Continuous time-reversal

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Let $(\mathbf{B}_t)_{t\in[0,T]}$ a d-dimensional Brownian motion associated with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, We define the process $(\mathbf{X}_t)_{t\in[0,T]}$ such that for any $t \in [0,T]$

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t b(s, \mathbf{X}_s)\, \mathrm{d}s + \mathbf{B}_t$$

In short, we write $\mathrm{d}\mathbf{X}_t = b(t, \mathbf{X}_t)\, \mathrm{d}t + \mathrm{d}\mathbf{B}_t$. We assume that $b \in \mathrm{C}^\infty\left(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d\right)$ and is bounded. The goal of this exercise is to show the time-reversal formula in continuous-time. More precisely, define $(\mathbf{Y}_t)_{t\in[0,T]} = (\mathbf{X}_{T-t})_{t\in[0,T]}$. We are going to show that

$$\mathrm{d}\mathbf{Y}_t = \{-b(T-t, \mathbf{Y}_t) + \nabla \log p_{T-t}(\mathbf{Y}_t)\}\, \mathrm{d}t + \mathrm{d}\mathbf{B}_t$$

In particular, we are going to show that for any $f \in \mathrm{C}_c^\infty\left(\mathbb{R}^d, \mathbb{R}\right), \left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale where for any $t \in [0,T]$

$$\mathbf{M}_t^f = f(\mathbf{Y}_t) - f(\mathbf{Y}_0) - \int_0^t \left\{\langle -b(T-s, \mathbf{Y}_s) + \nabla \log p_{T-s}(\mathbf{Y}_s), \nabla f(\mathbf{Y}_s)\rangle + \frac{1}{2}\Delta f(\mathbf{Y}_s)\right\} \mathrm{d}s,$$

where p_t is the density of the distribution of $\mathbf{X}_t$ (w.r.t. the Lebesgue measure) which is assumed to exist. We recall that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if

1. Finite expectation: for any $t \in [0,T], \mathbb{E}\left[\left\|\mathbf{M}_t^f\right\|\right] < +\infty$,

2. Conditional expectation [1] for any $s, t \in [0,T]$ with $s \leq t, \mathbb{E}\left[\mathbf{M}_t^f \mid \mathbf{Y}_s\right] = \mathbf{M}_s^f$

The fact that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale is equivalent to the fact that $(\mathbf{Y}_t)_{t\in[0,T]}$ is a weak solution to (1).

Remark: in what follows we assume that $(t,x) \mapsto \|\nabla \log p_t(x)\|$ has at most linear growth [2] and $(t,x) \mapsto p_t(x) \in \mathrm{C}^\infty\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$. In addition, we assume that $s, t, x_s, x_t \mapsto p_{t|s}(x_t \mid x_s) \in \mathrm{C}^\infty\left(\mathrm{A} \times \mathbb{R}^d \times \mathbb{R}^d\right)$ and is bounded, where $\mathrm{A} = \{(s,t) : s, t \in [0,T], t \geq s\}$. In addition, we have assume that for any $s, t \in \mathrm{A}$ and $x_t \in \mathbb{R}^d, x_s \mapsto \left\|\nabla_{x_s} \log p_{t|s}(x_t \mid x_s)\right\|$ has at most linear growth.

We recall the Itô formula. For any $\varphi \in \mathrm{C}^\infty\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$ such that $\|\nabla \log \varphi\|$ has linear growth we have for any $t, s \in [0,T]$

$$\mathbb{E}\left[\varphi\left(t, \mathbf{X}_t\right)-\varphi\left(s, \mathbf{X}_s\right) \mid \mathbf{X}_s\right]=\mathbb{E}\left[\int_s^t\left\{\partial_u \varphi\left(u, \mathbf{X}_u\right)+\left\langle b\left(u, \mathbf{X}_u\right), \nabla \varphi\left(u, \mathbf{X}_u\right)\right\rangle+\frac{1}{2} \Delta \varphi\left(u, \mathbf{X}_u\right)\right\} \mathrm{d} u \mid \mathbf{X}_s\right]$$

[1] Here I have assumed without proof that $\left(\mathbf{Y}_t\right)_{t \in[0, T]}$ is Markov

[2] A function $f:[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is said to have linear growth if there exists $C \geq 0$ such that for any $t \in[0, T]$ and $x \in \mathbb{R}^d,\|f(t, x)\| \leq C(1+\|x\|)$ We also recall the following result. For any $F \in \mathrm{C}_c^{\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and $g \in \mathrm{C}^{\infty}\left(\mathbb{R}^d, \mathbb{R}\right)$

$$\int_{\mathbb{R}^d}\langle F(x), \nabla g(x)\rangle \mathrm{d} x=-\int_{\mathbb{R}^d} g(x) \operatorname{div}(g)(x) \mathrm{d} x .$$

We denote by $\mathrm{C}_c^{\infty}\left(\mathbb{R}^d, \mathbb{R}\right)$, the set of infinitely differentiable continuous functions on $\mathbb{R}^d$ with compact support.

Question 1: Prove that $t \in [0, T], \mathbb{E}\left[\left\|\mathbf{M}_t^f\right\|\right] < +\infty$.

Answer: We need to show that the expectation of the norm of M_t^f is finite for all t in the interval $[0, T]$.

First, M_t^f is defined as:

$$M_t^f = f(Y_t) - f(Y_0) - \int_0^t \langle -b(T-s, Y_s) + \nabla \log p(T-s, Y_s), \nabla f(Y_s)\rangle ds - \frac{1}{2}\int_0^t \Delta f(Y_s) ds$$

where $Y_t = (X(T-t))$ for X being a process defined on $[0, T]$ and $p(T-s, Y_s)$ is the density of the distribution of $X(T-s)$.

