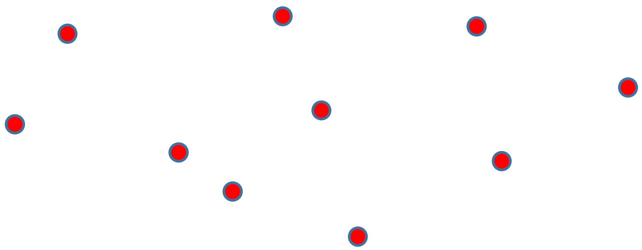
# Closest Pair

# P12. Closest pair of n points

I/P: Given $P=\{p_1, p_2, \ldots, p_n\}$ a set of n points. Each $p_i=(x_i, y_i)$.

O/P: Find the closest pair of points.



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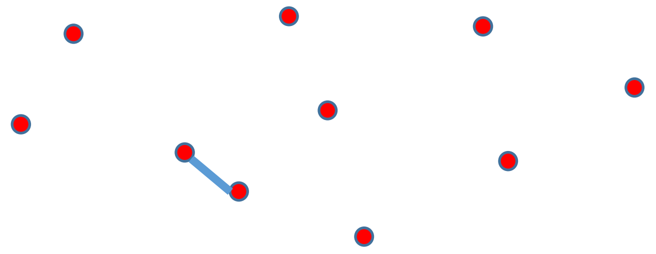

# Closest pair of n points

I/P: Given $P=\{p_1, p_2,\ldots, p_n\}$ a set of n points. Each $p_i=(x_i,y_i)$.

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Naïve algorithm:

1. For each point p in P, check all points in $P \setminus \{p\}$.
2. Compute the minimum distances with respect to p, i.e., min_p
3. Repeat step 1 and 2 for all points in P,
4. Finally return min of all min_p's.

# Let’s do the problem in 1-Dimension

y-coordinate of all points are same.

O(nlogn)

1.Sort each point with respect to x-coordinate.

2. For each point $p_i$ , check two neighbours $p_{i-1}$ and $p_{i+1}$.

3. Do it for all points and return the minimum.

O(n)

# Can we do via divide and conquer?

Here input is not sorted….. Means the point are not sorted with respect to x-coordinate…. ??

How to apply D&C??

Intuition is to divide points set in equal halves ??– But how??

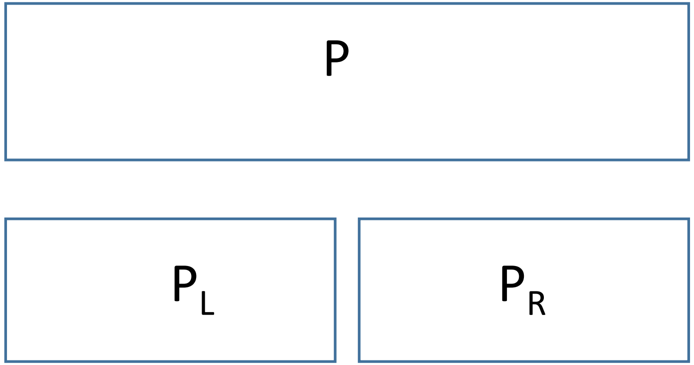
We need median to partition in two halves….??

Lets apply the unknown algorithm to partition point set in two halves.

P split in two halves $P_L$ and $P_R$.

Recurse on both halves.

$P_L$ contains the median point (W.L.O.G)…

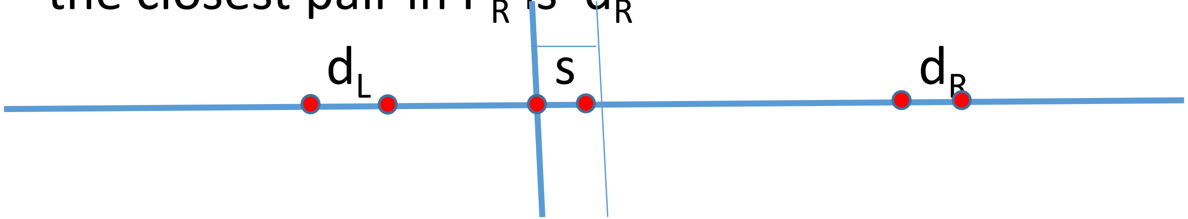



complexity??

