Fundamentals of digital systems

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February 19, 2024

Introduction

This document is designed to offer a LaTeX-styled overview of the Fundamentals of Digital Systems course, emphasizing brevity and clarity. Should there be any inaccuracies or areas for improvement, please do not hesitate to reach out to me at ali.elazdi@epfl.ch for corrections. Hoping to provide an experience similar to the one provided by past generations of students in other subjects. Also, for the latest version of this pdf file, as the update on drive may take some time, you can find it on my github repository at the following link:
`https://github.com/elazdi-al/FDS/blob/main/FDS.pdf`

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Chapter 1

Number Systems

1.1 Digital Representations

In a digital representation, a number is represented by an ordered n-tuple:

The n-tuple is called a **digit vector**, each element is a **digit**

The number of digits n is called the precision of the representation. (careful! leftward indexing)

$$X = (X_{n-1}, X_{n-1}, \ldots, X_0)$$

Each digit is given **a set of values** D_i (eg. For base 10 representation of numbers, $D_i = \{0, 1, 2, \ldots, 9\}$)

The **set size**, the maximum number of representable digit vectors is: $K = \prod_{i=0}^{n-1} |D_i|$

1.1.1 (Non)Redundant Number Systems

A number system is nonredundant if each digit-vector represents a different integer

1.1.2 Weighted Number Systems

The rule of representation if a Weighted (Positional) Number Systems is as follows :

$$\sum_{i=0}^{n-1} X_i W_i$$

where

$$W = (W_{n-1}, W_{n-2}, \ldots, W_0)$$

1.1.3 Radix Systems

When weights are in this format :

$$\begin{cases} W_0 = 1 \\ W_{i+1} = W_i R_i \quad \text{with } 1 \leq i \leq n-1 \end{cases}$$

Also written : $W_0 = 1$, $\prod_{j=0}^{i-1} R_j$

1.1.4 Fixed and Mixed-Radix Number Systems

In a **fixed-radix system**, all elements of the radix-vector have the same value r (*the radix*)

The weight vector in a fixed-radix system is given by:

$$W = \left(r^{n-1}, r^{n-2}, \ldots, r^2, r, 1\right)$$

and the integer x becomes:

$$x = \sum_{i=0}^{n-1} X_i \times r^i$$

In a **mixed-radix system**, the elements of the radix-vector differ

Example: Decimal Number System

The decimal number system has the following characteristics:

- Radix $r = 10$, it's a fixed-radix system.

The weight vector W is defined as:

$$W = \left(10^{n-1}, 10^{n-2}, \ldots, 10^2, 10, 1\right)$$

An integer x in this system is represented by:

$$x = \sum_{i=0}^{n-1} X_i \times 10^i$$

For example:

$$854703 = 8 \times 10^5 + 5 \times 10^4 + 4 \times 10^3 + 7 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$$

Examples of Fixed and Mixed radix systems

Fixed: The base of number systems.

- Decimal – radix 10
- Binary – radix 2
- Octal – radix 8
- Hexadecimal – radix 16

Mixed: An example of a mixed radix representation, such as time:

- Radix-vector $R = (24, 60, 60)$
- Weight-vector $W = (3600, 60, 1)$

1.1.5 Canonical Number Systems

In a **canonical number system**, the set of values for a digit D_i is with $|D_i| = R_i$, the corresponding element of the radix vector

$$D_i = \{0, 1, \ldots, R_i - 1\}$$

Canonical digit sets with fixed radix:

- Decimal: $\{0, 1, \ldots, 9\}$
- Binary: $\{0, 1\}$
- Hexadecimal: $\{0, 1, 2, \ldots, 15\}$

Range of values of x represented with n fixed-radix-r digits:

$$0 \leq x \leq r^n - 1$$

A system with fixed positive radix r and a canonical set of digit values is called a radix-r conventional number system.

1.2   Binary/Octal/Hexadecimal to/from Decimal Conversion Table

The hexadecimal system supplements 0-9 digits with the letters A-F.

Remark. Programming languages often use the prefix 0x to denote a hexadecimal number.

Conversion table up to 15.

Decimal	Binary	Octal	Hexadecimal
0	0000	00	0
1	0001	01	1
2	0010	02	2
3	0011	03	3
4	0100	04	4
5	0101	05	5
6	0110	06	6
7	0111	07	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

1.2.1   Convertion examples

Binary to Decimal:

To convert a binary number to decimal, multiply each bit by two raised to the power of its position number, starting from zero on the right.

Binary: $\underline{1011}$

Decimal: $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 8 + 0 + 2 + 1 = 11$

Decimal to Binary:

Let's convert the decimal number 25_{10} to binary.

$$
\begin{array}{rcll}
25 \div 2 & = & 12 & \text{remainder 1 (LSB)} \\
12 \div 2 & = & 6 & \text{remainder 0} \\
6 \div 2 & = & 3 & \text{remainder 0} \\
3 \div 2 & = & 1 & \text{remainder 1} \\
1 \div 2 & = & 0 & \text{remainder 1 (MSB)}
\end{array}
$$

Thus, the binary representation of 25_{10} is 11001_2 (reading the remainders in reverse).

Personal Remark The trick is always to try to answer the question, what's the biggest power of 2 I need to form the number?. For 157, the biggest power would be $2^7 = 128$, then $128+64$ is greater than 157, $128+32$ is still greater than 157, $128+16 = 144$, and so on to obtain : $128+16+8+4+1 = 157$ which can be written as $2^7+2^4+2^3+2^2+2^0 = 157$. Written in binary as 10011101_2

Octal to Decimal:

Each octal digit is converted to decimal by multiplying it by eight raised to the power of its position number, starting from zero on the right.

Octal: $\underline{257}$

Decimal: $2 \times 8^2 + 5 \times 8^1 + 7 \times 8^0 = 128 + 40 + 7 = 175$

Decimal to Octal:

To convert the decimal number 93_{10} to octal.

$$
\begin{array}{rcll}
93 \div 8 & = & 11 & \text{remainder 5} \\
11 \div 8 & = & 1 & \text{remainder 3} \\
1 \div 8 & = & 0 & \text{remainder 1}
\end{array}
$$

Thus, the octal representation of 93_{10} is 135_8 (reading the remainders in reverse).

Hexadecimal to Decimal:

To convert the hexadecimal number $1A3_{16}$ to decimal.

Hexadecimal: $\underline{1A3}$

Decimal:
$$1 \times 16^2 + A \times 16^1 + 3 \times 16^0$$
$$1 \times 256 + 10 \times 16 + 3 \times 1$$
$$256 + 160 + 3$$
$$419$$

Here, A in hexadecimal corresponds to 10 in decimal.

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Decimal to Hexadecimal:

To convert the decimal number 291_{10} to hexadecimal.

$$291 \div 16 \quad = \quad 18 \quad \text{remainder } 3$$
$$18 \div 16 \quad = \quad 1 \quad \text{remainder } 2$$
$$1 \div 16 \quad = \quad 0 \quad \text{remainder } 1$$

Thus, the hexadecimal representation of 291_{10} is 123_{16} (reading the remainders in reverse).

1.3 Octal/Hexadecimal to/from Binary

Bit-Vector Representation Summary

- Digit-vectors for binary, octal, and hexadecimal systems are represented using bit-vectors. In binary, 0 and 1 are directly represented as 0 and 1.

- In systems like octal or hexadecimal, a digit is a bit-vector of length k, where k is the number of bits needed to represent the base.

$$k = \log_2(r)$$

  with r the radix of the system (eg. 8 for octal convertion).

- For example, the hexadecimal digit B is represented as the bit-vector 1101 in binary. *We obtain a length 4 bit-vector because the base is 16 and* $\log_2(16) = 4$

Binary to Octal:

To convert a binary number to octal, group every three binary digits into a single octal digit, because $k = \log_2 8 = 3$.

Binary:	$\underline{010}\,\underline{000}\,\underline{100}\,\underline{110}$
Octal:	$2_8\,0_8\,4_8\,6_8$

Binary to Hexadecimal:

To convert a binary number to hexadecimal, group every four binary digits into a single hexadecimal digit, because $k = \log_2 16 = 4$.

Binary:	$\underline{1011}\,\underline{1110}\,\underline{1010}\,\underline{1101}$
Hexadecimal:	$B_{16}\,E_{16}\,A_{16}\,D_{16}$

Octal to Hexadecimal:

Convert the octal number to binary, then group the binary digits in sets of four and convert each group to its hexadecimal equivalent.

Octal:	$\underline{257}$
Binary:	$010\,101\,111$ (Octal to binary)
Binary grouped:	$\underline{0101}\,\underline{0111}$
Hexadecimal:	$5\,7$ (Binary to hexadecimal)

1.4   Representation of Signed Integers

1.4.1   Sign-Magnitude Representation (SM)

A signed integer x is represented by a pair (x_s, x_m), where x_s is the *sign* and x_m is the *magnitude* (positive integer).

The sign (positive, negative) is represented by a the most significant bit (MSB) of the digit vector:

$0 \rightarrow$ positive

$1 \rightarrow$ negative

The magnitude can be represented as any positive integer. In a conventional radix-r system, the range of n-digit magnitude is:

$$0 \leq x_m \leq r^n - 1$$

- Examples:

$$01010101_2 = +85_{10}$$
$$01111111_2 = +127_{10}$$
$$00000000_2 = +0_{10}$$
$$11010101_2 = -85_{10}$$
$$11111111_2 = -127_{10}$$
$$10000000_2 = -0_{10}$$

Note: The Sign-and-Magnitude representation is considered a redundant system because both 00000000_2 and 10000000_2 represent zero.

SM consists of an equal number of positive and negative integers.

An n-bit integer in sign-and-magnitude lies within the range *(because of 0's double representation and that MSB is used for the sign)*:

$$[-\left(2^{n-1}-1\right),+\left(2^{n-1}-1\right)]$$

Main disadvantage of SM: complex digital circuits for arithmetic operations (addition, subtraction, etc.).

1.5 True-and-Complement (TC)

1.5.1 Mapping

A signed integer x is represented by a positive integer x_R, C is a positive integer called the *complementation constant.*

$$x_R \equiv x \mod C$$

For $|x| < C$, by the definition of the modulo function, we have:

$$x_R = \begin{cases} x & \text{if } x \geq 0 \quad \text{(True form)} \\ C - |x| = C + x & \text{if } x < 0 \quad \text{(Complement form)} \end{cases}$$

1.5.2 Unambiguous Representation

To have an unambiguous representation, the two regions should not overlap, translating to the condition:
$$\max |x| < \frac{C}{2}$$

1.5.3 Converse Mapping

Converse mapping:

$$x = \begin{cases} x_R & \text{if } x_R < \frac{C}{2} \quad \text{(Positive values)} \\ x_R - C & \text{if } x_R > \frac{C}{2} \quad \text{(Negative values)} \end{cases}$$

When $x_R = \frac{C}{2}$, it is usually assigned to $x = -\frac{C}{2}$.

