Heat diffusion distance processes:
a statistical method to analyze graph data

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Forum des Jeunes Mathématicien.ne.s

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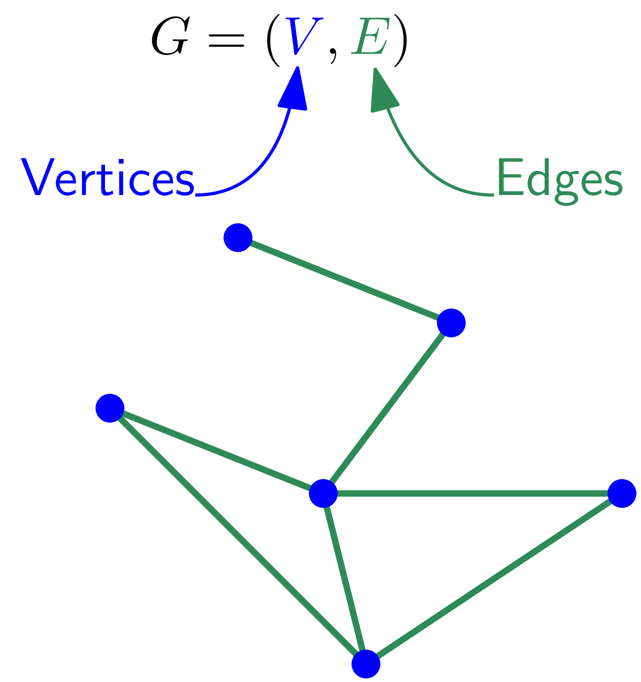
Introduction

Goals :

- analysis of graph samples

$$(G_1, \ldots, G_N)$$

- theoretical results (asymptotic in N)
- useable in practice



Requirements :

- take into account topological information
- graphs can be weighted
- graph sizes (same/different)
- node correspondance (known/unknown)

Outline

1. Tools
    - Heat Kernel Diffusion Processes
    - Heat Persistence Diffusion Processes

2. Theoretical Results
    - Functional Central Limit Theorem
    - Gaussian Approximation Rates

3. Simulations
    - Confidence Bands
    - Two-sample Tests

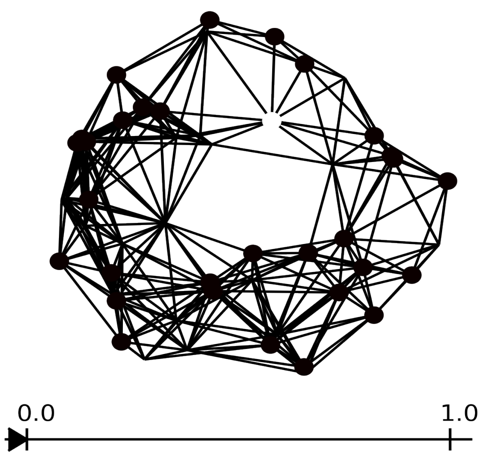
Comparing graphs

Assumption :
same sizes n & known node correspondance.

W, weight matrix, $W_{i,j}$: weight of $\{i, j\}$

D, degree matrix, $D_{i,i} = \sum_{j=1}^{n} W_{i,j}$

L = D - W, laplacian.



Heat Diffusion :

For $t \geq 0$, $u_t \in \mathbb{R}^n$: heat distribution,

$$\frac{d}{dt} u_t = -L u_t, \quad t \geq 0$$

$u_t = e^{-tL} u_0$,
e^{-tL}, heat kernel at time t.

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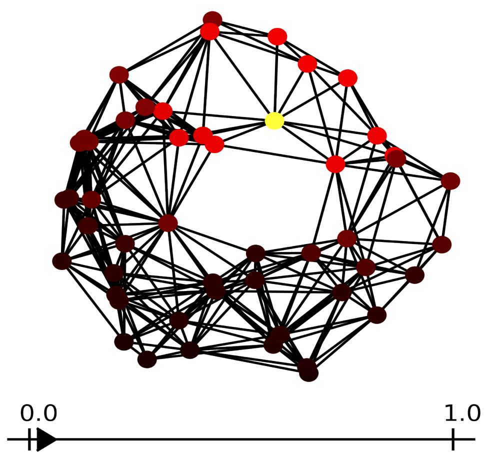

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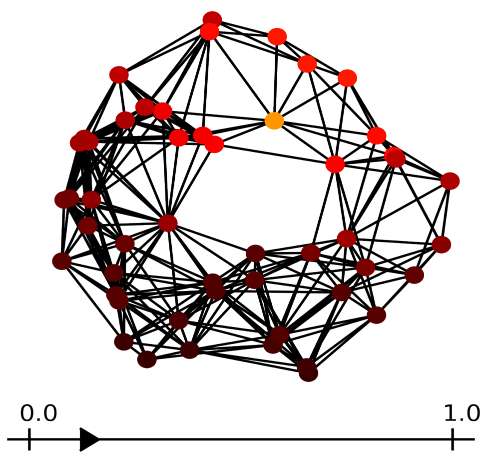