Using the Cauchy-Schwarz inequality, we can bound the norm of M_t^f as follows:

$$|M_t^f| \leq |f(Y_t)| + |f(Y_0)| + \int_0^t |\langle -b(T-s, Y_s) + \nabla \log p(T-s, Y_s), \nabla f(Y_s)\rangle| ds + \frac{1}{2}\int_0^t |\Delta f(Y_s)| ds$$

By assumption, f is a smooth function with compact support. Therefore, it is bounded and has finite moments. Also, since b and p are assumed to be smooth functions with bounded derivatives, they are also bounded. Hence, all terms in the above inequality are finite.

$\boxed{\text{Therefore, we can conclude that } \mathbb{E}[\|M_t^f\|] < +\infty \text{ for all } t \text{ in } [0, T].}$

Question 2: Prove that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if and only for any $g \in \mathrm{C}^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ and $t, s \in [0, T]$ with $t \geq s$

$$\mathbb{E}\left[\left(\mathbf{M}_t^f - \mathbf{M}_s^f\right) g\left(\mathbf{Y}_s\right)\right] = 0$$

Answer : Using the definition of M_t^f from Question 1, we can write this as:

$$\begin{aligned}
&\mathbb{E}[M_t^f - M_s^f M_{f,s}|Y_s] \\
&= f(Y_t) - f(Y_s) - \int_s^t \langle -b(T-u, Y_u) + \nabla \log p(T-u, Y_u), \nabla f(Y_u)\rangle du - \frac{1}{2}\int_s^t \Delta f(Y_u) du \\
&= g(X_t)f(X_t) - g(X_s)f(X_s) - \int_s^t g(X_u)f(X_u)\langle -b(T-u, X_u) + \nabla \log p(T-u, X_u), \nabla f(X_u)\rangle du \\
&- \frac{1}{2}\int_s^t g(X_u)\Delta f(X_u) du \\
&= \int_s^t g(X_u)\left(\langle -b(T-u, X_u) + \nabla \log p(T-u, X_u), \nabla f(X_u)\rangle - \frac{1}{2}\Delta f(X_u)\right) du.
\end{aligned}$$

Therefore, we can conclude that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if and only if for any $g \in \mathrm{C}^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ and $t, s \in [0,T]$ with $t > s$, we have

$$\int_s^t g(X_u)\left(\langle -b(T-u, X_u) + \nabla \log p(T-u, X_u), \nabla f(X_u)\rangle - \frac{1}{2}\Delta f(X_u)\right) du = 0.$$

This is true because of the integration by parts formula for stochastic integrals. Specifically, if f and g are smooth functions with compact support, then we have:

$$\int_s^t f(X_u)dg(X_u) = f(X_t)g(X_t) - f(X_s)g(X_s) - \int_s^t g(X_u)df(X_u)$$

where df and dg denote the Malliavin derivatives of f and g, respectively.

Using this formula with $f = g$, we can write:

$$\begin{aligned} &\int_s^t g(X_u)\left(\langle -b(T-u, X_u) + \nabla \log p(T-u, X_u), \nabla f(X_u)\rangle - \frac{1}{2}\Delta f(X_u)\right) du \\ &= \int_s^t g(X_u) dM_{f,u} \\ &= M_t^f - M_s^f \end{aligned}$$

Therefore, we have:

$$M_t^f - M_s^f = 0,$$

which implies that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if and only if for any $g \in \mathrm{C}^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ and $t, s \in [0,T]$ with $t > s$, we have:

$$\int_s^t g(X_u)\left(\langle -b(T-u, X_u) + \nabla \log p(T-u, X_u), \nabla f(X_u)\rangle - \frac{1}{2}\Delta f(X_u)\right) du = 0.$$

This is equivalent to the condition given in the question:

$$\boxed{\mathrm{E}\left[\left(\mathbf{M}_t^f - \mathbf{M}_s^f\right) g\left(\mathbf{Y}_s\right)\right] = 0}$$

for any $g \in \mathrm{C}^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ and $t, s \in [0,T]$ with $t \geq s$.

Question 3: Prove that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if and only for any $g \in \mathrm{C}^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ and $t, s \in [0,T]$ with $t \geq s$

$$\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right) - g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] = \mathbb{E}\left[g\left(\mathbf{X}_t\right)\int_s^t \left\{\langle b\left(u, \mathbf{X}_u\right) - \nabla \log p_u\left(\mathbf{X}_u\right), \nabla f\left(\mathbf{X}_u\right)\rangle - \frac{1}{2}\Delta f\left(\mathbf{X}_u\right)\right\} \mathrm{d}u\right]$$

For any $g \in \mathrm{C}_c^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ and $t \in [0,T]$, denote $h^{g,t} : [0,t] \times \mathbb{R}^d \to \mathbb{R}$ given for any $s \in [0,t]$ and $x \in \mathbb{R}^d$ by

$$h^{g,t}(s,x) = \mathbb{E}\left[g\left(\mathbf{X}_t\right) \mid \mathbf{X}_s = x\right].$$

In what follows, we fix $t \in [0,T]$ and $g \in \mathrm{C}_c^\infty\left(\mathbb{R}^d\right)$.

Answer: To prove that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if and only if the given condition holds, we need to show two things:

1. If the given condition holds, then $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale.
2. If $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale, then the given condition holds.

We will prove each of these statements in turn.

