Mathematica file for Electrodynamics course 1FA257
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Project

Fill in the sections below, you can copy-paste cells from the previous sections of this file, and change them as needed.

1. Field and potential of a charge distribution

Consider the following charge distribution

In[5]:=
$$\texttt{rho[\{x\_, y\_\}] :=} \frac{e^{\frac{1}{10}\left(-(-1+x)^2-y^2\right)}\left((-1+x)^4+y^4\right)}{\left((-1+x)^2+y^2\right)^2} - \frac{e^{\frac{1}{10}\left(-(1+x)^2-y^2\right)}\left((1+x)^4+y^4\right)}{\left((1+x)^2+y^2\right)^2}$$

a. Solve the Poisson equation to obtain the potential.

In[11]:=

```
charge = NIntegrate[rho[{x, y}], {x, -∞, ∞}, {y, -∞, ∞}]
```

Out[11]=

$$-8.25728\times10^{-16}$$

In[ ]:=

```
ϵ0 = 1;
PoissonEq = D[Phi[x, y], {x, 2}] + D[Phi[x, y], {y, 2}] == -rho[{x, y}] / ϵ0
```

Out[ ]=

$$\mathrm{Phi}^{(0,2)}[x,y] + \mathrm{Phi}^{(2,0)}[x,y] == -\frac{e^{\frac{1}{10}\left(-(-1+x)^2-y^2\right)}\left((-1+x)^4+y^4\right)}{\left((-1+x)^2+y^2\right)^2} + \frac{e^{\frac{1}{10}\left(-(1+x)^2-y^2\right)}\left((1+x)^4+y^4\right)}{\left((1+x)^2+y^2\right)^2}$$

In[ ]:=

$$\texttt{L1 = 50;}$$
$$\texttt{DirichletBC[x\_, y\_] := Phi[x, -L1] ==} \frac{1}{4\pi\epsilon0}\frac{\texttt{charge}}{\left(x^2+\texttt{L1}^2\right)^{1/2}}\ \texttt{\&\&}$$
$$\texttt{Phi[x, L1] ==} \frac{\texttt{charge}}{\left(x^2+\texttt{L1}^2\right)^{1/2}}\ \texttt{\&\& Phi[-L1, y] ==} \frac{\texttt{charge}}{\left(y^2+\texttt{L1}^2\right)^{1/2}}\ \texttt{\&\& Phi[L1, y] ==} \frac{\texttt{charge}}{\left(y^2+\texttt{L1}^2\right)^{1/2}}$$

```
sol = NDSolve[PoissonEq && DirichletBC[x, y], Phi, {x, -L1, L1}, {y, -L1, L1}]
Potential[{x_, y_}] := Phi[x, y] /. sol[[1]]
```

Out[ ]=

```
{{Phi → InterpolatingFunction[ Domain: {{-50., 50.}, {-50., 50.}} Output: scalar ]}}
```

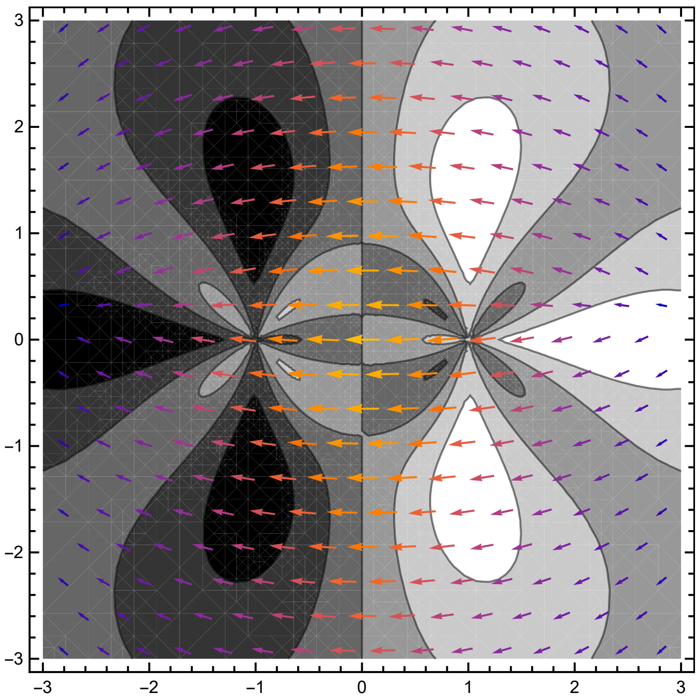
b. Take the gradient to obtain the electric field

```
EfieldPoisson[{x1_, y1_}] :=
 {-D[Potential[{x, y}], x], -D[Potential[{x, y}], y]} /. {x → x1, y → y1}
```