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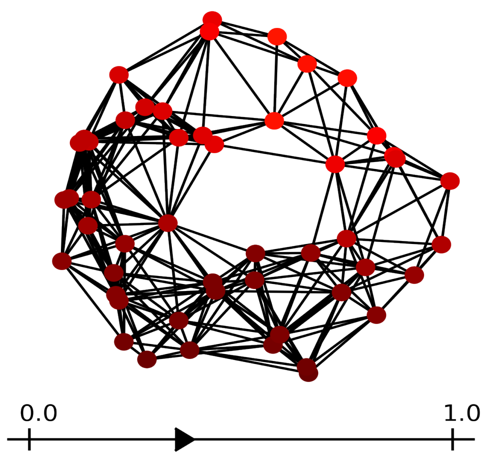

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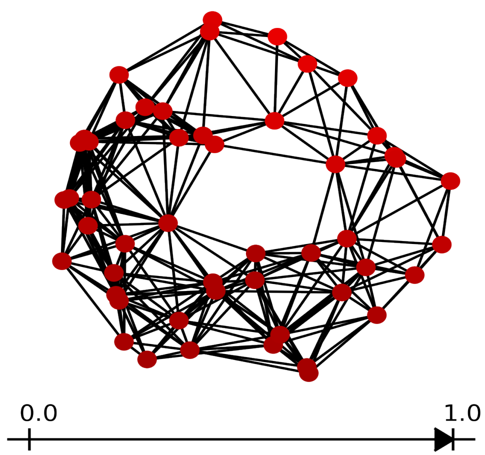

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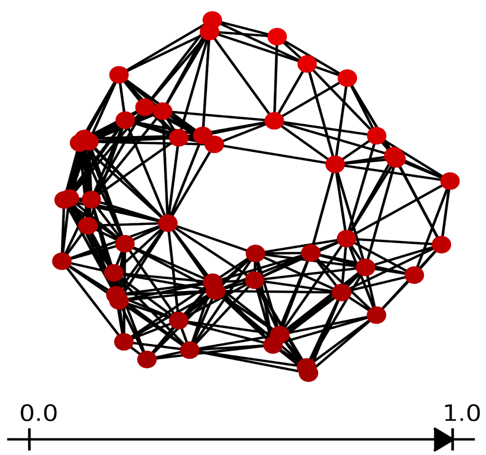

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Heat Kernel Distance : $D_t(G, G') = \|e^{-tL} - e^{-tL'}\|_F$ [HGJ13]

- respectful of the topology ✓
- t : scale parameter ✓

[HGJ13] : Graph diffusion distance: A difference measure for weighted graphs based on the graph Laplacian exponential kernel, Hammond, Gur, Johnson, 2013

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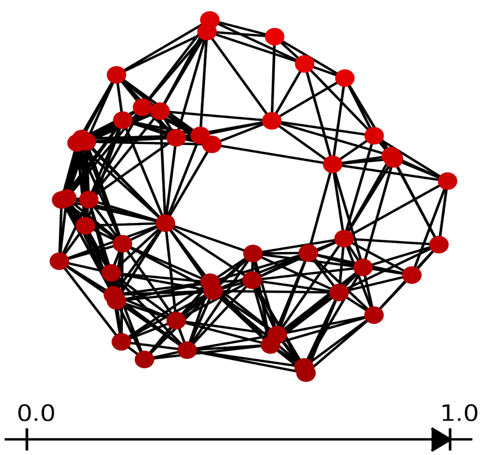