Proof of statement 1:

Assume that the given condition holds. We want to show that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale. To do this, we need to show that for any $s,t \in [0,T]$ with $s \le t$, we have

$$\mathbb{E}\left[\mathbf{M}_t^f - \mathbf{M}_s^f \,\middle|\, \sigma\{\mathbf{Y}_u : u \le s\}\right] = 0.$$

Using the definition of $\mathbf{M}_s^f$, we have

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{M}_t^f - \mathbf{M}_s^f \,\middle|\, \sigma\{\mathbf{Y}_u : u \le s\}\right] \\
&= \int_s^t g(\mathbf{X}_u)\langle -b(T-u,X_u) + \nabla \log p(T-u,X_u), \nabla f(X_u)\rangle - \frac{1}{2} g(X_u)\Delta f(X_u)du \\
&= \int_s^t g(\mathbf{X}_u)\langle -b(T-u,X_u) + \nabla \log p(T-u,X_u), \nabla f(X_u)\rangle du - \int_s^t g(\mathbf{X}_u)\frac{1}{2}\Delta f(X_u)du \\
&= \int_s^t g(\mathbf{X}_u) dM_{f,u}.
\end{aligned}$$

Here, we have used the fact that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a local martingale (which follows from the assumption that b is bounded and smooth, and f has compact support), so we can apply Itô's formula to obtain:

$$d(f(\mathbf{X}_u)g(\mathbf{X}_u)) = g(\mathbf{X}_u)dM_{f,u} + f(\mathbf{X}_u)d(g(\mathbf{X}_u)) + df(\mathbf{X}_u)d(g(\mathbf{X}_u)).$$

Integrating both sides from s to t, and taking expectations conditional on $\sigma\{\mathbf{Y}_u : u \le s\}$, we get:

$$\begin{aligned}
&\mathbb{E}\left[f(\mathbf{X}_t)g(\mathbf{X}_t) - f(\mathbf{X}_s)g(\mathbf{X}_s) \middle| \sigma\{\mathbf{Y}_u : u \le s\}\right] \\
&= \int_s^t \mathbb{E}[g(X_u)dM_{f,u}|\sigma\{\mathbf{Y}_v : v \le s\}] + \int_s^t \mathbb{E}[f(X_u)d(g(X_u))|\sigma\{\mathbf{Y}_v : v \le s\}] + \int_s^t \mathrm{d}u \int_{R^d} f(x)p(T-u,x)d(g(x)).
\end{aligned}$$

Since g has compact support, the second term on the right-hand side vanishes. Using Fubini's theorem and the fact that $(M_{f,t})_{t\in[0,T]}$ is a local martingale, we can write:

$$\begin{aligned}
&\mathbb{E}\left[f(\mathbf{X}_t)g(\mathbf{X}_t) - f(\mathbf{X}_s)g(\mathbf{X}_s) \middle| \sigma\{\mathbf{Y}_u : u \le s\}\right] \\
&= \int_s^t \mathbb{E}[g(X_u)dM_{f,u}|\sigma\{\mathbf{Y}_v : v \le s\}] + \int_s^t \mathrm{d}u \int_{R^d} f(x)p(T-u,x)d(g(x)) \\
&= \int_s^t \mathbb{E}[g(X_u)\left(\langle -b(T-u,X_u) + \nabla \log p(T-u,X_u), \nabla f(X_u)\rangle - \frac{1}{2}\Delta f(X_u)\right) du|\sigma\{\mathbf{Y}_v : v \le s\}] \\
&= \int_s^t g(\mathbf{X}_u)\left(\langle -b(T-u,X_u) + \nabla \log p(T-u,X_u), \nabla f(X_u)\rangle - \frac{1}{2}\Delta f(X_u)\right) du.
\end{aligned}$$

Here, we have used the fact that the given condition holds for $s < t$, so we can replace f with f and g with g in the integral.

Therefore, we have shown that if the given condition holds, then $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale.

Proof of statement 2:

Assume that $\left(\mathbf{M}_t^f\right)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale. We want to show that the given condition holds. To do this, we need to show that for any $g \in C^\infty(R^d, R)$ and $s, t \in [0,T]$ with $s < t$, we have

$$\mathbb{E}\left[g(\mathbf{X}_t) f(\mathbf{X}_t) - g(\mathbf{X}_t) f(\mathbf{X}_s)\right] = \mathbb{E}\left[g(\mathbf{X}_t) \int_s^t \left\{\langle b(u,\mathbf{X}_u) - \nabla \log p_u(\mathbf{X}_u), \nabla f(\mathbf{X}_u)\rangle - \frac{1}{2}\Delta f(\mathbf{X}_u)\right\} \mathrm{d}u\right].$$

Using the definition of $\mathbf{M}_t^f$, we have:

$$g(X_t)f(X_t) - g(X_s)f(X_s) = f(X_s)(g(X_t) - g(X_s)) + g(X_s)(f(X_t) - f(X_s)) + M_{f,t} - M_{f,s}.$$

Taking expectations and using the fact that $(M_{f,t})_{t\in[0,T]}$ is a $(Y_t)_{t\in[0,T]}$-martingale, we get:

$$\begin{aligned}
&g(X_t)f(X_t) - g(X_s)f(X_s) \\
&= f(X_s)(g(X_t) - g(X_s)) + g(X_s)(f(X_t) - f(X_s)) + E[M_{f,t} - M_{f,s}] \\
&= \int_s^t g(\mathbf{X}_u)dM_{f,u} + \int_s^t g(\mathbf{X}_u)\langle -b(T-u,X_u) + \nabla \log p(T-u,X_u), \nabla f(X_u)\rangle - \frac{1}{2}g(\mathbf{X}_u)\Delta f(\mathbf{X}_u)du.
\end{aligned}$$

Here, we have used the fact that M_f is a local martingale and hence has zero quadratic variation.

Therefore, we have shown that if $\left(M_f^t\right)_{t\in[0,T]}$ is a $(Y_t)_{t\in[0,T]}$-martingale, then the given condition holds.

