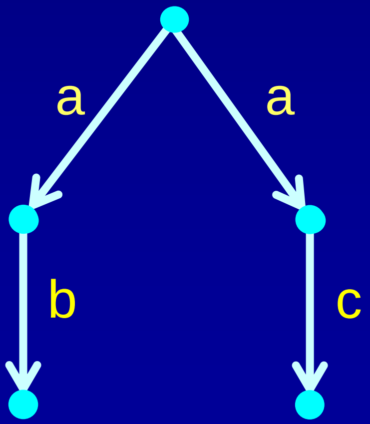


a b

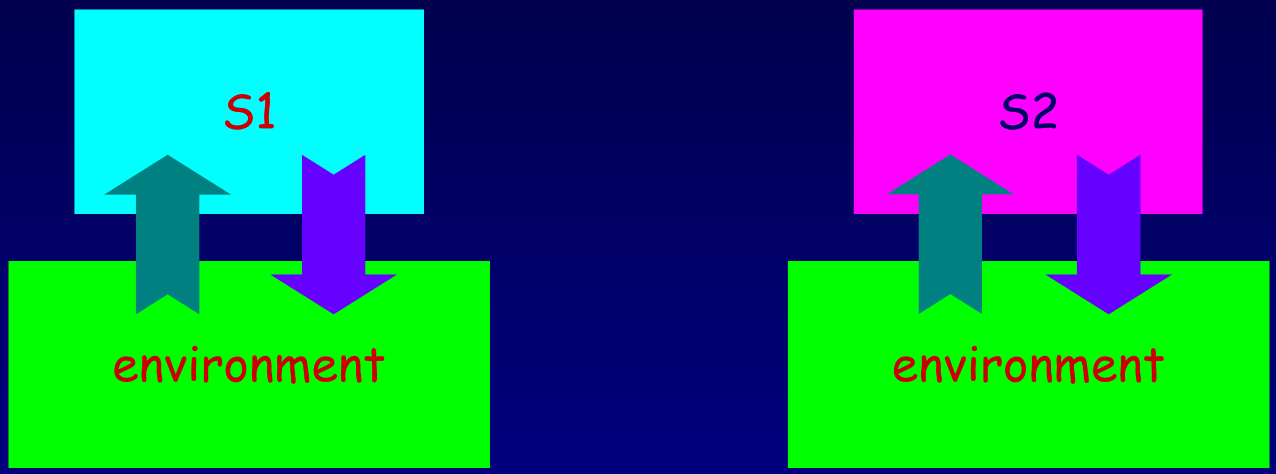
a c



$\approx_{tr}$



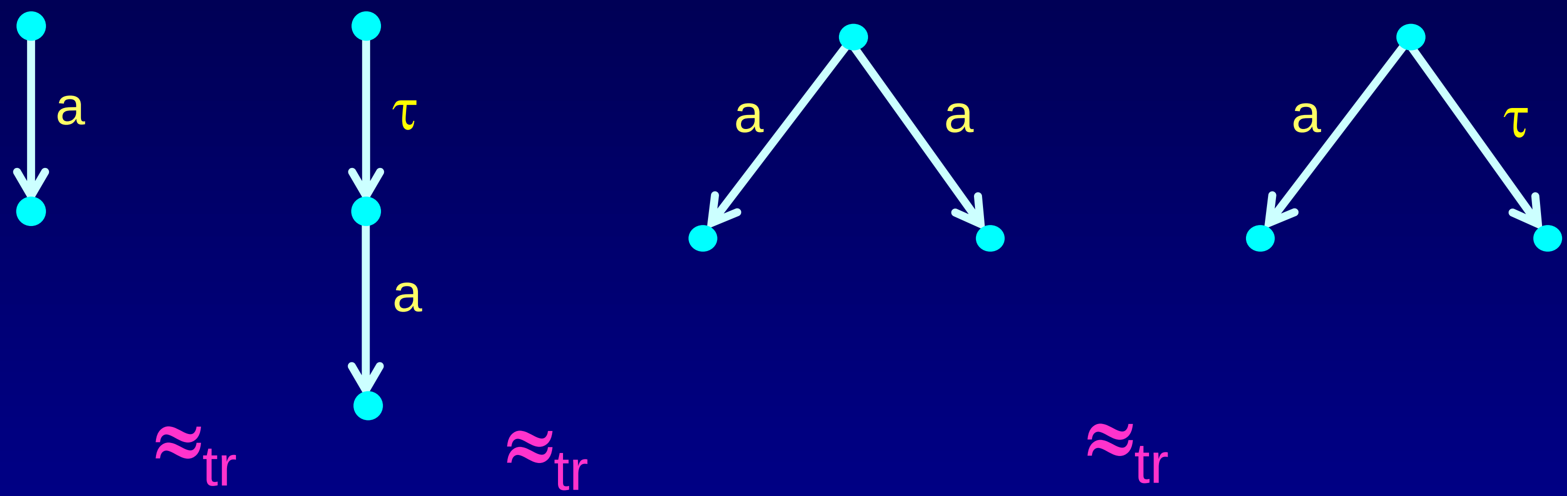
Trace Equivalence



$$S1 \;\approx_{tr}\; S2 \quad \Leftrightarrow \quad \text{traces}\,(\,S1\,) = \text{traces}\,(\,S2\,)$$

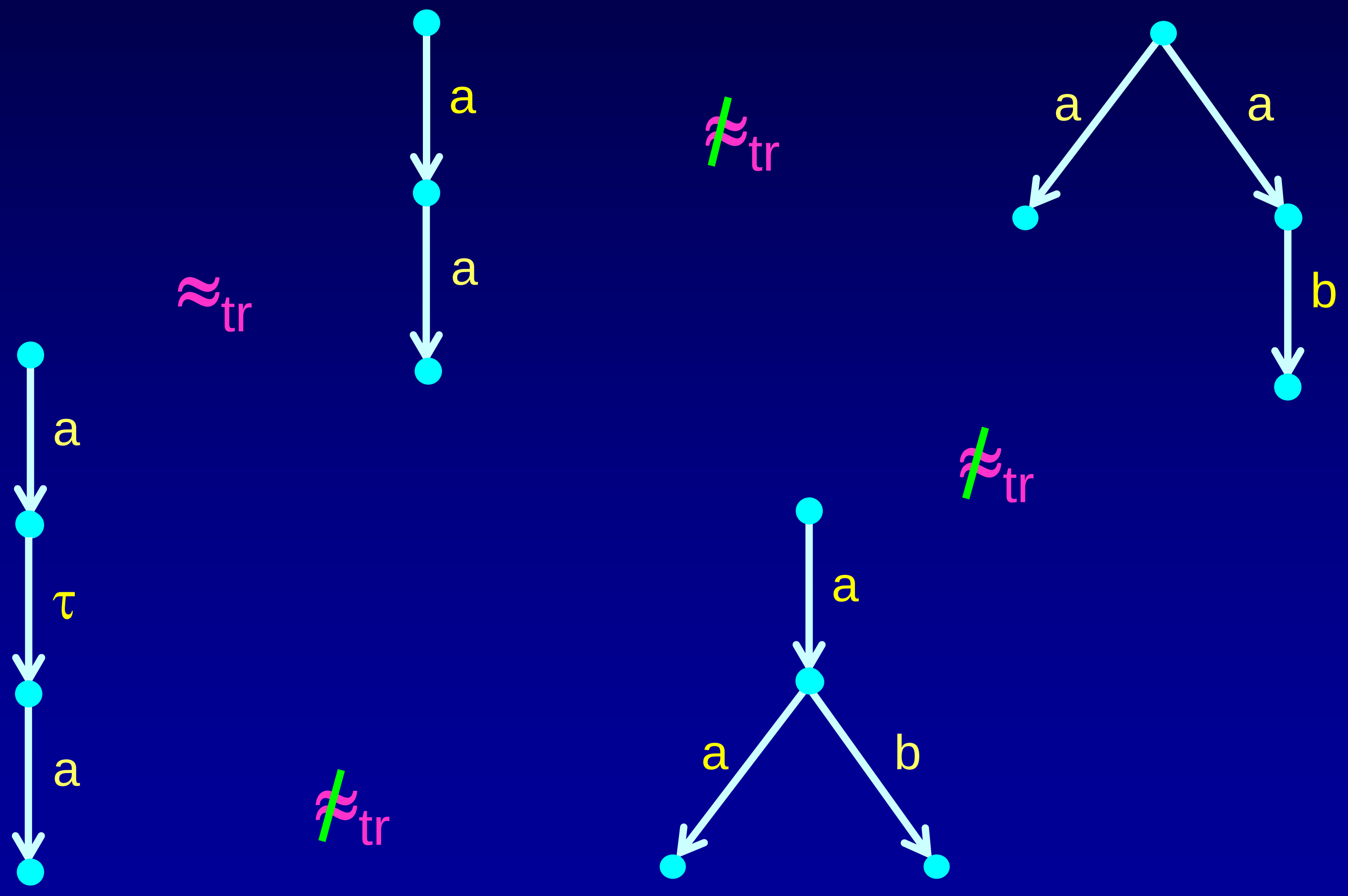
Traces: $\text{traces}\,(s) \;=\; \{\; \sigma \in L^* \mid s \stackrel{\sigma}{\Longrightarrow} \;\}$