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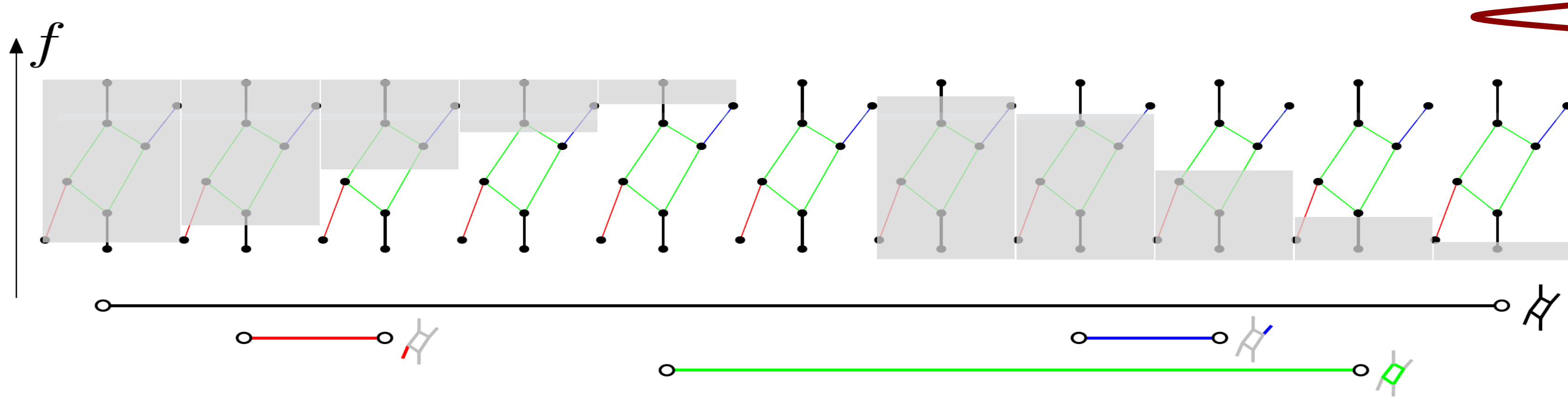
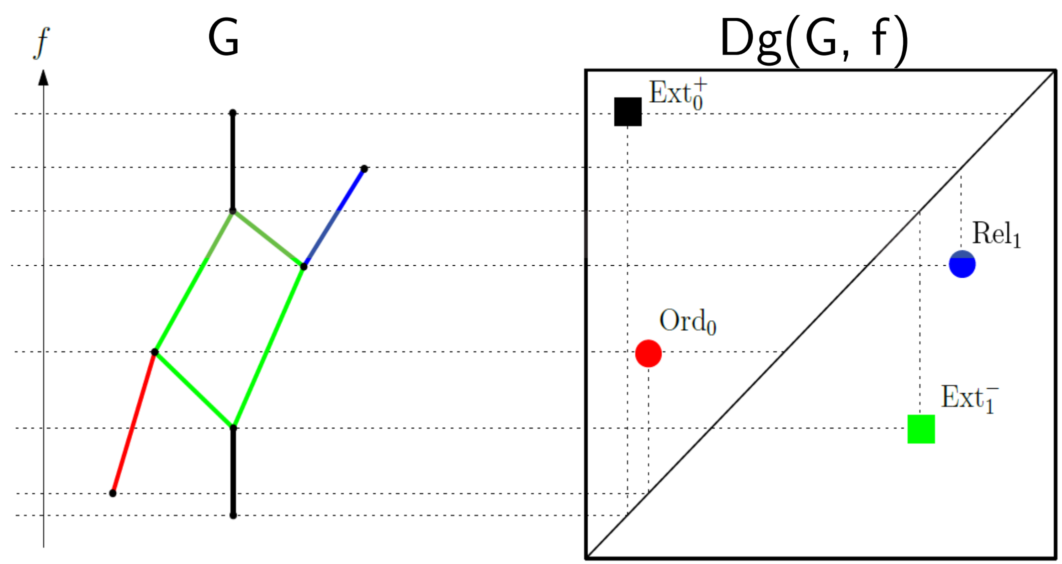
Using Topological Data Analysis

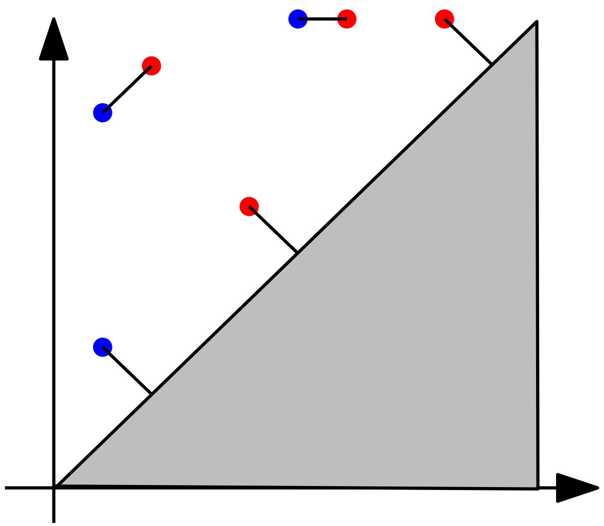




Figures from [CCIL+19]

[CCIL+19]: Perslay : A Neural Network Layer for Persistence Diagrams and New Graph Topological Signatures, Carriere, Chazal, Ike, Lacombe, Royer, Umeda, 2019

Comparing persistence diagrams



μ, ν : finite multisets of points in $\mathbb{R}^2$.
$\Delta = \{(a, a), \forall a \in \mathbb{R}\}$: diagonal

π : a matching from $\mu \cup \Delta$ to $\nu \cup \Delta$

$\Pi(\mu, \nu)$: set of all matchings

Bottleneck Distance :

$$d_B(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \sup_{x \in \mu \cup \Delta} \|x - \pi(x)\|_\infty$$

[EH10]

[EH10]: *An Introduction to Computational Topology*, Edelsbrunner and Harer, 2010.