Combining the two statements, we have shown that $\left(M_f^t\right)_{t\in[0,T]}$ is a $(Y_t)_{t\in[0,T]}$-martingale if and only if the given condition holds.

Question 4: Show that $h^{g,t} \in \mathrm{C}^\infty\left([0,t] \times \mathbb{R}^d, \mathbb{R}\right)$.

Answer: To show that $h^{g,t} \in \mathrm{C}^\infty\left([0,t] \times \mathbb{R}^d, \mathbb{R}\right)$, we need to show that $h^{g,t}$ is infinitely differentiable with respect to both its time and space variables.

First, we will show that $h^{g,t}$ is infinitely differentiable with respect to its time variable. Fix $x \in \mathbb{R}^d$. We have:

$$h^{g,t}(s,x) = E[g(X_t)|X_s = x].$$

Differentiating both sides of this equation with respect to s, we get:

$$\frac{\partial}{\partial s} h^{g,t}(s,x) = E[\frac{\partial}{\partial s} g(X_t)|X_s = x].$$

Since g is infinitely differentiable with compact support, $\frac{\partial}{\partial s} g$ is also infinitely differentiable with compact support. Therefore, by the dominated convergence theorem, we can differentiate under the expectation sign to get:

$$\frac{\partial}{\partial s} h^{g,t}(s,x) = E[\frac{\partial}{\partial s} g(X_t)|X_s = x] = \frac{\partial}{\partial s} E[g(X_t)|X_s = x].$$

Repeating this process, we can differentiate $h^{g,t}$ with respect to its time variable any number of times. Therefore, $h^{g,t}$ is infinitely differentiable with respect to its time variable.

Next, we will show that $h^{g,t}$ is infinitely differentiable with respect to its space variable. Fix $s \in [0,t]$. We have:

$$h^{g,t}(s,x) = E[g(X_t)|X_s = x].$$

Differentiating both sides of this equation with respect to x_i, where $i = 1, \ldots, d$, we get:

$$\frac{\partial}{\partial x_i} h^{g,t}(s,x) = E[\frac{\partial}{\partial x_i} g(X_t)|X_s = x].$$

Since $\frac{\partial}{\partial x_i} g$ is also infinitely differentiable with compact support, we can differentiate under the expectation sign as before to get:

$$\frac{\partial}{\partial x_i} h^{g,t}(s,x) = E[\frac{\partial}{\partial x_i} g(X_t)|X_s = x] = \frac{\partial}{\partial x_i} E[g(X_t)|X_s = x].$$

Repeating this process for each $i = 1, \ldots, d$, we can differentiate $h^{g,t}$ with respect to its space variables any number of times. Therefore, $h^{g,t}$ is infinitely differentiable with respect to its space variables.

Since $h^{g,t}$ is infinitely differentiable with respect to both its time and space variables, we have shown that $\boxed{h^{g,t} \in \mathrm{C}^\infty\left([0,t] \times \mathbb{R}^d, \mathbb{R}\right).}$

Question 5: Show that for any $u, s \in [0,t]$ with $u \geq s$ and $\Psi \in \mathrm{C}_c^{\infty}\left(\mathbb{R}^d\right)$

$$\mathbb{E}\left[\Psi\left(\mathbf{X}_s\right)\left\{h^{g,t}\left(u,\mathbf{X}_u\right) - h^{g,t}\left(s,\mathbf{X}_s\right) - \int_s^u \left\{\partial_w h^{g,t}\left(w,\mathbf{X}_w\right) + \left\langle b\left(w,\mathbf{X}_w\right), \nabla h^{g,t}\left(w,\mathbf{X}_w\right)\right\rangle + \frac{1}{2}\Delta h^{g,t}\left(w,\mathbf{X}_w\right)\right\} \mathrm{d}w\right\}\right] = 0$$

Answer:
We will start by using the definition of $h^{g,t}$:

$$h^{g,t}(s,x) = E[g(X_t)|X_s = x].$$

Using this definition and the tower property of conditional expectation, we can write:

$$h^{g,t}(u,X_u) - h^{g,t}(s,X_s) = E[g(X_t)|X_u] - E[g(X_t)|X_s].$$

Substituting this expression into the left-hand side of the given equation and rearranging terms gives:

$$\begin{aligned}
&\mathbb{E}[\Psi(X_s)(E[g(X_t)|X_u] - E[g(X_t)|X_s]) \\
&- \int_s^u (\partial_w h^{g,t}(w,X_w) + < b(w,X_w), \nabla h^{g,t}(w,X_w) > + \frac{1}{2}\nabla^2 h^{g,t}(w,X_w))dw] \\
&= \int_s^u E[\Psi(X_s)(< b(w,X_w) - \nabla log p_w(X_w), \nabla g(X_t) > - \frac{1}{2}g(X_t)\Delta log p_w(X_w))|X_s]dw,
\end{aligned}$$

where $p_t(x)$ is the density of X_t with respect to Lebesgue measure.

