Optimal investment and benefit strategies for a target benefit pension plan where the risky assets are jump diffusion processes

Ricardo Josa Fombellida[1], Paula López Casado[1]

[1]Dpto. Estadística e Investigación Operativa e IMUVA, Universidad de Valladolid

July 10, 2024

ISDG 2024 - Valladolid

Contents

Contents

Classification of the pension plans

Two types of funds attending to the attribution of risks:

- Defined Contribution (DC): the contribution is fixed in advance and the benefits are the result of the investment decisions; the risk is assumed by the employees.
- Defined Benefits (DB): the benefits promised are fixed in advance; the manager select the contributions and the investments in order to fulfill with this obligation; the risk is assumed by her/him.

✓ Both types of plans have been studied in the literature with dynamic programming techniques.

✓ The target benefit plan (TBP) is a new type of collective pension plan that blends elements of the DB and DC plans to provide benefits at retirement that are linked to how well the pension plan performs.

TBP

Consider a **aggregated target benefit pension plan** where, at every instant of time, active participants coexist with retired participants.

- Contribution rate is proportional to the size of the fund.
- Benefit is a control variable for the fund manager that also adjusts the investment
- Main objective of the manager is to increase the benefits as much as possible in order to make the pension plan more attractive to the participants
- Fund surplus is invested in a financial market composed of one riskless asset and several risky assets, modeled by Brownian motions and Poisson processes

Notation

T: Planning horizon, end date of the pension plan, $0 < T < \infty$;

$F(t)$: value of fund assets at time t

$P(t)$: benefits promised to the participants at time t

$C(t)$: contribution proportion of the fund wealth made by the manager at time t to the funding process; it is a deterministic function

ρ: positive constant rate of discount or time preference of the manager

r: constant risk-free market interest rate

$\pi(t)$: vector of investments in the risky assets

γ: risk aversion

Contribution

Contribution proportion has an exponential form:

$$C(t) = c_1 e^{c_2 t}, c_1 > 0$$

We consider three interesting particular cases:

1. When $c_2 = 0$ the contribution is constantly indexed to the fund wealth: $C(t) = c_1$, with $c_1 > 0$.
2. When $c_2 > 0$, we assume a salary growth that is materialized in the contribution, Roch (2022)
3. When $c_2 < 0$, the manager allow reduce the contribution proportion without fund increase, in order to make the pension plan more attractive to the participants, Zhao and Wang (2022)

Roch, O. (2022). Continuous-time optimal pension indexing in pay-as-you-go systems. Applied Stochastic Models in Business and Industry, 38, 458-474.

Zhao, H., Wang, S. (2022). Optimal investment and benefit adjustment problem for a target benefit pension plan with Cobb-Douglas utility and Epstein-Zin recursive utility. European Journal of Operational Research, 301 (3), 1166-1180.

Contents

Financial market

$(\Omega, \mathscr{F}, \mathbb{P})$, $\mathscr{F}_t = \sigma\{w(s); 0 \leq s \leq t\}$ continuous and right continuous filtration generated by $w = (w_1, \ldots, w_n)^\top$, n–dimensional standard Brownian motion, and $(\Omega^N, \mathscr{F}^N, \mathbb{P}^N)$, $\mathscr{F}_t^N = \sigma\{(N(s)); 0 \leq s \leq t\}$, a m–dimensional Poisson process $N = (N_1, \ldots, N_m)^\top$ with constant intensity $\lambda = (\lambda_1, \ldots, \lambda_m)^\top$, $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$

- As in Merton's model, the financial market is composed by a bond and n risky assets which are Geometric Brownian Motion with Poisson jumps:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1,$$

$$dS^i(t) = S^i(t-)\Big(b_i dt + \sum_{j=1}^{l} \sigma_{ij} dw_j(t) + \sum_{k=1}^{m} \varphi_{ik} dN_k(t)\Big), S^i(0) = s_i.$$

$r, b_i, \sigma_{ij} > 0, \varphi_{ij} > -1$, $b_i > r > 0$, for all $i = 1, \ldots, n$.

Denote $\sigma = (\sigma_{ij})$, $\varphi = (\varphi_{ij})$, $b = (b_1, b_2, \ldots, b_n)^\top$, $\bar{1} = (1, 1, \ldots, 1)^\top$, $\Sigma = \sigma\sigma^\top$. Suppose there exists Σ^{-1}, that is, σ^{-1}.

The Sharpe ratio of the portfolio is $\theta = \sigma^{-1}(b - r\bar{1})$.