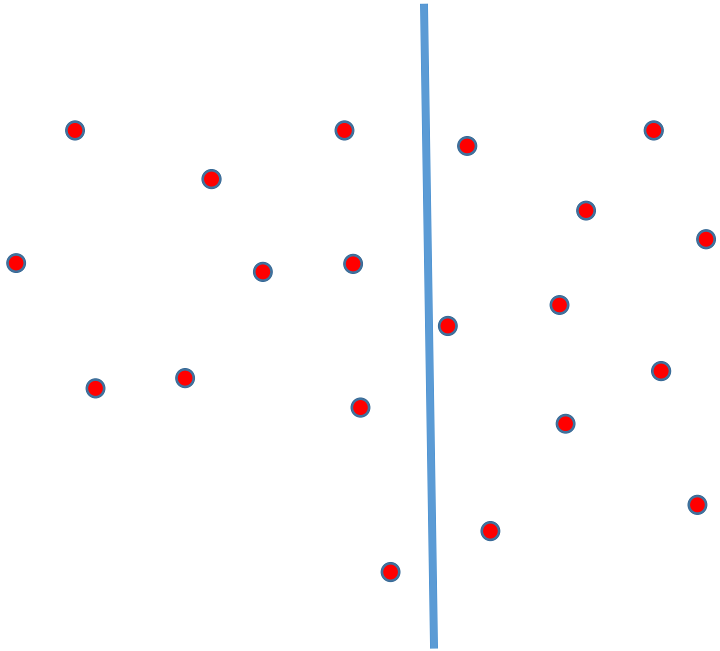
Suppose the closest pair in $P_L$ is $d_L$

the closest pair in $P_R$ is $d_R$

$s = \min(d_L, d_R)$



# In 2D



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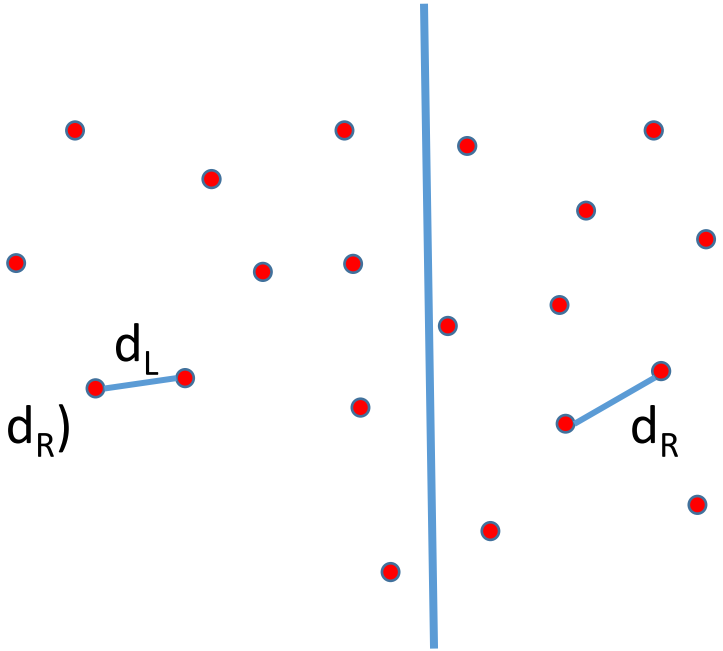