Trace Equivalence

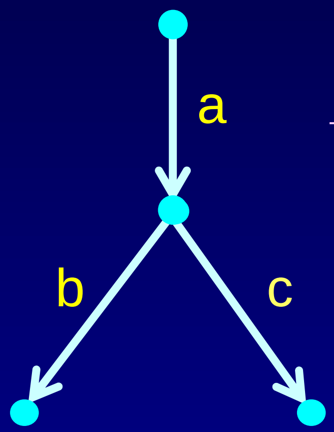


for all: traces (_) **=** { **ε, a** }

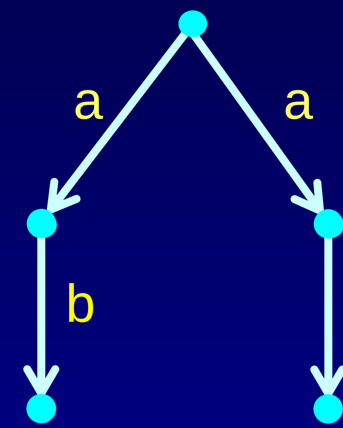
Trace Equivalence

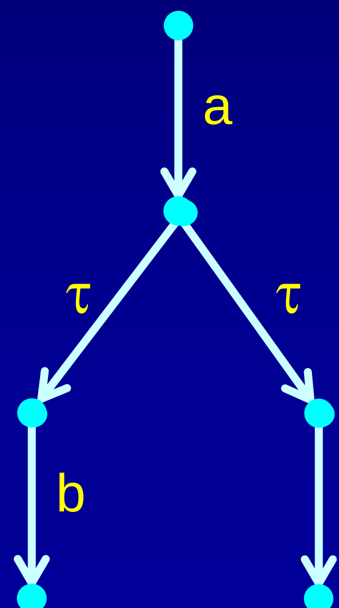


Trace Equivalence



$\approx_{tr}$



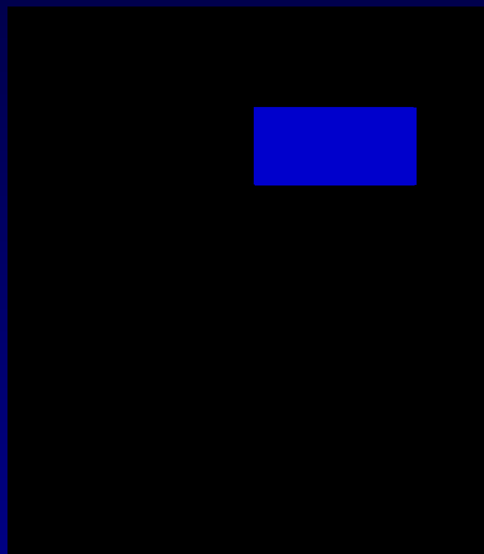


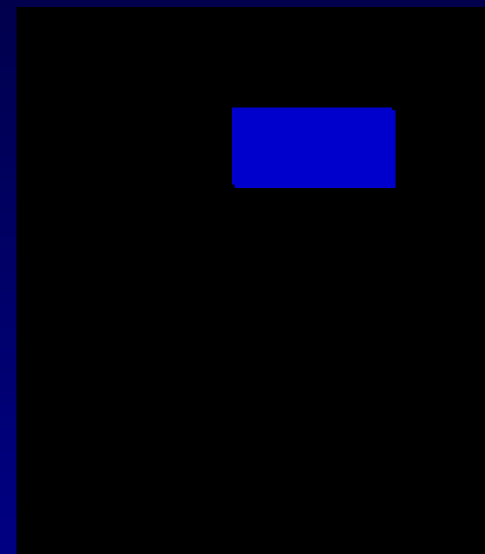
traces = { ε , a , a b , a c }

Equivalence of Transition Systems

a

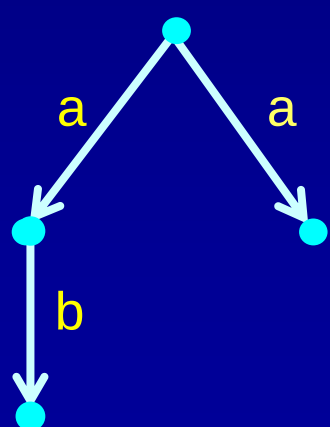
a b



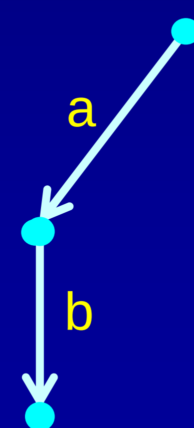


a

a b



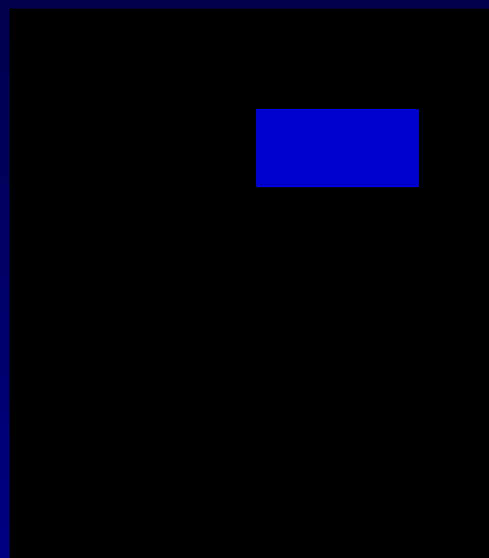
$\approx_{tr}$

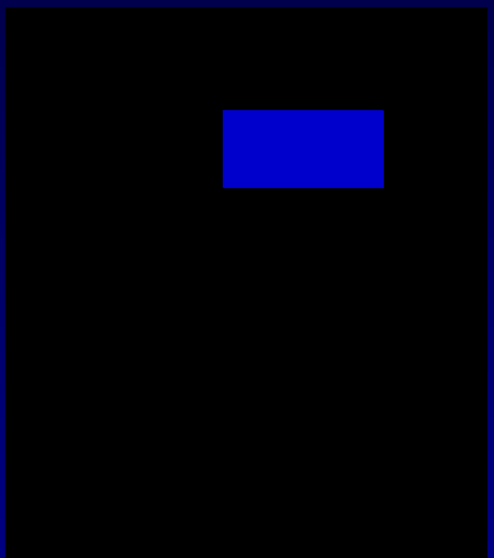


Equivalence of Transition Systems

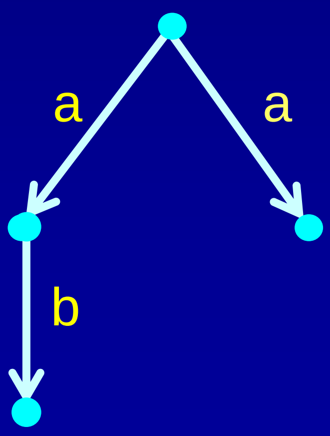