Risky asset evolution $S(t)$

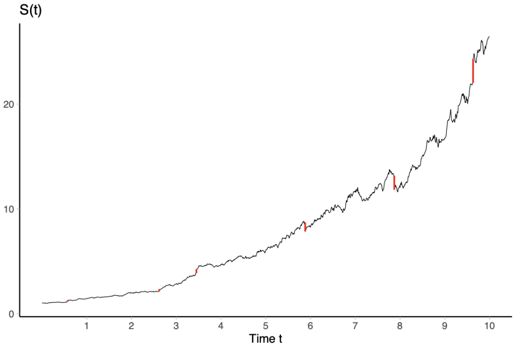

Figure: Risky asset evolution for $b = 0.15$, $\sigma = 0.10$, $r = 0.01$, $\lambda_1 = 0.5$, $\lambda_2 = 0.25$, $\varphi_1 = \pm 0.1$, $\varphi_2 = \pm 0.1$ and $S(0) = 1$.

The fund wealth

- $\pi = (\pi_1, \ldots, \pi_n)^\top$, where π_i is the amount of surplus to be invested in the risky asset $S^i, i = 1, \ldots, n)$
- $X - \sum_{i=1}^n \pi_i$ is invested in the bond S^0

Dynamics of the fund F is driven by

$$dF(t) = \sum_{i=1}^{n} \pi_i(t) F(t) \frac{dS^i(t)}{S^i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t)\right) F(t) \frac{dS^0(t)}{S^0(t)} + (C(t)F(t) - P(t))\, dt$$

$$dF(t) = \Big(rF(t) + \pi(t)^\top (b - r\bar{1}) F(t) + C(t)F(t) - P(t) \Big) dt + \pi(t)^\top F(t) \sigma\, dw(t) + \pi(t)^\top F(t) \varphi\, dN(t)$$

with the initial condition $F(0) = F_0 > 0$.

Optimal strategies when the isoelastic utility is maximized

The objective functional to be maximized over the class of admissible controls $\mathcal{A}$, is given by

$$J((t,F);(P,\pi)) = \mathbb{E}_{t,F}\Big\{ \int_t^T e^{-\rho(s-t)}\, U(P(s))ds + e^{-\rho(T-t)} U(F(T))\Big\},$$

where

- U is a utility function of the benefit and the fund
- The time preference of the sponsor is given by $\rho > 0$
- The aim is to maximize the expected utility of the benefit along the planning interval and of the fund wealth at the end of the plan
- Note that we are considering P and Π as control variables.

We consider a CRRA utility function:

$$U(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1.$$

Optimal benefit and optimal investment proportions

Theorem

The optimal benefit and the optimal investment proportions in the risky assets are given by

$$P^* = e^{-\int_t^T a(s)ds}\left(1+\int_t^T e^{-\int_s^T a(u)du}ds\right)^{-1}F,$$

where π is solution of the equation:

$$b_i - r - \gamma\sum_{j=1}^{l} a_{ij}\pi_j + \sum_{k=1}^{m}\lambda_k\left(1+\Pi^\top\varphi_k\right)^{-\gamma}\varphi_{ik} = 0, \quad i = 1,\ldots,n, \tag{1}$$

$$a(t) = -\frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\Big(C(t)+\Psi(\Pi^*,\gamma)\Big),$$

Optimal Benefit and Optimal Investment Proportions

Theorem

$$dF^*(t) = \Big(r + \Pi^{*\top}(b - r\bar{1}) + C(t) - e^{-\int_t^T a(s)ds}\Big(1 + \int_t^T e^{-\int_s^T a(u)du} ds\Big)^{-1}\Big)$$
$$+ \Pi^{*\top}\sigma F^*(t)\, dw(t) + \Pi^{*\top}\varphi F^*(t)\, dN(t),$$

Given a solution $\Pi(1)$, it is straightforward to obtain the optimal benefit P^* and the evolution of the optimal fund F^*.
From F^*, the expected optimal fund is given by

$$\mathrm{E}F^*(t) = F_0 \exp\Big\{\Big(r + \Pi^{*\top}(b - r\bar{1}) + \sum_{k=1}^{m} \lambda_k \Pi^{*\top}\varphi_k\Big)t$$
$$+ \int_0^t \Big(C(v) - e^{-\int_v^T a(s)ds}\Big(1 + \int_v^T e^{-\int_s^T a(u)du} ds\Big)^{-1}\Big) dv\Big\}.$$

Particular cases

Particular cases:

- Exponential contribution proportion
- Model without jumps
- Infinite horizon
- Logarithmic utility

