A linear-time parameterized algorithm for computing the width of a DAG

Manuel Cáceres, Massimo Cairo, Brendan Mumey, Romeo Rizzi and Alexandru I. Tomescu

24.06.2021, WG

Basics

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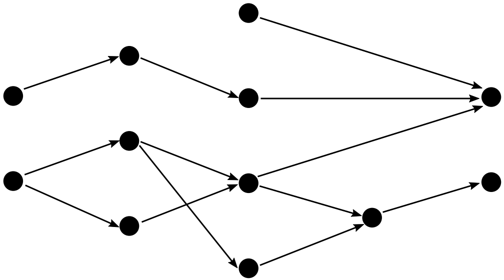
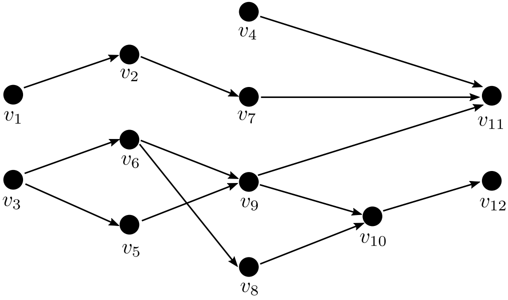
- Directed acyclic graph (DAG) $G = (V, E)$



Basics

- Topological ordering: $O(|V| + |E|)$ [15, 18]

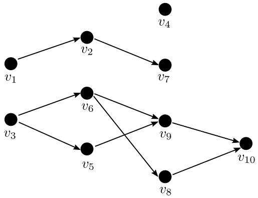
$v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}$



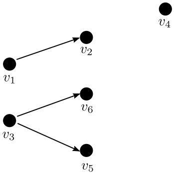
Basics

- Topologically induced subgraph, $G_i := G[\{v_1, \ldots, v_i\}]$

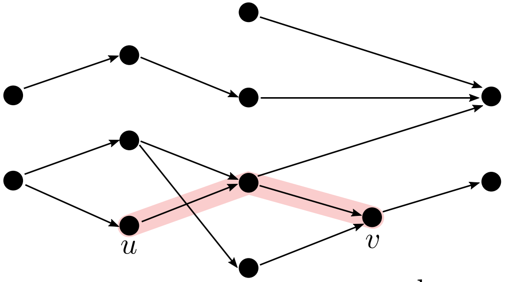
$G_{10} :=$



$G_6 :=$



- Constant-time reachability queries



u reaches v

Basics

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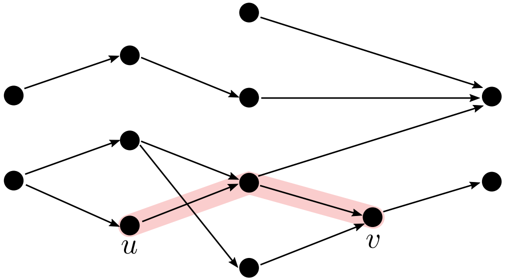
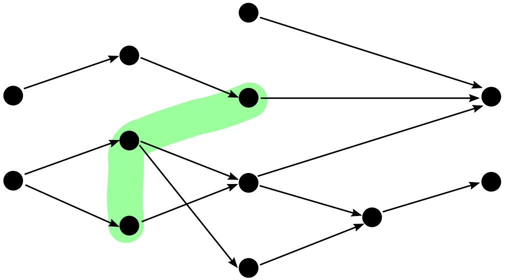
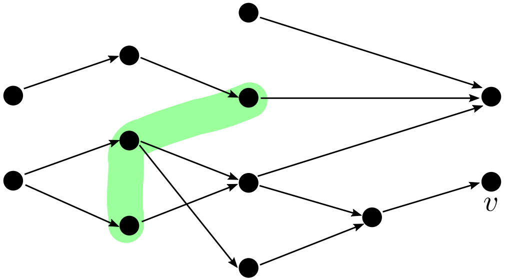


$\mathcal{S}.reaches(u, v)$ in $O(1)$

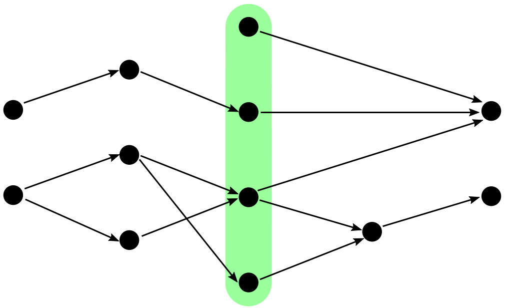
- Antichain



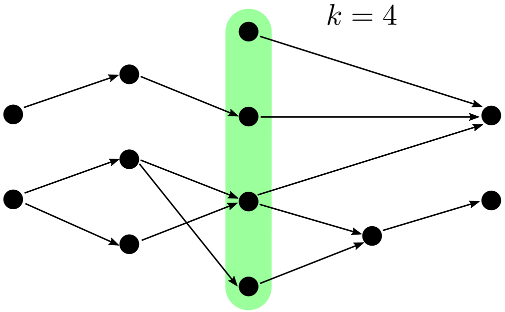
- Antichain reaches v



- Maximum antichain



- Width of DAG



Applications

Applications of computing the width

- Bioinformatics [1, 11]