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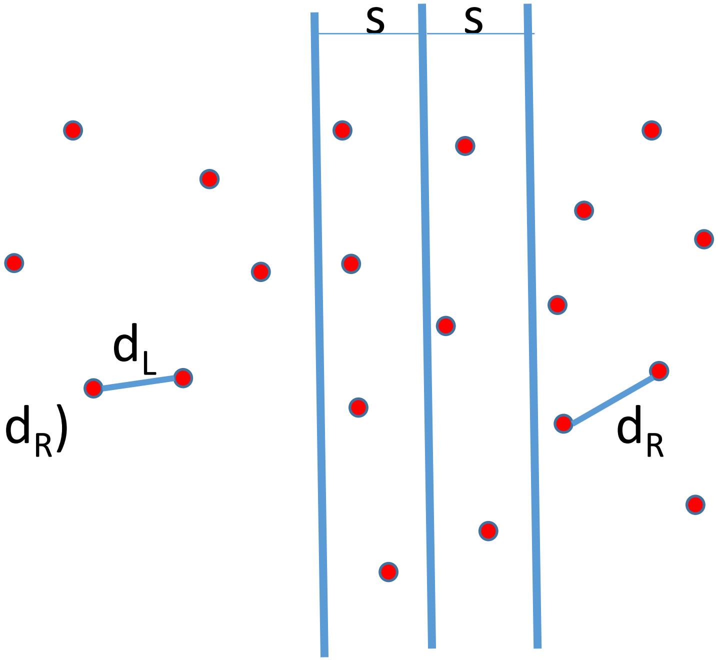


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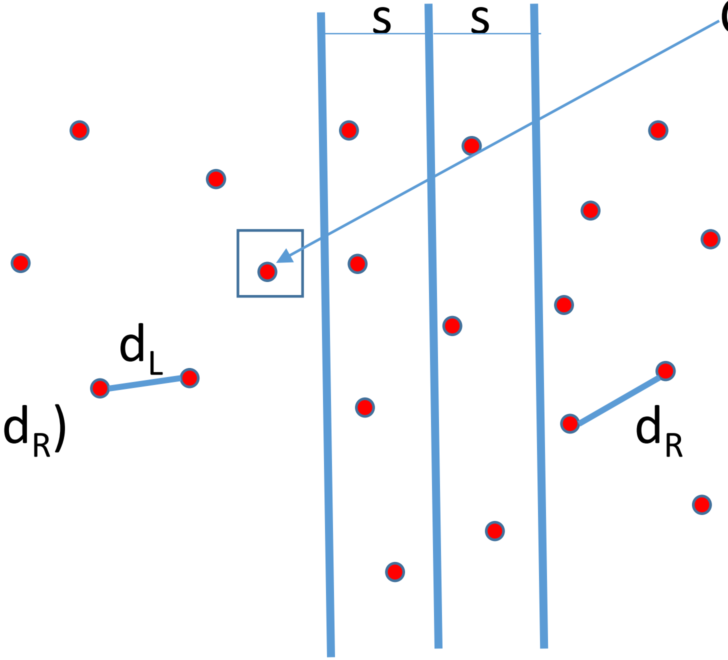
# In 2D

