Bubble Sort[1]

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BubbleSort(A = (a_1, a_2, ..., a_n))
for i=1 to n-1:
    for j=1 to n-i:
        if (a_j > a_{j+1}), then swap (a_j, a_{j+1})
    endfor
endfor
Return
```

There are two loops in this algorithm: To prove it correct, we need two loop invariants.

1. Firstly, we prove the loop invariant for the inner loop.
2. We will use the loop invariant of the inner loop to prove the invariant of the outer loop.

> ***Inner Loop invariant:*** *After iteration j of the inner loop, the maximum element of $(a_1, a_2, \ldots, a_{j+1})$ is in position $j+1$.*

Proof: (by induction)

Base case (j=0): Iteration j=0 is prior to the loop. In this step, the maximum element of $(a_1 \ldots a_1)$ is in position 1.

Ind Hyp: Assume prior to iteration $j+1$ that the maximum element of $(a_1, a_2, \ldots, a_{j+1})$ is in position $j+1$.

Ind Step: We need to show that after iteration $j+1$ the maximum element of $(a_1, a_2, \ldots, a_{j+2})$ is in position $j+2$. We know by Ind Hyp that the maximum element of $(a_1, a_2, \ldots, a_{j+1})$ is in position $j+1$. Either this is the maximum element or element a_{j+2}. In $j+1$-th iteration, we compare a_{j+1} to a_{j+2}, and swap them if $a_{j+1} > a_{j+2}$ (Algorithm's logic). So, after this iteration the maximum element of $(a_1, a_2, \ldots, a_j + 2)$ is in position $j+2$.

Thus, by induction, we proved that after iteration j of the loop, the maximum element of $(a_1, a_2, \ldots, a_{j+1})$ is in position $j+1$.

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Outer Loop invariant: *After iteration i of the outer loop, the i maximum elements of $(a_1, a_2, \ldots, a_n)$ are in ascending order in positions $n - i + 1$ to n.*

Proof: (by induction) Base case: (i=0) Prior to the loop, the 0 maximum elements of $(a_1, a_2, \ldots, a_n)$ are in ascending order in no positions.

Ind Hyp: Assume prior to iteration $i + 1$ that the i maximum elements of $(a_1, a_2, \ldots, a_n)$ are in ascending order in positions $n - i + 1$ to n.

Ind Step: By ind hyp, we know that the i maximum elements of $(a_1, a_2, \ldots, a_n)$ are in ascending order in position $n - i + 1$ to n. We need to show that after iteration $i + 1$, the maximum element of $(a_1, a_2, \ldots, a_{n-i})$ is in position $n - i$.

In particular, the outer loop iteration $i + 1$ ends after iteration of $n - i - 1$ of the inner loop. So, after iteration $n - i - 1$ of the inner loop, the maximum element of $(a_1, a_2, \ldots, a_{n-i-1+1})$ is in position $n - i - 1 + 1 = n - i$ (by Inner loop ivariant).

Thus, by induction, we proved that after iteration i of the outer loop, the i maximum elements of $(a_1, a_2, \ldots, a_n)$ are in ascending order in positions $n-i+1$ to n.

Proof of Correctness: *The Bubble Sort algorithm outputs the sorted order.*

Proof: Using outer loop invariant, we can prove this.