

Algorithms - CheatSheet

IN BA4 - Ola Nils Anders Svensson

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This is a cheat sheet for the Algorithms midterm exam. For suggestions, contact me on Telegram ([elazdi_al](#)) or via EPFL email (ali.elazdi@epfl.ch).

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<p>Master Theorem</p> <p>If $T(n) = aT(\frac{n}{b}) + f(n)$, $a \geq 1$, $b > 1$, and $f(n)$ asymptotically positive.</p> <p>Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.</p> <p>Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.</p> <p>Case 3: If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, and if $a f(\frac{n}{b}) \leq c f(n)$ for some $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.</p> <p>Common case - if $f(n) = \Theta(n^d)$ for some exponent d:</p> <ul style="list-style-type: none">- If $\frac{d}{b} < 1$ (or $d < \log_b a$), then $T(n) = \Theta(n^d)$.- If $\frac{d}{b} = 1$ (or $d = \log_b a$), then $T(n) = \Theta(n^d \log n)$.- If $\frac{d}{b} > 1$ (or $d > \log_b a$), then $T(n) = \Theta(n^{\log_b a})$.	
<p>Queue Operations</p> <p>Queue-Empty(Q): Time: $O(1)$, Auxiliary Space: $O(1)$</p> <ol style="list-style-type: none">Returns TRUE if the queue is empty ($Q.head = Q.tail$).Returns FALSE otherwise. <p>Enqueue(Q, x): Time: $O(1)$, Auxiliary Space: $O(1)$</p> <ol style="list-style-type: none">Adds element x to the rear of queue Q. <p>Dequeue(Q): Time: $O(1)$ (or wrap around if using circular array)</p> <ol style="list-style-type: none">$Q.tail \leftarrow Q.tail + 1$ (or wrap around if using circular array)If Queue-Empty(Q), return error "underflow".Otherwise, remove and return the element at the front.$x \leftarrow Q[Q.head]$$Q.head \leftarrow Q.head + 1$ (or wrap around) <p>Return x</p> <p>Queue Implementation:</p> <ol style="list-style-type: none">Q.head: Index of the front elementQ.tail: Index where next element will be insertedIn a circular array, indices wrap aroundLeave one slot empty to distinguish full/empty states <p>Overall Space Complexity: $O(n)$ for a queue of capacity n</p>	
<p>Merge Sort</p> <ol style="list-style-type: none">Divide: Split the array evenly into two smaller subarrays, and continue dividing recursively.Sort (Recursively): Apply merge sort recursively on each subarray until each has only one element (base case). <p>MERGE-SORT(A, p, r)</p> <pre>if p < r q ← ⌊(p+r)/2⌋ MERGE-SORT(A, p, q) MERGE-SORT(A, q+1, r) MERGE(A, p, q, r)</pre> <p>3. Merge: Combine the two sorted subarrays into a single sorted array:</p> <ol style="list-style-type: none">Initializing pointers at the start of each subarray.Comparing the elements pointed to, and appending the smaller one into a new array.Advancing the pointer in the subarray from which the element was chosen.Repeating this process until all elements in both subarrays are merged into the sorted array. <p>Merge Cost Complexity: $O(n)$ per merge operation.</p> <p>Time Complexity: $O(n \log n)$</p> <p>Space Complexity: $O(n)$</p>	<p>MERGE(A, p, q, r)</p> <pre>n1 ← q - p + 1 n2 ← r - q let L[1...n1 + 1] and R[1...n2 + 1] be new arrays for i ← 1 to n1 do L[i] ← A[p + i - 1] for j ← 1 to n2 do R[j] ← A[q + j] L[n1 + 1] ← ∞ R[n2 + 1] ← ∞ i ← 1 j ← 1 for k ← p to r do if L[i] ≤ R[j] A[k] ← L[i] i ← i + 1 else A[k] ← R[j] j ← j + 1</pre>
<p>Priority Queue</p> <p>Maintains a dynamic set of elements with associated priority values (keys).</p> <p>Maximum(S): Return element of S with highest priority (return A[1], complexity $O(1)$)</p> <p>Insert(S, x): Insert element x into set S</p> <ol style="list-style-type: none">Increment the heap sizeInsert a new node in the last position in the heap, with key $-\infty$Increase the $-\infty$ value to key using Heap-Increase-Key <p>Extract-Max(S): Remove and return element of S with highest priority</p> <ol style="list-style-type: none">Make sure heap is not emptyMake a copy of the maximum element (the root)Make the last node in the tree the new rootRe-heapify the heap, with one fewer nodeReturn the copy of the maximum element <p>Increase-Key(S, x, k): Increase the value of element x's key to the new value k</p> <ol style="list-style-type: none">Make sure key $\geq A[i]$Update A[i]'s value to keyTraverse the tree upward comparing new key to the parent and swapping if necessary <p>Time Complexity: Insert, Extract-Max, Increase-Key: $O(\log n)$ Maximum: $O(1)$</p> <p>Space Complexity: $O(n)$</p>	
<p>Stack Operations</p> <p>Stack-Empty(S): Time: $O(1)$, Auxiliary Space: $O(1)$</p> <ol style="list-style-type: none">Returns TRUE if the stack is empty.Returns FALSE otherwise. <p>Push(S, x): Time: $O(1)$, Auxiliary Space: $O(1)$</p> <ol style="list-style-type: none">Adds element x to the top of stack S.Increments the stack pointer. <p>Pop(S): Time: $O(1)$, Auxiliary Space: $O(1)$</p> <ol style="list-style-type: none">If Stack-Empty(S), return error "underflow".Otherwise, remove and return the top element.Decrements the stack pointer. <p>Stack Implementation:</p> <ol style="list-style-type: none">Elements are stored in a simple arrayS.top: Index of the topmost elementAn empty stack has S.top = 0 or S.top = -1 (implementation dependent) <p>Overall Space Complexity: $O(n)$ for a stack of size n</p>	
<p>Dynamic Programming</p> <p>Problem: Optimal solutions to overlapping subproblems</p> <p>Fibonacci Sequence:</p> <p>Top-Down (Memoization):</p> <ol style="list-style-type: none">Create memo array $F[0..n]$ initialized to NILBase cases: $F[0] = 0$, $F[1] = 1$Recursive with memo: <p>- Return $F[n]$ if already computed</p> <p>- Otherwise compute</p> <p>$F[n] = F[n-1] + F[n-2]$</p> <p>- Store result in $F[n]$ and return</p> <p>MEMOIZED-FIB(n)</p> <pre>Let r = [0..n] be a new array for i = 0 to n do r[i] ← 0 return MEMOIZED-FIB-AUX(n, r)</pre> <p>MEMOIZED-FIB-AUX(n, r)</p> <pre>if r[n] ≥ 0 return r[n] if n = 0 or n = 1 ans ← 1 else ans ← MEMOIZED-FIB-AUX(n-1, r) + MEMOIZED-FIB-AUX(n-2, r) r[n] ← ans return r[n]</pre> <p>Bottom-Up (Tabulation):</p> <ol style="list-style-type: none">Create array $F[0..n]$Base cases: $F[0] = 0$, $F[1] = 1$For $i = 2$ to n: - Compute $F[i] = F[i-1] + F[i-2]$Return $F[n]$ <p>BOTTOM-UP-FIB(n)</p> <pre>Let r be a new array of size n + 1 r[0] ← 0 r[1] ← 1 for i = 2 to n do r[i] ← r[i-1] + r[i-2] return r[n]</pre>	<p>Cut-Rod Problem:</p> <p>Find optimal way to cut rod to maximize revenue</p> <p>Top-Down (Memoization):</p> <ol style="list-style-type: none">Create memo array $r[0..n]$ with $r[0] = 0$For uncalculated $r[j]$, compute: $r[j] = \max_{1 \leq i \leq j} (p[i] + r[j-i])$Return $r[n]$ <p>MEMOIZED-CUT-ROD-AUX(p, n, r)</p> <pre>if r[n] ≥ 0 return r[n] if n = 0 q ← 0 else for i = 1 to n do q ← max{ q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n-i, r) } r[n] = q return q</pre> <p>Memoized-CUT-ROD(p, n)</p> <pre>let r[0..n] be a new array for i = 0 to n do r[i] ← 0 return MEMOIZED-CUT-ROD-AUX(p, n, r)</pre> <p>Bottom-Up (Tabulation):</p> <ol style="list-style-type: none">Create array $r[0..n]$ with $r[0] = 0$For $j = 1$ to n: - Compute $r[j] = \max_{1 \leq i \leq j} (p[i] + r[j-i])$Return $r[n]$ <p>EXTENDED-BOTTOM-UP-CUT-ROD(p, n)</p> <pre>let r[0..n] and s[0..n] be new arrays r[0] ← 0 for i ← 1 to n do q ← -∞ for j ← 1 to i do if q < p[j] + r[i-j] q ← p[j] + r[i-j] s[i] ← j r[i] ← q return r[n]</pre>
<p>Time Complexity: $O(n)$</p> <p>Space Complexity: $O(n)$</p> <p>(includes input and memo array)</p>	<p>Time Complexity: $O(n^2)$</p> <p>Space Complexity: $O(n)$</p> <p>(includes input prices and array)</p>