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  - Perfect Phylogeny Haplotyping

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  - Perfect Phylogeny Haplotyping
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  - Dimension of a game
- Distributed computation [13, 19]
  - K mutual exclusion violation

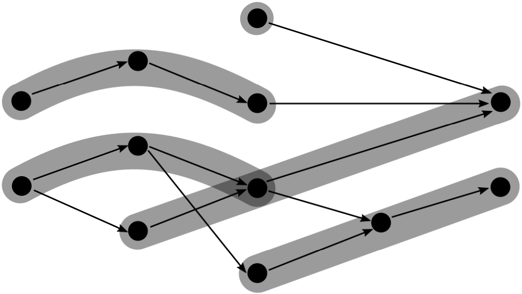
Algorithms parameterized by the width k

Why algorithms parameterized by k?

- Natural parameter
- Some applications: small k (pan-genomes [16])
- FPT-algorithms [20, 5, 2, 10]

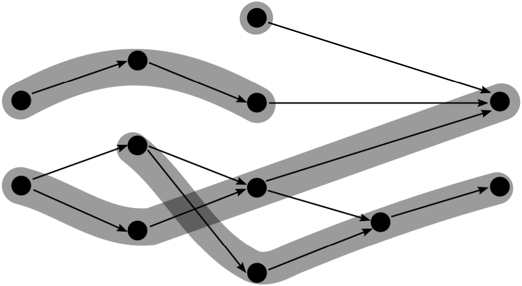
(Most) State-of-the-art: *Minimum Path Cover*

- Path cover



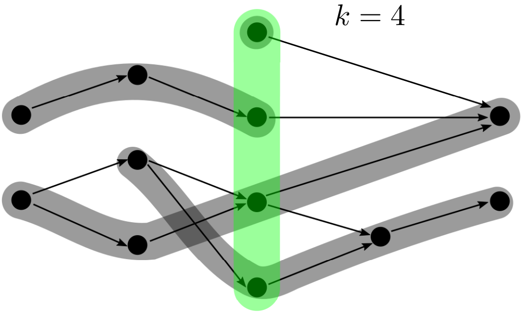
Basics

- Minimum Path Cover (MPC)



Basics

- Dilworth's Theorem [6]



MPC algorithms

Maximum Matching	Minimum Flow
– $O(\sqrt{\lvert V\rvert}\lvert E\rvert)$ [9, 12] (posets)	– $O(\lvert V\rvert\lvert E\rvert)$ [17, 8]
– $O(\lvert V\rvert^2 + k\sqrt{k}\lvert V\rvert)$ [3]	– $O(k\lvert E\rvert \log \lvert V\rvert)$ [16]
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Felsner et al. [7] recognize posets:

- $O(|V|)$, for $k \leq 3$.
- $O(|V| \log |V|)$, for $k = 4$.
- "the case $k = 5$ already seems to require an unpleasantly involved case analysis" [7, p. 359]

Our result

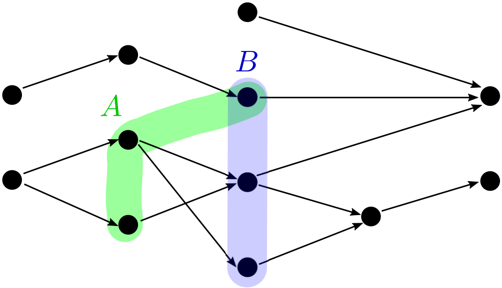
$O(f(k)(|V|| + |E|))$ time algorithm

Maximum Antichain

Antichain domination

Definition 1 (Dominates)

Antichain B *dominates* antichain A if $|A| = |B|$ and for each $b \in B$, A reaches b



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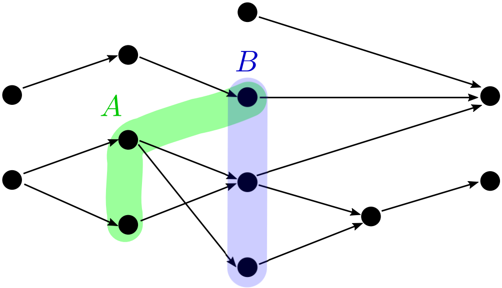


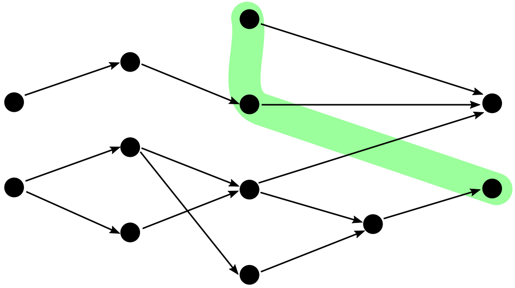
Lemma 1

Domination is a partial order on antichains of G.