Asymmetrical representation simplifies sign detection.

1.6 Two's Complement System

This is the True-and-Compelement system with $C = 2^n$, where n is the number of bits used to represent the integer.

Range is asymmetrical:
$$-2^{n-1} \leq x \leq 2^{n-1} - 1$$

The representation of zero is unique.

1.6.1 Sign Detection in Two's Complement System

Since $|x| < C/2$ and assuming the sign is 0 for positive and 1 for negative numbers:

$$\text{sign}(x) = \begin{cases} 0 & \text{if } x_R < C/2 \\ 1 & \text{if } x_R \geq C/2 \end{cases}$$

Therefore, the sign is determined from the most-significant bit:

$$\text{sign}(x) = \begin{cases} 0 & \text{if } x_{n-1} = 0 \\ 1 & \text{if } x_{n-1} = 1 \end{cases} \quad \text{equivalent to} \quad \text{sign}(x) = x_{n-1}$$

1.6.2 Mapping from Bit-Vectors to Values

The value of an integer represented by a bit-vector $b_{n-1}b_{n-2}\ldots b_1 b_0$ can be universally expressed as:
$$\text{Value} = (-2^{n-1} \cdot b_{n-1}) + \sum_{i=0}^{n-2} b_i \cdot 2^i$$

where b_{n-1} is the MSB (sign bit) and is 0 for non-negative numbers and 1 for negative numbers.

Examples

$$X = 011011_2 = 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 8 + 2 + 1 = 27_{10}$$

$$X = 11011_2 = -1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = -16 + 8 + 2 + 1 = -5_{10}$$

$$X = 10000000_2 = -1 \cdot 2^7 = -128_{10}$$

$$X = 10000011_2 = -1 \cdot 2^7 + 1 \cdot 2^1 + 1 \cdot 2^0 = -128 + 2 + 1 = -125_{10}$$

1.6.3 Change of Sign in Two's Complement System

The two's complement system represents negative numbers by inverting the bits of their positive counterparts and adding one. This process is equivalent to subtracting the number from 2^n.

For an n-bit number x:

$$z = -x = (\sim x) + 1 = C - x_R$$

where $(\sim x)$ is the bitwise NOT of x and x_R is the decimal representation of x.

Examples

Converting $+17$ to -17:

$$+17_{10} = 00010001_2$$
$$-17_{10} = \overline{00010001_2} + 1 = 11101111_2$$
$$2^8 - 17 = 256 - 17 = 239 = 11101111_2$$

Converting -99 to $+99$:

$$-99_{10} = 10011101_2$$
$$+99_{10} = \overline{10011101_2} + 1 = 01100011_2 \qquad (TC)$$
$$2^8 - 99 = 256 - 99 = 157 = 01100011_2 \; (Substracting \; 99 \; from \; 256)$$

1.7 Range Extension and Arithmetic Shifts

1.7.1 Range Extension

Performed when a value x represented by a digit-vector of n bits needs to be represented by a digit-vector of m bits, where $m > n$.

x is equal to z with

$$X = (X_{n-1}, X_{n-2}, \ldots, X_1, X_0)$$
$$Z = (Z_{m-1}, Z_{m-2}, \ldots, Z_1, Z_0)$$

1.8 Range Extension Algorithm in SM

In sign-and-magnitude system, the range-extension algorithm is defined as:

$$z_s = x_s \; (\text{sign bit})$$
$$Z_i = 0 \quad \text{for } i = m-1, m-2, \ldots, n$$
$$Z_i = X_i \quad \text{for } i = n-1, \ldots, 0$$

Example: Consider $X = 11010101_2 = -45_{10}$ and $X = 100101101_2 = -45_{10}$.

The algorithm extends the range of X by adding zeros to the left of the most significant bit, preserving the sign bit.

1.8.1 Arithmetic Shifts

Two elementary transformations often used in arithmetic operations are scaling (multiplying and dividing) by the radix.

In the conventional radix-2 number system for integers:

Left arithmetic shift: multiplication by 2, expressed as $z = 2x$.

Right arithmetic shift: division by 2, expressed as $z = 2^{-1}x - \varepsilon$, where $|\varepsilon| < 1$ and the value of ε is such that it makes z an integer. *The value of ε is the remainder of the division.*

1.8.2 Left Arithmetic Shift in Sign-and-Magnitude System

Algorithm (assuming the overflow does not occur):

$$z_s = x_s \text{ (sign bit retained)}$$
$$Z_{i+1} = X_i, \quad \text{for } i = 0, \ldots, n-2$$
$$Z_0 = 0 \text{ (insert zero at the least significant bit)}$$

Example:

Given $X = 100101101_2 = -45_{10}$,

The left arithmetic shift $SL(X)$ would be $101011010_2 = -90_{10}$.

1.8.3 Right Arithmetic Shift in Sign-and-Magnitude System

Algorithm:

$$z_s = x_s \text{ (sign bit retained)}$$
$$Z_{i-1} = X_i, \quad \text{for } i = 1, \ldots, n-1$$
$$Z_{n-1} = 0 \text{ (insert zero at the most significant bit)}$$

Example:

Given $X = 100101101_2 = -45_{10}$,

The right arithmetic shift $SR(X)$ would be $100010110_2 = -22_{10}$.

1.8.4 Left Arithmetic Shift in Two's Complement System

Algorithm (assuming that overflow does not occur):

$$Z_{i+1} = X_i, \quad \text{for } i = 0, \ldots, n-2$$
$$Z_0 = 0 \text{ (insert zero at the least significant bit)}$$

Examples:

Given $X = 00110101_2 = 13_{10}$,

The left arithmetic shift $SL(X)$ is $01101010_2 = 26_{10}$.

Given $Y = 11010101_2 = -11_{10}$,

The left arithmetic shift $SL(Y)$ is $10101010_2 = -22_{10}$.

1.8.5 Right Arithmetic Shift in Two's Complement System

Algorithm (assuming that overflow does not occur):

$$Z_{n-1} = X_{n-1}$$
$$Z_{i-1} = X_i, \quad \text{for } i = 1, \ldots, n-1$$

The most significant bit (MSB) is duplicated to keep the sign of the number the same.

Examples:

For $X = 001101_2 = 13_{10}$, the right arithmetic shift is $SR(X) = 000110_2 = 6_{10}$.

For $Y = 110101_2 = -11_{10}$ (in two's complement), the right arithmetic shift is $SR(Y) = 111010_2 = -6_{10}$.

1.9 Hamming Weight and Distance

1.9.1 Hamming Weight (HW)

The Hamming weight of a binary sequence is the number of symbols that are equal to one (1s).

For example, the Hamming weight of 11010101 is 5, as there are five 1s in the bit sequence.

1.9.2 Hamming Distance (HD)

The Hamming distance between two binary sequences of equal length is the number of positions at which the corresponding symbols are different.

For example, the Hamming distance between 11010101 and 01000111 is 3, as they differ in three positions.

1.10   Binary Coded Decimal (BCD)

Binary Coded Decimal (BCD) represents decimal numbers where each decimal digit is encoded as a four-bit binary number. This method allows decimal numbers to be represented in a format that is easy for digital systems to process.

1.10.1   BCD Encoding

- In BCD, each of the decimal digits 0 through 9 is represented by a four-bit binary number, ranging from 0000 to 1001.

- Binary values from $1010_2(10_{10})$ to $1111_2(15_{10})$ are not used in standard BCD encoding.

- For example, 25 is represented as $\underline{0010\ 0101}_2$.

1.10.2   Conversion Algorithms

From BCD to Decimal

To convert a BCD-encoded number to its decimal representation:

```
1. Initialize i to the highest index of BCD digits (n-1), D to 0.
2. While i is greater than or equal to 0:
   a. Multiply D by 10.
   b. Add the decimal value of the BCD digit at index i to D.
   c. Decrement i.
```

From Decimal to BCD

To convert a decimal number to its BCD representation:

```
1. Initialize i to 0, D to the decimal number.
2. While D is not equal to 0:
   a. Calculate D mod 10 and store it as the current BCD digit.
   b. Divide D by 10.
   c. Increment i.
```

1.11 Gray Code Conversion Algorithm

Rule for Conversion:

For bit i in the Gray code, look at bits i and $i+1$ in the binary code (bit n in binary is zero if $i+1 = n$).

If bits i and $i+1$ in the binary are the same, bit i in the Gray code is 0.

If they are different, bit i in the Gray code is 1.

1.11.1 Example

To convert the binary number 1011 to Gray code.

let: $\underline{1011}_2 = b_3 b_2 b_1 b_0$.

Apply the conversion rule:

$g_3 = b_3$ since there is no b_4 (assume $b_4 = 0$).

$g_2 = b_3 \oplus b_2$.

$g_1 = b_2 \oplus b_1$.

$g_0 = b_1 \oplus b_0$.

Calculate the Gray code bits:

$g_3 = 1 \oplus 0 = 1$ as b_4 doesn't exist and is thus a 0.

$g_2 = 1 \oplus 0 = 1$.

$g_1 = 0 \oplus 1 = 1$.

$g_0 = 1 \oplus 1 = 0$.

The Gray code is: $g_3 g_2 g_1 g_0 = \underline{1110}_{\text{gray code}}$.

Chapter 2

Number Systems (Part II)

2.1 Addition and Subtraction of Unsigned Integers

Personal Remark. In case this is not clear, this video explains it pretty good:
https://www.youtube.com/watch?v=sJXTo3EZoxM

2.1.1 Addition of Binary Numbers

To add binary numbers, follow these rules:

$$0 + 0 = 0$$

$$0 + 1 = 1$$

$$1 + 0 = 1$$

$$1 + 1 = 10 \text{ (0 and carry 1 to the next higher bit)}$$

$$1 + 1 + 1 = 11 \text{ (1 and carry 1 to the next higher bit)}$$

Example:

Adding 1101_2 and 1011_2:

		1	1	1	0	(Carry)
		1	1	0	1	
+		1	0	1	1	
1	1	0	0	0		

2.1.2   Subtracting Two Binary Numbers

Works exactly like subtracting decimal numbers, but with a borrow of 2 instead of 10. The rules for binary subtraction include:

$$0 - 0 = 0$$

$$1 - 0 = 1$$

$$1 - 1 = 0$$

$$0 - 1 = 1 \text{ (with a borrow from the next higher bit)}$$

Example:

Subtracting 1000_2 from 1101_2:

$$
\begin{array}{ccccc}
 & 1 & 1 & 0 & 1 \\
- & 1 & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 0 & 1 \\
\end{array}
$$

Therefore, the difference between 1101_2 and 1011_2 is 0010_2.

