QUESTION 4

Theorem every odd natural number is one of the forms $4n+1$ or $4n+3$ where $n \in \mathbb{Z}$.

Proof: by induction, considering each case.

Base case: $1 = 4n + 1$ where $n = 0$.

Inductive step: If k_i is an odd natural number, the next odd natural number is given by $k_{i+1} = k_i + 2$. There are two cases to consider.

(i) Assume $k_i = 4n + 1$. Then:

$$\begin{aligned} k_{i+1} &= k_i + 2 \\ &= 4n + 1 + 2 && \text{(by induction hypothesis)} \\ &= 4n + 3 \end{aligned}$$

Therefore, k_{i+1} is of the required form.

(ii) Assume $k_i = 4n + 3$. Then:

$$\begin{aligned} k_{i+1} &= k_i + 2 \\ &= 4n + 3 + 2 && \text{(by induction hypothesis)} \\ &= 4n + 4 + 1 \\ &= 4(n+1) + 1 \end{aligned}$$

Therefore, k_{i+1} is again of the required form.

Therefore every odd natural number is one of the forms $4n+1$ or $4n+3$ where $n \in \mathbb{Z}$. $\square$