Frontier Antichains

Definition 2 (Frontier)

- Maximal elements of domination relation
- Antichains only dominated by themselves



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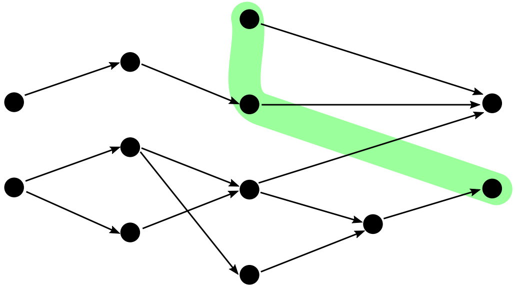

Lemma 3

G has at most 2^k frontier antichains

If A is a frontier antichain of G we also say that A is G-frontier

Classification of G_i-frontiers

We classify G_i-frontiers A into two categories:

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Type 1 : $v_i \in A$

Lemma 4

If A is type 1 G_i-frontier, then $A \setminus \{v_i\}$ is G_{i-1}-frontier

Type 2 : $v_i \notin A$

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Lemma 4

If A is type 1 G_i-frontier, then $A \setminus \{v_i\}$ is G_{i-1}-frontier

Lemma 6

If B is G_{i-1}-frontier and B does not reach v_i, then $B \cup \{v_i\}$ is type 1 G_i-frontier

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Lemma 5

If A is type 2 G_i-frontier, then A is G_{i-1}-frontier

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Type 2 : $v_i \notin A$

Lemma 5

If A is type 2 G_i-frontier, then A is G_{i-1}-frontier

Lemma 2

If B is G_{i-1}-frontier but not G_i-frontier, then B is dominated by type-1 G_i-frontier

The Algorithm

(for posets)

Algorithm (simplified)

for $v_i \in v_1, \ldots, v_{|V|}$ *in topological order* **do**
    for $A \in G_{i-1}$*-frontiers* **do**
        if A *does not reach* v_i **then**
            Store A as type 1 G_i-frontier
    for $A \in G_{i-1}$*-frontiers* **do**
        if $\forall B \in$ *type 1* G_i*-frontiers,* B *does not dominate* A **then**
            Store A as type 2 G_i-frontier
return Largest frontier

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$O(k^2 4^k |V|)$: with constant-time reachability queries (posets)

The Algorithm

(Maintain constant-time reachability queries)

The Support

Observation 1

When computing G_i-frontiers we only need reachability among vertices of G_{i-1}-frontiers and v_i

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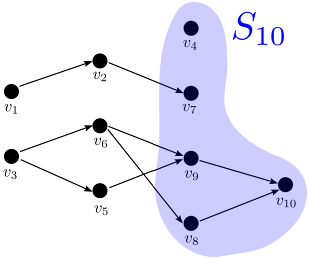
Definition 3 (Support)

$$S_i := \bigcup_{A \in G_i\text{-frontiers}} A$$

$G_{10} :=$



The Support

Observation 1

When computing G_i-frontiers we only need reachability among vertices of G_{i-1}-frontiers and v_i

Definition 3 (Support)

$$S_i := \bigcup_{A \in G_i\text{-}\mathit{frontiers}} A$$

Lemma 7 and 8 (Informal)

A vertex v_i only belongs to a topologically adjacent sequence of supports $S_i, \ldots, S_j$

$\Rightarrow$ Theorem 2 and Theorem 3

Algorithm

Observation 1

When computing G_i-frontiers we only need reachability among vertices of $S_{i-1} \cup \{v_i\}$

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- Reduced to maintain reachability **from** vertices in S_{j-1} **to** v_j for each $j \leq i$ **(Theorem 2)**

Algorithm

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- Reduced to maintain reachability **from** vertices in S_{j-1} **to** v_j for each $j \leq i$ **(Theorem 2)**
- Compute inductively reachability **from** vertices in S_{i-1} **to** v_i, in $O(k2^k)$ per edge incoming to v_i **(Theorem 3)**

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Theorem 1

Given a DAG $G = (V, E)$ of width k, we can compute a maximum antichain of it in time $O(k^2 4^k |V| + k2^k |E|)$

Acknowledgments

We thank the anonymous reviewers for their useful suggestions, and you for listening until the end 😊

Funding

This work was partially funded by the ERC Starting Grant 851093 (SAFEBIO)

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