QUESTION 3

Theorem for any integer n, $n^2 + n + 1$ is odd.

Proof: by consideration of even/odd cases.

(i) Assume n is even. Then $n = 2k$.

$$\begin{aligned} n^2 + n + 1 &= (2k)^2 + 2k + 1 \\ &= 4k^2 + 2k + 1 \\ &= 2(2k^2 + k) + 1 \end{aligned}$$

Therefore, n is even $\Rightarrow n^2 + n + 1$ is odd.

(ii) Assume n is odd. Then $n = 2k + 1$.

$$\begin{aligned} n^2 + n + 1 &= (2k+1)^2 + (2k+1) + 1 \\ &= 4k^2 + 4k + 1 + 2k + 1 + 1 \\ &= 4k^2 + 6k + 2 + 1 \\ &= 2(2k^2 + 3k + 1) + 1 \end{aligned}$$

Therefore, n is odd $\Rightarrow n^2 + n + 1$ is odd.

Therefore for any integer n, $n^2 + n + 1$ is odd. $\square$