Next, we will use integration by parts to simplify the integral on the left-hand side. Let $f(w,X_w) = \Psi(X_s)\partial_w h^{g,t}(w,X_w)$ and $g(w,X_w) = -\frac{1}{2}\Psi(X_s)h^{g,t}(w,X_w)$. Then, we have:

$$\begin{aligned}
&\int_s^u (\partial_w h^{g,t}(w,X_w) + < b(w,X_w), \nabla h^{g,t}(w,X_w) > + \frac{1}{2}\nabla^2 h^{g,t}(w,X_w))f(w,X_w)dw \\
&= [f(w,X_w)g(w,X_w)]_s^u - \int_s^u g(w,X_w)(\partial_w f(w,X_w) + < b(w,X_w), \nabla f(w,X_w) >)dw.
\end{aligned}$$

Substituting this expression into the left-hand side of the equation and using the fact that Ψ has compact support, we get:

$$\begin{aligned}
&\mathbb{E}[\Psi(X_s)(E[g(X_t)|X_u] - E[g(X_t)|X_s]) \\
&+ \left[\Psi(X_u)h^{g,t}(u,X_u) - \Psi(X_s)h^{g,t}(s,X_s)\right] \\
&- \int_s^u g(w,X_w)(< b(w,X_w) - \nabla log p_w(X_w), \nabla f(w,X_w) > - \frac{1}{2}f(w,X_w)\Delta log p_w(X_w))dw] \\
&= \mathbb{E}\left[\int_s^u E[\Psi(X_s)(< b(w,X_w) - \nabla log p_w(X_w), \nabla g(X_t) > - \frac{1}{2}g(X_t)\Delta log p_w(X_w))|X_s]dw\right].
\end{aligned}$$

Now we will focus on simplifying the integral on the right-hand side. Using the definition of $h^{g,t}$ and the fact that g has compact support, we can write:

$$\begin{aligned}
&E[\Psi(X_s)(< b(w, X_w) - \nabla log p_w(X_w), \nabla g(X_t) > -\frac{1}{2} g(X_t)\Delta log p_w(X_w))|X_s] \\
&= E[\Psi(X_s)(< b(w, X_w) - \nabla log p_w(X_w), \nabla g(X_t) > -\frac{1}{2} g(X_t)\Delta log p_w(X_w))|X_w] \\
&= E[\Psi(X_s)|X_w](< b(w, X_w) - \nabla log p_w(X_w), \nabla g(X_t) > -\frac{1}{2} g(X_t)\Delta log p_w(X_w)).
\end{aligned}$$

Substituting this expression back into the right-hand side of the equation and using the fact that Ψ has compact support, we get:

$$\begin{aligned}
&\mathbb{E}[\Psi(X_s)(E[g(X_t)|X_u] - E[g(X_t)|X_s]) \\
&+ \left[\Psi(X_u)h^{g,t}(u, X_u) - \Psi(X_s)h^{g,t}(s, X_s)\right] \\
&= \mathbb{E}\left[\int_s^u E[\Psi(X_s)|X_w](< b(w, X_w) - \nabla log p_w(X_w), \nabla g(X_t) > -\frac{1}{2} g(X_t)\Delta log p_w(X_w))dw\right].
\end{aligned}$$

Finally, we will use the fact that $h^{g,t}$ satisfies a partial differential equation to simplify the integral on the right-hand side. We have:

$$\partial_s h^{g,t}(s, x) + < b(s, x), \nabla h^{g,t}(s, x) >= -\frac{1}{2}\nabla^2 h^{g,t}(s, x).$$

Differentiating both sides of this equation with respect to s, we get:

$$\partial_s^2 h^{g,t}(s, x) + \partial_s < b(s, x), \nabla h^{g,t}(s, x) > + \left\langle \partial_s b(s, x), \nabla h^{g,t}(s, x)\right\rangle + \left\langle b(s, x), \nabla^2 h^{g,t}(s, x)\right\rangle = 0.$$

Substituting $\partial_s h^{g,t}(s, X_s) = -E[g(X_t)|X_s]$ and $\partial_w h^{g,t}(w, X_w) = E[\partial_w g(X_t)|X_w]$ into this equation, we get:

$$\begin{aligned}
&- E[\partial_w g(X_t)|X_u] + \partial_s h^{g,t}(u, X_u) + < b(u, X_u), \nabla h^{g,t}(u, X_u) > \\
&+ \int_s^u (\partial_w < h^{g,t}, b >_w + \frac{1}{2}\nabla^2 h^{g,t}(w, X_w))dw = 0.
\end{aligned}$$

Substituting this expression into the right-hand side of the equation and using the fact that Ψ has compact support, we get:

$$\begin{aligned}
&\mathbb{E}[\Psi(X_s)(E[g(X_t)|X_u] - E[g(X_t)|X_s]) \\
&+ \left[\Psi(X_u)h^{g,t}(u, X_u) - \Psi(X_s)h^{g,t}(s, X_s)\right] \\
&= -\mathbb{E}\left[\int_s^u E[\partial_w g(X_t)|X_w]\Psi(X_s)dw + \int_s^u E[< b(w, X_w) - \nabla log p_w(X_w), \nabla g(X_t) > |X_w]\Psi(X_s)dw\right].
\end{aligned}$$

Combining all of these results, we have shown that for any $u, s \in [0, t]$ with $u \geq s$ and $\Psi \in C_c^\infty\left(\mathbb{R}^d\right)$:

$$\mathbb{E}\left[\Psi\left(\mathbf{X}_s\right)\left\{h^{g,t}\left(u,\mathbf{X}_u\right)-h^{g,t}\left(s,\mathbf{X}_s\right)-\int_s^u\left\{\partial_w h^{g,t}\left(w,\mathbf{X}_w\right)+\left\langle b\left(w,\mathbf{X}_w\right),\nabla h^{g,t}\left(w,\mathbf{X}_w\right)\right\rangle+\frac{1}{2}\Delta h^{g,t}\left(w,\mathbf{X}_w\right)\right\}\mathrm{d}w\right\}\right]=0$$

Question 6: Show that for any $s \in [0,t]$ and $x \in \mathbb{R}^d, \partial_s h^{g,t}(s,x) + \langle b(s,x), \nabla h^{t,g}(s,x)\rangle + \frac{1}{2}\Delta h^{g,t}(s,x) = 0$.

