Exercise ①: Box - Muller and Marsaglia - bray

. Let R be a random variable with Rayleigh distribution with param 1 on Θ with uniform dist on $[0, 2\pi]$

. $R \perp\!\!\!\perp \Theta$

. $\forall r \in \mathbb{R}$. $f_R(r) = r\exp\left(-\frac{r^2}{2}\right)\mathbb{1}_{\mathbb{R}_+}(r)$

1- Let $X = R\cos(\theta)$ $Y = R\sin(\theta)$

P $\begin{cases} \text{We know that : } X, Y \text{ two random variables.} \\ \qquad h \text{ continued and bounded} \\ \text{then. } E(h(X,Y)) = \int h(x,y) f_X(x) f_Y(y)\,dx\,dy \\ \text{then } X \perp\!\!\!\perp Y \end{cases}$

For that let's consider h a continuous and bounded function

$E(h(X,Y)) = E(h(R\cos\theta, R\sin(\theta)))$

$= \int_{\mathbb{R}^2} h(r\cos(\theta), r\sin(\theta)) f_{r,\theta}(r,\theta)\,dr\,d\theta.$

$R \perp\!\!\!\perp \Theta$ so the $f_{r,\theta} = f_r \times f_\theta.$

$E(h(X,Y)) = \int_{\mathbb{R}^2} h(r\cos(\theta), r\sin(\theta)) \frac{1}{2\pi} \mathbb{1}_{[0,2\pi]}(\theta)\, r e^{-\frac{r^2}{2}}\,dr\,d\theta$

Let's consider $\varphi : (r,\theta) \longrightarrow (r\cos(\theta), r\sin(\theta))$

and $\det(J_{\varphi}) = r$

then. $E(h(x,y)) = \int_{\mathbb{R}^2} h(x,y) \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$

$= \int_{\mathbb{R}^2} h(x,y) \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dx\,dy$

$= \int_{\mathbb{R}^2} h(x,y) f_X(x) \cdot f_Y(y)\,dx\,dy$

hence from P and $\begin{cases} X \perp\!\!\!\perp Y \\ X \sim N(0,1) \\ Y \sim N(0,1) \end{cases}$

2- Let's write an algorithm to sample $N(0,1)$

we use the question ①

we have $X, Y \sim N(0,1)$ with $X = R\cos(\theta)$ $Y = R\cos(\theta)$

hence $R, \theta \longrightarrow R\cos\theta$, $R\sin(\theta)$

we have $R = F^{-1}(u)$ with $u \sim U([0,1])$
with $u = F_R(r)$

$= \int_0^r x e^{-x^2/2} dx$

$= 1 - e^{-r^2/2}$

$u = 1 - e^{-r^2/2} \Rightarrow r = \sqrt{-2\log(1-u)}$

hence we can define uniform distributions U_1, U_2
$U_1, U_2 \sim U([0,1]) \Rightarrow \begin{cases} R = \sqrt{2\log(1-U_1)} \\ \Theta = 2\pi U_2 \end{cases}$

$\Rightarrow X = R\cos(\Theta) \quad Y = R\sin(\Theta)$

3. Marsaglia-Bray algorithm

1. Let's consider $E = \{(v_1, v_2) \in [-1,1]^2 \mid v_1^2 + v_2^2 \leq 1\}$
The distribution of $V = (v_1, v_2)$ is:

$f_V(v_1, v_2) = \mathbb{1}_{v_1^2 + v_2^2 \leq 1}(v_1, v_2) \frac{1}{|E|}$

hence $V_1, V_2 \sim U_{D(0,1)}$

2. Let's set

$T_1 = \frac{V_1}{\sqrt{V_1^2 + V_2^2}} \qquad T_2 = \frac{V_2}{\sqrt{V_1^2 + V_2^2}}$

$V = V_1^2 + V_2^2$

Let's prove: $\begin{cases} T \perp\!\!\!\perp V \\ V \sim U([0,1]) \\ (T_1, T_2) \sim (\cos\Theta, \sin\Theta) \quad \Theta \sim U[0, 2\pi] \end{cases}$

Like Q1

$E(h(T_1, V)) = \int_{\mathbb{R}^2} h(t, v) f_{T_1, V}(t, v) \, dt \, dv$

$= \int_{\mathbb{R}^2} h\left(\frac{v_1}{\sqrt{v_1^2 + v_2^2}}, v_1^2 + v_2^2\right) \frac{1}{\pi} \mathbb{1}_{v_1^2 + v_2^2 \leq 1}(v_1, v_2) \, dv_1 \, dv_2$

$\begin{cases} \psi : v_1, v_2 \to (t, v) \\ \det(J_{ac}\, \psi) = \frac{1}{2\sqrt{1-t^2}} \end{cases}$

hence $E(h(T_1, V)) = \int_{\mathbb{R}^2} h(t, v) \frac{1}{\pi} \mathbb{1}_{[-1,1]}(t) \mathbb{1}_{[0,1]}(v) \, dt \, dv$