2.2   Overflow and Underflow in Unsigned Binary Arithmetic

2.2.1   Overflow

Overflow in unsigned binary arithmetic occurs when the sum of two numbers exceeds the maximum value that can be represented by the given number of bits. For an n-bit unsigned number, the maximum value that can be represented is $2^n - 1$. If the result of an addition is greater than this maximum value, the system experiences overflow, leading to an incorrect result.

Example: Consider adding two 4-bit unsigned numbers 1111_2 and 0001_2:

$$
\begin{array}{cccccc}
 & & 1 & 1 & 1 & 1 \\
+ & & 0 & 0 & 0 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
$$

The result 10000_2 is a 5-bit number, but only the 4 least significant bits 0000_2 are kept in a 4-bit system, leading to overflow.

2.2.2   Underflow

Underflow in unsigned binary arithmetic occurs when the result of a subtraction is less than 0, which is not representable in unsigned arithmetic. Since unsigned numbers cannot represent negative values, any operation that would result in a negative value causes underflow.

Example: Consider subtracting a larger 4-bit unsigned number 1010_2 from a smaller one 0100_2:

$$
\begin{array}{ccccc}
 & 0 & 1 & 0 & 0 \\
- & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

underflow

Since the result would be negative, which cannot be represented in unsigned arithmetic, this situation is considered underflow.

2.3   Two's Complement Addition and Subtraction

Graphical Representation

In two's complement arithmetic, a circular graphical representation can be used to illustrate the addition and subtraction of numbers:

Circular representation of two's complement

- Moving **clockwise** from 0 represents the *addition* of positive numbers.

- Moving **counterclockwise** represents the *subtraction* of positive numbers.

- Crossing the line where the sign changes indicates a *change of sign* from positive to negative or vice versa.
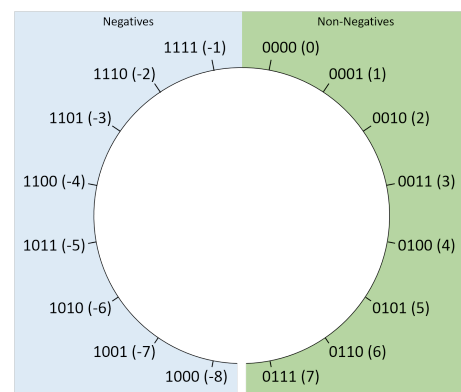
Examples:

The addition of two positive numbers, such as $2 + 3$, is shown by moving clockwise from 0 to 2 and then moving 3 more steps clockwise, resulting in 5.

The subtraction of a smaller number from a larger number, such as $5 - 3$, is shown by moving clockwise from 0 to 5 and then moving 3 steps counterclockwise, resulting in 2.

2.3.1 Addition and Subtraction

2.3.2 Addition

Given two n-bit numbers A and B, their sum in two's complement arithmetic is obtained by directly adding them together as binary numbers:

$$\text{Sum} = A + B \tag{2.1}$$

If there is an overflow, i.e., a carry out of the most significant bit (MSB), it is discarded. The result is also represented in n bits.

2.3.3 Subtraction

To subtract one n-bit number B from another A using two's complement arithmetic, convert B to its two's complement and then add it to A:

1. Find the two's complement of B, denoted as $\bar{B}$, by inverting all the bits of B and adding 1.

2. Add A to the two's complement of B:

$$\text{Difference} = A + \bar{B} \tag{2.2}$$

As with addition, discard any overflow from the MSB.

2.4 Binary Multiplication

Proof. Let X and Y be two numbers, then their product can be represented as:

$$X \cdot Y = X \cdot \left(-Y_{n-1} \cdot 2^{n-1} + X \sum_{i=0}^{n-2} Y_i \cdot 2^i \right)$$

$$= -X \cdot Y_{n-1} \cdot 2^{n-1} + X \sum_{i=0}^{n-2} Y_i \cdot 2^i$$

$$= -Y_{n-1} \cdot X \cdot 2^{n-1} + Y_{n-2} \cdot X \cdot 2^{n-2} + \ldots + Y_2 \cdot X \cdot 2^2 + Y_1 \cdot X \cdot 2^1 + Y_0 \cdot X \cdot 2^0$$

$$\square$$

Binary multiplication operates similarly to decimal multiplication but is performed bit by bit. Here is a clearer example illustrating the multiplication of two binary numbers:

Multiplicand	1101	(This is the number to be multiplied)
Multiplier	$\times$ 0011	(This number multiplies the multiplicand)

	1101	(Multiply by 1, the least significant bit of the multiplier)
+	11010	(Multiply by 1, add one zero to the right, (left shift, $<< 1$))
+	000000	(Multiply by 0, add two zeros to the right, (left shift, $<< 2$))
+	0000000	(Multiply by 0, add three zeros to the right, (left shift, $<< 3$))

	100111	(Sum of the partial products)

Chapter 3

Number Systems (Part III)

3.1 Fixed-Point Number Representation

Let x be an integer : $x = x_{int} + xfr$, with x_{int} the integer part and x_{fr} the fractional part.

Let X be a digit-vector:

$$X = (X_{m-1}X_{m-2}\ldots X_1X_0 \,.\, X_{-1}X_{-2}\ldots X_{-f})$$

where

X_{m-1} to X_0 represent the integer component

X_{-1} to X_{-f} represent the fractional component

The dot (.) represents the radix point (assumed to be fixed)

For unsigned numbers:

$$x = \sum_{i=-f}^{m-1} X_i \cdot 2^i$$

For signed numbers in two's complement:

$$x = -X_{m-1} \cdot 2^{m-1} + \sum_{i=-f}^{m-2} X_i \cdot 2^i$$

3.1.1 Examples of Fixed-Point Numbers

Decimal Numbers

Decimal number system with $m = 5, f = 5$

Example decimal digit vector

$$x = (10077.01690)$$

$$x = 1 \cdot 10^4 + 7 \cdot 10^1 + 7 \cdot 10^0 + 1 \cdot 10^{-2} + 6 \cdot 10^{-3} + 9 \cdot 10^{-4}$$
$$= 10000 + 70 + 7 + 0.01 + 0.006 + 0.0009$$
$$= 10077.0169$$

Most negative (min):

$$x_{\min} = -99999.99999 = \frac{-9999999999}{10^5}$$

Largest number (max, positive):

$$x_{\max} = +99999.99999 = \frac{+9999999999}{10^5}$$

Unsigned Binary Numbers

Unsigned with $m = 3, f = 4$

Example binary digit vector

$$X = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} + 1 \cdot 2^{-4} = 5.4375$$

Smallest number (min):
$$x_{\min} = 000.0000_2 = 0$$

Largest number (max):

$$x_{\max} = 111.1111_2 = 7 + \frac{15}{16} = 7,9375$$

Sign-and-Magnitude Binary Numbers

Sign-and-magnitude with $m = 5, f = 3$

Example binary digit vector

$$X = (-1)^1 \cdot (1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3}) = -5.75$$

Most negative number (min):

$$x_{\min} = 11111.111_2 = -15 + \frac{7}{8} = -15.875$$

Largest number (max, positive):

$$x_{\max} = 01111.111_2 = 15 + \frac{7}{8} = 15.875$$

3.2 Concepts of Finite Precision Math

3.2.1 Precision

Let X be a digit-vector:

$$X = (X_{m-1}X_{m-2}\ldots X_1 X_0 \, . \, X_{-1}X_{-2}\ldots X_{-f})$$

Precision is the maximum number of non zero bits.

$$Precision(x) = m + f$$

with m the number of bits for the integer part and f the number of bits for the fractional part.

3.2.2 Resolution

Resolution is the smallest non-zero number that can be represented.

$$Resolution(x) = 2^{-f}$$

3.2.3 Range

The range of a fixed-point number is the difference between the largest and smallest numbers that can be represented.

$$Range(x) = x_{\max} - x_{\min}$$

In two's complement, the range is given by:

$$Range(x) = \sum_{i=-f}^{m-2} 2^i - \left(-2^{m-1}\right)$$

3.2.4 Accuracy

Accuracy is the maximum difference between a real value and the represented value.

$$Accuracy(x) = \frac{Resolution(x)}{2}$$

3.2.5 Dynamic Range

The dynamic range is the ratio of the largest and the smallest positive number that can be represented.

$$Dynamic\ Range(x) = \frac{x_{\max}}{x_{\min}}$$

In two's complement :

$$Dynamic\ Range(x) = \frac{2^{m-1}}{2^{-f}} = 2^{m-1+f}$$

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3.3 Floating-Point Number Representation

Personal Remark. Please take the time to really understand this part. It is crucial for the understanding of the rest of the course. Take some time to understand the vocabulary and its meaning.

A real number that is exactly representable in a computer is called a **floating-point number**.