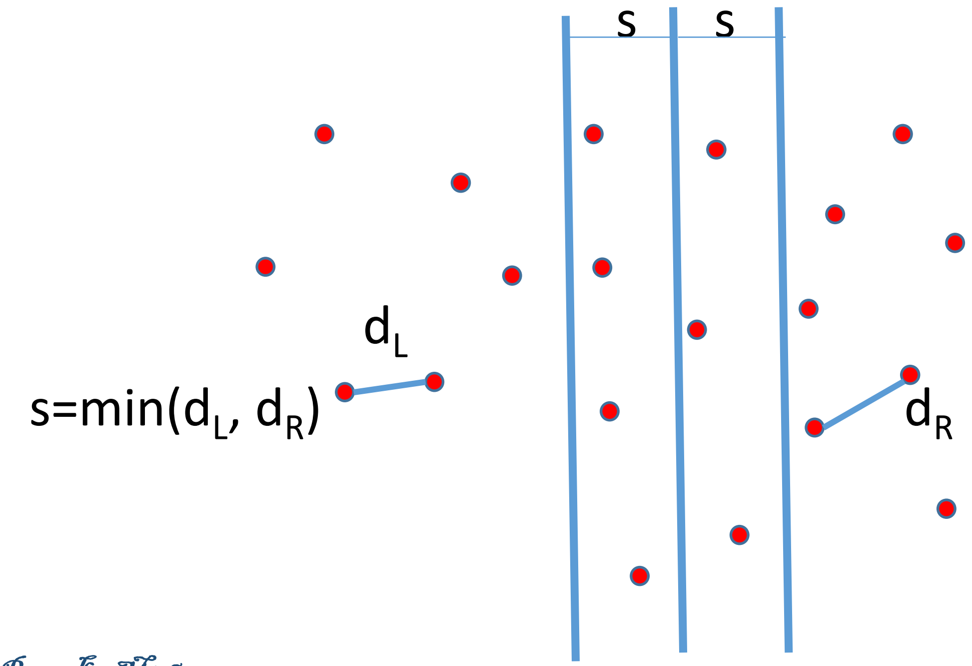



Can this point be used ?

No, so check points in strip.

$s=\min(d_L, d_R)$

# In 2D

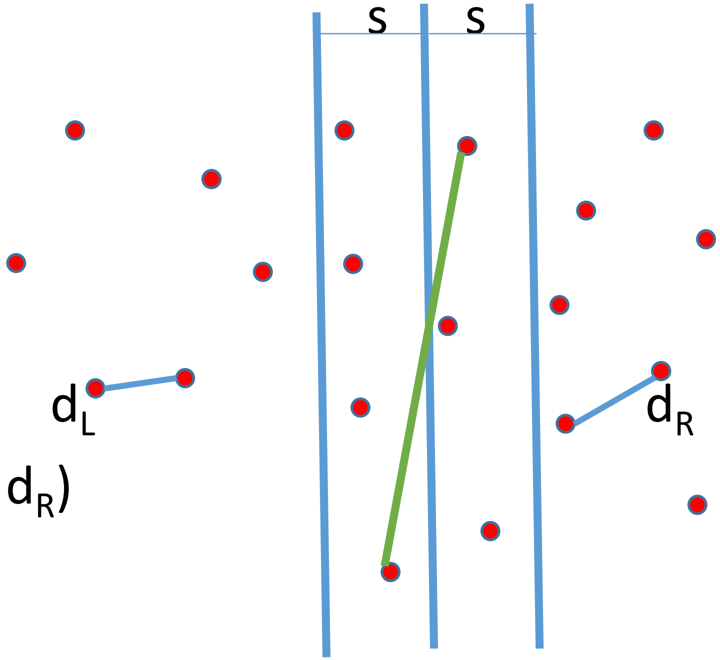



So, we need to find the closest pair in strip only.