hence $V \sim U[0,1]$

we have $f_{T_1}(t) = \frac{1}{\pi\sqrt{1-t^2}} \mathbb{1}_{[-1,1]}(t)$

$$E(h(T_1)) = \int_{-1}^{1} h(t) \cdot \frac{1}{\pi\sqrt{1-t^2}}\, dt$$

we consider $\theta = \arccos t$ $\quad d\theta = \frac{-1}{\sqrt{1-t^2}}\, dt$

hence $E(h(T_1)) = \int_{-\pi}^{0} - \frac{h(\cos\theta)}{\pi}\, d\theta$

$$= \int_{0}^{2\pi} \frac{h(\cos\theta)}{2\pi}\, d\theta.$$

hence $T_1 \sim \cos\theta$ with $\theta \sim U[0,2\pi]$

similarly $f_{T_2}(t) = \frac{1}{\pi\sqrt{1-t^2}} \mathbb{1}_{[-1,1]}(t)$ $\quad \theta = \arcsin t \quad d\theta = \frac{1}{\sqrt{1-t^2}}\, dt$

hence $T_2 \sim \sin\theta$ with $\theta \sim U([0,2\pi])$

3. we set $S = \sqrt{-2\log(V_1^2+V_2^2)}$

$$X = S \frac{V_1}{\sqrt{V_1^2+V_2^2}} = S.T_1$$

$$Y = S.T_2$$

with $T_1, T_2 \sim \cos(\theta), \sin\theta \quad \theta \sim U([0,2\pi])$
from question 2 it's gaussian distribution on $[0,1]$

Exercice ②: Invariant distribution

1. Let's define $(X_n)_{n\geq 0}$ a markov chain:

- if $X_n = \frac{1}{m}$: $\begin{cases} X_{n+1} = \frac{1}{m+1} & \text{with } p = 1 - X_n^2 \quad (1.1) \\ X_{n+1} \sim U([0,1]) & \text{with } p \, X_n^2 \quad (1.2) \end{cases}$
- if not $X_{n+1} \sim U([0,1])$ ②

Let's prove that

$$P(x,A) = \begin{cases} x^2 \int_{A\cap[0,1]} dt + (1-x^2)\delta_{\frac{1}{n+1}}(A) & \text{if } x = \frac{1}{m} \\ \int_{A\cap[0,1]} dt & \text{otherwise} \end{cases}$$

First we know that the transitional kernel corresponds to the kernel of the distribution of $X_{n+1} | X_n$

Let's consider $D = \{1/m, m \in \mathbb{N}^*\}$

- Case 1: $X_n \notin D$, h a bounded function.

$$E[h(X_{n+1}|X_n)] = \int_{\mathbb{R}} h(y) f_{(X_{n+1}|X_n)}(y)\, dy$$

$$= \int_{\mathbb{R}} h(y) \mathbb{1}_{[0,1]}(y)\, dy \quad (2)$$

hence

$$P(x,A) = \int_A p(x,y)\,dy = \int_{A\cap[0,1]} dy$$

- Case 2: $X_n \in D$ $\quad X_n = \frac{1}{m} \quad m \in \mathbb{N}^*$

$$E[h(X_{n+1}|X_n = \frac{1}{m})] = \int_{\mathbb{R}} h(y) f_{X_{n+1}|X_n}(y)\,dy$$

$$= \int_{\mathbb{R}} h(y)\left[(1-X_n^2)\delta_{\frac{1}{m+1}}(y) + X_n^2 \mathbb{1}_{[0,1]}\right]dy$$

$$= \int_{\mathbb{R}} h(y)\left[\left(1-\frac{1}{m^2}\right)\delta_{1/m+1}(y) + \frac{1}{m^2}\mathbb{1}_{[0,1]}\right]dy$$

hence $P(x,A) = \frac{1}{m^2}\int_{A\cap[0,1]} dy + \left(1-\frac{1}{m^2}\right)\delta_{1/m+1}(A)$.

$$\boxed{\text{Thus} \quad P(x,A) = \begin{cases} x^2 \cdot \int_{A\cap[0,1]} dt + (1-x^2)\delta_{1/m+1}(A) & \text{if } x = 1/m \\ \int_{A\cap[0,1]} dt \end{cases}}$$

2. Let's prove that the uniform distribution is invarient with P

We denote this invariant distribution Π on $[0,1]$

hence $(\Pi P)(A) = \int_{\mathbb{R}} \Pi(dx) P(x, A)$

$$= \int_{\mathbb{R}} 1\!\!1_{[0,1]}(dx) P(x, A)$$

$$= \int_{D} 1\!\!1_{[0,1]}(dx) \left[x^2 \int_{A \cap [0,1]} dy + (1+x^q) \delta_{\frac{1}{m+1}}(A) \right]$$

$$+ \int_{\mathbb{R} \setminus D} 1\!\!1_{[0,1]}(dx) \left[\int_{A \cap [0,1]} dy \right] dx$$

but $\int_D 1\!\!1_{[0,1]}(dx) P(x, A) = 0$

hence $(\Pi P)(A) = \int_{\mathbb{R} \setminus D} 1\!\!1_{[0,1] \cap A}(x) dx = \Pi(A)$

Finally Π is invariant with the transition kernel P.

3. Let $x \notin D$ Let's compute $Pf(x) = E[f(X_1) \mid X_0 = x]$ for f a bounded measure

$$Pf(x) = E[f(X_1) \mid X_0 = x]$$

$$= \int_{\mathbb{R}} 1\!\!1_{[0,1]}(dx) P(x, dx) f(x) \quad X_1 \sim U[0,1]$$

$$= \int_{\mathbb{R}} \Pi(dx) P(x, dx) f(x)$$

$$= \Pi P f(x)$$

$$= \Pi f(x)$$

$$= 1\!\!1_{[0,1]}(f(x))$$