c. Plot both the potential and the electric field

```
ChargeDistPlot =
 ContourPlot[rho[{x, y}], {x, -3, 3}, {y, -3, 3}, ColorFunction → GrayLevel]
step = 0.5;
MeshPts = Table[{x, y}, {x, -3, 3, step}, {y, -3, 3, step}];

VectorFieldPoints = Table[Table[
    {MeshPts[[i, j]], EfieldPoisson[MeshPts[[i, j]]]}
    , {j, 1, Length[MeshPts[[i]]]}], {i, 1, Length[MeshPts]}];

Show[
 ChargeDistPlot,  (*plot the charge distribution*)
 ListVectorPlot[VectorFieldPoints,
  (*The list of values of the vector field at each point of the mesh*)
  VectorScaling → Automatic, Frame → False,
  PlotRange → {{-3, 3}, {-3, 3}}  (*options for the plot*)
 ]
 , PlotRange → {{-3, 3}, {-3, 3}}
]
```

Out[ ]=

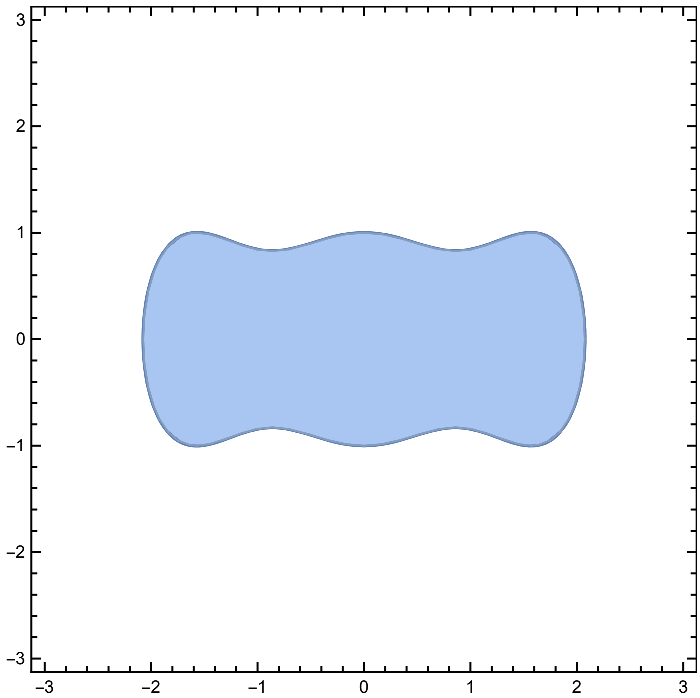


2. Green's function

Consider the region defined by

In[ ]:=
```
g2[{x_, y_}] := x^2 Cos[x]^2 + y^2 - 1
S2 = ImplicitRegion[g2[{x, y}] == 0, {{x, -3, 3}, {y, -3, 3}}];
V2 = ImplicitRegion[g2[{x, y}] ≤ 0.001, {{x, -3, 3}, {y, -3, 3}}];
Show[
 ContourPlot[g2[{x, y}] == 0, {x, -3, 3}, {y, -3, 3}],
 Region[V2],
 Region[S2]
]
```