Choice of f

Heat Kernel Signature (HKS) : [SOG09] [HRG14]

$$h_t(G) : i \to (e^{-tL})_{i,i}$$

"Remaining heat at node i"

Heat Persistence Distance (HPD) :

$$\boxed{H_t(G, G') = \max_{D_g} \; d_B\left(Dg\left(G, h_t(G)\right), Dg\left(G', h_t(G')\right)\right)}$$

[SOG09]: A concise and provably informative multiscale signature based on heat diffusion, Sun, Ovsjanikov, Guibas, 2009

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How to choose t ?

Don't Choose !

Functional Point of View

$$D_{\cdot}(G,G') : \begin{array}{ccc} [0,T] & \mapsto & \mathbb{R} \\ t & \mapsto & D_t(G,G') \end{array} \quad \text{or} \quad H_{\cdot}(G,G') : \begin{array}{ccc} [0,T] & \mapsto & \mathbb{R} \\ t & \mapsto & H_t(G,G') \end{array}$$

Empirical Process Point of View

$$\{D_t((G,G')), \quad t \in [0,T]\} \quad \text{or} \quad \{H_t((G,G')), \quad t \in [0,T]\}$$

$$\mathcal{F}_{HKD} = \{D_t(\cdot), \quad t \in [0,T]\} \quad \text{or} \quad \mathcal{F}_{HPD} = \{H_t(\cdot), \quad t \in [0,T]\}$$

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(G_0, G'_0) (G_1, G'_1)

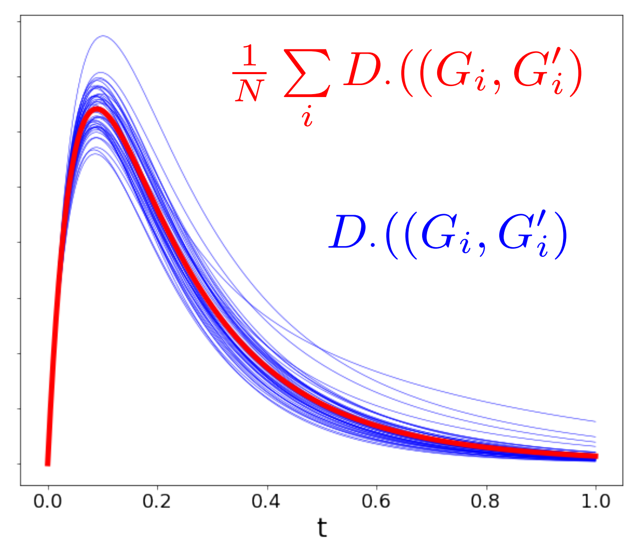
(G_2, G'_2) $(G_{\dots}, G'_{\dots})$

(G_N, G'_N)

Dataset

$D_{\cdot}(\cdot)$

or $H_{\cdot}(\cdot)$



Lipschitz continuity

Proposition. *Fix n and $w_{\max} > 0$.*
For all (G, G') of size n, with weights in $[0, w_{\max}]$,

$$\begin{array}{ccc} [0,T] & \rightarrow & \mathbb{R}^+ \\ t & \rightarrow & D_t(G,G') \end{array} \quad \text{is } (n^{3/2} w_{\max})\text{-lipschitz continuous.}$$

Proposition. *Fix n and $w_{\max} > 0$.*
For all (G, G') of size **at most** *n, with weights in $[0, w_{\max}]$,*

$$\begin{array}{ccc} [0,T] & \rightarrow & \mathbb{R}^+ \\ t & \rightarrow & H_t(G,G') \end{array} \quad \text{is } (2n w_{\max})\text{-lipschitz continuous.}$$

Functional central limit theorem

- $(G_1, G'_1), \ldots, (G_N, G'_N) \sim P$ (i.i.d sample)
- P_N : empirical measure

$$P_N D_t = \frac{1}{N} \sum_{i=1}^{N} D_t((G_i, G'_i)) \qquad PD_t = \mathbb{E}_P \left[D_t((G, G')) \right]$$

Theorem. *Fix n and $w_{\max} > 0$. For all distribution P over pairs of graphs of size n, with weights in $[0, w_{\max}]$,*
*the family $\mathcal{F}_{HKD} = \{D_t(\cdot),\ t \in [0,T]\}$ is P-***Donsker**

$\{\sqrt{N}(P_N - P)D_t,\ t \in [0,T]\} \xrightarrow{weak}$ Gaussian Process $\mathbb{G}$