- naïve way: $O(n^2)$

$T(n)=T(n/2)+O(n^2)$

# In 2D

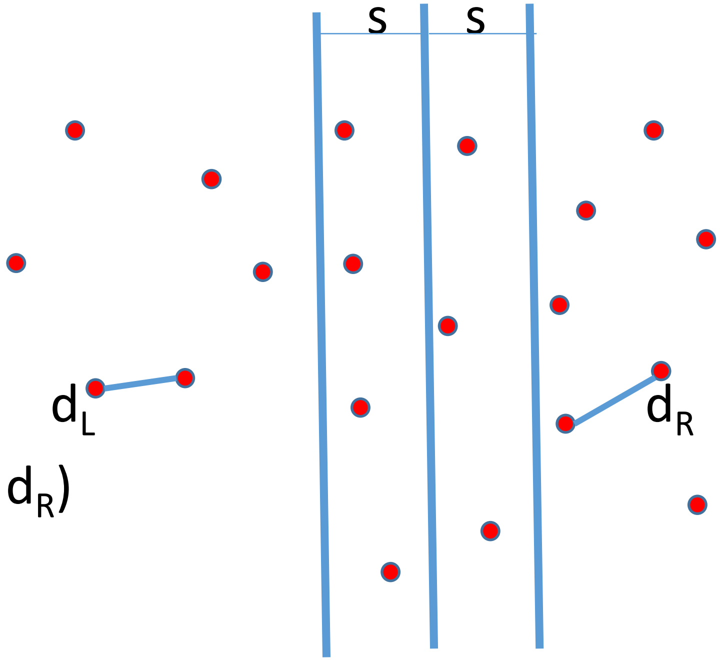



$s = \min(d_L, d_R)$

Do we need to compare this pair?

NO!!

# In 2D

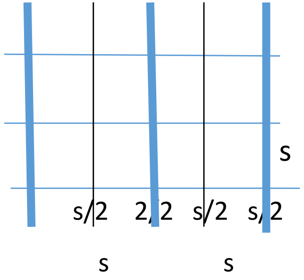


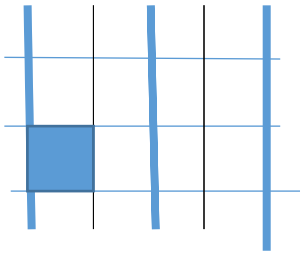

$s=\min(d_L, d_R)$

Do we need to compare this pair?

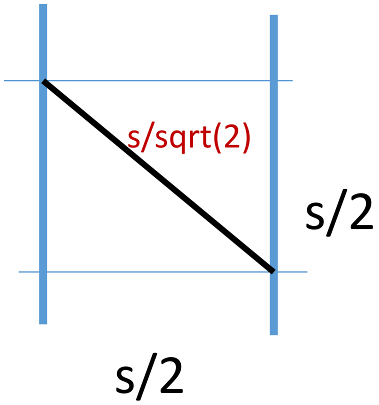
NO!!

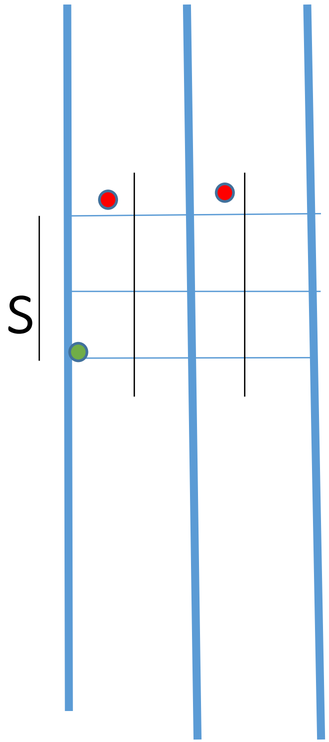
Let's dissects the strip.





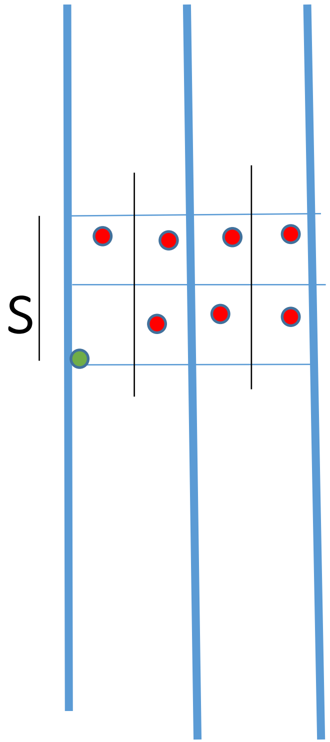
so, in a box, can have one point.






for green point, do we need to check these red points ?