Out[ ]=

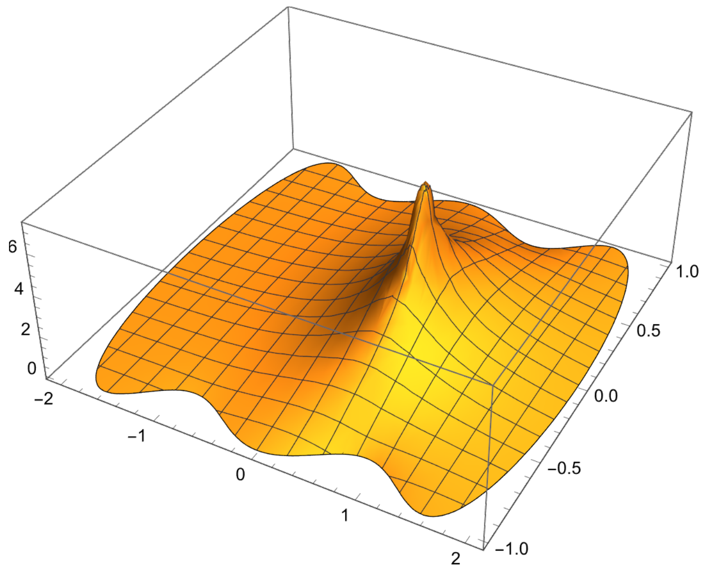


a.

Compute the Green function subject to **Dirichlet boundary conditions** at surface S2. Plot the green’s function at {x,y}={1/2,0} as a function of {x’,y’}

In[ ]:=
```
peakfactor = 10^3;
G1x_,y_ := NDSolveValue[{
   ∇²{x2,y2} f[x2, y2] == -4 π (peakfactor/π E^(-peakfactor ((x-x2)^2+(y-y2)^2)))
   (*Poisson eq., with a Gaussian replacing the 2d Dirac Delta*),
   DirichletCondition[f[x2, y2] == 0, True] (* Dirichlet boundary condition *)
  }, f, {x2, y2} ∈ V2]
f1 = G1_{1/2,0};
Plot3D[f1[x, y], {x, y} ∈ V2, PlotRange → All]
```

Out[ ]=



b.

Consider the following charge configuration

q_1= 1 $\{x_1, y_1\}$={2,1}

q_2= 1 $\{x_2, y_2\}$={0,2}

q_3= -2 $\{x_3, y_3\}$={-1,-2}

Compute the exact potential using Coulomb's formula

Apply Green's theorem with Dirichlet boundary condition to compute the potential at a point inside the region V2, by supplying :

- the charge distribution above
- the boundary values for the potential

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')\, G_D(\mathbf{x},\mathbf{x}')\, d^3x - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')\, da' . \quad (1.40)$$

In[∘]:=
```
ϵ0 = 1;
```

$$\text{PhiPoint}[q0\_, \{x0\_, y0\_\}] := \frac{1}{4\pi\epsilon 0}\frac{q0}{\left((x-x0)^2+(y-y0)^2\right)^{1/2}}$$

```
ρ[x_, y_] := DiracDelta[x - 2, y - 1] + DiracDelta[x, y - 2] + DiracDelta[x + 1, y + 2]
PhiTot = PhiPoint[1, {2, 1}] + PhiPoint[1, {0, 2}] + PhiPoint[-2, {-1, -2}]
```

Out[∘]=

$$\frac{1}{4\pi\sqrt{x^2+(-2+y)^2}}+\frac{1}{4\pi\sqrt{(-2+x)^2+(-1+y)^2}}-\frac{1}{2\pi\sqrt{(1+x)^2+(2+y)^2}}$$

In[∘]:=

$$f1 = G1_{0,0};$$

```
df1dx = D[f1[x, y], x];
df1dy = D[f1[x, y], y];
df1dn = Normalize[{x, y}].{df1dx, df1dy};
```

$$-\frac{1}{4\pi}\,\text{NIntegrate}[\text{PhiTot df1dn}, \{x, y\} \in \text{S2}]\ //\ \text{Quiet}$$

```
PhiTot /. {x → 0, y → 0} // N
```

Out[∘]=

```
0.00551499
```

Out[∘]=

```
0.00420061
```

c.