$\forall h : \mathcal{C}([0,T]) \to \mathbb{R}$, continuous and bounded,
$\lim_{N\to\infty} \mathbb{E}\left[h\Big(\sqrt{N}(P_N - P)D.\Big) \right] = \mathbb{E}\left[h(\mathbb{G}) \right]$

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Theorem. *Fix n and $w_{\max} > 0$. For all distribution P over pairs of graphs of size* **at most** *n, with weights in $[0, w_{\max}]$,*
*the family $\mathcal{F}_{HPD} = \{H_t(\cdot),\ t \in [0,T]\}$ is P-***Donsker**

Consequences: consistent confidence bands and two-sample tests

Results

Gaussian Approximation with rate r_N :

$\forall \lambda > 1$, $\exists C$ s.t. $\forall N \geq 1$,
one can construct on the same probability space both X_N and a version of the Gaussian process $\mathbb{G}^{(N)}$, s.t.

$$\mathbb{P}\left(\|X_N - \mathbb{G}^{(N)}\|_\infty > C.r_N\right) \leq N^{-\lambda}.$$

$\left\{\sqrt{N}(P_N - P)D_t, \quad t \in [0,T]\right\}$ and $\left\{\sqrt{N}(P_N - P)H_t, \quad t \in [0,T]\right\}$
admit Gaussian Approximations with rate :

$$r_N = N^{-1/7} \log(N)^{9/14}.$$

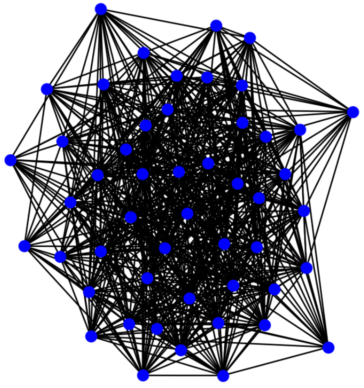
Remark: r_N is independent of n (graph size)

Simulations : Stochastic Models

Erdös-Renyi (ER)

$n = 50$

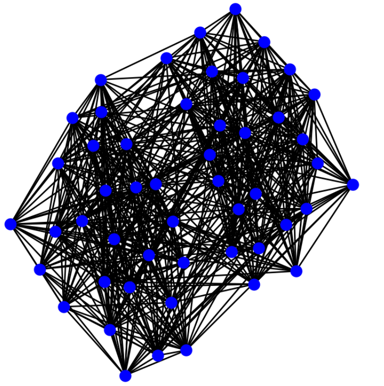
$p = 0.5$



Stochastic Block Model (SBM)

$n_1 = n_2 = 25$

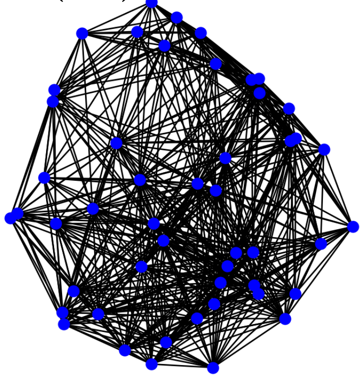
$p = \begin{pmatrix} 0.75, & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$



Geometric (Disk)

$n = 50$ or $\mathcal{P}(50)$

$p = 0.5$



Geometric (Annulus)