3.3.1 Significand, Base, Exponent

A floating-point number consits of a **significand** (or mantissa), a **base** (or radix), and an **exponent**.

the signed *significand* (also called *mantissa*) M^*

the signed *exponent* E

where b is a constant called the *base*

$$x = M^* \times b^E$$

This reminds us of the usual scientific notation, base 10:

$+35200 = +3.52 \times 10^4$ (Coefficient)

$-0.099 = -9.9 \times 10^{-2}$ (Exponent)

3.3.2 Benefits

1. *Consider 32-bit two's complement signed integers*

   The dynamic range for a 32-bit two's complement signed integer can be expressed as:
   $$\text{Dynamic Range}_1(x) = \frac{|x|_{\text{max}}}{|x_{\text{positive, nonzero}}|_{\text{min}}} = \frac{|-2^{32-1}|}{2^0} = 2^{31} \approx 2 \cdot 10^9$$

2. *Consider 32-bit floating-point number, with 24-bits significand and 8-bit exponent in two's complement*

   $\text{Dynamic Range}_2(x) = \frac{|x|_{\text{max}}}{|x_{\text{positive, nonzero}}|_{\text{min}}}$

   $$= \frac{(2^{23} - 1) \cdot 2^{(2^{8-1})-1}}{2^{-128}} = \frac{(2^{23} - 1) \cdot 2^{127}}{2^{-128}} = (2^{23} - 1) \cdot 2^{255} \approx 5 \cdot 10^{83}$$

3. *Dynamic range increase*

   Comparing the two dynamic ranges:

   $$\frac{\text{Dynamic Range}_2(x)}{\text{Dynamic Range}_1(x)} \approx 10^{74}$$

1. *Consider a fixed-point number with 8 fractional bits*

$$\text{Resolution}_1(x) = 2^{-8}$$

2. *Consider 32-bit floating-point number, with 24-bits significand in sign-and-magnitude and 8-bit exponent in two's complement*

$$\text{Resolution}_2(x) = 2^0 \cdot 2^{-2(8-1)} = 2^{-2^7} = 2^{-128}$$

3. *Improved resolution*

$$\frac{\text{Resolution}_2(x)}{\text{Resolution}_1(x)} = \frac{2^{-128}}{2^{-8}} = 2^{-120}$$

3.3.3   Representation

We will be focusing on the following digit-vector:

$$X = (\underbrace{S}_{\text{Sign}} E_{m-1} E_{m-2} \ldots E_1 E_0 \underbrace{M_{n-1} M_{n-2} \ldots M_0}_{\text{Magnitude}})$$

The floating-point representation becomes

$$x = (-1)^s \times M \times b^E$$

where $s \in \{0, 1\}$ is the **sign**, and M is the **magnitude** of the signed significant

In the rest of the lecture, we assume significand is represented in sign-and-magnitude.

Normalization

The goal of normalization is to represent the number such that the magnitude (M) is within the range $1 \leq M < 2$. This means that the leading digit before the binary point is always 1. Here are examples to demonstrate this process:

Positive Example:

Given: $+1010.1000_2$

Normalize: $+1.1010_2 \times 2^3$

Decimal Conversion: $1.3125 \times 8 = 10.5$

Explanation: The binary number 1010.1000_2 is normalized by shifting the binary point such that the first digit is 1, and adjusting the exponent (2^3) accordingly. The equivalent decimal number is 10.5.

Negative Example:

Given: $-(0.00000011)_2$

Normalize: $-1.1_2 \times 2^{-7}$

Decimal Conversion: $-(1.5)_{10} \times 2^{-7} = -0.01171875$

Explanation: The binary number 0.00000011_2 is normalized by shifting the binary point to get a leading 1, and adjusting the exponent (2^{-7}) to reflect the shift. The equivalent decimal number is -0.01171875.

Why Normalize?

Normalization removes redundancy from floating-point representation, making it unique. Consider these examples to understand the redundancy in non-normalized representations:

$$+ (1010)_2 \times 2^{-2} = 10 \times 2^{-2} = 2.5;$$
$$+ (1.01)_2 \times 2^1 = 1.25 \times 2^1 = 2.5;$$
$$+ (0101)_2 \times 2^{-1} = 5 \times 2^{-1} = 2.5;$$

In these cases, different representations lead to the same decimal value, illustrating redundancy. Normalization ensures that each number has a unique floating-point representation.

Conclusion: Floating-point representation is redundant unless it is normalized. By normalizing, we ensure a unique and efficient representation for computational purposes.

In normalized floating-point representation, the leading digit is always 1 and is omitted as a hidden bit to save space, with the remaining digits representing the fraction part of the significand :

$$+(101.001)_2 \times 2^{-4} \Rightarrow +(1.01001)_2 \times 2^{-2} \Rightarrow +(.01001)_2 \times 2^{-2}$$

3.3.4   Biased Representation

Given a binary number with n bits, the value of the biased representation can be calculated as follows:

$$x = (\sum_{i=0}^{n-1} X_i) \cdot 2^i - B$$

Where X_i represents the ith bit of the binary number (starting from 0 for the least significant bit), and B is the bias, which is calculated by:

$$B = 2^{(n-1)} - 1$$

The exponent thus becomes :

$$e = (\sum_{i=0}^{n-1} E_i \cdot 2^i) - (2^{(n-1)} - 1)$$

For instance, with a 3-bit binary number, the bias B is $2^{(3-1)} - 1 = 3$. Therefore, the biased representation maps binary numbers to the integer range from $-B$ to $2^n - 1 - B$.

Example

For a 3-bit binary number, the biased representations would be:

Decimal	Binary	Biased Decimal
7	111	4
6	110	3
5	101	2
4	100	1
3	011	0
2	010	-1
1	001	-2
0	000	-3

Note that the minimum exponent in the biased representation is zero so that the representation of FP zero value is all zeros (zero sign, exponent, and mantissa).

Summary of the Floating-Point Representation

Let the binary vector :

$$X = (SE_{m-1}E_{m-2}\ldots E_1E_0 \ . \ M_{n-1}M_{n-2}\ldots M_0)$$

(m)-bit exponent

- Biased, $B = 2^{m-1} - 1$

$(n+1)$-bit significand

- Sign-and-magnitude
- Normalized, one hidden bit

$$x = (-1)^S \times \left(1 + \sum_{i=1}^{n} M_{n-i}2^{-i}\right) \times 2^{\left(\sum_{j=0}^{m-1} E_j 2^j\right) - (2^{m-1}-1)}$$

3.3.5   Rounding

The result of a floating-point operation is a real number that, to be represented exactly, might require a significand with an infinite number of digits.

For a representation close to the exact result, we perform **rounding**.

Consider the real number x_{real} and the consecutive floating-point numbers F_1 and F_2, such that $F_1 \leq x_{\text{real}} \leq F_2$.

We can perform several types of rounding:

Round to nearest (tie to even)
Round towards zero (truncation)
Round towards plus or towards minus infinity

- Round to nearest (tie to even)

$$R_{\text{near}}(x_{\text{real}}) = \begin{cases} F_1, & \text{if } |x_{\text{real}} - F_1| < |x_{\text{real}} - F_2| \\ F_2, & \text{if } |x_{\text{real}} - F_1| > |x_{\text{real}} - F_2| \\ \text{even}(F_1, F_2), & \text{if } |x_{\text{real}} - F_1| = |x_{\text{real}} - F_2| \end{cases}$$

- Round towards zero (truncation)

$$R_{\text{zero}}(x_{\text{real}}) = \begin{cases} F_1, & \text{if } x_{\text{real}} \geq 0 \\ F_2, & \text{if } x_{\text{real}} < 0 \end{cases}$$

- Round towards plus or minus (negative) infinity

$$R_{\text{pinf}}(x_{\text{real}}) = F_2$$

$$R_{\text{ninf}}(x_{\text{real}}) = F_1$$

Examples of Rounding Methods

Let's consider $x_{\mathrm{real}} = 2.5$, $F_1 = 2$, and $F_2 = 3$ for our examples.

Round to nearest (tie to even)

$$R_{\mathrm{near}}(2.5) = \mathrm{even}(2, 3) = 2$$

Since both F_1 and F_2 are equidistant from x_{real}, we choose the even number which is 2.

Round towards zero (truncation)

$$R_{\mathrm{zero}}(2.5) = 2$$

Since x_{real} is positive, we round towards zero, resulting in F_1.

Round towards plus infinity

$$R_{\mathrm{pinf}}(2.5) = 3$$

When rounding towards plus infinity, we choose F_2.

Round towards minus infinity

$$R_{\mathrm{ninf}}(2.5) = 2$$

When rounding towards minus infinity, we choose F_1.
Now let's consider a negative value $x_{\mathrm{real}} = -2.5$, $F_1 = -3$, and $F_2 = -2$.

Round towards zero (truncation) for negative value

$$R_{\mathrm{zero}}(-2.5) = -2$$

Since x_{real} is negative, we round towards zero, resulting in F_2.

Round towards plus infinity for negative value

$$R_{\mathrm{pinf}}(-2.5) = -2$$

When rounding towards plus infinity for a negative number, we choose the larger number, which is F_2.

Round towards minus infinity for negative value

$$R_{\mathrm{ninf}}(-2.5) = -3$$

When rounding towards minus infinity for a negative number, we choose the smaller number, which is F_1.

3.4 IEEE 754 Standard

FP Format in IEEE 754

Exactly what we described:

$(n + 1)$-bit significand

- Sign-and-magnitude
- Normalized, one hidden bit

m-bit exponent

- Biased, $B = 2^{m-1} - 1$

Let X a digit-vector represented as:

$$X = (SE_{m-1}E_{m-2}\ldots E_1E_0.M_{n-1}M_{n-2}\ldots M_1M_0)$$

Basic and extended formats:

+ Basic formats:

    - Single precision (32 bits)

        * Sign S: 1 bit
        * Exponent E: 8 bits
        * Fraction F: 23 bits

    - Double precision (64 bits)

        * Sign S: 1 bit
        * Exponent E: 11 bits
        * Fraction F: 52 bits

+ Default round to nearest (ties to even)

3.4.1 Special Values

The IEEE 754 standard defines special values with unique bit patterns:

Floating-point zero: is represented by all zeros in both the exponent and the significand fields.

<div align="center">

Positive zero: 0   00000000   00000000000000000000000

Negative zero: 1   00000000   00000000000000000000000

</div>

The most significant bit (the sign bit) differentiates between positive and negative zero.

Positive and negative infinity: are represented by all ones in the exponent field and all zeros in the significand field.

<div align="center">

Positive infinity: 0   11111111   00000000000000000000000

Negative infinity: 1   11111111   00000000000000000000000

</div>

NaN (Not a Number): is represented by all ones in the exponent field and a non-zero significand field.

<div align="center">

NaN (example): −   11111111   10000000000000000000000

</div>

NaN values represent indeterminate or undefined results, such as the square root of a negative number. The sign bit can be either 0 or 1, but the significand must not be all zeros.

3.4.2 Overflow, underflow, and others

Overflow: Occurs when the rounded value is too large to be represented by the floating-point format.

- The result is set to positive or negative infinity, depending on the sign.

Underflow: Happens when the rounded value is too small to be represented.

- Typically, the result is set to a denormalized number or zero.

Division by zero: Occurs when a finite non-zero number is divided by zero.

- The result is set to positive or negative infinity, based on the sign of the numerator.

Inexact result: Occurs when the result of an operation is not an exact floating-point number.

- The result is rounded to the nearest representable value.

Invalid: This flag is set when the result of an operation is not a real number (NaN).

- Examples include the square root of a negative number or the indeterminate form 0/0.

