QUESTION 9

Given an infinite collection $A_n, n = 1, 2, ...$ of intervals of the real line, their *intersection* is defined to be $\cap_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\}$.

Give an example of a family of intervals $A_n, n = 1, 2, ...$ such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n = \varnothing$. Prove that your example has the stated property.

Such an example is the family of intervals $A_n = (0, \frac{1}{n})$.

Proof of 1st property: by inspection.

It is clear that $(0, \frac{1}{n+1}) \subset (0, \frac{1}{n})$ since both intervals have the same left-open endpoint 0, and considering the right-open endpoints, $\frac{1}{n+1} < \frac{1}{n}$.

Therefore $(\forall n)(A_{n+1} \subset A_n)$.

Proof of 2nd property: by consideration of $\lim_{n\to\infty} \frac{1}{n}$.

Assume $\lim_{n\to\infty} \frac{1}{n} = L$. Therefore for every real number $\epsilon > 0$, there exists a natural number N such that:

$$(\forall n > N)(|\frac{1}{n} - L| < \epsilon) \tag{1}$$

It is clear that

$$(\forall n > N)(\frac{1}{n} < \frac{1}{N})$$

and

$$(\forall \epsilon \in \mathbb{R}, \epsilon > 0)(\frac{1}{\lceil \frac{1}{\epsilon} \rceil} \leq \epsilon)$$

Therefore, choosing $N = \lceil \frac{1}{\epsilon} \rceil$,

$$(\forall n > N)(|\frac{1}{n} - 0| < \frac{1}{\lceil \frac{1}{\epsilon} \rceil} \leq \epsilon)$$

This is the definition of the limit in (1) with $L = 0$. Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore $A_\infty = (0, 0) = \varnothing$, and hence $\cap_{n=1}^{\infty} A_n = \varnothing$. $\square$