Generative Grammars

Generative Grammars Informally

- Fix a vocabulary
  - A set of symbols
  - Some of these symbols, called **terminals**, play the tokens of the output stream of lexical analysis

- E.g. take the vocabulary {S, a, b} where “a” and “b” are terminals

Generative Grammars

- One non-terminal symbol of the vocabulary is chosen as **start symbol**

- E.g., S in {S, a, b}

Generative Grammars

- Fix a set of **productions**
  - Rules for rewriting strings into strings
  - Constraint:
    - the string to be replaced must contain at least a non-terminal

- E.g. $\{S \to aSb,\ S \to ab\}$

Generative Grammars

- These are the ingredients of a **generative grammar**
- A **language** of words of terminals can be generated from the start symbol:
  - Apply the rewriting rules in any possible way, as many times as possible
  - Each rewriting is called a **derivation step**

Notation for the derivation relation

$\{ \quad \to aSb,\ S \to ab\}$

$S \Rightarrow a\,b$

- Is a one-step derivation from S
- "ab" is made up of terminals only
- Hence "ab" **belongs** to the language generated by the given grammar

$$\{S \rightarrow aSb,\ S \rightarrow ab\}$$

$S \Rightarrow aSb \Rightarrow aabb$

- Is a two-step derivation from S
- "aabb" is made up of terminals only
- Hence "aabb" **belongs** to the language generated by the given grammar

$\{S \rightarrow aSb,\ S \rightarrow ab\}$

$S \Rightarrow aSb \Rightarrow aaSbb$

- Is a two-step derivation of a string from S
- But "aaSbb" contains a non-terminal
- Hence "aaSbb" **does not belong** to the language generated by the given grammar

$\{S \rightarrow aSb,\ S \rightarrow ab\}$

- Which is the language generated by this grammar?

- $\{a^n b^n \mid n>0\}$

Notation

- Capital letters for non-terminals

Convention

- Special character epsilon (ε) used to denote the empty word

- Length of ε is 0
  - $\varepsilon \equiv \varepsilon\ \varepsilon$
  - $\varepsilon \equiv b^0$ for every terminal b

Example 1

$S \rightarrow aAb$

$aA \rightarrow aaAb$

$A \rightarrow \varepsilon$

- Generated language: $\{a^n b^n \mid n>0\}$
- OBSERVE: Different grammars can generate the same language

Example 2

$S \to AB$

$A \to aA$

$A \to a$

$B \to Bb$

$B \to b$

- Generated language: $\{a^n b^m \mid n,m>0\}$

Example 3

$S \rightarrow aSBc$

$S \rightarrow abc$

$cB \rightarrow Bc$

$bB \rightarrow bb$

- Generated language: $\{a^n b^n c^n \mid n>0\}$

Example 4

$S \to AB$

$A \to a$

- Generated language: $\emptyset$

Example 5

$S \rightarrow \varepsilon$

- Generated language: $\{\varepsilon\}$
- $\{\varepsilon\} \neq \emptyset$

Example 6

$S \to aSb$

$S \to \varepsilon$

- Generated language:
  - $\{a^n b^n \mid n>0\} \cup \{\varepsilon\} = \{a^n b^n \mid n \geq 0\}$

Examp

Notation for more productions with same left-hand side

$S \to CD$

$C \to aCA \mid bCB$

$AD \to aD$

$BD \to bD$

$Aa \to aA$

$Ab \to bA$

$Ba \to aB$

$Bb \to bB$

$C \to \varepsilon$

$D \to \varepsilon$

- Generated language?