IEEE 754

Example: Converting single-precision FP to decimal

Find the decimal equivalent of

$$X = (1\ 01111100\ 01000000000000000000000)$$

Solution:

- Sign $S = 1$, hence negative.

- Exponent $E = 01111100$ represents the biased exponent. To find the actual exponent:
$$E_{\text{actual}} = 124 - (2^7 - 1) = 124 - 127 = -3$$

- Mantissa (including the hidden bit) $M = 1.01$ in binary represents:
$$M = 1 + 2^{-2} = 1.25$$

- Result:
$$x = -1.25 \times 2^{-3} = -0.15625$$

Example: Converting decimal to single-precision FP

Find the single-precision FP equivalent of $x = -0.8125$
Solution:

- Sign $= 1$, negative.

- Fraction bits can be obtained using multiplication by 2.

- Converting 0.8125 to binary by successive multiplication:

$$0.8125 \times 2 = 1.625 \rightarrow 1$$
$$0.625 \times 2 = 1.25 \rightarrow 1$$
$$0.25 \times 2 = 0.5 \rightarrow 0$$
$$0.5 \times 2 = 1.0 \rightarrow 1$$

Stop when the fractional part becomes zero.

- Mantissa M in binary is 0.1101, normalized is 1.101×2^{-1}.

- Exponent adjustment:
$$E_{\text{actual}} = -1$$
$$E = E_{\text{actual}} + B = -1 + (2^7 - 1) = 126$$

- Result:
$$X = (1\ 01111110\ 10100000000000000000000)$$

Chapter 4

Number Systems (Part IV)

4.1 Fixed Point Arithmetic

4.1.1 Addition and Subtraction

Let x and y be two fixed-point numbers with the same number of integer and fractional bits. The sum and difference of x and y can be calculated as follows:

$$x + y = x_{int} + y_{int} + x_{fr} + y_{fr}$$

$$x - y = x_{int} - y_{int} + x_{fr} - y_{fr}$$

4.1.2 Multiplication

	010.11	Multiplicand
$\times$	011.01	Multiplier
	000000	First partial product (always zero), sign-extended
+	001011	1 x multiplicand, sign-extended
	001011	Intermediate result, sign-extended
+	00000	0 x multiplicand, left-shifted by 1 place and sign-extended
	00001011	Intermediate result, sign-extended convert to fixed-point
+	001011	1 x multiplicand, left-shifted by 2 places and sign-extended
	00010111	Intermediate result, sign-extended
+	001011	1 x multiplicand, left-shifted by 3 places and sign-extended
	001001111	Intermediate result, sign-extended
+	000000	0 x multiplicand, left-shifted by 4 places and sign-extended
	0010001111	Result, integer

4.2 In two's complement

Given two fixed-point numbers x and y, the addition or subtraction in two's complement is given by:

$$x \pm y = \left(-X_{(m_x-1)}2^{(m_x-1)} + \sum_{i=-f_x}^{m_x-2} X_i 2^i\right) \pm \left(-Y_{(m_y-1)}2^{(m_y-1)} + \sum_{i=-f_y}^{m_y-2} Y_i 2^i\right) \qquad (4.1)$$

Where:

m_x and m_y are the total number of bits for the integer components of x and y respectively.

f_x and f_y are the number of bits for the fractional components of x and y respectively.

$X_{(m_x-1)}$ and $Y_{(m_y-1)}$ are the sign bits of x and y.

The largest integer-part exponent is $\max(m_x-1, m_y-1)$. Consequently, the number of bits for the resulting integer component is $\max(m_x, m_y) + 1$.

The smallest fractional-part exponent is $\min(-f_x, -f_y)$. Consequently, the number of bits for the resulting fractional component is $\max(f_x, f_y)$.

Multiplication in Two's Complement

Multiplication on two binary numbers $x(m_x, f_x)$ and $y(m_y, f_y)$

$$x \cdot y = (x_{\text{int}} + x_{\text{fr}}) \cdot (y_{\text{int}} + y_{\text{fr}})$$

In two's complement:

$$x \cdot y = \left(-X_{m_x-1}2^{m_x-1} + \sum_{i=-f_x}^{m_x-2} X_i 2^i\right) \cdot \left(-Y_{m_y-1}2^{m_y-1} + \sum_{i=-f_y}^{m_y-2} Y_i 2^i\right)$$

The largest integer-part exponent: $(m_x - 1) + (m_y - 1)$. Consequently: $m_{xy} = m_x + m_y$

The smallest fractional-part exponent: $(-f_x) + (-f_y)$. Consequently: $f_{xy} = f_x + f_y$

4.3 Floating-Point Arithmetic

Let x and y be represented as (S_x, M_x, E_x) and (S_y, M_y, E_y)

The significands $M^* = (-1)^S M$ are normalized

Addition/subtraction result is z, also represented as (S_z, M_z, E_z):

$$z = x \pm y = M_x^* \times 2^{E_x} \pm M_y^* \times 2^{E_y}$$

The significand of the result is also normalized

$$z = M_z^* \times 2^{E_z}$$

Four main steps to compute the result of floating-point addition/subtraction:

1. **Add/Subtract significand and set exponent:**

   – Align the significands by shifting the one with the *smaller* exponent.

   – Perform addition/subtraction on the aligned significands.

   $$M_z^* = \begin{cases} (M_x^* + (M_y^* \times 2^{(E_y - E_x)})) \times 2^{E_x} & \text{if } E_x \geq E_y \\ ((M_x^* \times 2^{(E_x - E_y)}) + M_y^*) \times 2^{E_y} & \text{if } E_x < E_y \end{cases}$$

   $$E_z = \max(E_x, E_y)$$

2. **Normalize the result and update the exponent, if required:**

   – Check if the result's significand is within the normalized range.

   – If not, shift the significand to the right or left until it is normalized, adjusting the exponent accordingly to maintain the value.

3. **Round the result, normalize, and adjust exponent, if required:**

   – Apply rounding rules to the significand to fit within the precision limits.

   – After rounding, if the significand overflows (e.g., carries out during addition), normalize the result again and adjust the exponent.

4. **Set flags for special values, if required:**

   – Check for overflow or underflow conditions and set flags accordingly.

   – Identify and mark results that are special values (e.g., infinity, NaN) based on the operation and input values.

Step 1: Floating-Point Addition/Subtraction Detailed Algorithm

Algorithm steps:

– Subtract exponents $d = E_x - E_y$.

– Align significands:

   * Compare the exponents of the two operands.

   * Shift right d positions the significand of the operand with the smallest exponent.

   * Select as the exponent of the result the largest exponent.

– Add/subtract signed significands and produce the sign of the result.

Floating-point operations based on the signs of the operands.

FP operation	Signs of the operands	Effective operation
+	=	add
+	$\neq$	subtract
−	=	subtract
−	$\neq$	add

Step 2: Normalize the Result

After the initial addition or subtraction, the result may not always be in the normalized form required by floating-point representation standards. Normalization ensures that the significand (mantissa) is within a specific range, usually just below 1 (for binary floating-point numbers, this means the leading bit is just to the right of the decimal point).

Steps for normalization:

– If the result of the operation causes the significand to exceed its predefined size (overflow), the significand is shifted to the right, and the exponent is increased accordingly.

– Conversely, if the operation results in a significand that's too small (underflow), the significand is shifted to the left, and the exponent is decreased.

– This process ensures that the floating-point number is as close to its true value as possible within the limits of the representation.

Step 3: Round the Result

After normalization, the next step is to round the result to fit within the target floating-point format's precision. Rounding is crucial because it affects the accuracy and representation of the result.

Steps for rounding:

– Evaluate the significand's precision and compare it with the format's limit.

– If the significand exceeds the precision limit, round it according to a rounding rule (e.g., round to nearest, round towards zero, round towards positive/negative infinity).

– Common rounding strategies include:

  * **Round to Nearest:** Round to the nearest value, with ties going to the nearest even number.

  * **Round Down (Towards Zero):** Always round towards zero, truncating any fractional part.

* **Round Up (Away from Zero):** Always round away from zero, increasing the magnitude of the result.

– After rounding, if there's an overflow in the significand (e.g., a carry into a new digit), normalize the result again. This may involve shifting the significand and adjusting the exponent.

Step 4: Set Flags for Special Values

The final step in floating-point addition or subtraction involves handling special cases and setting flags accordingly. Special values include infinity, not-a-number (NaN), and potential overflow or underflow conditions.

Handling special values:

– **Infinity:** If the result of the operation is too large to be represented in the given floating-point format, set the result to infinity. The sign of infinity depends on the operation and operands.

– **NaN (Not-a-Number):** If the operation involves invalid operations (e.g., $0/0$, $\infty - \infty$), set the result to NaN. NaN propagates through most floating-point operations.

– **Overflow:** If the result exceeds the maximum representable value, set an overflow flag. The result is typically set to infinity with the appropriate sign.

– **Underflow:** If the result is too small to be represented (closer to zero than the smallest representable value), set an underflow flag. The result may be set to zero or the smallest denormalized number, depending on the format and flags.

4.3.1 An Example in Binary

Let's add two binary floating-point numbers: 1.01×2^3 and 1.1×2^2. Here's how we do it step by step:

1. **Line Up the Dots:** First, we need to align the exponents. We'll adjust the second number to have the same exponent as the first, by increasing its exponent and shifting its significand to the right:

$$1.1 \times 2^2 = 0.11 \times 2^3$$

Now, both numbers are 1.01×2^3 and 0.11×2^3.

2. **Add Them Up:** With the exponents aligned, we can now add the significands:

$$1.01 + 0.11 = 10.00$$

The result in binary is 10.00. Since we're working in binary, 10.00 is actually 2 in decimal.

3. **Make It Look Right:** The result 10.00×2^3 is already in the correct format, but let's note that if our result was something like 1.000×2^4, we would need to adjust it to keep it in normalized form.

4. **Round It Off:** Our result doesn't need rounding in this case, but if we had more digits than we could store, we'd round off to the nearest value we could represent.

5. **Check for Special Cases:** There are no special cases here, as our result is a regular binary floating-point number.

So, our final result is 10.00×2^3 in binary, which is 8 in decimal.

Chapter 5

Number Systems (Part V)

This small chapter concludes the Number Systems chapter with a some modern applications of low precision computing.

5.1 Low precision computer arithmetic

AI is taking on an increasingly important role

Deep Neural Networks (DNNs) are the most widespread

E.g., Large Language Models (LLM) generate human-like content

Challenge: exponential size growth

GPT3 has 175 billion parameters

Large models mean

A lot of data, any computations

...and we want the result quickly!

