Common Useful Formulas & Identities

1 List of physical constants

Physical Constant	**Symbol**	**Precise value**	**Rounded value**
Speed of light	c	299,792,458 m/s	3×10^{8} m/s
Planck's constant	h	6.62607015×10^{-34} J·Hz	6.63×10^{-34}
Standard gravity	g or g_0	$9.80665\,\mathrm{m/s^2}$	$9.81\,\mathrm{m/s^2}$
Elementary charge	e	$1.602176634\times10^{-19}$ C	1.60×10^{-19} C
Electron mass	m_e	$9.1093837139\times10^{-31}$ kg	9.11×10^{-31} kg
Proton mass	m_p	$1.67262192595\times10^{-27}$ kg	1.67×10^{-27} kg
Universal gravitational constant	G	$6.67430\times10^{-11}\,\mathrm{m^3\,kg^{-1}s^{-2}}$	$6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}s^{-2}}$
Vacuum permittivity (electric constant)	ε_0	$8.8541878188\times10^{-12}\,\mathrm{C^2\,kg^{-1}m^{-3}\,s^2}$	$8.85\times10^{-12}\,\mathrm{C^2\,kg^{-1}m^{-3}\,s^2}$
Vacuum permeability (magnetic constant)	μ_0	$1.25663706127\times10^{-6}\,\mathrm{N\cdot A^{-2}}$	$1.26\times10^{-6}\,\mathrm{N\cdot A^{-2}}$
Avogadro's number	N_A	$6.02214076\times10^{23}\,\mathrm{mol^{-1}}$	$6.02\times10^{23}\,\mathrm{mol^{-1}}$
Ideal gas constant	R	$8.31446261815324\,\mathrm{J\cdot K^{-1}\cdot mol^{-1}}$	$8.31\,\mathrm{J\cdot K^{-1}\cdot mol^{-1}}$
Boltzmann constant	k_B	$1.380649\times10^{-23}\,\mathrm{J\cdot K^{-1}}$	$1.38\times10^{-23}\,\mathrm{J\cdot K^{-1}}$

Table 1. Common universal physical constants

2 Vectors & Geometry

- Position vector: $\vec{r}=\langle x,y,z\rangle$
- Basis vectors: $\hat{\imath},\hat{\jmath},\hat{k}$ (also written $\hat{x},\hat{y},\hat{z}$)
- Cross product: $\vec{A}\times\vec{B}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A & A & A\\ B & B & B\end{vmatrix}$ (3D case), $\vec{A}\times\vec{B}=(A_xB_y-A_yB_x)\hat{k}$ (2D case),

  $\vec{A}\times\vec{B}=0$ when $\vec{A},\vec{B}$ are **parallel**
- Dot product: $\vec{A}\cdot\vec{B}=A_x\,B_x+A_y\,B_y+A_z\,B_z$, $\vec{A}\cdot\vec{B}=0$ when $\vec{A},\vec{B}$ are **perpendicular**
- Cylindrical coordinates: (ρ,ϕ,z) where ρ is distance from z-axis, ϕ is rotation around z-axis and z is elevation.

  Conversions to Cartesian: $x=\rho\cos\phi, y=\rho\sin\phi, z=z$
- Spherical coordinates: (r,θ,ϕ) where r is distance from origin, θ is elevation angle, and ϕ is rotation around the vertical axis (physics convention)

  Conversions to Cartesian: $x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta$

3 Vector calculus

3.1 Partial derivatives

- Scalar-valued functions: $f(\vec{r})=f(x,y,z)$

- Vector-valued functions: $\vec{F}(\vec{r}) = \vec{F}(x,y,z)$
- Chain rule for operators: $\dfrac{\partial}{\partial s} = \dfrac{\partial x}{\partial s}\dfrac{\partial}{\partial x}$
- Chain rule for functions – given scalar-valued $f(v(x,y),w(x,y))$:
  $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x}, \quad \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial y}$
- General chain rule: $\dfrac{\partial f}{\partial x_i} = \sum_j \dfrac{\partial f}{\partial x_j}\dfrac{\partial x_j}{\partial x_i}$