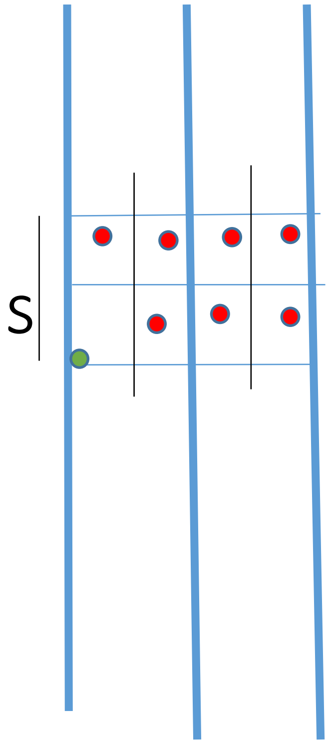
NO? Why?




for green point, we need to check these red points ?

Yes? Why?

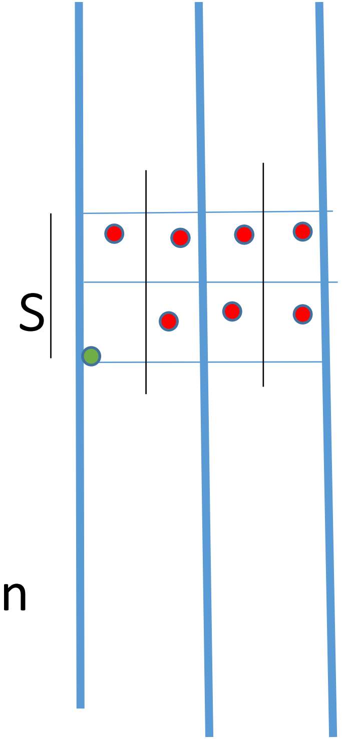
7 points




how to find seven points for each point?

$O(n\log^2 n)$

$T(n)=2T(n/2)+n\log n$




for green point, we need to check these red points ?

Yes? Why?

7 points

so, for each point, we need to check at most next 7 points.

To get next 7, need to sort

# 2 dimensions, divide and conquer

- Split set of points into two halves by vertical line
- Recursively compute closest pair in left and right half
- Need to then compute closest pairs across separating line
- How can we do this efficiently?

# Sorting points by x and y

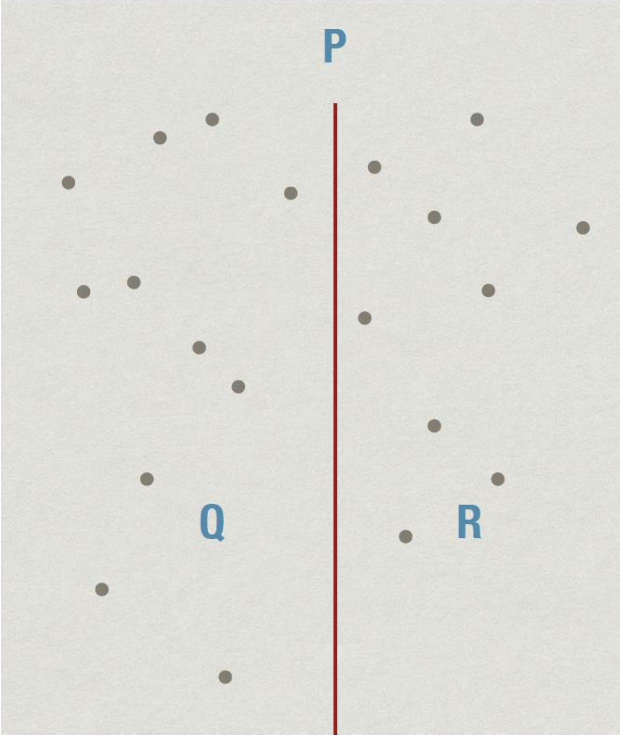
Given n points $P = \{p_1, p_2, \ldots, p_n\}$, compute

- $P_x$ : P sorted by x coordinate
- $P_y$ : P sorted by y coordinate

Divide P by vertical line into equal size sets Q and R.