Give the numerical value of the potential obtained from the Green function at the points {x,y}={0,0}, {1/2,0},{0,1/2},{1/2,1/2}

Also give the value of the true potential (Coulomb's formula) at each these points

```
points = {{0.0, 0.0}, {0.5, 0.0}, {0.0, 0.5}, {0.5, 0.5}}
GreenPhi[{x0_, y0_}] :=
```

$$\frac{1}{4\pi}\,\text{NIntegrate}[\text{PhiTot Normalize}[\{x, y\}].\{D[G1_{x0,y0}[x, y], x], D[G1_{x0,y0}[x, y], y]\},$$

```
    {x, y} ∈ S2] // Quiet
TruePhi[{x0_, y0_}] := PhiTot /. {x → x0, y → y0}
Greenlist = Table[
  Append[p, GreenPhi[p]]
  , {p, points}]
Coulomblist = Table[
  Append[p, TruePhi[p]]
  , {p, points}]
```

Out[∘]=

```
{{0., 0.}, {0.5, 0.}, {0., 0.5}, {0.5, 0.5}}
```

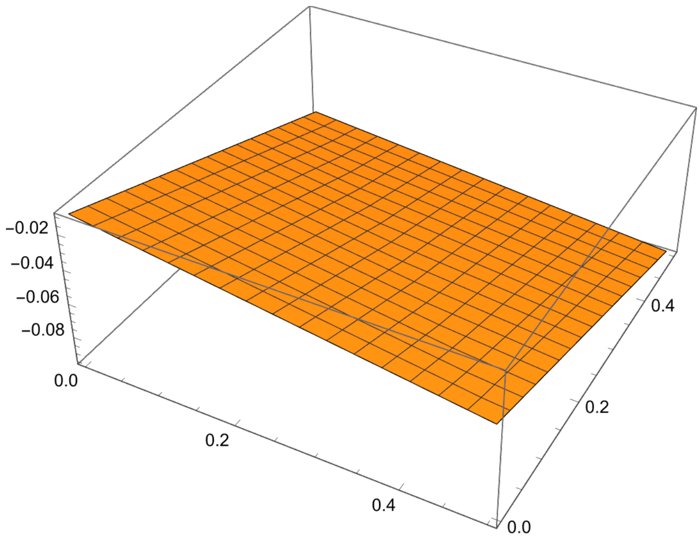
Out[∘]=

```
{{0., 0., -0.00551499}, {0.5, 0., -0.0230307},
 {0., 0.5, -0.0394259}, {0.5, 0.5, -0.0487821}}
```

Out[∘]=

```
{{0., 0., 0.00420061}, {0.5, 0., 0.0190804}, {0., 0.5, 0.0325437}, {0.5, 0.5, 0.0460687}}
```

```
In[ ]:= Δlist = Table[
          Append[p, GreenPhi[p] - TruePhi[p]]
          , {p, mesh}]
        ListPlot3D[Δlist, PlotRange → All]
Out[ ]=
        {{0., 0., -0.0097156}, {0.5, 0., -0.0421111},
         {0., 0.5, -0.0719696}, {0.5, 0.5, -0.0948509}}
Out[ ]=
```



3. Waveguides

Consider a rectangular waveguide with cross section of size [0,a] x [0,b] in the x, y directions.

a. Plot the values of $k_\lambda / \omega_\lambda$ for the first five TE modes. Set μ=ϵ=1 for simplicity.
Fix (a=1,b=1) then repeat with (a=1.5,b=1). What do you observe?