3.2 Vector differential operators

$\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$ (del operator)

grad $f = \nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle$ (gradient operator)

- in cylindrical coordinates: $\nabla f = \left\langle \dfrac{\partial f}{\partial \rho}, \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}, \dfrac{\partial f}{\partial z} \right\rangle$
- in spherical coordinates: $\nabla f = \left\langle \dfrac{\partial f}{\partial r}, \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}, \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi} \right\rangle$

div $\vec{F} = \nabla \cdot \vec{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$ (divergence operator)

- in cylindrical coordinates: $\nabla \cdot \vec{F} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho F_\rho) + \dfrac{1}{\rho}\dfrac{\partial F_\phi}{\partial \phi} + \dfrac{\partial F_z}{\partial z}$
- in spherical coordinates: $\nabla \cdot \vec{F} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 F_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta F_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial F_\phi}{\partial \phi}$

curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F & F & F \end{vmatrix} = \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)\hat{k}$ (curl operator)

2D case for curl: $\nabla \times \vec{F} = \left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)\hat{k}$

$\nabla^2 f = \nabla \cdot (\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ (scalar Laplacian)

- in cylindrical coordinates: $\nabla^2 f = \dfrac{1}{\rho}\left(\dfrac{\partial f}{\partial \rho} + \rho\dfrac{\partial^2 f}{\partial \rho^2}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2}$
- in spherical coordinates: $\nabla^2 f = \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r}\right) f + \dfrac{1}{r^2 \sin\theta}\left[\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial}{\partial \theta}\right) f\right] + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 f}{\partial \varphi^2}$

$\nabla^2 \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F}) = \dfrac{\partial^2 \vec{F}}{\partial x^2} + \dfrac{\partial^2 \vec{F}}{\partial y^2} + \dfrac{\partial^2 \vec{F}}{\partial z^2}$ (vector Laplacian)

3.3 Vector derivative identities

$(\vec{F} \cdot \nabla)\phi = \vec{F} \cdot (\nabla \phi)$ for scalar-valued ϕ and vector-valued $\vec{F}$

$(\vec{F}\times\nabla)\phi=\vec{F}\times(\nabla\phi)$ for scalar-valued ϕ and vector-valued $\vec{F}$

$\nabla(\phi\varphi)=\phi\nabla\varphi+\varphi\nabla\phi$ for scalar-valued ϕ,φ

$\nabla\cdot(\phi\vec{F})=\phi(\nabla\cdot\vec{F})+(\nabla\phi)\cdot\vec{F}$ for scalar-valued ϕ and vector-valued $\vec{F}$

$\nabla\times(\phi\vec{F})=\phi(\nabla\times\vec{F})+(\nabla\phi)\times\vec{F}$ for scalar-valued ϕ and vector-valued $\vec{F}$

$\nabla\cdot(\nabla\times\vec{F})=0$ for any vector-valued $\vec{F}$

$\nabla\times(\nabla\phi)=0$ for any scalar-valued ϕ

Line integral: $\int_C f(\vec{r})\,ds=\int f(t)\,|\vec{r}|dt$ (scalar), $\int_C\vec{F}\cdot d\vec{r}=\int\vec{F}\cdot\vec{r}'(t)\,dt$ (vector)

Area integral: $\iint f\,dA$

- in Cartesian coordinates: $\iint f\,dA=\int_{x_1}^{x_2}\int_{y_1}^{y_2}f(x,y)dydx$
- in cylindrical coordinates: $\iint f\,dA=\int_{z_1}^{z_2}\int_0^{2\pi}f(\phi,z)\rho d\phi dz$
- in spherical coordinates: $\iint f\,dA=\int_0^{2\pi}\int_0^{\pi}f(\theta,\phi)\,r^2\sin\theta d\theta d\phi$

Surface (flux) integral: $\iint\vec{F}\cdot d\vec{A}$, $d\vec{A}=\hat{n}\,dA$

Volume integral: $\iiint f\,dV$

- in Cartesian coordinates: $\iiint f\,dV=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}f(x,y,z)\,dzdydx$
- in cylindrical coordinates: $\iiint f\,dV=\int_{z_1}^{z_2}\int_0^{2\pi}\int_0^R f(\rho,\phi,z)\,\rho d\rho d\phi dz$
- in spherical coordinates: $\iiint f\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^R f(r,\theta,\phi)\,r^2\sin\theta drd\theta d\phi$