a √

a b √

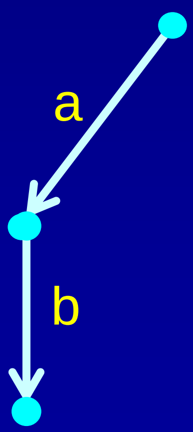




a b √



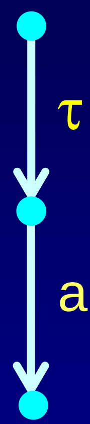
$\not\approx_{ctr}$



Completed Trace Equivalence

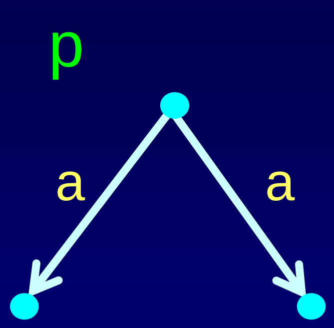
$\approx_{tr}$

$\approx_{ctr}$



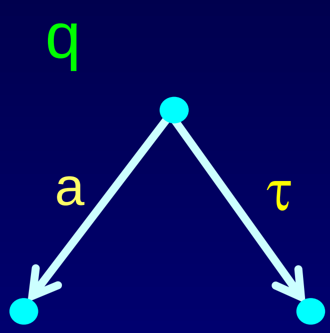
$\approx_{tr}$

$\approx_{ctr}$



$\approx_{tr}$

$\not\approx_{ctr}$



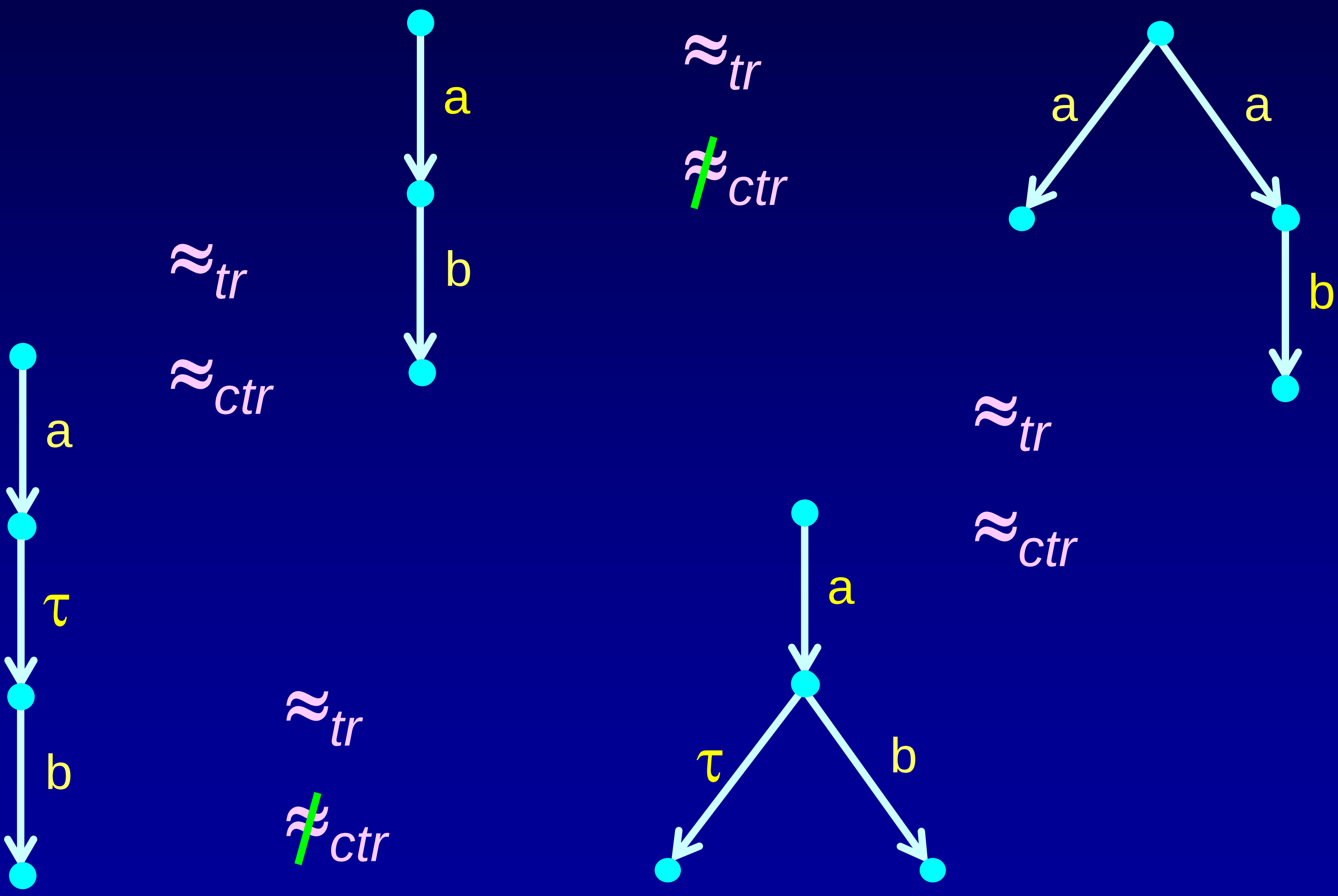
p after a refuses L

~~p after ε refuses L~~

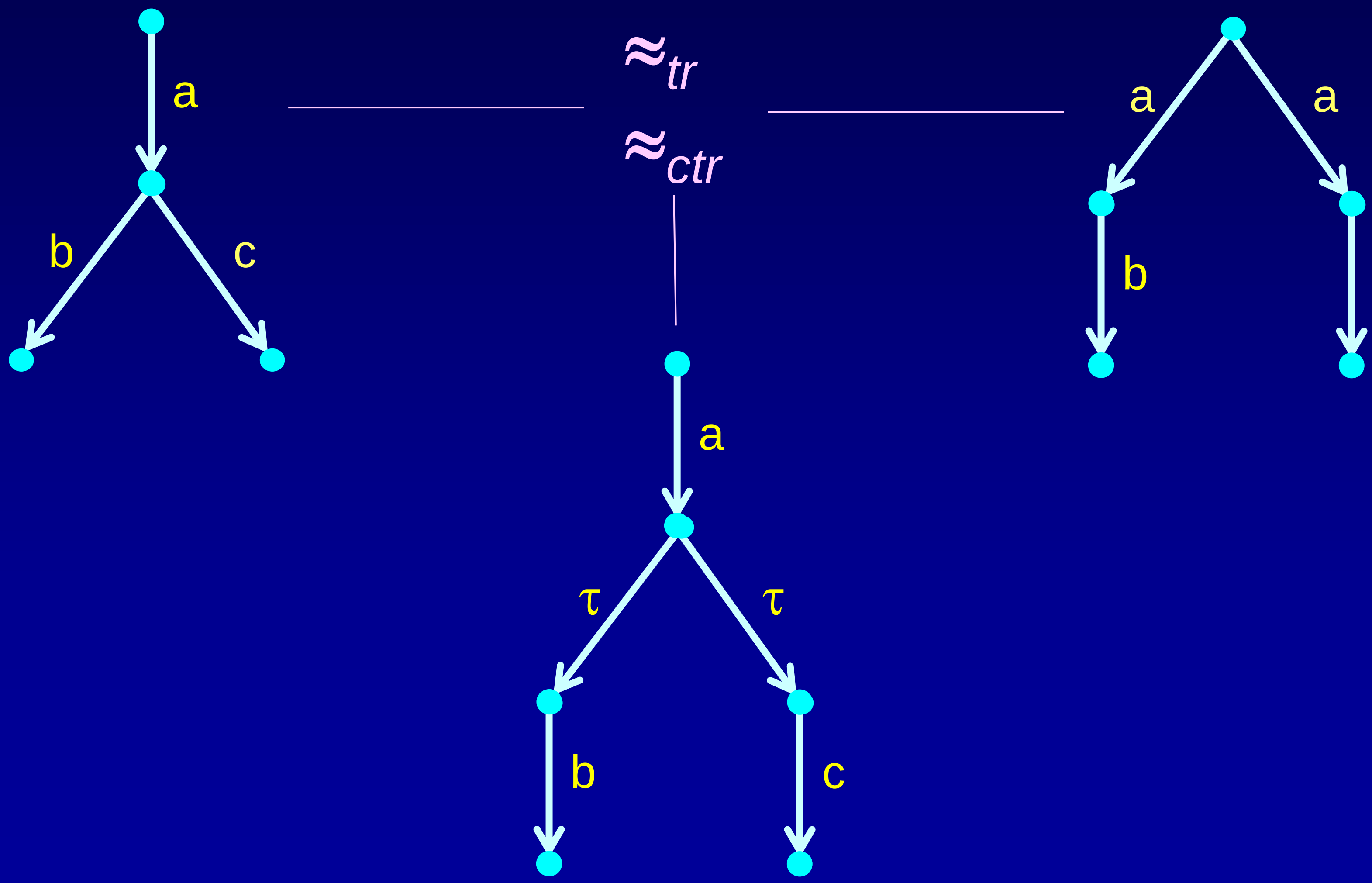
q after a refuses L

q after ε refuses L

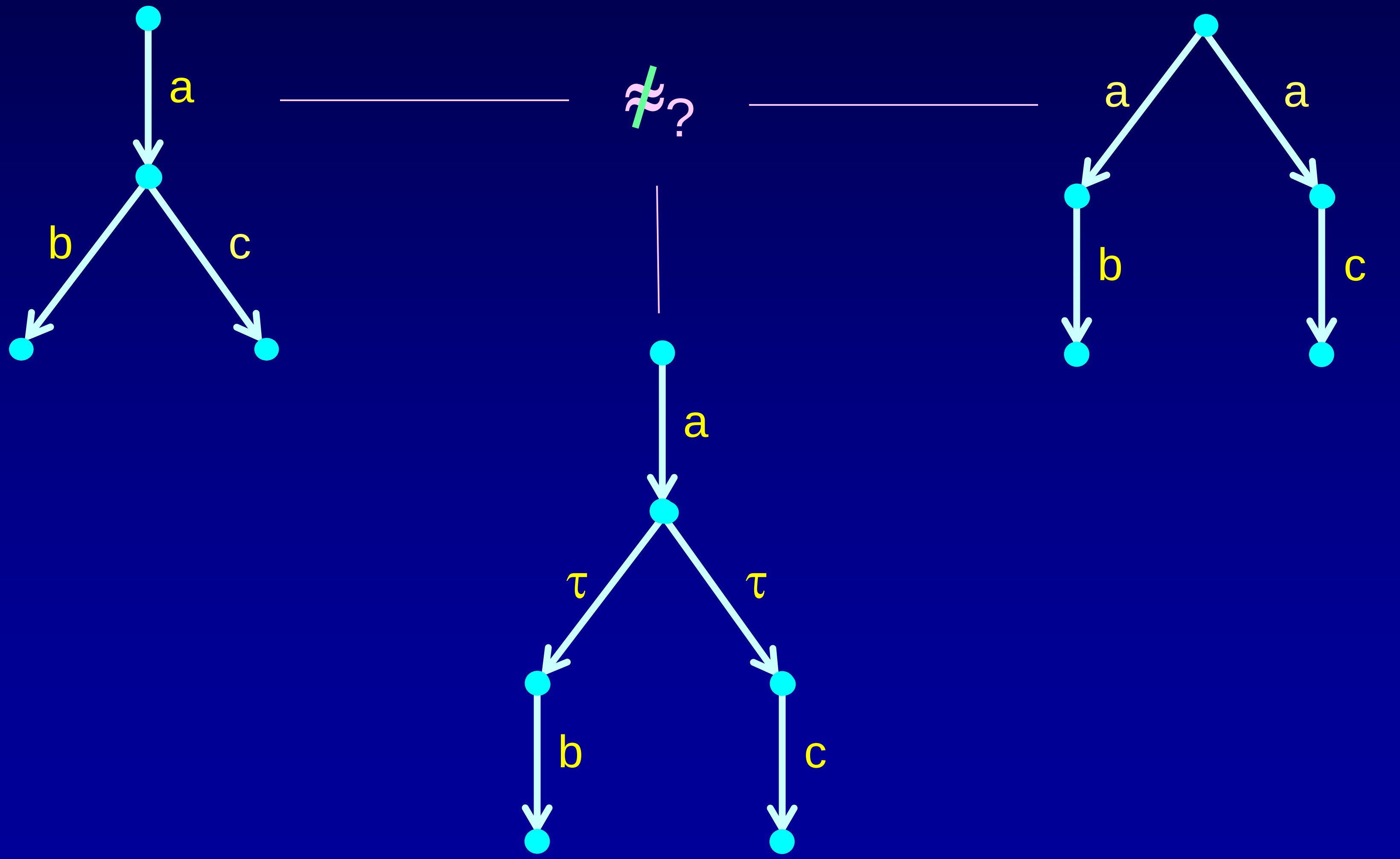
(Completed) Trace Equivalence



Completed Trace Equivalence



(Completed) Trace Equivalence : Others ?