In[ ]:=
```
μ = 1;
ϵ = 1;
a = 1;
b = 1;
```

$$\text{OmegaComputation[\{m\_, n\_, a\_, b\_\}]} := \frac{\pi}{\text{Sqrt}[\mu\,\epsilon]}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2}$$

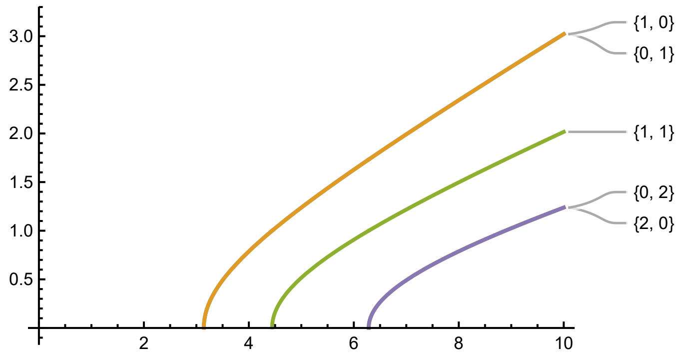
$$\text{KComputation[\{m\_, n\_, a\_, b\_, }\omega\text{\_\}]} := \text{Sqrt}\left[\mu\,\epsilon\,\omega^2 - \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\right]$$

```
FirstFiveModes = {{0, 1}, {1, 0}, {1, 1}, {2, 0}, {0, 2}}
Ratio[{{m_, n_}, ω_}] := KComputation[{m, n, a, b, ω}] / OmegaComputation[{m, n, a, b}]
Components[{x_}] := Table[Ratio[{p, x}], {p, FirstFiveModes}]


Plot[Evaluate@Components[{x}], {x, 0, 10}, PlotLabels → FirstFiveModes]
a = 1.5;
Components[{x_}] := Table[Ratio[{p, x}], {p, FirstFiveModes}]

Plot[Evaluate@Components[{x}], {x, 0, 10}, PlotLabels → FirstFiveModes]
```