Gradient theorem: $\int_C\nabla f\cdot d\vec{s}=f(b)-f(a)$, $\oint\nabla f\cdot d\vec{s}=0$

Stoke's theorem: $\iint_S(\nabla\times\vec{F})\cdot d\vec{A}=\int_C\vec{F}\cdot d\vec{s}$ where curve C is the boundary of surface S

Note: when $\vec{F}$ is a 2D vector field this is called Green's theorem.

Divergence theorem: $\iiint_V(\nabla\cdot\vec{F})\cdot dV=\iint_S\vec{F}\cdot d\vec{A}$ where surface S is the boundary of volume V

4 Tensors

Indices: usually either Roman letters (e.g. i,j,k) or Greek letters (e.g. μ,ν,λ), typically roman letters range from 1 to 3 and roman letters range from 0 to 3

- Basis vectors: written e_i, in Cartesian $e_1 = \hat{x}, \quad e_2 = \hat{y}, \quad e_3 = \hat{z}$
- Vectors: written V^i, in Cartesian $V^1 = V^x, \quad V^2 = V^y, \quad V^3 = V^z$
- Dot product: written $\sum_i A_i B^i = A_1 B^1 + A_2 B^2 = A_3 B^3 = A_x B^x + A_y B^y + A_z B^z$

Note: when using the *Einstein summation convention*, we assume a summation sign whenever there is the same letter used in a lower and an upper index. So instead of $\sum_i A_i B^i$ we can just write $A_i B^i$. We will use this convention for the rest of this sheet.

- Metric tensor: g_{ij} which is a square matrix (3×3 for Euclidean space and 4×4 for spacetime)
- Converting between upper and lower indices: $V_i = g^{ij} V_j$ and $V_i = g_{ij} V^j$
- Line element: $ds^2 = g_{ij}\, dx^i\, dx^j$

Note: in Cartesian coordinates, there is *no difference* between upper & lower indices

- Gradient: $\partial_i f$ where f is a scalar field
- Divergence: $\partial_i F^i$ where F^i is a vector field
- Laplacian: $\partial^i \partial_i f$ where $\partial^i = g^{ij} \partial_j$. In Cartesian coordinates $\partial^i = \partial_i$.
- Kronecker delta: δ_{ij}, essentially the identity matrix (it is equal to 1 when $i = j$ and 0 otherwise)
- Contraction: $B_i^i = B_1^1 + B_2^2 + B_3^3$ (self-contraction), with another tensor: $A^j B_{ij} = C_i$

5 Electromagnetics

- Poisson's equation: $\nabla^2 V = \rho / \varepsilon_0$
- Maxwell's equations:
  - Gauss's law for electricity: $\nabla \cdot \vec{E} = \rho / \varepsilon_0$
  - Gauss's law for magnetism: $\nabla \cdot \vec{B} = 0$
  - Faraday's law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
  - Maxwell-Ampere law: $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$
- Conservation of charge (continuity equation): $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$
- Electromagnetic wave equation:
  - For electric fields: $\dfrac{\partial^2 \vec{E}}{\partial t^2} = c^2 \nabla^2 \vec{E} = c^2 \left(\dfrac{\partial^2 \vec{E}}{\partial x^2} + \dfrac{\partial^2 \vec{E}}{\partial y^2} + \dfrac{\partial^2 \vec{E}}{\partial z^2} \right)$

- For magnetic fields: $\dfrac{\partial^2 \vec{B}}{\partial t^2} = c^2 \nabla^2 \vec{B} = c^2 \left(\dfrac{\partial^2 \vec{B}}{\partial x^2} + \dfrac{\partial^2 \vec{B}}{\partial y^2} + \dfrac{\partial^2 \vec{B}}{\partial z^2} \right)$