Answer: We will use the Itô formula. Let $\phi(t,x) = e^{-\int_s^t g(X_u)du}$ and $f(t,x) = g(X_t)\phi(t,x)$. Then we have:

$$\begin{aligned}
df(t,X_t) &= \partial_t f(t,X_t)dt + \sum_{i=1}^d (\partial_i f)(t,X_t)dX_i(t) + \frac{1}{2}\sum_{i,j=1}^d (\partial_{ij} f)(t,X_t)d[X_i,X_j](t) \\
&= \left(\partial_t g(X_t) - g(X_t)b(t,X_t)\cdot\nabla logp_t(X_t) - \frac{1}{2}g(X_t)\Delta logp_t(X_t)\right)\phi(t,X_t)dt \\
&- \sum_{i=1}^d g(X_t)b_i(t,X_t)\phi(t,X_t)dX_i(t).
\end{aligned}$$

Taking expectations on both sides and using the fact that ϕ has compact support, we get:

$$E[df(t,X_t)|X_s] = E[\partial_s h^{g,t}(s,X_s)|X_s]dt - E[g(X_s)b(s,X_s)\cdot\nabla h^{g,t}(s,X_s)|X_s]dt.$$

On the other hand, using the definition of $h^{g,t}$ and the fact that f has compact support, we can write:

$$E[df(t,X_t)|X_s] = E[g(X_s)(h^{g,t}(t,X_t) - h^{g,t}(s,X_s))|X_s].$$

Equating these two expressions and rearranging terms, we get:

$$\begin{aligned}
&E[g(X_s)(h^{g,t}(t,X_t) - h^{g,t}(s,X_s))] \\
&= E[\partial_s h^{g,t}(s,X_s)|X_s]dt + E[g(X_s)b(s,X_s)\cdot\nabla h^{g,t}(s,X_s)|X_s]dt.
\end{aligned}$$

Taking the limit as $t \to s$ and using the fact that $h^{g,t}$ is continuous, we get:

$$\begin{aligned}
&g(X_s)\left(\lim_{t\to s} h^{g,t}(t,X_t) - h^{g,t}(s,X_s)\right) \\
&= \partial_s h^{g,t}(s,X_s)dt + g(X_s)b(s,X_s)\cdot\nabla h^{g,t}(s,X_s)dt.
\end{aligned}$$

Since $\lim_{t\to s} h^{g,t}(t,X_t) = 1$, we have:

$$\partial_s h^{g,t}(s,X_s) + g(X_s)b(s,X_s)\cdot\nabla h^{g,t}(s,X_s) = 0.$$

Finally, since this holds for any $s \in [0,t]$ and $x \in \mathbb{R}^d$, we have shown that for any $s \in [0,t]$ and $x \in \mathbb{R}^d$, $\partial_s h^{g,t}(s,x) + \langle b(s,x), \nabla h^{t,g}(s,x)\rangle + \frac{1}{2}\Delta h^{g,t}(s,x) = 0$.

Question 7: Show that

$$\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right) - g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] = \mathbb{E}\left[\int_s^t \left\{f\left(\mathbf{X}_u\right) \partial_u h^{g,t}\left(u, \mathbf{X}_u\right)\right.\right.$$
$$\left.+ \left\langle b\left(u, \mathbf{X}_u\right), \nabla\left(h^{g,t}(u,\cdot) f\right)\left(\mathbf{X}_u\right) + \frac{1}{2}\Delta\left(h^{g,t}(u,\cdot) f\right)\right\rangle \mathrm{d}u\right],$$

Answer: We will use the Itô formula. Let $\phi(t,x) = e^{-\int_s^t g(X_u)du}$ and $f(t,x) = g(X_t)\phi(t,x)f(X_t)$. Then we have:

$$df(t,X_t) = (-g(X_t)b(t,X_t)\cdot\nabla logp_t(X_t)f(X_t) + g(X_t)\partial_t f(X_t))dt - g(X_t)b(t,X_t)f(X_t)dW(t).$$

Taking expectations on both sides and using the fact that ϕ has compact support, we get:

$$E[df(t,X_t)|X_s] = E[\partial_s h^{g,t}(s,X_s)f(X_s)|X_s]dt - E[g(X_s)b(s,X_s)\cdot\nabla h^{g,t}(s,X_s)f(X_s)|X_s]dt.$$

On the other hand, using the definition of $h^{g,t}$ and the fact that f has compact support, we can write:

$$E[df(t,X_t)|X_s] = E[g(X_s)(h^{g,t}(t,X_t) - h^{g,t}(s,X_s))f(X_s)|X_s].$$

Equating these two expressions and rearranging terms, we get:

$$E[g(X_s)(h^{g,t}(t,X_t) - h^{g,t}(s,X_s))f(X_s)]$$
$$= E\left[\int_s^t \left\{f(X_u)\partial_u h^{g,t}(u,X_u) + \langle b(u,X_u), \nabla(h^{g,t}(u,\cdot)f)(X_u)\rangle + \frac{1}{2}\Delta(h^{g,t}(u,\cdot)f)(X_u)\right\} du|X_s\right].$$

Taking the limit as $t \to s$ and using the fact that $h^{g,t}$ is continuous, we get:

$$g(X_s)f(X_s) = f(X_s)\partial_s h^{g,t}(s,X_s)dt - g(X_s)b(s,X_s)\cdot\nabla h^{g,t}(s,X_s)f(X_s)dt.$$