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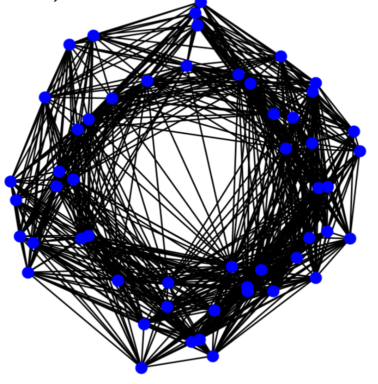
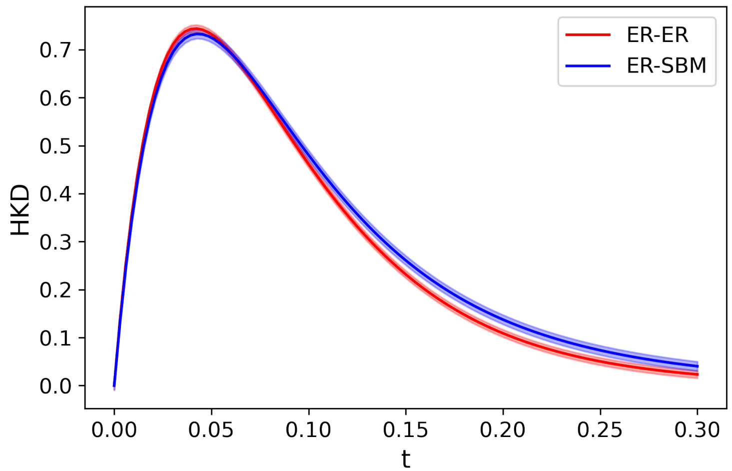
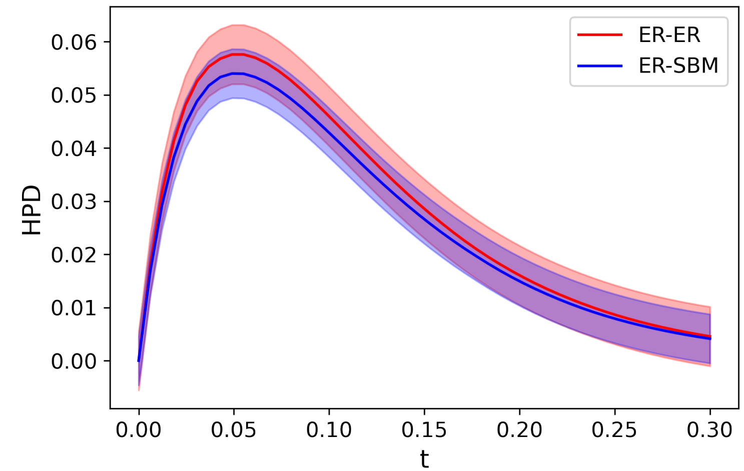
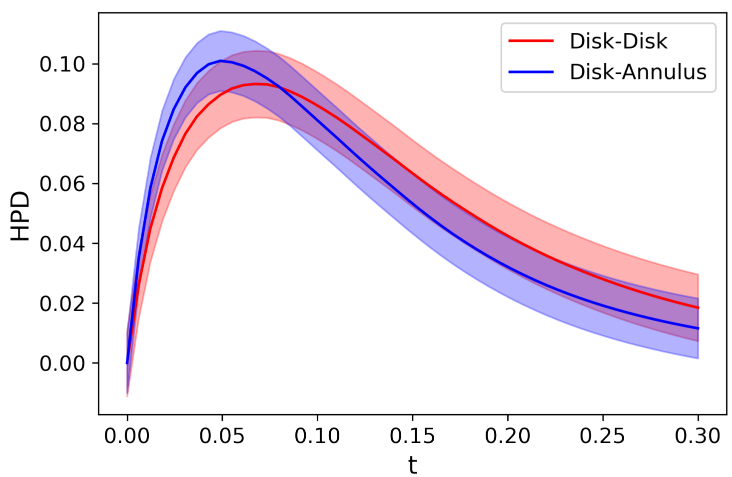
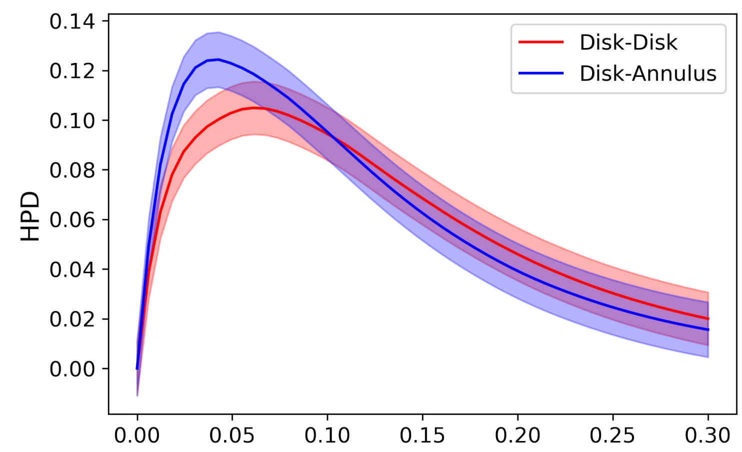


Simulations : Confidence Bands







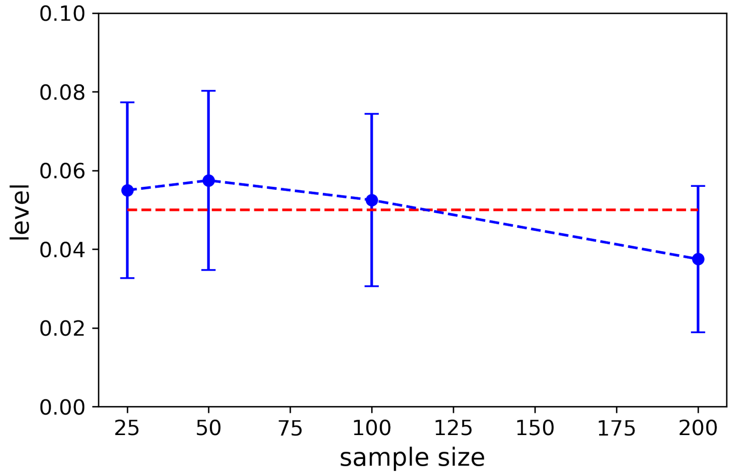


Graphs with random size.