(Completed) Trace Equivalence

Traces: $\text{traces}(s) = \{ \sigma \in L^* \mid s \xRightarrow{\sigma} \}$

Trace equivalence: $p \approx_{tr} q \iff \text{traces}(p) = \text{traces}(q)$

Reachable states: $s \text{ after } \sigma = \{ s' \mid s \xRightarrow{\sigma} s' \}$

Refusal: $s \text{ refuses } A \iff \forall a \in A \cup \{\tau\}: s \stackrel{a}{\not\longrightarrow}$

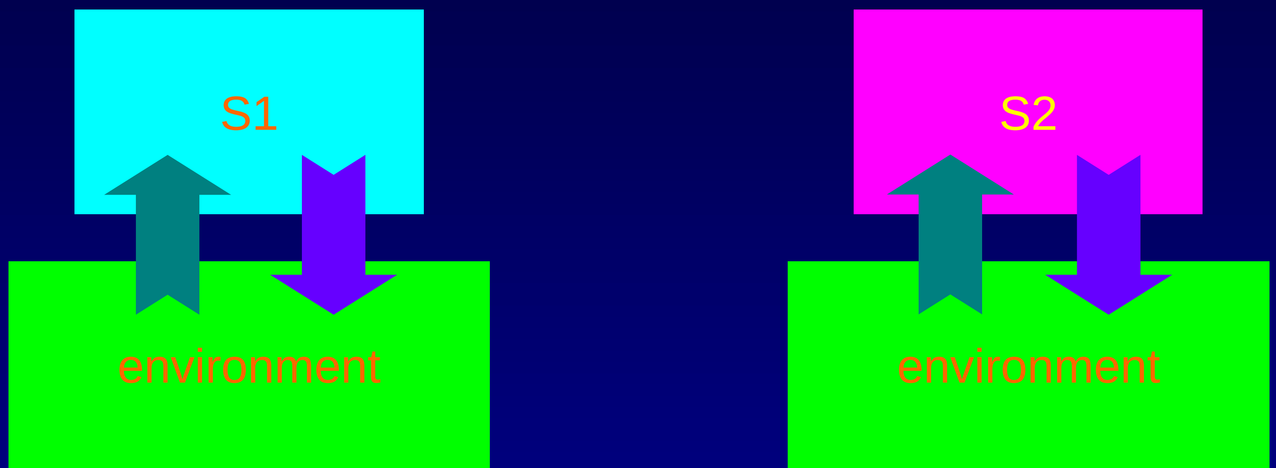
Refusals: $s \text{ after } \sigma \text{ refuses } A \iff \exists s' \in s \text{ after } \sigma : s' \text{ refuses } A$

Completed traces: $\text{Ctraces}(s) = \{ \sigma \in L^* \mid s \text{ after } \sigma \text{ refuses } L \}$

Completed trace equivalence:

$p \approx_{ctr} q \iff \text{Ctraces}(p) = \text{Ctraces}(q)$ and $\text{traces}(p) = \text{traces}(q)$

Comparing Systems : Testing Equivalence



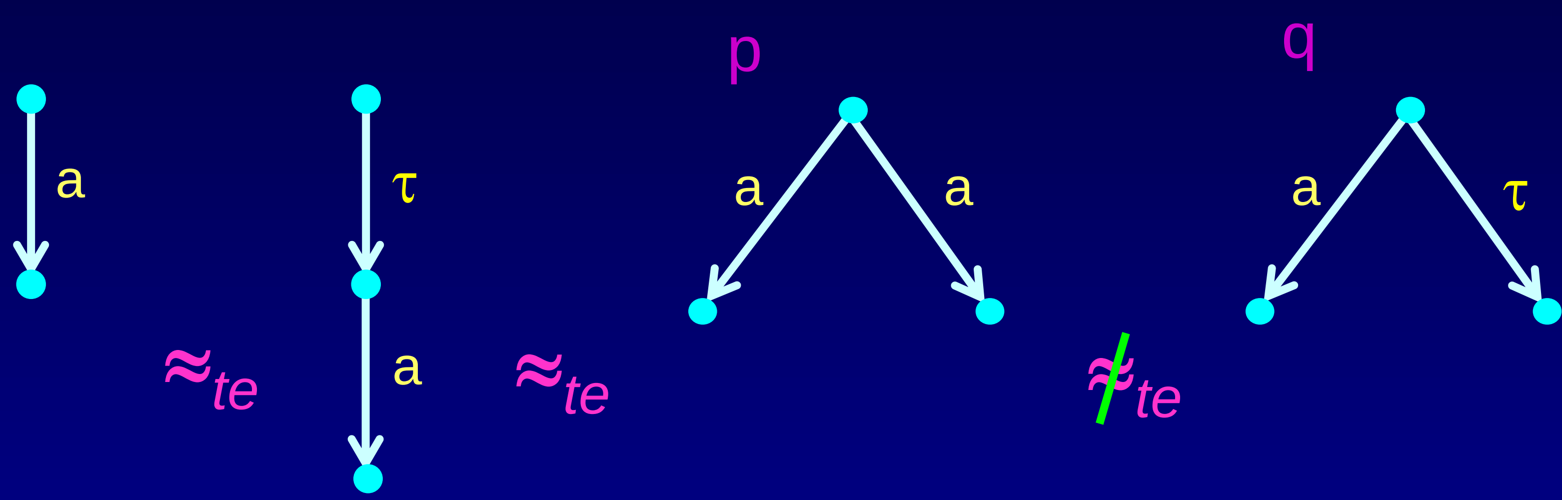
$$S1 \;\approx_{te}\; S2 \quad \Leftrightarrow \quad \forall\, e \in E\, .\;\; obs\,(\,e, S1\,) \;=\; obs\,(e, S2\,)$$

$\downarrow \quad \downarrow$

? ?

LTS(L) Ctraces (e||s)

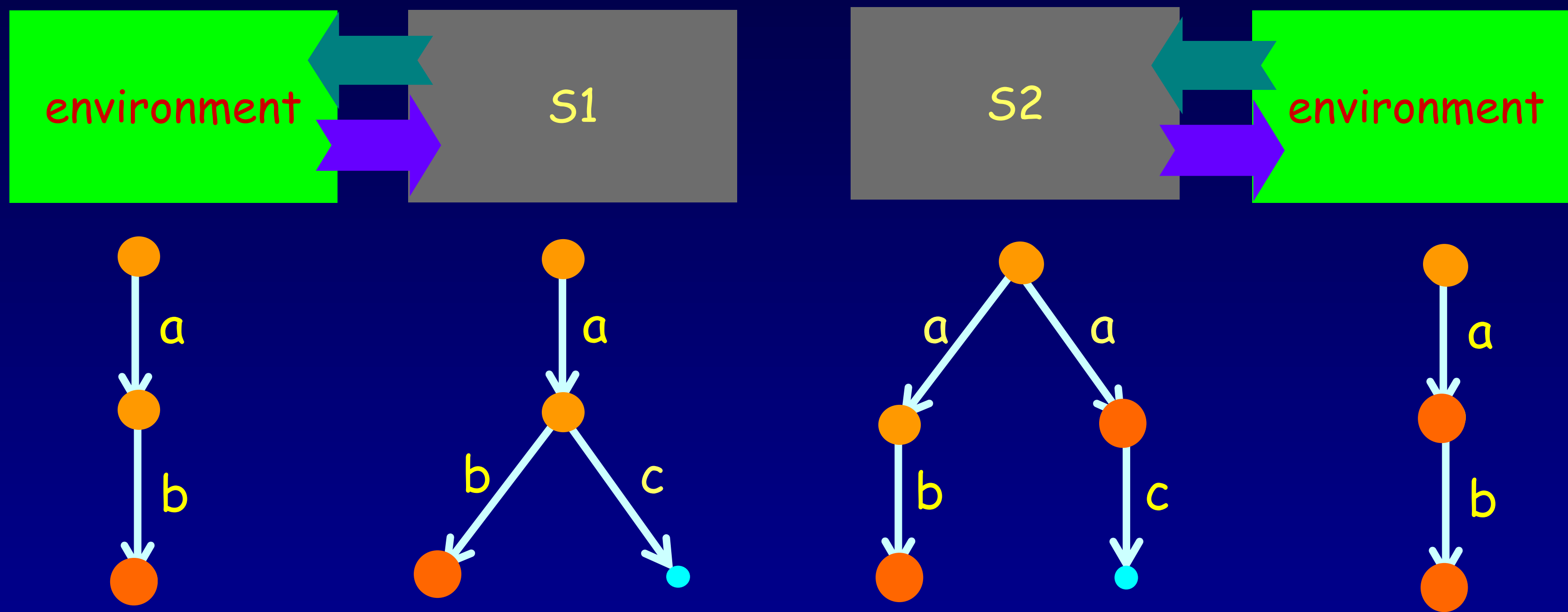
Testing Equivalence



Environment e :