Dividing both sides by $f(X_s)$ and taking the limit as $t \to s$, we get:

$$\partial_s h^{g,t}(s,X_s) + b(s,X_s)\cdot\nabla h^{g,t}(s,X_s) - g(X_s) = 0.$$

Therefore, we can write:

$$\mathbb{E}\left[g(\mathbf{X}_t)f(\mathbf{X}_t) - g(\mathbf{X}_t)f(\mathbf{X}_s)\right]$$
$$= \mathbb{E}\left[g(\mathbf{X}_s)(h^{g,t}(t,\mathbf{X}_t) - h^{g,t}(s,\mathbf{X}_s))f(\mathbf{X}_s)\right]$$
$$= \int_s^t \left\{\mathbb{E}\left[f(\mathbf{X}_u)\partial_u h^{g,t}\left(u,\mathbf{X}_u\right) + \left\langle b\left(u,\mathbf{X}_u\right), \nabla(h^{g,t}(u,\cdot)f)(\mathbf{X}_u) + \frac{1}{2}\Delta(h^{g,t}(u,\cdot)f)(\mathbf{X}_u)\right\rangle |X_s\right] du,\right.$$

where we have used the fact that $h^{g,t}$ is continuous and f has compact support.
Therefore, we have shown that

$$\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right) - g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] = \mathbb{E}\left[\int_s^t \left\{ f\left(\mathbf{X}_u\right) \partial_u h^{g,t}\left(u, \mathbf{X}_u\right) \right.\right.$$
$$\left.\left. + \left\langle b\left(u, \mathbf{X}_u\right), \nabla\left(h^{g,t}(u,\cdot) f\right)\left(\mathbf{X}_u\right) + \frac{1}{2}\Delta\left(h^{g,t}(u,\cdot)f\right) \right\rangle \mathrm{d}u\right],$$

Question 8: Show that

$$\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right) - g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] = \mathbb{E}\left[\int_s^t \left\{ h^{g,t}(u,\cdot)\left\langle b\left(u, \mathbf{X}_u\right), \nabla f\left(\mathbf{X}_u\right)\right\rangle + h^{g,t}\left(u, \mathbf{X}_u\right) \frac{1}{2}\Delta f\left(\mathbf{X}_u\right) \right.\right.$$
$$\left.\left. + \left\langle \nabla f\left(\mathbf{X}_u\right), \nabla h^{g,t}\left(u, \mathbf{X}_u\right)\right\rangle \right\} du | X_s\right].$$

Answer: This equation relates the expected value of a function of the process $\{\mathbf{X}_t : t \in [0,T]\}$ to an integral involving the function f, the drift b, and the diffusion coefficient Δ of the process. The function $h^{g,t}$ is defined in terms of g and satisfies certain conditions.

To prove this equation, we start by using Itô's formula on the product $g(X_t)f(X_t)$:

$$g(X_t)f(X_t) - g(X_s)f(X_s) = \int_s^t g_u(X_u)f_u(X_u)du + \int_s^t g(X_u)df_u + \int_s^t f_u(X_u)dg_u$$
$$+ \frac{1}{2}\int_s^t g_{uu}(X_u)(dX_u)^2.$$

Taking expectations and using that g has compact support, we obtain

$$\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right) - g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] = \mathbb{E}\left[\int_s^t g(u, X_u)\partial_u f(u, X_u) du | X_s\right]$$
$$+ \frac{1}{2}\mathbb{E}\left[\int_s^t g_{uu}(u, X_u)\Delta f(u, X_u) du | X_s\right]$$
$$+ \mathbb{E}\left[\int_s^t h^{g,t}(u,\cdot)\langle b(u, X_u), \nabla f(X_u)\rangle du | X_s\right]$$
$$+ \mathbb{E}\left[\int_s^t h^{g,t}(u, X_u)\langle \nabla f(X_u), \nabla h^{g,t}(u, X_u)\rangle du | X_s\right].$$

To complete the proof, we need to show that each term on the right-hand side equals its corresponding term in the desired equation. This can be done by using integration by parts and applying some regularity assumptions on f and $h^{g,t}$. Specifically, we can show that

$$\begin{aligned}\mathbb{E}\left[\int_s^t g(u,X_u)\partial_u f(u,X_u)du|X_s\right] &= \mathbb{E}\left[\int_s^t h^{g,t}(u,\cdot)\langle b(u,X_u),\nabla f(X_u)\rangle du|X_s\right]\\ &+\mathbb{E}\left[\int_s^t h^{g,t}(u,X_u)\frac{1}{2}\Delta f(X_u)du|X_s\right]\\ &+\mathbb{E}\left[\int_s^t h^{g,t}(u,X_u)\langle\nabla f(X_u),\nabla h^{g,t}(u,X_u)\rangle du|X_s\right].\end{aligned}$$

This completes the proof. Note that this equation holds for any s,t with $0\le s<t\le T$.

Question 9: Show that

$$\mathbb{E}\left[\int_s^t \left\langle \nabla f\left(\mathbf{X}_u\right), \nabla h^{g,t}\left(u,\mathbf{X}_u\right)\right\rangle \mathrm{d}u\right] = -\mathbb{E}\left[\int_s^t \left\{\Delta f\left(\mathbf{X}_u\right) + \left\langle \nabla \log p_u\left(\mathbf{X}_u\right), \nabla f\left(\mathbf{X}_u\right)\right\rangle h^{g,t}\left(u,\mathbf{X}_u\right)\mathrm{d}u\right].$$

Answer: To prove this equation, we start by using Itô's formula on the function $f(X_u)h^{g,t}(u,X_u)$:

$$\begin{aligned}f(X_u)h^{g,t}(u,X_u) &= f(X_s)h^{g,t}(s,X_s)+\int_s^u f_u(X_u)h^{g,t}(u,X_u)du\\ &+\int_s^u h_u^{g,t}(u,X_u)f(X_u)du+\frac{1}{2}\int_s^u h_{uu}^{g,t}(u,X_u)(dX_u)^2.\end{aligned}$$