Need to efficiently compute

$Q_x$, $Q_y$, $R_x$, $R_y$



# Sorting points by x and y

- Need to efficiently compute

$Q_x$, $Q_y$, $R_x$, $R_y$

➢ $Q_x$ is first half of $P_x$

➢ $R_x$ is second half of $P_x$

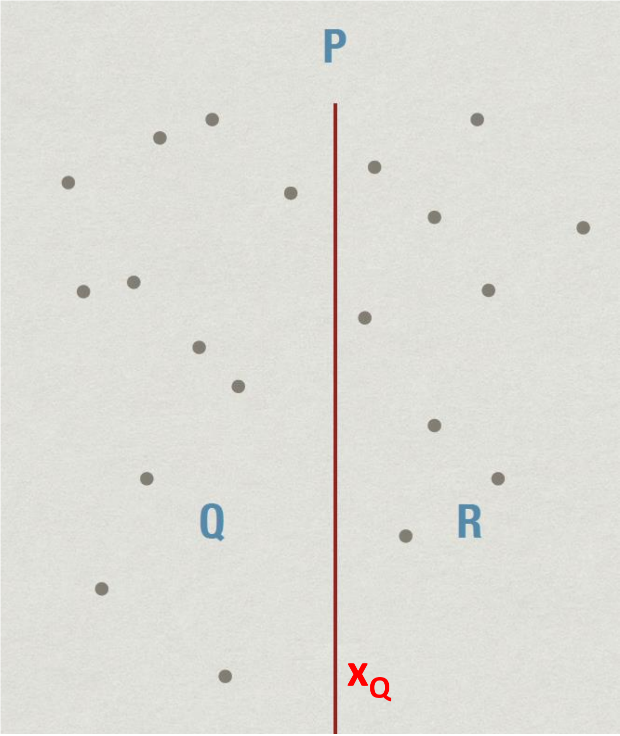
- When splitting $P_x$,

note the largest x coordinate in Q, $x_Q$

- Separate $P_y$ as $Q_y$, $R_y$ by checking x

coordinate with $x_Q$.

- All O(n)



# 2 dimensions, divide and conquer

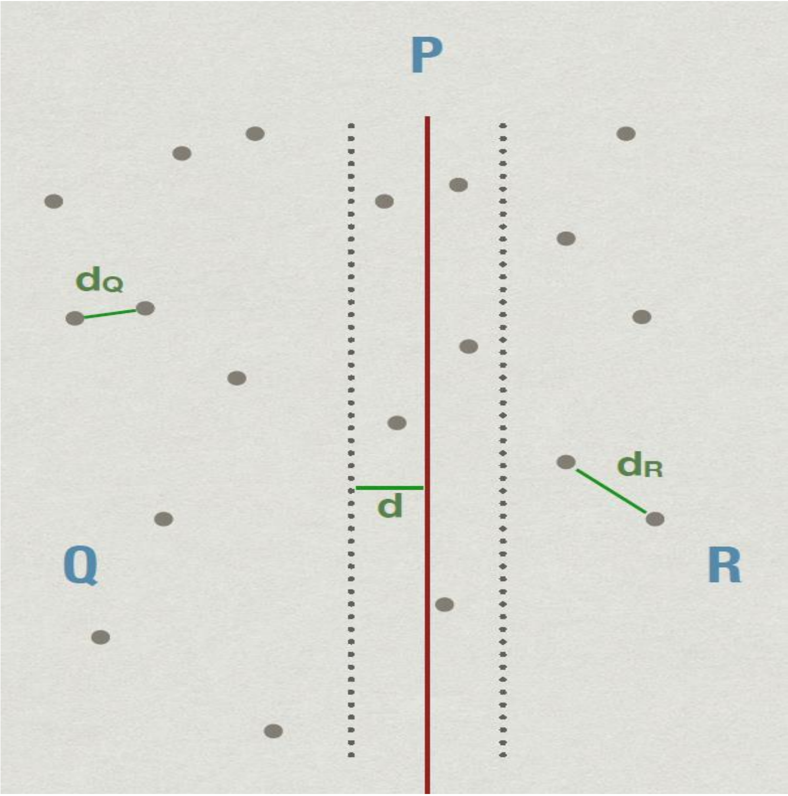
- Basic recursive call is ClosestPair($P_x$, $P_y$)