- Helmholtz equation: $\nabla^2 \vec{E} = k^2 \vec{E}$, $\nabla^2 \vec{B} = k^2 \vec{B}$ (eigenvalue problem for ∇^2 operator)

- Poynting vector: $\vec{S} = \dfrac{1}{\mu_0} \vec{E} \times \vec{B}$

- Intensity of EM radiation: $I = |\vec{S}|$ where $\vec{S}$ is Poynting vector

- Electromagnetic energy density: $u = \dfrac{1}{2} \varepsilon_0 |\vec{E}|^2 + \dfrac{1}{2\mu_0} |\vec{B}|^2$

6 Quantum physics

6.1 General quantum physics

- Time-dependent Schrödinger equation: $i \dfrac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$ where $\Psi = \Psi(\vec{r}, t)$

- Schrödinger equation: $\hat{H} |\psi\rangle = E\, |\psi\rangle$ where $|\psi\rangle$ is the quantum state. In position space we can also write this as $\hat{H}\psi = E\psi$ with a wavefunction $\psi(x, y, z)$.

- $\vec{\Psi}(\vec{r}) = \sum_i c_i \psi_i e^{-iEt/\hbar}$ – relationship between time-dependent and time-independent wavefunctions and $\sum_i |c_i|^2 = 1$

- Classical Hamiltonian: $\hat{H} = \hat{K} + \hat{V} = \dfrac{2}{2m} \nabla^2 + V(\vec{r})$ where $\vec{V}(\vec{r})$ is the potential energy

- Normalization condition: $\langle \Psi | \Psi \rangle = \int_{-\infty}^{\infty} \Psi(x) \Psi^*(x)\, dx = 1$

- Expectation (average) value of operator $\hat{A}$: $\langle \Psi | \hat{A} | \Psi \rangle = \int_{-\infty}^{\infty} \Psi(x) \hat{A} \Psi(x)\, dx$

6.2 Laser physics

- Time-dependent Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$ where $\hat{H}_0$ is the time-independent Hamiltonian (Hamiltonian at $t = 0$)

- Two-state system with ground state $|1\rangle$ and excited state $|2\rangle$: $|\Psi\rangle = c_1 e^{-i\omega t} |1\rangle + c_2 e^{-i\omega t} |2\rangle$

- Einstein coefficients: $A_{12} = \dfrac{\omega_{12}^3 e^2}{3\pi \varepsilon h c^3} |\langle 1 | \vec{r} | 2 \rangle|^2$ where $\omega_{12} = \dfrac{E_1 - E_2}{}$ and $B_{12} = A_{12} \dfrac{\pi^2 c^3}{\omega_{12}^3}$

- Rate equations: $\dfrac{dN_2}{dt} = -\dfrac{dN_1}{dt} = B_{12}\, \rho(\nu) N_1$ where N_1, N_2 is the population (number of atoms in) each state

7 Orbital mechanics

- Universal gravitation: $\vec{g} = \dfrac{d^2 \vec{r}}{dt^2} = \sum_i \dfrac{G m_i}{|\vec{r} - \vec{r}_i|} \vec{r} - \vec{r}_i$ where r_i is location of a gravitating body

- Poisson's equation for the gravitational potential: $\nabla^2 \phi = 4\pi G \rho$ where ϕ is the gravitational potential

- Gravitational field: $\vec{g} = -\nabla\phi$
- Gauss's law for gravity: $\nabla\cdot\vec{g} = -4\pi G\rho$
- Two-body orbit equation (for small mass m orbiting big(ger) mass M) $r(\theta) = \frac{\ell^2}{m^2\mu}\frac{1}{1+e\cos\theta}$ where e is the eccentricity of the orbit, $\mu = G(M+m) \approx GM$, and ℓ is the angular momentum (a conserved quantity)

8 Fluid & heat dynamics

- Stefan-Boltzmann law: $I = \sigma T^4$

  where I is the intensity, σ is the Sefan-Bolzmann constant, and T is the absolute temperature in Kelvin

- Planck's law: $\rho(\nu,T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\left(\frac{h\nu}{k}T-1\right)}}$

  where u is the energy density, $\rho(\nu)$ is the spectral energy density, and T is also the absolute temperature in Kelvin