Luckily, ML models are tolerant to small errors

5.2 Challenges and limitations

32-bit or 64-bit floating-point formats

Arithmetic units are large (many bits $\Rightarrow$ high area, high energy)

We can put fewer units per chip (e.g., less compute power in GPU)

Poor arithmetic density (in number of ops / 1mm^2)

Fewer units, fewer computations

The model predictions are accurate, but it takes a long time to compute them

Fixed-point or integer formats

Arithmetic units are smaller and faster (approx. 10x area savings)

Better arithmetic density and lower delays

The errors due to limited dynamic range are too significant for most ML models; the accuracy of their predictions suffers

New number formats are needed: the best of both worlds

5.3   Block Floating Point

Block Floating Point

Imagine a block (vector) of binary numbers in FP (Floating Point) format, where each vector element has its own S (Sign), M (Mantissa), and E (Exponent).

S_1	E_1	M_1
S_2	E_2	M_2
$\vdots$	$\vdots$	$\vdots$
S_n	E_n	M_n

If the exponents in the block are not too different, we could use a single shared exponent per block.

E_{block} (shared)

S_1	M_1
S_2	M_2
S_n	M_n
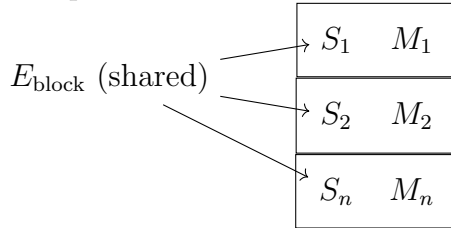
In BFP with shared Exponent:

Find the largest exponent in the block of FP numbers. This will be the shared exponent E_{block}.

Calculate the difference $d_i = E_{\text{block}} - E_i$ between the shared exponent and each of the other exponents E_i in the block.

Adjust the mantissa by right-shifting the signed mantissa of each number by d_i. As a result of these adjustments, the mantissa in BFP cannot be normalized, and there is no hidden bit.
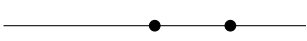
Chapter 6

Digital Logic Circuits

6.1 Introduction to Digital Logic Circuits

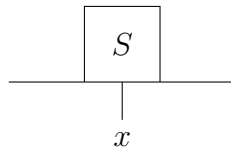The smallest unit of Digital Information is a binary value 1 or 0.
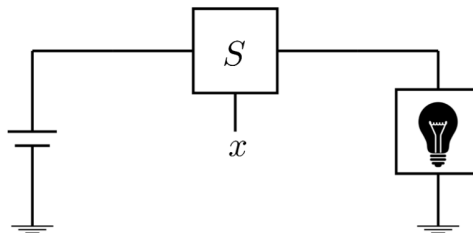
6.1.1 The simplest binary logic element

Similarly, a switch can either be open or closed. Let x an input variable.

- Open $(x = 0)$

- Closed $(x = 1)$

The symbol for a switch controlled by an input variable:
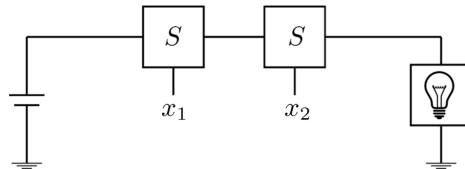
6.1.2 The simplest binary logic element

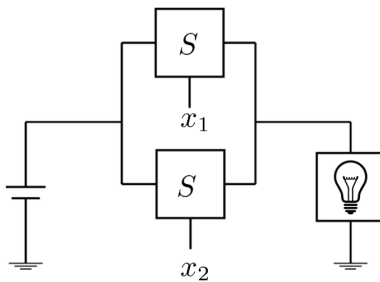Light controlled by a single switch

Two-Variable Logic Functions

Series and Parallel Connections

Remark. The choice of symbols is not random $\cdot, +$, respectively similar to multiplication and addition.
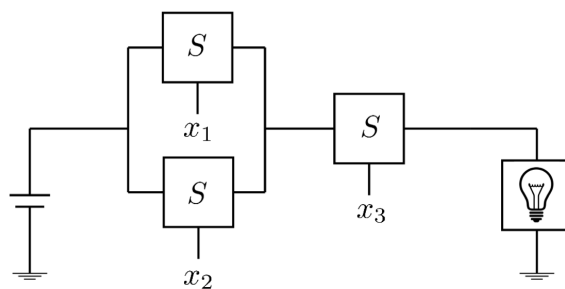

Logic AND function
$L(x_1, x_2) = x_1 \cdot x_2$, with $\cdot$ the AND operator
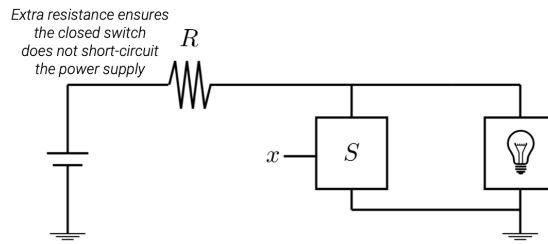

Circular representation of two's complement
$L(x_1, x_2) = x_1 + x_2$, with $+$ the OR operator

A series-parallel connection of three switches
The corresponding Logic Function is $L(x_1, x_2, x_3) = (x_1 + x_2) \cdot x_3$
Smaller circuits separated by parentheses

NOT GATE

The corresponding Logic Function is $L(x) = \bar{x}$, with $\bar{x}$ the complement of x.

6.2 Truth Tables

Logical operations can be defined in the form of a truth table

AND
$$L(x_1, x_2) = x_1 \cdot x_2$$
where $L = 1$ if $x_1 = 1$ and $x_2 = 1$, $L = 0$ otherwise.

OR
$$L(x_1, x_2) = x_1 + x_2$$
where $L = 1$ if $x_1 = 1$ or $x_2 = 1$, or if $x_1 = x_2 = 1$, $L = 0$ if $x_1 = x_2 = 0$.

x_1	x_2	AND	OR
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

For n logical variables we get 2^n rows.

For example :

AND
$$L(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3$$
where $L = 1$ if $x_1 = x_2 = x_3 = 1$, $L = 0$ otherwise.

OR
$$L(x_1, x_2, x_3) = x_1 + x_2 + x_3$$
where $L = 0$ if $x_1 = x_2 = x_3 = 0$, $L = 1$ otherwise.

x_1	x_2	x_3	AND	OR
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

NOT, with $L(x) = \bar{x}$

x	$\bar{x}$	NOT
0	0	0
0	1	0

Precedence Table for Logic Operations

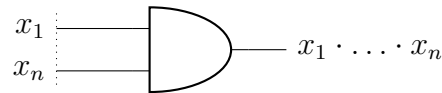| Precedence (Priority)	Operator	Operation	Description
1	$\bar{x}$	NOT	Negation of A
2	$x_1 \cdot x_2$	AND	Conjunction of A and B
3	$x_1 + x_2$	OR	Disjunction of A and B

6.3 Logic Gates

6.3.1 AND GATE
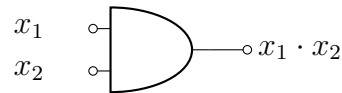
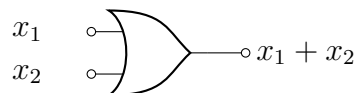$$f(x_1, x_2) = x_1 \cdot x_2$$

6.3.2 AND GATE (n-variables)

$$f(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n$$
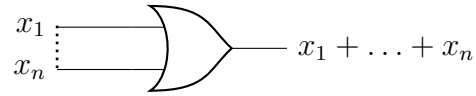
6.3.3 OR GATE

True if at least one input is true.
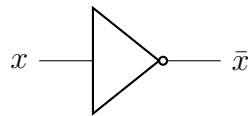

$$f(x_1, x_2) = x_1 + x_2$$

6.3.4 OR GATE (n-variables)

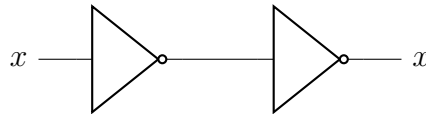$$f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$$

6.3.5 NOT GATE

True if input is false, and vice versa.

$$f(x) = \bar{x}$$

6.3.6 DOUBLE NOT GATE (BUFFER)

A buffer is a gate that does not change the input signal.
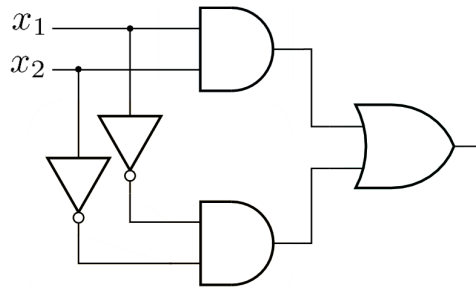

$$f(x) = x = \bar{\bar{x}}$$

6.3.7 NAND GATE

6.3.8 NOR GATE

6.3.9 Example of Complex Logic Circuit

$$f(x_1, x_2) = x_1 x_2 + \bar{x}_1 \bar{x}_2$$

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	0
1	0	0
1	1	1

Corresponding logic table

6.4 Analysis of a Logic Network

Here we'll be looking at sequences of 0s and 1s sent by the input variables. :
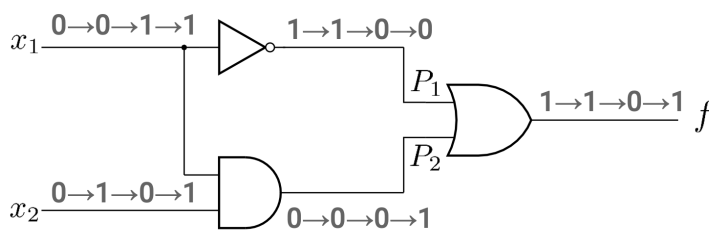
$$f(x_1, x_2) = \bar{x}_1 + x_1 x_2$$

Timing diagram representing the variations in electrical current magnitude over time

Checking for Equivalence of Logic Networks

Two logic networks, represented by functions f and g, are equivalent if:

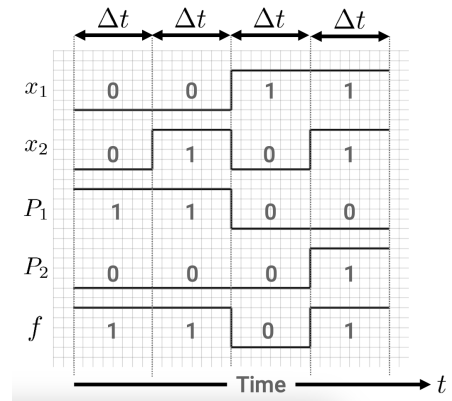> Their truth tables are identical, which is a verification through perfect induction.