obs(e,p) = { a √ }   obs(e,q) = { a √, √ }

Comparing Systems : Testing Equivalence



a b √

$\not\approx_{te}$

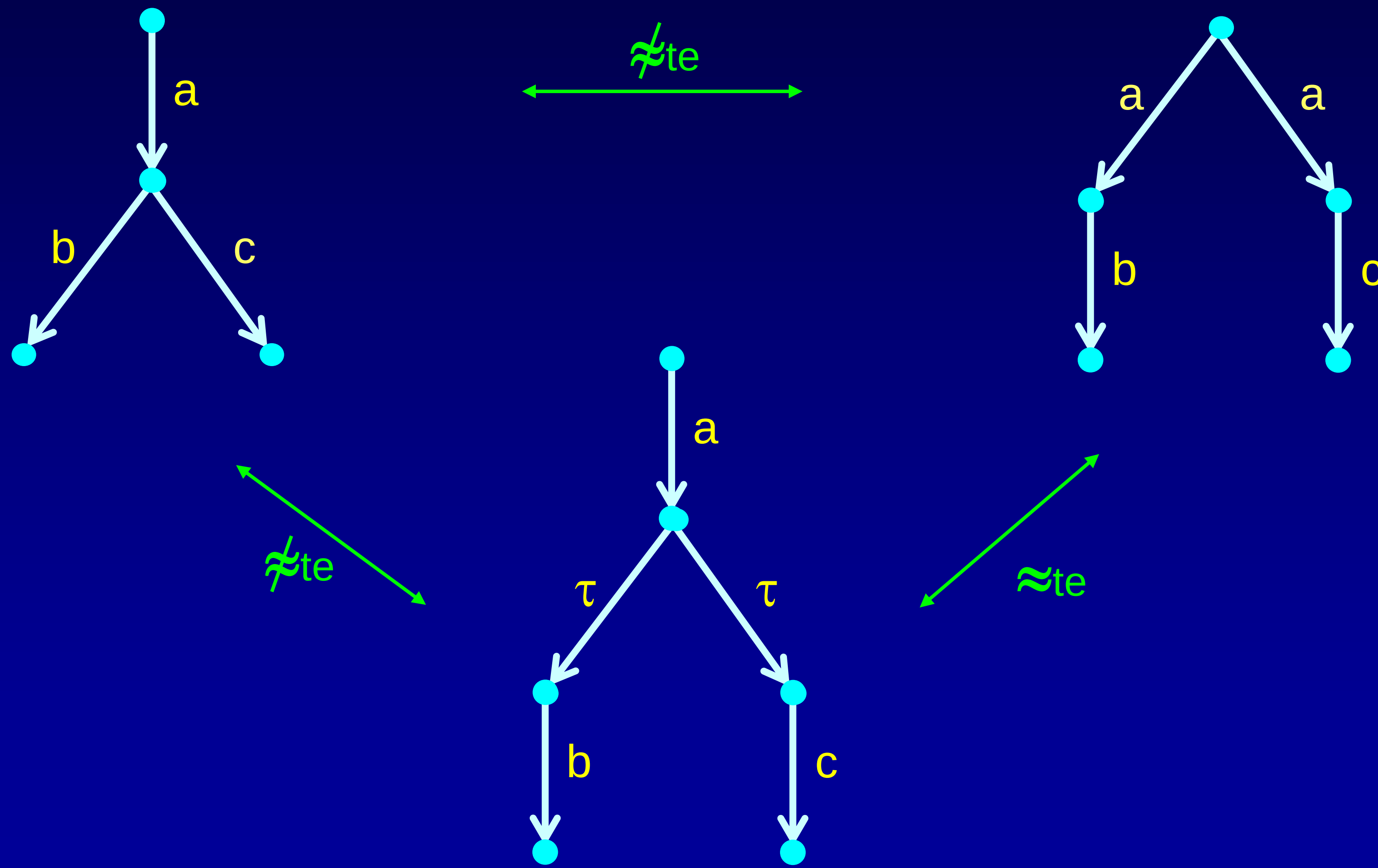
a b √

a √

S1 after ~~a refuses~~ {b}

S2 after a refuses {b}

Testing Equivalence

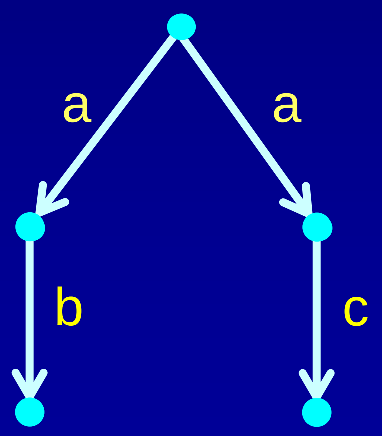


Testing Equivalence

$$p \;\approx_{te}\; q \quad \Leftrightarrow$$

$$\forall\, A \subseteq \mathbf{L},\ \forall\, \sigma \in \mathbf{L}^* \ :$$

$$p \text{ after } \sigma \text{ refuses } A \;\Leftrightarrow\; q \text{ after } \sigma \text{ refuses } A$$



p afer a refuses {c}

p afer a b refuses L

p afer ε refuses ∅

p afer a ~~refuses~~ {b,c}

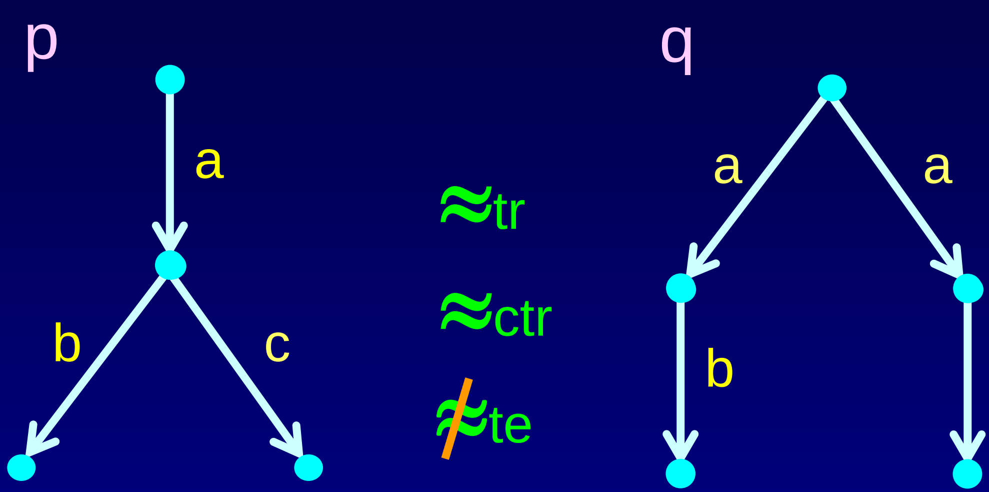
p afer a a ~~refuses~~ ∅

p afer ε ~~refuses~~ L

Testing Equivalence

a b √

a c √



a b √

a c √

traces (p) = traces (p) = { ε, a, a b, a c }

Ctraces (p) = Ctraces (p) = { a b, a c }

p after a b refuses L

p after a ~~refuses~~ L

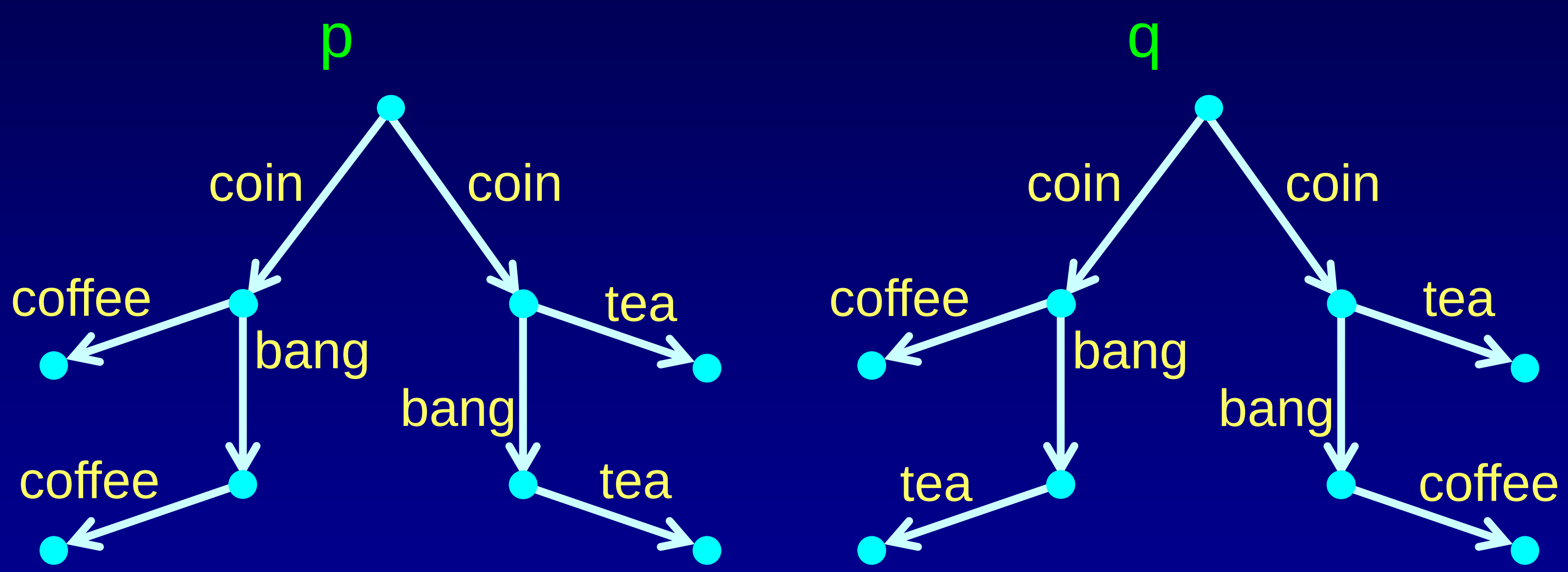
p after a ~~refuses~~ {a,c}

q after a b refuses L

q after a ~~refuses~~ L

q after a refuses {a,c}

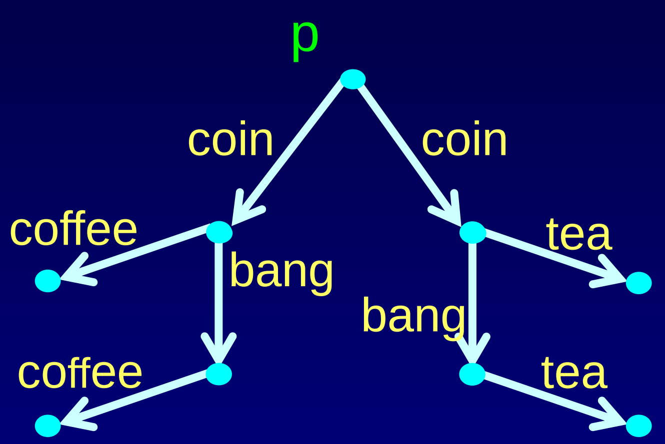
Testing Equivalence

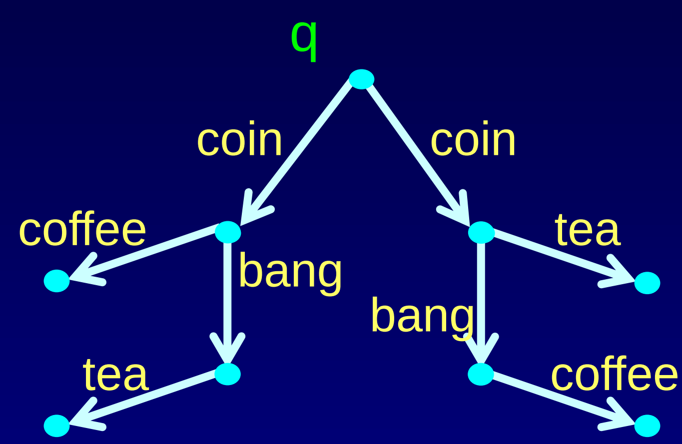


p $\approx_{te}$ q

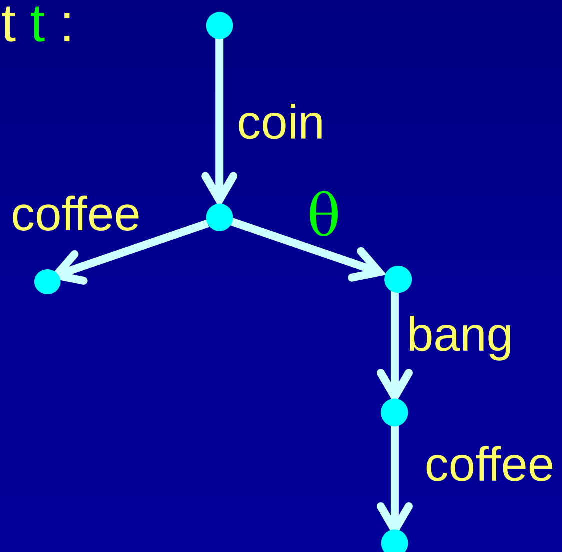
But: if you want coffee you will eventually always succeed in q but not p !?