Contents

Optimal Investment Proportion Π

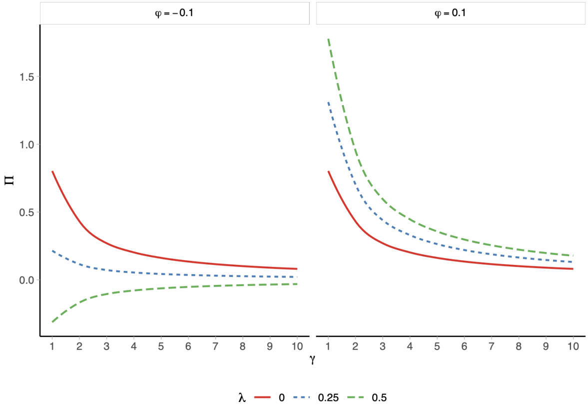

Figure: Optimal investment proportion by risk aversion, jump intensity and uncertainty Poisson parameter for $c_2 = -0.05$ under a bear regime

Optimal benefit without fund effect

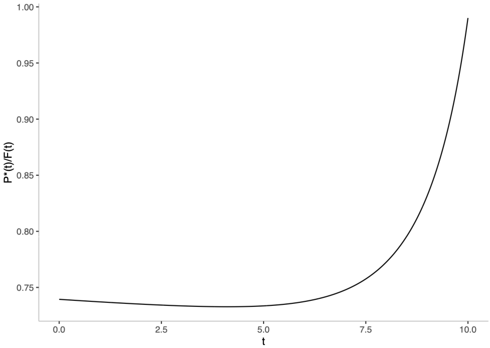

Figure: Optimal relative benefit over time for $\gamma = 3$, $\lambda = 0.5$, $\varphi = 0.1$ and $c_2 = -0.05$ under bull regime

Optimal Relative Benefit over time $P^*(t)/F^*(t)$

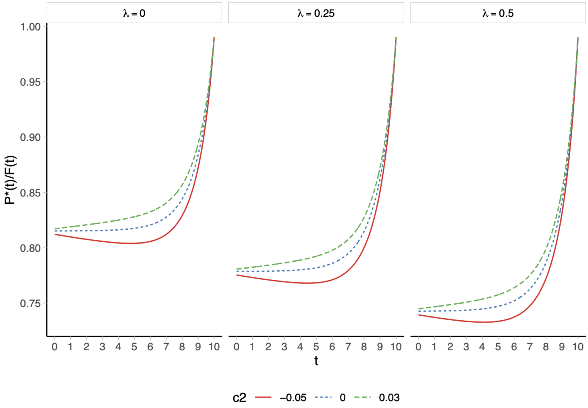

Figure: Optimal relative benefit over time by the rate of contribution and the jump intensity over time for $\gamma = 3$ and $\varphi = 0.1$ under a bull regime

Expected Fund Evolution *EF*

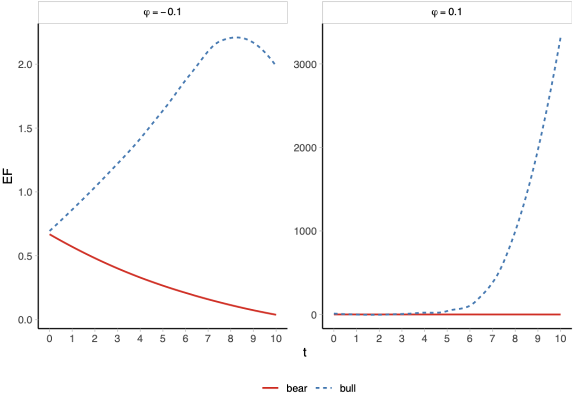

Figure: Expected fund evolution by jump uncertainty parameter for $\gamma = 3$, $\lambda = 0.5$, $c_2 = 0.05$ under bear and bull regimes

Contents

Final comments

- Higher levels of risk aversion result in a more conservative investment approach, with a reduced proportion allocated to risky assets
- Contributions influence the trajectory of optimal benefits. A negative contribution growth rate leads to a temporary decline in benefits, followed by recovery, while positive contributions result in a continuous upward trajectory
- There is a proof of jump effect on the market dynamics. Positive jumps contribute significantly to the growth of the fund, while negative jumps can impede fund ascent

Further research:

- Other utility functions can be considered, such as CARA
- Consider as a stochastic function the contribution proportion of the fund wealth instead of deterministic, adding value and complexity to the model
- Other aim is to consider minimizing the quadratic deviations between benefit and terminal fund and its target values