Tutorial 3, Design and Analysis of Algorithms, 2023

1. Note that we can multiply two binomials $(ax+b)(cx+d) = acx^2+(ad+bc)x+bd$, where we need 4 multiplications: ac, ad, bc, bd. Similarly to the Karatsuba's algorithm, we can improve this by only: doing 3 multiplications $ac, bd, (a+b)(c+d)$. So, using the same idea solve the following problem. Devise a Divide and Conquer algorithm to multiple two trinomials in $O(n^{\log_3 5})$ time. Justify the time complexity.

2. Given a set of n intervals $I = [a_1, b_1], [a_2, b_2], \ldots [a_n, b_n]$. Here $a_i < b_i$ for all $i = 1$ to n. Devise a Divide and Conquer algorithm to compute the length of the biggest overlap between any two intervals in $O(n \log n)$ time. Justify the time complexity. For eg, $[1, 7]$ overlaps with $[3, 9]$, and the length of the overlap between them is $7 - 3 + 1 = 4$.

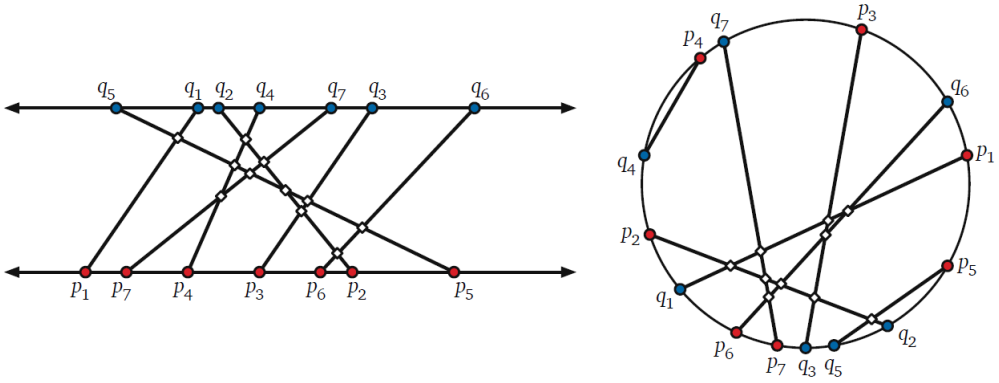
3. Given $n = 2^k - 1$ elements, construct a binary min heap using Divide and Conquer in $O(n)$ time. *(Note that, in general, we can construct binary min heap in $O(n)$ time without using DnC.)*

4. Given a set $P = \{p_1, p_2, \ldots p_n\}$ of n points in 2-D, and distance between two points is Manhattan distance(not the Euclidean distance). Find the closest pair of points using divide and conquer in $O(n \log n)$ time. Justify the time complexity. *(Hint: Find number of points to be checked for a point in a strip)*

5. Given a set $P = \{p_1, p_2, \ldots p_n\}$ of n points, construct a convex hull using divide and conquer in in $O(n \log n)$ time. Justify the time complexity.

6. Suppose you are given two sets of n points, one set $p_1, p_2, \ldots p_n$ on the line $y = 0$ and the other set $q_1, q_2, \ldots q_n$ on the line $y = 1$. Create a set of n line segments by connect each point p_i to the corresponding point q_i. Describe and analyze a divide-and-conquer algorithm to determine how many pairs of these line segments intersect, in $O(n \log n)$ time. *(See the left image of the figure below to understand the problem).*

7. Now suppose you are given two sets$p_1, p_2, \ldots p_n$ and $q_1, q_2, \ldots q_n$ of n points on the unit circle. Connect each point p_i to the corresponding q_i. Describe and analyze a divide-and-conquer algorithm to determine how many pairs of these line segments intersect in $O(n \log^2 n)$ time. *(See the right image of the figure below to understand the problem). Hint Use sorting with respect to polar coordinate.*



8. Let S be a set of n-distinct real numbers and let $k \leq n$ be a positive integer (k may not be a constant). Design an algorithm, running in $O(n)$ time, that determines the k numbers in S that are closest to the median of S. Justify the time complexity.

9. For n distinct elements $x_1, x_2, \ldots x_n$ with positive weights $w_1, w_2, \ldots, w_n$ such that $\sum_{i=1}^{n} w_i = 1$, the weighted median is the element x_k satisfying $\sum_{x_i < x_k} w_i < 1/2$ and $\sum_{x_i > x_k} w_i \leq 1/2$. Show how to compute the weighted median of n elements in $O(n)$ time using a linear-time median algorithm discussed in the class.

10. Need more of ☕ ☕ ☕ to solve the question number 7 in $O(n \log n)$ time ? Try this after you solve all the questions. Good luck and become algo 😈.