Refusal Equivalence





Test t :



θ only possible
if nothing else is possible

coin θ bang coffee $\surd$ $\notin$ obs (p || t)

coin θ bang coffee $\surd$ $\in$ obs (q || t)

$p \not\approx_{rf} q$

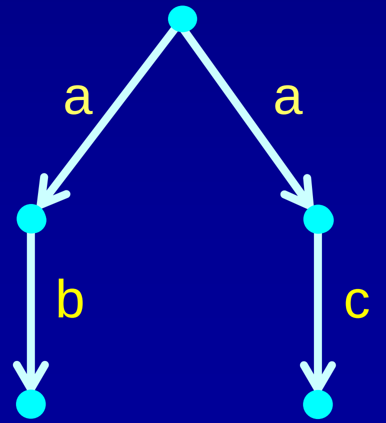
Refusal Equivalence

Failure A : $s \xrightarrow{A} s \iff \forall \mu \in A \cup \{\tau\} : s \overset{\mu}{\not\longrightarrow}$

Failure trace $\boldsymbol{\sigma}$: $\boldsymbol{\sigma} \in (L \cup \wp (L))^* : s \overset{\sigma}{\Longrightarrow}$

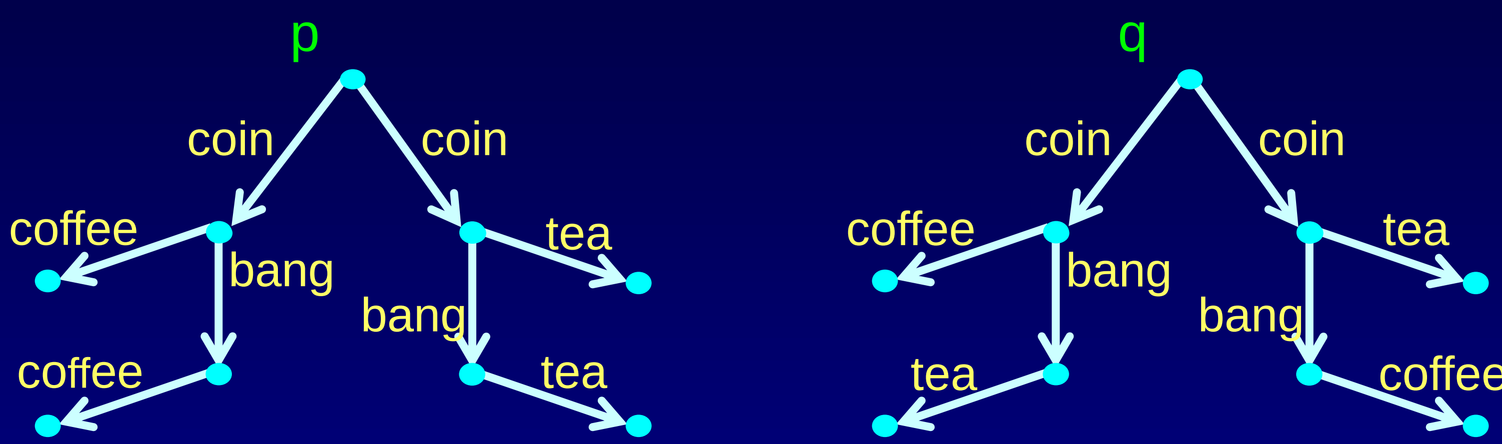
Failure traces of p : $Ftraces(p) = \{ \sigma \in (L \cup \wp (L))^* \mid p \overset{\sigma}{\Longrightarrow} \}$

Failure trace equivalence
= refusal equivalence : $p \approx_{rf} q \iff Ftraces(p) = Ftraces(q)$



Ftraces :	Not Ftraces :
{b,c} a {a,c} b L	{a,b,c} a {a,c} b L
a {c} b {a} {b} {c}	a {c} c L
∅ a {b} {b} c	a a ∅

Refusal Equivalence



$p \not\approx_{rf} q$

Ftrace of p :

coin {coffee} bang {coffee} tea

Not an Ftrace of q :

coin {coffee} bang {coffee} tea

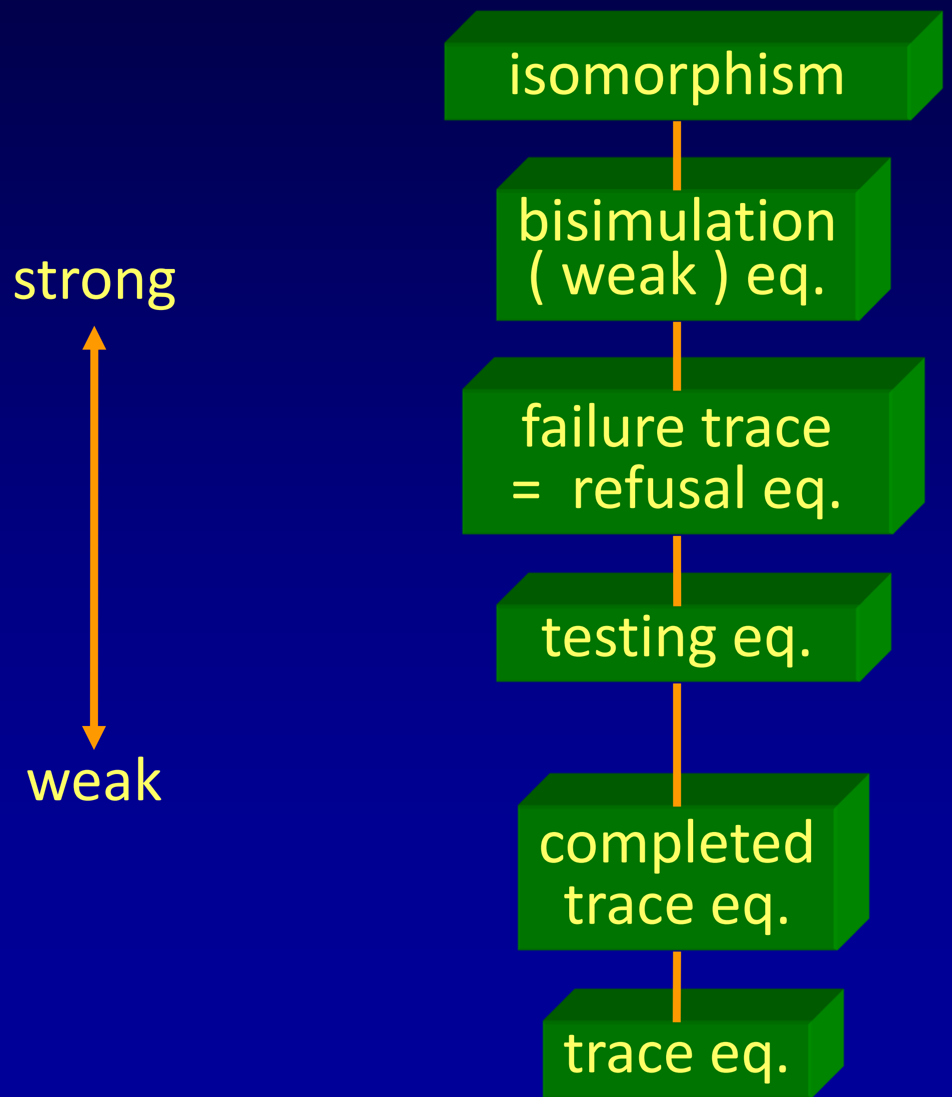
Not an Ftrace of p :

coin {coffee} bang coffee

An Ftrace of q :

coin {coffee} bang coffee

Equivalences on Transition Systems



Isomorphism

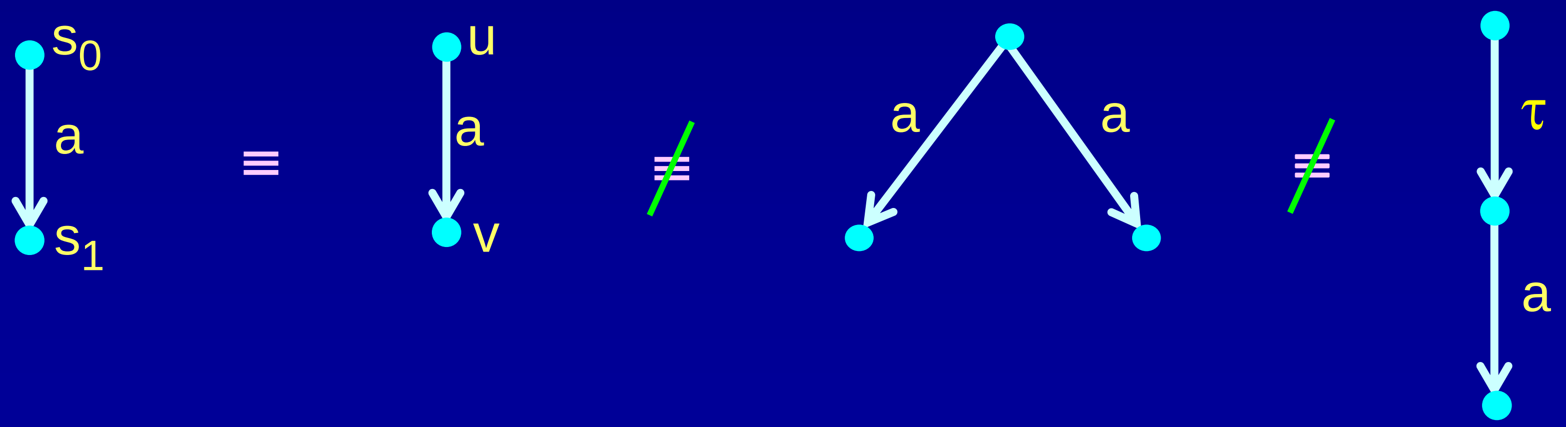
Isomorphism: $p \equiv q \iff \exists$ bijection $f : S_p \to S_q$:

$\forall\, s_1, s_2 \in S_p,\ \ \forall\, \mu \in L \cup \{\tau\}$:

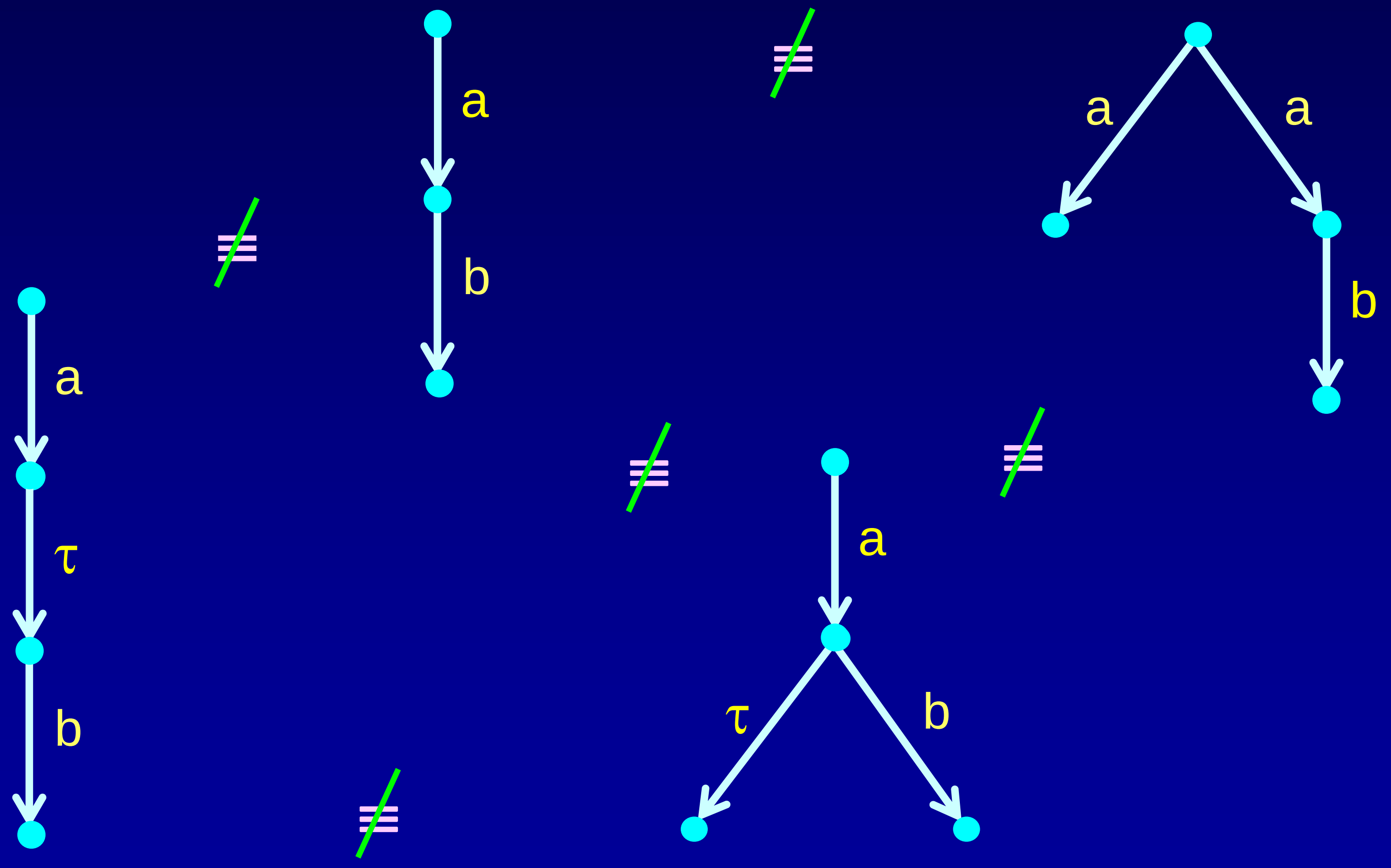
$$s_1 \xrightarrow{\mu} s_2 \iff f(s_1) \xrightarrow{\mu} f(s_2)$$

$$f(s_{0p}) = s_{0q}$$

"p and q
are exactly the same
modulo state names"



Isomorphism



Bisimulation

Bisimulation: $p \approx_b q \iff$

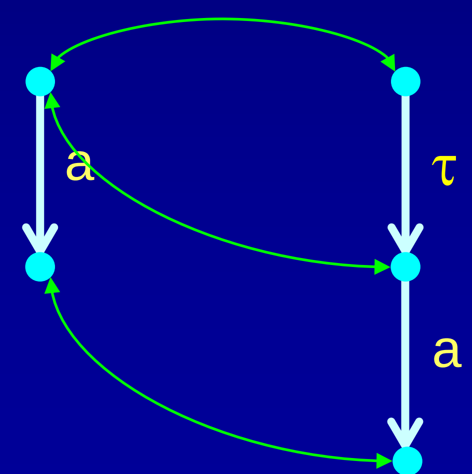
$\exists\, \mathfrak{R} \subseteq S_p \times S_q : \quad \langle s_{0p}, s_{0q} \rangle \in \mathfrak{R}$ and

$\forall\, \langle s_1, s_2 \rangle \in \mathfrak{R}, \quad \forall\, \sigma \in L^* :$

whenever $s_p \xRightarrow{\sigma} s'_p$ then $s_q \xRightarrow{\sigma} s'_q$ and $\langle s'_p, s'_q \rangle \in \mathfrak{R}$

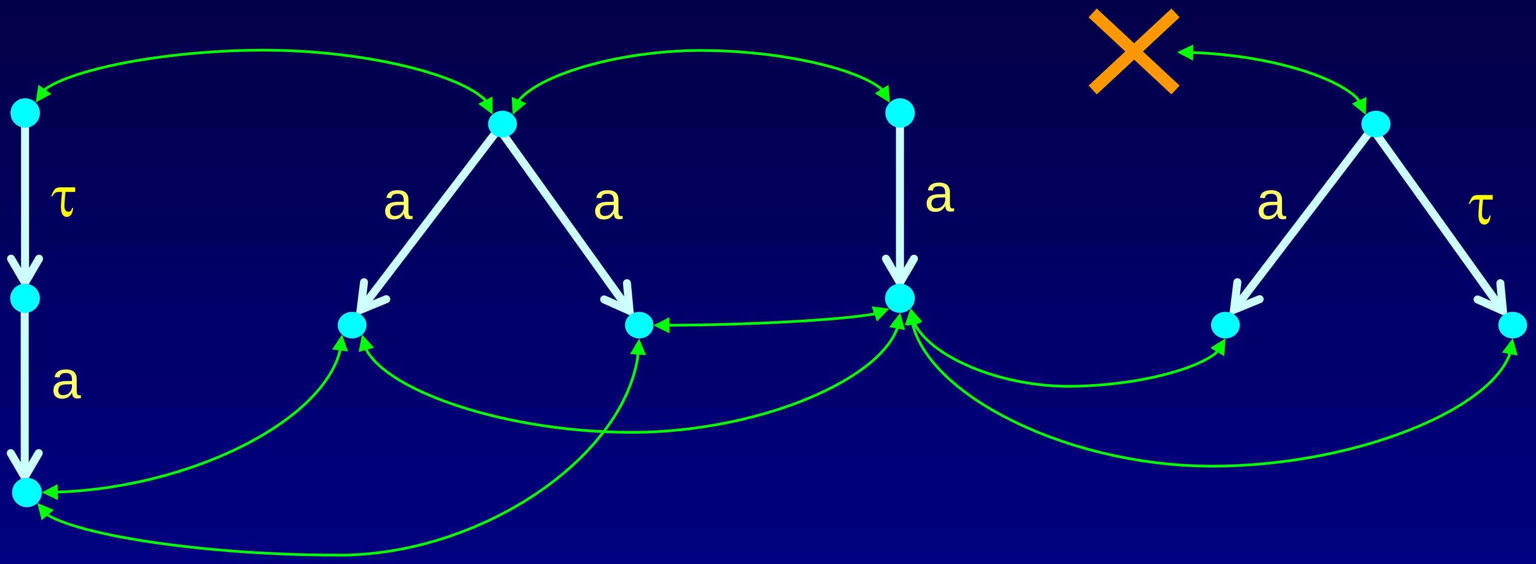
whenever $s_q \xRightarrow{\sigma} s'_q$ then $s_p \xRightarrow{\sigma} s'_p$ and $\langle s'_p, s'_q \rangle \in \mathfrak{R}$

"p and q simulate each other and go to states from where they can simulate each other again"