> A sequence of algebraic manipulations using Boolean algebra can transform one logic expression into the other.

> Their Venn diagrams are the same, providing a simple visual aid for equivalence.

Finding the Best Equivalent Network (Out of Scope)

The best equivalent logic network is the simplest and cheapest in terms of the number of logic gates used. The process of finding the best equivalent expression is called *minimization* and can be achieved through:

> A sequence of algebraic transformations, although it's not always clear which transformations to apply.

> Using the Karnaugh map, which is simpler than a truth table but becomes unmanageable by hand for more than 4 inputs.

> Automated techniques in synthesis tools (software) which can handle more complex networks efficiently.

6.5   The Venn Diagram

A Venn diagram is a visual representation of the relationships between sets. Each set is represented by a circle. The overlapping parts of circles represent common elements.
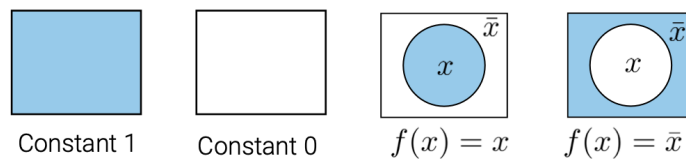
Basic Venn diagram representations

6.6 Network Equivalence Verification

Venn Diagram Approach

$$xy + \overline{x}z + yz \overset{?}{=} xy + \bar{x}z$$
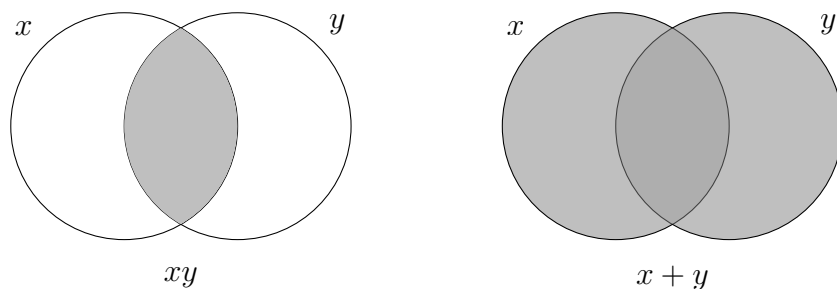

$$xy \qquad \bar{x}z \qquad yz$$
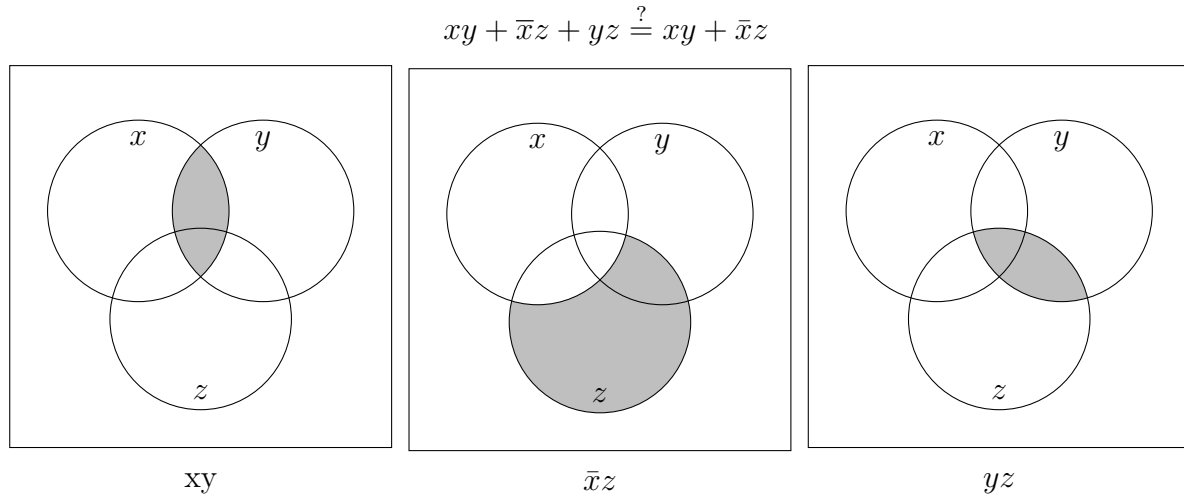
Thus, the two expressions are equivalent.

$$\equiv$$

$$xy + \overline{x}z + yz \qquad\qquad\qquad xy + \bar{x}z$$
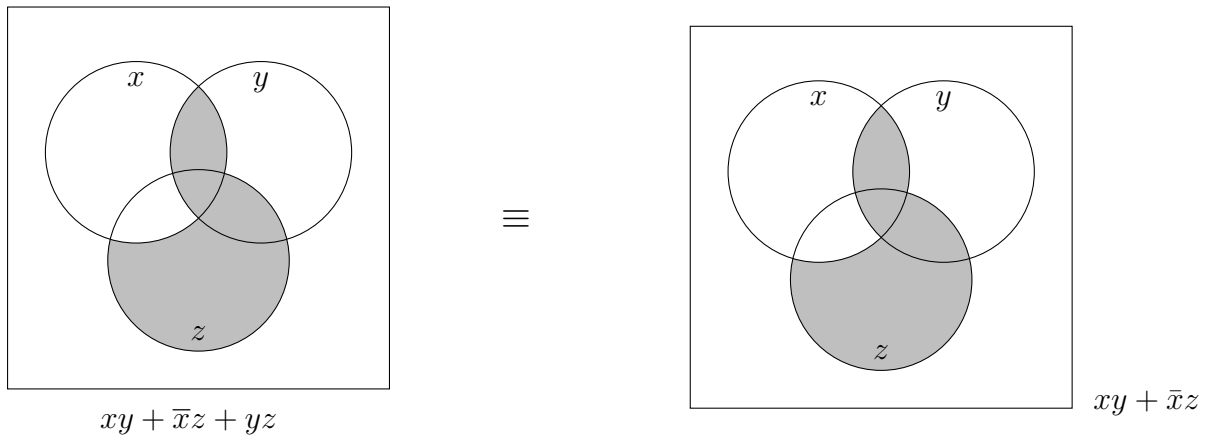
6.7 Boolean Algebra

6.7.1 Axioms

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$

2a. $1 \cdot 1 = 1$

2b. $0 + 0 = 0$

3a. $0 \cdot 1 = 1 \cdot 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$

4a. If $x = 0$, then $\bar{x} = 1$

4b. If $x = 1$, then $\bar{x} = 0$

5a. $x \cdot 0 = 0$

5b. $x + 1 = 1$

6a. $x \cdot 1 = x$

6b. $x + 0 = x$

7a. $x \cdot x = x$

7b. $x + x = x$

8a. $x \cdot \bar{x} = 0$

8b. $x + \bar{x} = 1$

9. $\bar{\bar{x}} = x$

10. Commutative Property:

    (a) Multiplication: $x \cdot y = y \cdot x$

    (b) Addition: $x + y = y + x$

11. Associative Property:

    (a) Multiplication: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

    (b) Addition: $x + (y + z) = (x + y) + z$

12. Distributive Property:

    (a) Multiplication over Addition: $x \cdot (y + z) = x \cdot y + x \cdot z$

    (b) $x + y \cdot z = (x + y) \cdot (x + z)$

13. Absorption (covering):

    (a) $x + x \cdot y = x$

    b. $x \cdot (x + y) = x$

14. Combining:

    (a) $x \cdot y + x \cdot \bar{y} = x$

    (b) $(x + y) \cdot (x + \bar{y}) = x$

15. DeMorgan's theorem:

    (a) $\bar{x} \cdot \bar{y} = \overline{x + y}$

    (b) $\bar{x} + \bar{y} = \overline{x \cdot y}$

16. Redundancy:

    (a) $x + \bar{x} \cdot y = x + y$

    (b) $x \cdot (\bar{x} + y) = x \cdot y$

17. Consensus:

    (a) $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

    (b) $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

6.8   Logic Synthesis

6.8.1   Minterms and Maxterms

Personal Remark. I changed the expression of the function for the product as it was clearer to my sens

Personal Remark.2. This process ressembles a lot what we've seen in AICC I, with CNF and DNF.

These two functions correspond to ways of representing binary numbers like we're used but using n-variable vectors. Here i is the number we're representing m_i the corresponding "variable" minterm representation and M_i maxterm's

Let a product of n variables $f(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} x_j = x_1 \times x_2 \ldots \times x_n = i = m_i$ in which each of the n variables only appear once, this is called a **minterm**.

Let a sum of n variables $f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_j = x_1 + x_2 + \ldots + x_n = i = M_i$ in which each of the n variables only appear once, this is called a **maxterm**.

6.8.2   Examples

Minterms

To make it simple, $1 \to x$, $0 \to \bar{x}$
Let $n = 3, i = 5$:
$5 = \underline{101}_2$ thus $m_5 = x_1 \bar{x}_2 x_3$
Let $n = 5, i = 3$:
$3 = \underline{00011}_2$ thus $m_3 = \bar{x}_1 \bar{x}_2 x_3 \bar{x}_4 \bar{x}_5$

Maxterms

Maxterms are calculated as $M_i = \overline{m_i}$
Let $n = 3, i = 5$:
Using De Morgan's Law :

We have :

$$m_5 = x_1 \overline{x_2} x_3$$

thus :

$$M_5 = \overline{m_5} = \overline{x_1 \overline{x_2} x_3} = \overline{x_1} + x_2 + \overline{x_3}$$

Let $n = 5, i = 3$:

We have :

$$m_3 = \overline{x_1} \overline{x_2} x_3 \overline{x_4} \overline{x_5}$$

thus :

$$M_3 = \overline{m_3} = \overline{\overline{x_1} \overline{x_2} \overline{x_3} x_4 x_5} = x_1 + x_2 + x_3 + \overline{x_4} + \overline{x_5}$$

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x_1}\overline{x_2}\overline{x_3}$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x_1}\overline{x_2}x_3$	$M_1 = x_1 + x_2 + \overline{x_3}$
2	0	1	0	$m_2 = \overline{x_1}x_2\overline{x_3}$	$M_2 = x_1 + \overline{x_2} + x_3$
3	0	1	1	$m_3 = \overline{x_1}x_2x_3$	$M_3 = x_1 + \overline{x_2} + \overline{x_3}$
4	1	0	0	$m_4 = x_1\overline{x_2}\overline{x_3}$	$M_4 = \overline{x_1} + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x_2}x_3$	$M_5 = \overline{x_1} + x_2 + \overline{x_3}$
6	1	1	0	$m_6 = x_1x_2\overline{x_3}$	$M_6 = \overline{x_1} + \overline{x_2} + x_3$
7	1	1	1	$m_7 = x_1x_2x_3$	$M_7 = \overline{x_1} + \overline{x_2} + \overline{x_3}$

6.8.3 Sum-of-Product (SOP) Form and Product-of-Sum (POS) Form

For a function f represented by a table, the rows where $f = 1$ are represented by a sum of minterms, and the rows where $f = 0$ are represented by a product of maxterms.
Remark This is not the most optimal implementation of a function.