Example 7: Derivation 1

$S \to CD$

$C \to aCA \mid bCB$

$AD \to aD$

$BD \to bD$

$Aa \to aA$

$Ab \to bA$

$Ba \to aB$

$Bb \to bB$

$C \to \varepsilon$

$D \to \varepsilon$

$S \Rightarrow CD$

$CD \Rightarrow D$

$D \Rightarrow \varepsilon$

Example 7: Derivation 2

$S \to CD$

$C \to aCA \mid bCB$

$AD \to aD$

$BD \to bD$

$Aa \to aA$

$Ab \to bA$

$Ba \to aB$

$Bb \to bB$

$C \to \varepsilon$

$D \to \varepsilon$

$S \Rightarrow CD$

$CD \Rightarrow aCAD$

$aCAD \Rightarrow aCaD$

$aCaD \Rightarrow aaD$

$aaD \Rightarrow aa$

Example 7: Derivation 3

$S \rightarrow CD$

$C \rightarrow aCA \mid bCB$

$AD \rightarrow aD$

$BD \rightarrow bD$

$Aa \rightarrow aA$

$Ab \rightarrow bA$

$Ba \rightarrow aB$

$Bb \rightarrow bB$

$C \rightarrow \varepsilon$

$D \rightarrow \varepsilon$

$S \Rightarrow CD$

$CD \Rightarrow aCAD$

$aCAD \Rightarrow abCBAD$

$abCBAD \Rightarrow abCBA$

$abCBA \Rightarrow abBA$

Example 7: Derivation 4

$S \rightarrow CD$

$C \rightarrow aCA \mid bCB$

$AD \rightarrow aD$

$BD \rightarrow bD$

$Aa \rightarrow aA$

$Ab \rightarrow bA$

$Ba \rightarrow aB$

$Bb \rightarrow bB$

$C \rightarrow \varepsilon$

$D \rightarrow \varepsilon$

$S \Rightarrow CD$

$CD \Rightarrow aCAD$

$aCAD \Rightarrow abCBAD$

$abCBAD \Rightarrow abCBaD$

$abCBaD \Rightarrow abCaBD$

$abCaBD \Rightarrow abCabD$

$abCabD \Rightarrow ababD$

$ababD \Rightarrow abab$

Generative Grammars Formally

- A grammar is a tuple
  (V,T,S,P)

  - V vocabulary of terminals and non-terminals
  - T set of terminals
  - S start symbol in (V\T)
  - P set of productions

Not

Zero or more repetitions of elements in the base set

- Uppercase, early in the alphabet
  - $A, B, \ldots. \in (V \setminus T)$
- Uppercase, late in the alphabet
  - $X, Y, \ldots \in V$
- Lowercase, early in the alphabet
  - $a, b, \ldots. \in T$
- Lowercase, early in Greek alphabet
  - $\alpha, \beta, \ldots \in V^*$
- Strings of terminals
  - $w, w_0, \ldots.$

Productions

- General form:

One or more repetitions of elements in the base

$$\delta \rightarrow \beta$$

- $\delta \in V^+$
- δ contains at least a non-terminal
- δ called **driver** of the production
- β called **body** of the production

Generated Languages

- $G = (V,T,S,P)$

- **$L(G) = \{ w \mid w \in T^* \text{ and } S \Rightarrow^* w \}$**

T* because w may just be ϵ

Hierarchy of Grammars

- Depending on the shape of productions

- **Context-free grammars**, or just **free grammars**:

$$A \rightarrow \beta$$

Context-free Languages

- L is a **context-free language**
- Iff
- There exists a context-free grammar G such that L=L(G)

Context-free Languages

Our focus

Canonical Derivations

- **Rightmost (Leftmost)** derivation step:
  - Replace the rightmost (**leftmost**) non-terminal
- **Canonical derivations** of words in the language:
  - Either every step is rightmost
  - Or every step is leftmost

Derivation Trees

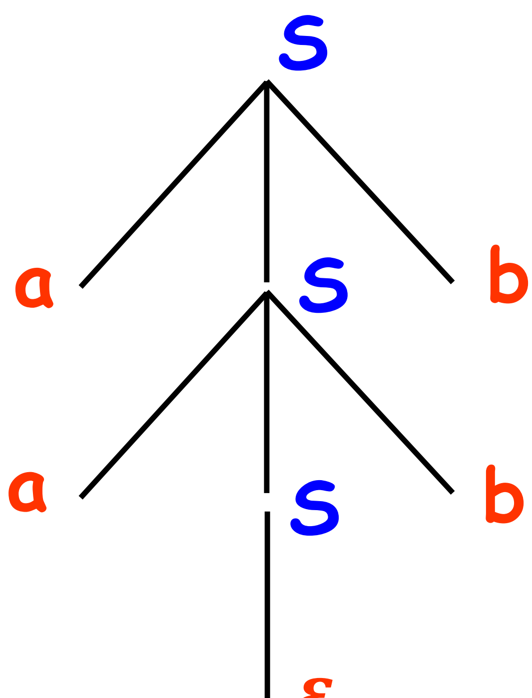
- Start symbol is the root
- For every derivation step under the production
- $A \rightarrow X_1\ X_2\ \ldots\ X_n$
- Generate children $X_1\ X_2\ \ldots\ X_n$ for node A
- Terminals are the leaves (and so is ε)