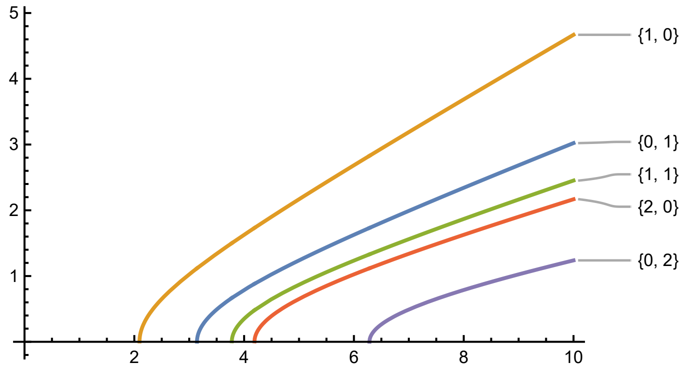
Out[ ]=

{{0, 1}, {1, 0}, {1, 1}, {2, 0}, {0, 2}}

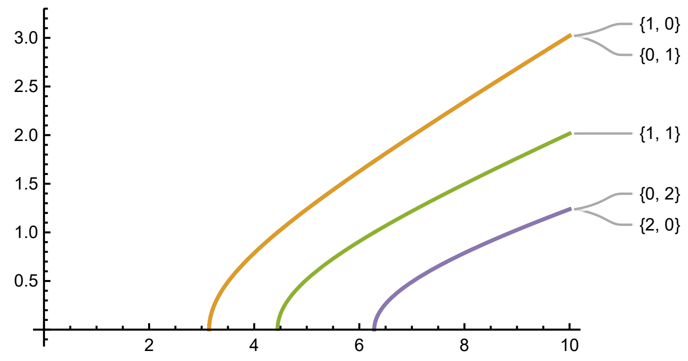
Out[ ]=



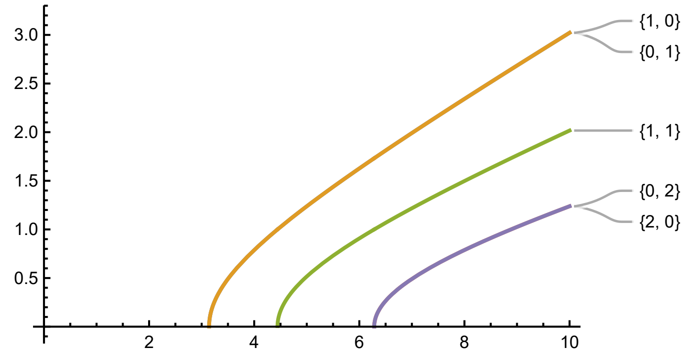
Out[ ]=



In[ ]:=



Out[ ]=

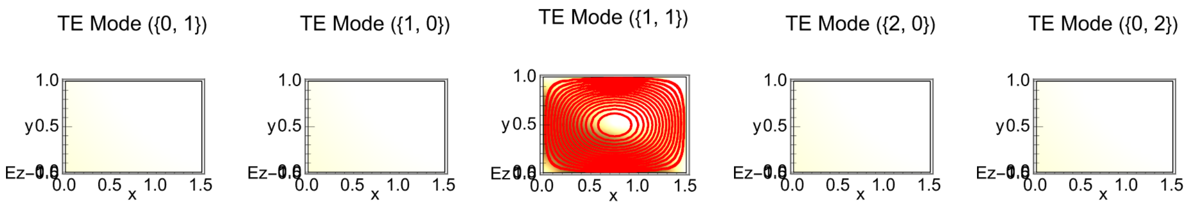


We can observe that by breaking the symmetry of the waveguide by using b different than a, the degeneracy in the TE modes is lifted: in the first graph, only 3 lines are shown since the values obtained are the same for the {0,1} and {1,0} pairs.

b. Fix a=1.5 and b=1. Plot the transverse electric field of the lowest five modes.

In[ ]:=
```
TEField[{m_, n_}, x_, y_] :=
 Module[{kx, ky, kz, ω, Ez},
  kx = (m π) / a;
  ky = (n π) / b;
  kz = Sqrt[kx^2 + ky^2];
  ω = kz;
  Ez = Sin[kx x] Sin[ky y];
  Ez]


PlotRangeAll = All;

plots = Table[Plot3D[TEField[p, x, y], {x, 0, a}, {y, 0, b}, PlotRange → PlotRangeAll,
    PlotLabel → StringForm["TE Mode (`1`)", p], AxesLabel → {"x", "y", "Ez"},
    MeshFunctions → {#3 &}, MeshStyle → {{Thick, Red}}, BoxRatios → {1.5, 1, 0.6},
    Lighting → "Neutral", ViewPoint → {0, 0, Infinity}], {p, FirstFiveModes}];

(*Show all the plots together*)
Grid[Partition[plots, 5], Spacings → {1, 1}]
```