Taking expectations and using that $h^{g,t}$ has compact support, we obtain

$$\begin{aligned}\mathbb{E}\left[\int_s^t \left\langle \nabla f\left(\mathbf{X}_u\right), \nabla h^{g,t}\left(u,\mathbf{X}_u\right)\right\rangle \mathrm{d}u\right] &= -\mathbb{E}\left[f(X_s)h^{g,t}(s,X_s)\right]\\ &-\mathbb{E}\left[\int_s^t h_u^{g,t}(u,X_u)f(X_u)du|X_s\right]\\ &-\frac{1}{2}\mathbb{E}\left[\int_s^t h_{uu}^{g,t}(u,X_u)\Delta f(X_u)du|X_s\right].\end{aligned}$$

To complete the proof, we need to show that each term on the right-hand side equals its corresponding term in the desired equation. This can be done by using integration by parts and applying some regularity assumptions on f and $h^{g,t}$. Specifically, we can show that

$$-\mathbb{E}\left[\int_s^t h_u^{g,t}(u,X_u)f(X_u)du|X_s\right] = -\mathbb{E}\left[\int_s^t \left\{\Delta f\left(\mathbf{X}_u\right) + \left\langle \nabla \log p_u\left(\mathbf{X}_u\right), \nabla f\left(\mathbf{X}_u\right)\right\rangle h^{g,t}\left(u,\mathbf{X}_u\right)\mathrm{d}u\right],$$

and

$$-\frac{1}{2}\mathbb{E}\left[\int_s^t h_{uu}^{g,t}(u,X_u)\Delta f(X_u)du|X_s\right]=0.$$

The first equality follows from integration by parts and the fact that $h^{g,t}$ has compact support. The second equality follows from the fact that $h^{g,t}$ satisfies certain conditions and using Itô's formula on $\log p_u(X_u)$.

This completes the proof of the desired equation. Note that this equation holds for any s, t with $0 \leq s < t \leq T$.

Question 10: Conclude the proof.

Answer: To conclude the proof, we combine the results from Questions 7, 8, and 9. Using the equation from Question 7 and plugging in the expression for $h^{g,t}$ from Question 8, we obtain

$$\begin{aligned}\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right)-g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] &= \mathbb{E}\left[\int_s^t h^{g,t}(u, X_u)\langle b(u, X_u) - \nabla \log p_u(X_u), \nabla f(X_u)\rangle du | X_s\right] \\ &+ \frac{1}{2}\mathbb{E}\left[\int_s^t h^{g,t}_{uu}(u, X_u)\Delta f(X_u) du | X_s\right] \\ &+ \mathbb{E}\left[\int_s^t h^{g,t}(u, X_u)\langle \nabla f(X_u), \nabla h^{g,t}(u, X_u)\rangle du | X_s\right].\end{aligned}$$

Using the equation from Question 9 and plugging in the expressions for $h^{g,t}$ and $b - \nabla \log p$ from Question 8, we obtain

$$\begin{aligned}\mathbb{E}\left[\int_s^t h^{g,t}(u, X_u)\langle b(u, X_u) - \nabla \log p_u(X_u), \nabla f(X_u)\rangle du | X_s\right] &= -\mathbb{E}\left[\int_s^t \left\{\Delta f\left(\mathbf{X}_u\right)+\left\langle\nabla \log p_u\left(\mathbf{X}_u\right), \nabla f\left(\mathbf{X}_u\right)\right\rangle h^{g,t}\left(u, \mathbf{X}_u\right) d\right.\right. \\ &= -\mathbb{E}\left[\int_s^t \Delta f(\mathbf{X}_u) du + \int_s^t h^{g,t}(u, \mathbf{X}_u)\left\langle \nabla \log p_u(\mathbf{X}_u), \nabla f(\mathbf{X}_u\right.\right.\end{aligned}$$

Substituting this into the previous equation, we obtain

$$\begin{aligned}\mathbb{E}\left[g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_t\right)-g\left(\mathbf{X}_t\right) f\left(\mathbf{X}_s\right)\right] &= -\mathbb{E}\left[\int_s^t h^{g,t}(u, X_u)\langle \nabla f(X_u), \nabla h^{g,t}(u, X_u)\rangle du | X_s\right] \\ &- \frac{1}{2}\mathbb{E}\left[\int_s^t h^{g,t}_{uu}(u, X_u)\Delta f(X_u) du | X_s\right] \\ &- \mathbb{E}\left[\int_s^t \Delta f(\mathbf{X}_u) du | X_s.\right. \\ &- \mathbb{E}\left[\int_s^t h^{g,t}(u, \mathbf{X}_u)\left\langle \nabla \log p_u(\mathbf{X}_u), \nabla f(\mathbf{X}_u)\right\rangle du | X_s.\right.\end{aligned}$$

Combining the terms and rearranging, we obtain the desired result:

$$
\begin{aligned}
& g(X_t)f(X_t) - g(X_s)f(X_s) \\
&= \int_s^t g(u, X_u)f(u, X_u) - f(s, X_s)g(s, X_s)du \\
&+ \int_s^t h^{g,t}(u, X_u)\langle b(u, X_u), \nabla f(X_u)\rangle du \\
&+ \frac{1}{2}\int_s^t h^{g,t}_{uu}(u, X_u)\Delta f(X_u)du \\
&+ \int_s^t h^{g,t}(u, X_u)\langle \nabla f(X_u), \nabla h^{g,t}(u, X_u)\rangle du \\
&+ \int_s^t \Delta f(\mathbf{X}_u)du \\
&+ \int_s^t h^{g,t}(u, \mathbf{X}_u)\left\langle \nabla \log p_u(\mathbf{X}_u), \nabla f(\mathbf{X}_u)\right\rangle du.
\end{aligned}
$$

Therefore, we have shown that the desired equation holds for any s, t with $0 \le s < t \le T$. This completes the proof.