Derivation Trees

- The derived word is at the **frontier** of the tree

Example

$S \to aSb \mid \varepsilon$



Ambiguity in Natural Languages

- L'uomo guarda la donna con il binocolo

Ambiguity

- Grammar G is **ambiguous**
- Iff
- There exists $w \in L(G)$ that can be generated by two distinct canonical derivations, either both rightmost or both leftmost

Arithmetic Expressions

$E \rightarrow E{+}E \mid E{*}E \mid n$

- Ambiguous?

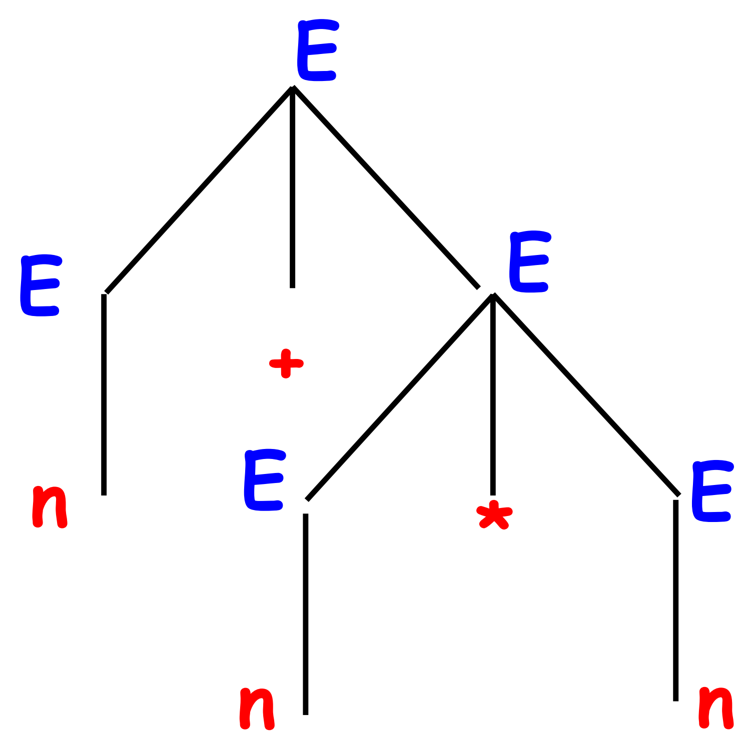
Arithmetic Expressions

$E \rightarrow E{+}E \mid E{*}E \mid n$

- Take $w = n{+}n{*}n$

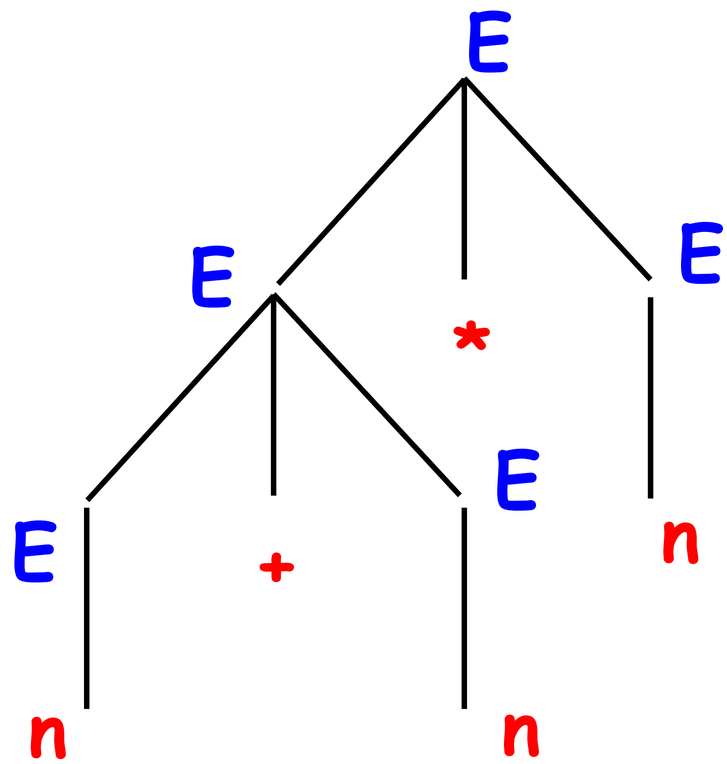
Arithmetic Expressions

E
$\Rightarrow E + E$
$\Rightarrow n + E$
$\Rightarrow n + E * E$
$\Rightarrow n + n * E$
$\Rightarrow n + n * n$