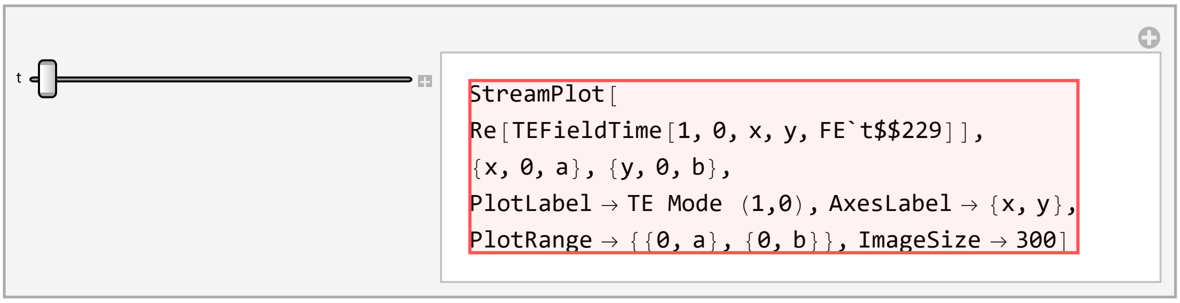
Out[ ]=



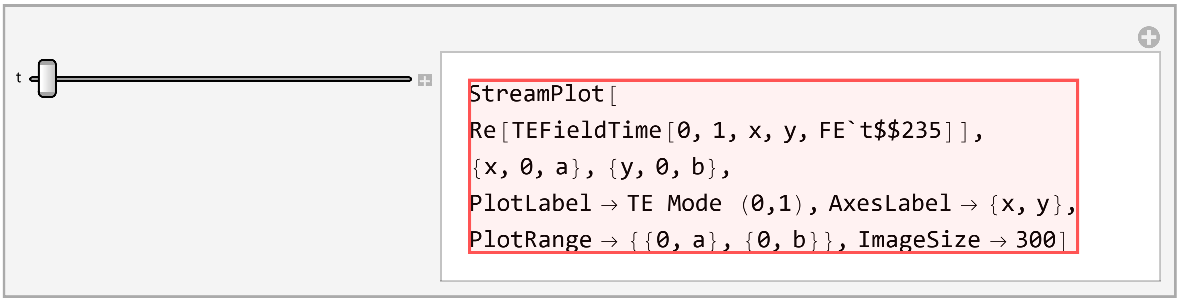
c. Now introduce time, and plot the behavior of the transverse electric field at z=0 for the (1,0), (0,1) and (1,1) modes as a function ot time (use the function Manipulate[StramPlot[...] , {t,0,10}]).
Repeat with the linear superposition of modes (1,0) + (0,1)

In[ ]:=
```
TEFieldTime[m_, n_, x_, y_, t_] :=
 Module[{kx, ky, kz, ω, Ex, Ey}, kx = (m π) / a;
  ky = (n π) / b;
  kz = Sqrt[kx^2 + ky^2];
  ω = kz;
  Ex = Cos[kx x] Sin[ky y] Exp[I ω t];
  Ey = -Sin[kx x] Cos[ky y] Exp[I ω t];
  {Ex, Ey}]



Manipulate[StreamPlot[Re[TEFieldTime[1, 0, x, y, t]], {x, 0, a},
  {y, 0, b}, PlotLabel → "TE Mode (1,0)", AxesLabel → {"x", "y"},
  PlotRange → {{0, a}, {0, b}}, ImageSize → 300], {t, 0, 10, 0.1}]

Manipulate[StreamPlot[Re[TEFieldTime[0, 1, x, y, t]], {x, 0, a},
  {y, 0, b}, PlotLabel → "TE Mode (0,1)", AxesLabel → {"x", "y"},
  PlotRange → {{0, a}, {0, b}}, ImageSize → 300], {t, 0, 10, 0.1}]

Manipulate[StreamPlot[Re[TEFieldTime[1, 1, x, y, t]], {x, 0, a},
  {y, 0, b}, PlotLabel → "TE Mode (1,1)", AxesLabel → {"x", "y"},
  PlotRange → {{0, a}, {0, b}}, ImageSize → 300], {t, 0, 10, 0.1}]

Manipulate[StreamPlot[Re[TEFieldTime[1, 0, x, y, t] + TEFieldTime[0, 1, x, y, t]],
  {x, 0, a}, {y, 0, b}, PlotLabel → "TE Modes (1,0) + (0,1)", AxesLabel → {"x", "y"},
  PlotRange → {{0, a}, {0, b}}, ImageSize → 300], {t, 0, 10, 0.1}]
```

Out[ ]=



Out[ ]=



Out[ ]=

t

```
StreamPlot[
Re[TEFieldTime[1, 1, x, y, FE`t$$241]],
{x, 0, a}, {y, 0, b},
PlotLabel → TE Mode (1,1), AxesLabel → {x, y},
PlotRange → {{0, a}, {0, b}}, ImageSize → 300]
```

Out[ ]=

t

```
StreamPlot[Re[TEFieldTime[1, 0, x, y, FE`t$$247] +
TEFieldTime[0, 1, x, y, FE`t$$247]], {x, 0, a},
{y, 0, b}, PlotLabel → TE Modes (1,0) + (0,1),
AxesLabel → {x, y},
PlotRange → {{0, a}, {0, b}}, ImageSize → 300]
```