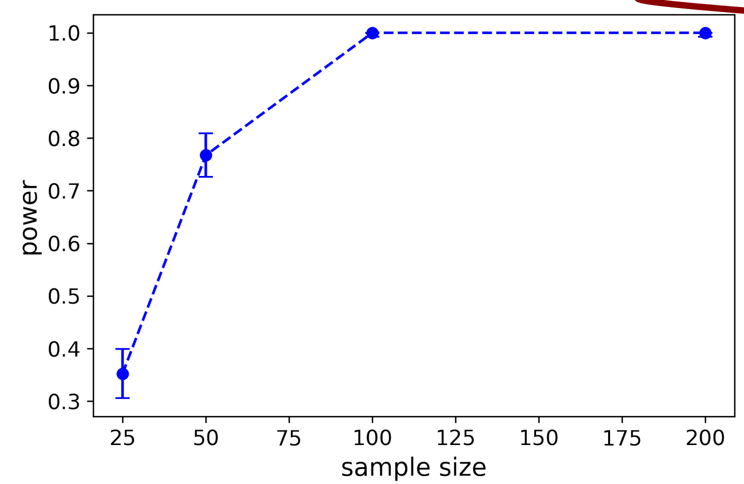
Confidence level 99%, sample size : 100, bootstrap sample size : 1000.

Simulations : Two-sample Tests

HKD



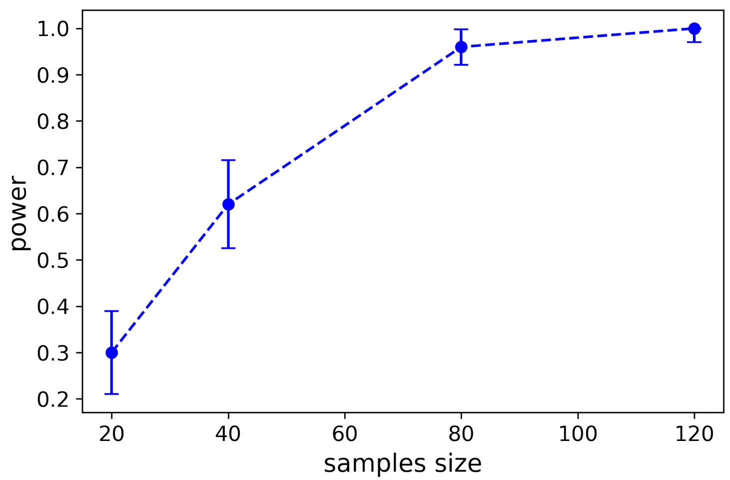
ER-ER



ER-ER vs ER-SBM

Level 95%, bootstrap sample size : 1000, number of tests : 400

HPD



Disk-Disk vs Disk-Annulus

Level 95%, bootstrap sample size : 1000, number of tests : 100

Simulations : Two-sample Tests

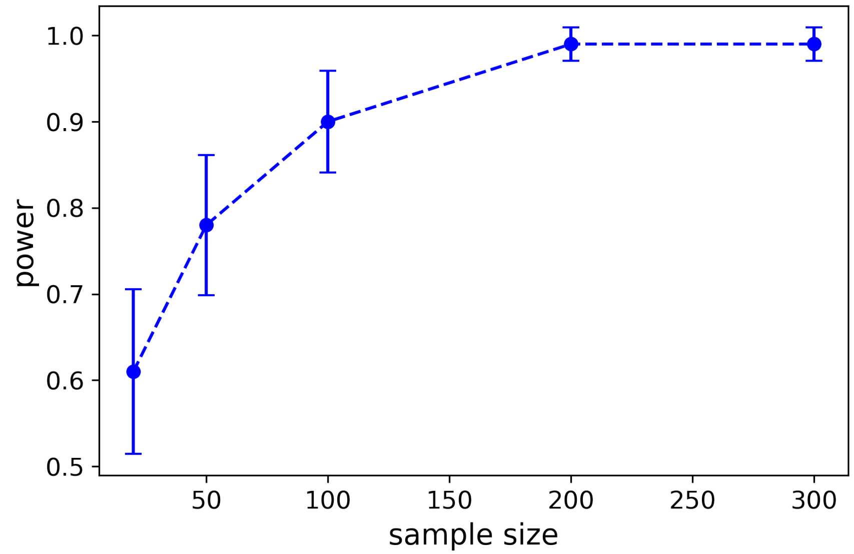
Neyman-Pearson regime :

sample of size N

Neyman-Pearson test : $ER(p_1(N))$ vs $ER(p_2(N))$

$$|p_1(N) - p_2(N)| \gg 1/\sqrt{N}$$

$$n = 50 \qquad p_1(N) \to 0.5, \quad p_2(N) \to 0.5 \qquad |p_1(N) - p_2(N)| \sim \log(N)/\sqrt{N}$$



Conclusion

L. (2021) Heat diffusion distance processes: a statistically founded method to analyze graph data sets. arXiv preprint arXiv:2109.13213.