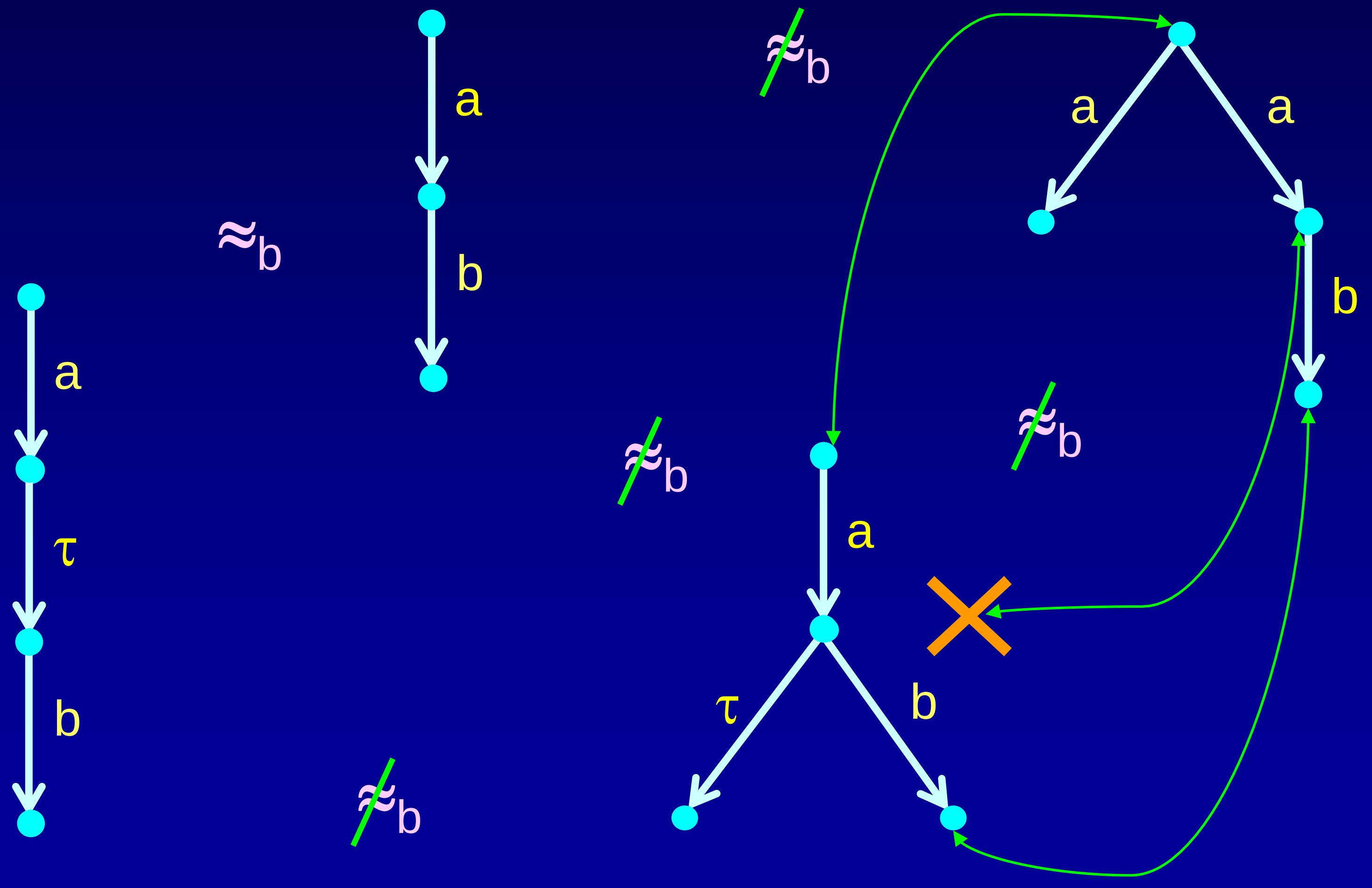
$\approx_b$

Bisimulation

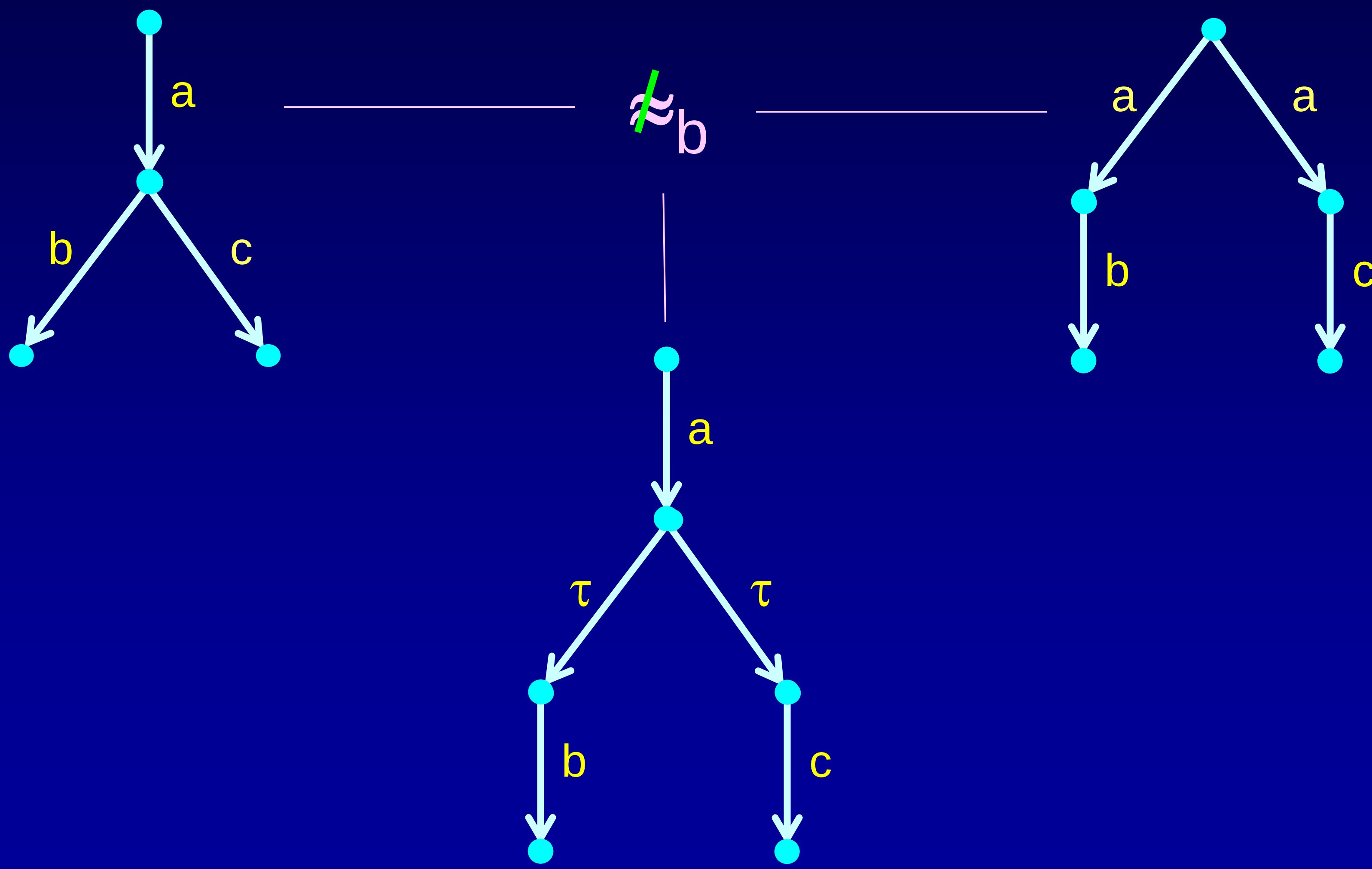


$\approx_b$ $\approx_b$ $\not\approx_b$

Bisimulation



Bisimulation



Bisimulation

Equivalences on Transition Systems

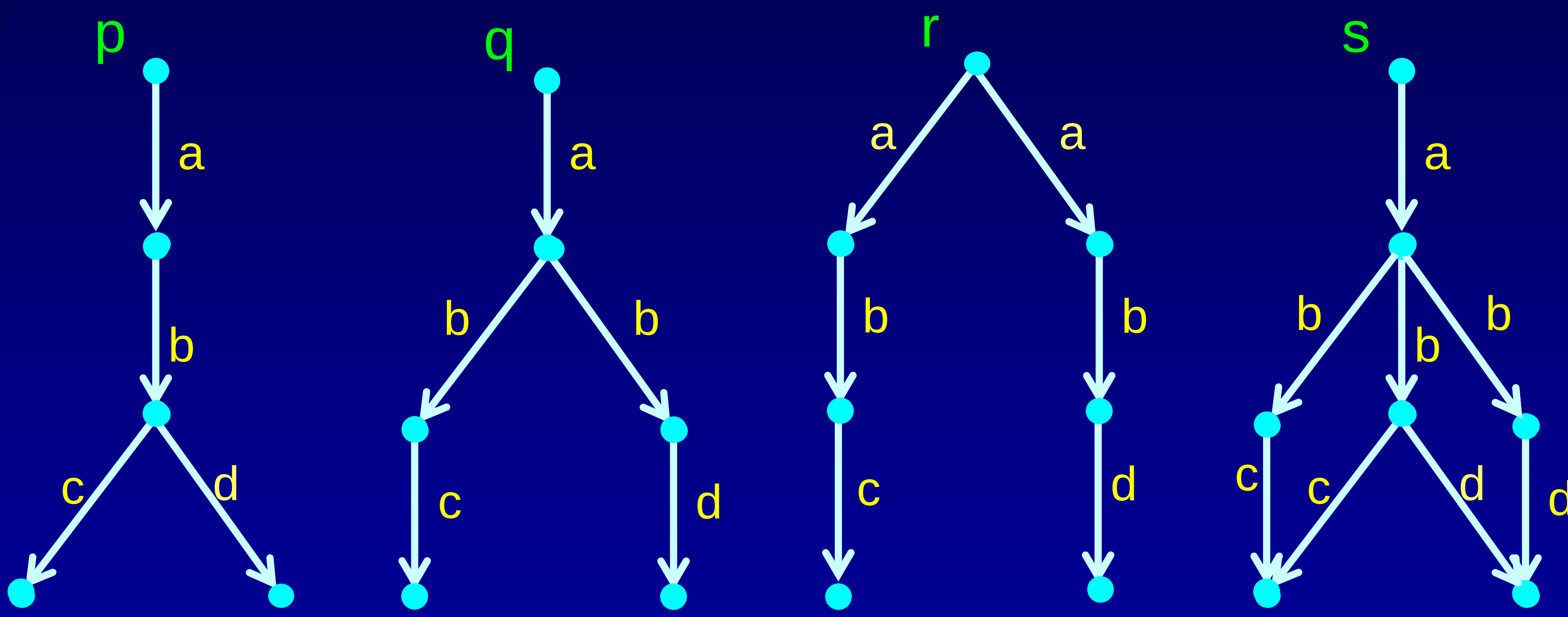
strong

↕

weak

isomorphism	now you need to observe τ's ……
bisimulation (weak)	test an LTS with another LTS, and undo, copy, repeat as often as you like
failure trace = refusal	test an LTS with another LTS, and try again (continue) after failure
failures = testing	test an LTS with another LTS
completed trace	observing sequences of actions and their end
trace	observing sequences of actions

Equivalences : Examples



Equivalences : Examples