Matrix Chain Multiplication:

Find optimal parenthesization to minimize multiplications

Bottom-Up (Tabulation):

1. Create table $m[1..n, 1..n]$ with $m[i, i] = 0$
- $m[i, j]$ stores minimal cost of multiplying matrices i through j
2. For $l = 2$ to n (chain length):
3. For $i = 1$ to $n - l + 1$:
 - Set $j = i + l - 1$
 - Compute $m[i, j] = \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\}$
 - Store k in $s[i, j]$ that achieved minimum cost
4. Return $m[1, n]$

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MATRIX-CHAIN-ORDER( $p$ )
 $n \leftarrow p.\text{length} - 1$ 
let  $m[1..n, 1..n]$  and  $s[1..n, 1..n]$  be new tables
for  $i = 1$  to  $n$  do
     $m[i, i] \leftarrow 0$ 
for  $\ell = 2$  to  $n$  do //  $\ell$  is the chain length
    for  $i = 1$  to  $n - \ell + 1$  do
         $j \leftarrow i + \ell - 1$ 
         $m[i, j] \leftarrow \infty$ 
        for  $k = i$  to  $j - 1$  do
             $q \leftarrow m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
            if  $q < m[i, j]$ 
                 $m[i, j] \leftarrow q$ 
                 $s[i, j] \leftarrow k$ 
return  $m, s$ 
```

Complexity:

- Time: $O(n^3)$
- Space: $O(n^2)$ (includes matrix dimensions and tables)
- Table $s[i, j]$ stores index of last matrix in first parenthesized group