- Set up recursive calls ClosestPair($Q_x$, $Q_y$)
  and ClosestPair($R_x$, $R_y$) for left and right half of P in time O(n)
- How to combine these recursive solutions?

# Combining solutions

- Let $d_Q$ be closest distance in Q and $d_R$ be closest distance in R
- Let d be min($d_Q$, $d_R$)
- Only need to consider points across the separator at most distance d from separator

➢ Any pair outside this strip cannot be closest pair overall

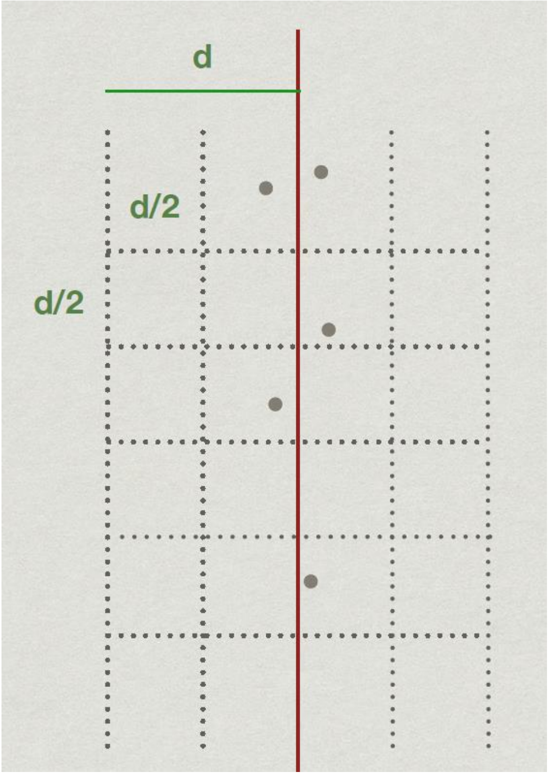





Prepared by Pawan K. Mishra

# Combining solutions

- From $Q_y$, $R_y$, extract $S_y$,
  points in d-band sorted by y coordinate
- Scan $S_y$ from bottom to top,
  comparing each point against next
  7 points in $S_y$
- Linear scan



# Algorithm

ClosestPair($P_x$,$P_y$)

if ($|P_x|$ <= 3)

compute pairwise distances and

return the closest pair and distance

Construct ($Q_x$, $Q_y$, $R_x$, $R_y$)

($d_Q$,$q_1$, $q_2$) = ClosestPair($Q_x$, $Q_y$)

($d_R$ ,$r_1$, $r_2$) = ClosestPair($R_x$, $R_y$)

Construct $S_y$ and scan to find ($d_S$,$s_1$, $s_2$)

Return ($d_Q$,$q_1$, $q_2$) , ($d_R$ ,$r_1$, $r_2$) , ($d_S$,$s_1$, $s_2$) depending on which among ($d_Q$,$d_R$, $d_S$) is minimum

# Analysis

- Computing ( $P_x$, $P_y$ ) from P takes O(n log n) (before recursion call)
- Recursive algorithm

➢ Setting up ($Q_x$, $Q_y$, $R_x$, $R_y$) from ( $P_x$, $P_y$ ) is O(n)

➢ Setting up $S_y$ from $Q_y$, $R_y$ is O(n)

➢ Scanning $S_y$ is O(n)

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$