Sum-of-Product (SOP) Form

Consider a function f of $n = 3$ variables and the truth table below
Canonical SOP form:

$$f(x_1, x_2, x_3) = \sum (m_1, m_4, m_5, m_6) = \sum m(1, 4, 5, 6)$$

x_1	x_2	x_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Thus the canonical SOP form :

$$f(x_1, x_2, x_3) = \overline{x_1 x_2} x_3 + x_1 \overline{x_2 x_3} + x_1 \overline{x_2} x_3 + x_1 x_2 \overline{x_3}$$
$$= (\overline{x_1} + x_1) x_2 x_3 + \overline{x_1} (\overline{x_2} + x_2) x_3$$
$$= 1 \cdot x_2 x_3 + \overline{x_1} \cdot 1 \cdot x_3$$
$$= x_2 x_3 + \overline{x_1} x_3$$

Product-of-Sum (POS) Form

Consider a function f of $n = 3$ variables and the truth table below

x_1	x_2	x_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

$$f(x_1, x_2, x_3) = \prod (M_0, M_2, M_3, M_7)$$

$$f(x_1, x_2, x_3) = M_0 \cdot M_2 \cdot M_3 \cdot M_7$$

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)(x_1 + \overline{x_2} + x_3)(\overline{x_1} + x_2 + \overline{x_3})$$

Complementing f :

$$\overline{f}(x_1, x_2, x_3) = m_0 + m_2 + m_3 + m_7 = \overline{M_0} + \overline{M_2} + \overline{M_3} + \overline{M_7}$$

$$\overline{f}(x_1, x_2, x_3) = \overline{M_0 \cdot M_2 \cdot M_3 \cdot M_7}$$
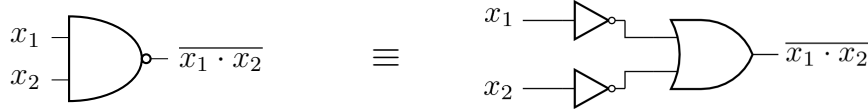
By De Morgan's theorem

$$f(x_1, x_2, x_3) = m_0 + m_2 + m_3 + m_7 = \overline{f}$$

Personal Remark. Don't forget the objective here, these are just two ways, equivalent ways, to represent a function only using ORs and ANDs.

6.9   NAND and NOR Logic Networks

6.9.1   NAND GATE

$$f(x_1, x_2) = \overline{x_1 \cdot x_2} = \overline{x_1} + \overline{x_2}$$

Logic Network with NAND Gates

Implement the following function in the SOP form with NAND gates:

$$f = x_2 + \overline{x_1 x_3} \tag{6.1}$$

Algorithm: start by applying double inversion and, then, De Morgan's theorem to simplify the expression:

$$f = x_2 + \overline{x_1 x_3} \tag{6.2}$$

$$= \overline{\overline{x_2 + x_1 \overline{x_3}}} \tag{6.3}$$

$$= \overline{\overline{x_2} \cdot \overline{x_1 \overline{x_3}}} \tag{6.4}$$

6.9.2   NOR GATE

$$f(x_1, x_2) = \overline{x_1 + x_2} = \overline{x_1} \cdot \overline{x_2}$$

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Logic Network with NOR Gates

Implement the following function in the POS form with NOR gates:

$$f = (x_1 + x_2)(x_2 + x_3) \tag{6.5}$$

Algorithm: start by applying double inversion and, then, De Morgan's theorem to simplify the expression:

$$f = (x_1 + x_2)(x_2 + \overline{x_3}) \tag{6.6}$$

$$= \overline{\overline{(x_1 + x_2)(x_2 + \overline{x_3})}} \tag{6.7}$$

$$= \overline{\overline{(x_1 + x_2)} + \overline{(x_2 + \overline{x_3})}} \tag{6.8}$$

6.10 Incompelitely Defined Functions

6.10.1 Don't Care Condition

Don't care conditions occur in logic design when certain input combinations never occur, or their corresponding outputs do not affect the system behavior.

6.10.2 Example

Consider x_1 and x_2 as inputs controlling two doors to a lion's cage. Here, x_1 is the control for the outer door and x_2 for the inner door. The truth table is as follows:

x_1	x_2	f	Behavior
0	0	0	Doors closed
0	1	1	Outer closed, inner open
1	0	1	Outer open, inner closed
1	1	X	Doors open (don't care)

The condition where both doors are open is a 'don't care' because it should never happen for safety reasons.

6.10.3 Incomplete Functions

A logic function with 'don't care' conditions is termed 'incompletely defined'. These conditions can be exploited to optimize the circuit by choosing don't care outputs to simplify the logic expression.

6.10.4 Sum of Products (SOP) Example

For the lion's cage control, with 'don't cares' the function can be expressed as:

$$f = \sum m(1, 2) + D(3)$$

- Assuming $D(3) = 0$, the function simplifies to:

$$f = \overline{x_1}x_2 + x_1\overline{x_2}$$

using 5 gates and 8 inputs.

- Assuming $D(3) = 1$, the function simplifies further to:

$$f = x_1\bar{x}_2 + x_1 = x_1 + x_2$$

using an OR gate and redundancy rules.

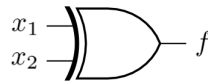*Adding don't care values sometimes allows for a simpler implementation.*

6.11   Even and Odd Detectors (XNOR and XOR Gates)

6.11.1   XOR Gate

True when one and only one input is True The exclusif OR (XOR) gate is represented with the table below :

x_1	x_2	f
0	0	0
0	1	1
1	0	1
0	1	0

and looks like:

$$f = x_1 \oplus x_2$$

6.11.2   XNOR Gate

The exclusif NOR (XNOR) gate is represented with the table below : *True when both inputs are the same.*

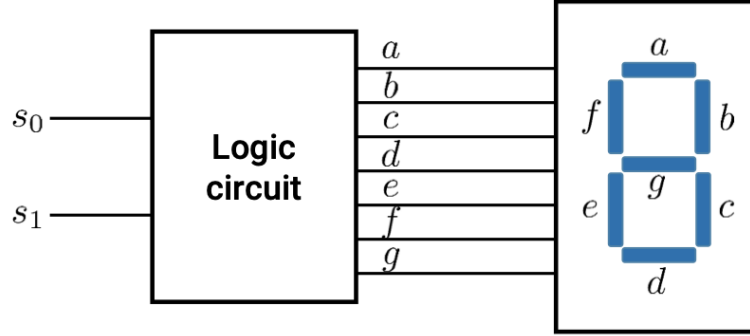| x_1	x_2	f
0	0	1
0	1	0
1	0	0
1	1	1

and looks like:

$$f = x_1 \odot x_2$$

6.12 Design Example

6.12.1 Number Display

Here we will be designing a logic circuit to drive a seven-segment display .

We can derive one logic function per output of the table below:

					Output			
S_1	S_0	a	b	c	d	e	f	g
0	0	1	1	1	1	1	1	0
0	1	0	1	1	0	0	0	0
1	0	1	1	0	1	1	0	1
1	1	1	1	1	1	0	0	1

$$a(s_0, s_1) = M_1 = s_1 + \overline{s_0}$$
$$b(s_0, s_1) = 1$$
$$c(s_0, s_1) = M_2 = s_1 + s_0$$
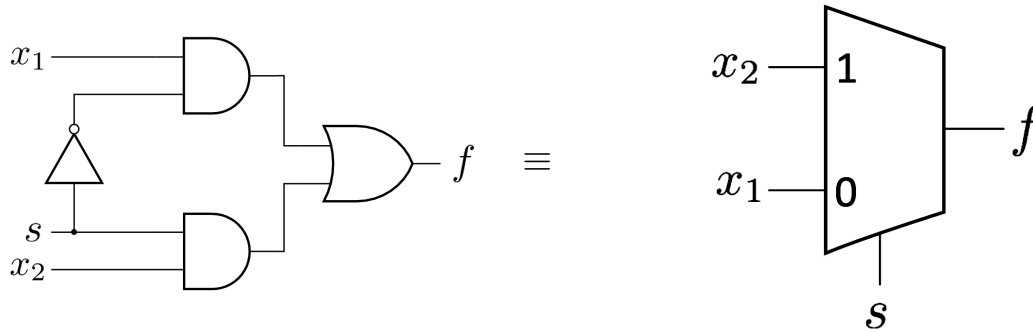$$d(s_0, s_1) = M_1 = s_1 + \overline{s_0} = a(s_0, s_1)$$
$$e(s_0, s_1) = M_1 \cdot M_3 = m_0 + m_2 = s_1\overline{s_0} + s_1 s_0 = \overline{s_0}$$
$$f(s_0, s_1) = m_0 = \overline{s_1 s_0}$$
$$g(s_0, s_1) = M_0 \cdot M_1 = m_2 + m_3 = s_1\overline{s_0} + s_1 s_0 = s_1$$

6.12.2 Multiplexer

A *multiplexer*, or MUX, is a circuit that selects one of several inputs to pass to the output based on the value of one or more select inputs.

Reading the truth table, we can derive the following logic functions:

s	x_1	x_2	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$f(s, x_1, x_2) = \bar{s}x_1x_2 + \bar{s}x_1\overline{x_2} + s\overline{x_1}x_2 + sx_1x_2$$

$$f(s, x_1, x_2) = \bar{s}x_1(\overline{x_2} + x_2) + s(\overline{x_1} + x_1)x_2$$

$$= \bar{s}x_1 + sx_2$$

If there are n inputs to select from, the number of select signals required in a MUX can be determined by:

$$\text{Number of select signals} = \lceil \log_2 n \rceil$$

Examples:

Two inputs: one select signal to choose between inputs indexed as '0' and '1'

Four inputs: two select signals (indices 0, 1, 2, 3)

Eight inputs: three select signals (indices 0, 1, 2, ..., 7)