Future work:

- Applications to real datasets (activation graphs from NN)
- Learning tasks : classification, change point detection, ...
- Relationship between n and N

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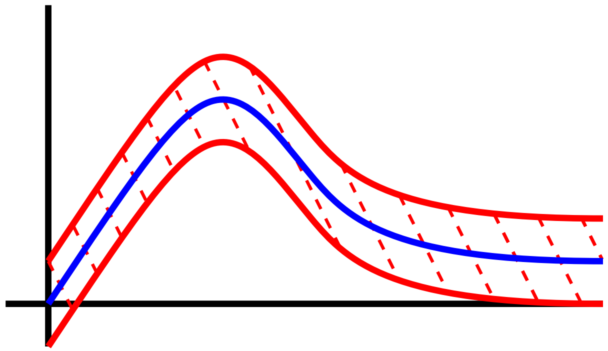
Thank you for your attention!

Confidence bands

$\mathcal{D} = (G_1, G'_1), \ldots, (G_N, G'_N) \sim P$
$P_N = N^{-1} \sum_i \delta_{(G_i, G'_i)}$
$\alpha \in]0, 1[$

$$\mathbb{P}\left(\|P_N D. - PD.\|_\infty \geq T_{\alpha,P}\right) \leq \alpha$$

unknown



From the Donsker property:

$$\sqrt{N}(P_N D. - PD.) \xrightarrow{weak} \mathbb{G} \xleftarrow{weak} \sqrt{N}(\hat{P}_N D. - P_N D.) \mid \mathcal{D}$$

$\tilde{c}_\alpha$ Monte Carlo estimator of c_α, s.t

$$\mathbb{P}\left(\|\hat{P}_N D. - P_N D.\|_\infty \geq c_\alpha/\sqrt{N} \mid \mathcal{D}\right) \leq \alpha.$$

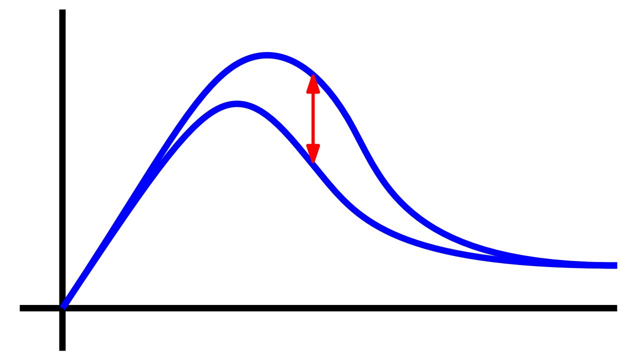
$$\lim_{N\to\infty} \mathbb{P}\left(\|P_N D. - PD.\|_\infty \geq \tilde{c}_\alpha/\sqrt{N}\right) \leq \alpha$$

Two-sample Tests

$X_1, \dots, X_N \sim P$ a sample
$P_N = N^{-1} \sum_i \delta_{X_i}$

$Y_1, \dots, Y_M \sim Q$ a sample
$Q_M = M^{-1} \sum_i \delta_{Y_i}$

$$\mathcal{H}_0 : P = Q \quad \text{or} \quad \mathcal{H}_1 : P \neq Q$$



Idea : compute $T_{N,M} = \|P_N D. - Q_M D.\|_\infty$.

- reject $\mathcal{H}_0$, if $T_{N,M} > T$
- accept $\mathcal{H}_0$, otherwise

$$\boxed{\mathbb{P}_{\mathcal{H}_0}\left(T_{N,M} > T\right) \leq \alpha}$$

$\tilde{c}$: Monte-Carlo estimator of c, s.t.

$$\mathbb{P}\left(\|\hat{P}_N D. - \hat{Q}_M D.\|_\infty \geq c\frac{\sqrt{N+M}}{\sqrt{NM}} \mid \mathcal{D}\right) \leq \alpha.$$

resampled from
$Z = (X_1, \dots, X_N, Y_1, \dots, Y_M)$

$$\lim_{N,M\to\infty} \mathbb{P}_{\mathcal{H}_0}\left(T_{N,M} \geq \tilde{c}\frac{\sqrt{N+M}}{\sqrt{NM}}\right) \leq \alpha$$

if $PD. \neq QD.$, $\displaystyle\lim_{N,M\to\infty} \mathbb{P}_{\mathcal{H}_1}\left(T_{N,M} \geq \tilde{c}\frac{\sqrt{N+M}}{\sqrt{NM}}\right) = 1$