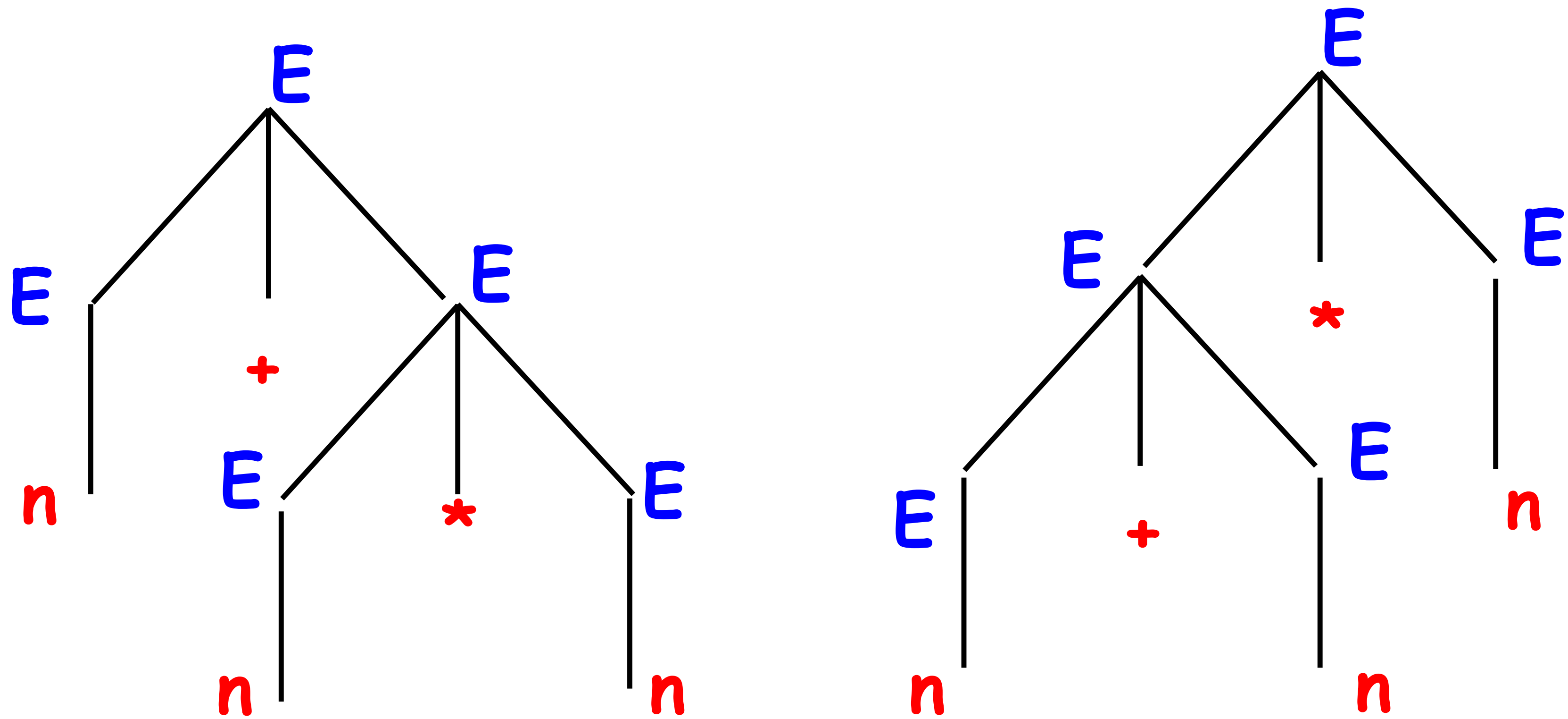


Arithmetic Expressions

But also



Arithmetic Expressions



Arithmetic Expressions

$E \rightarrow E+E \mid E*E \mid n$

- Ambiguous!

Dangling Else

$S \rightarrow$ if b then S | if b then S else S | other

- Ambiguous?

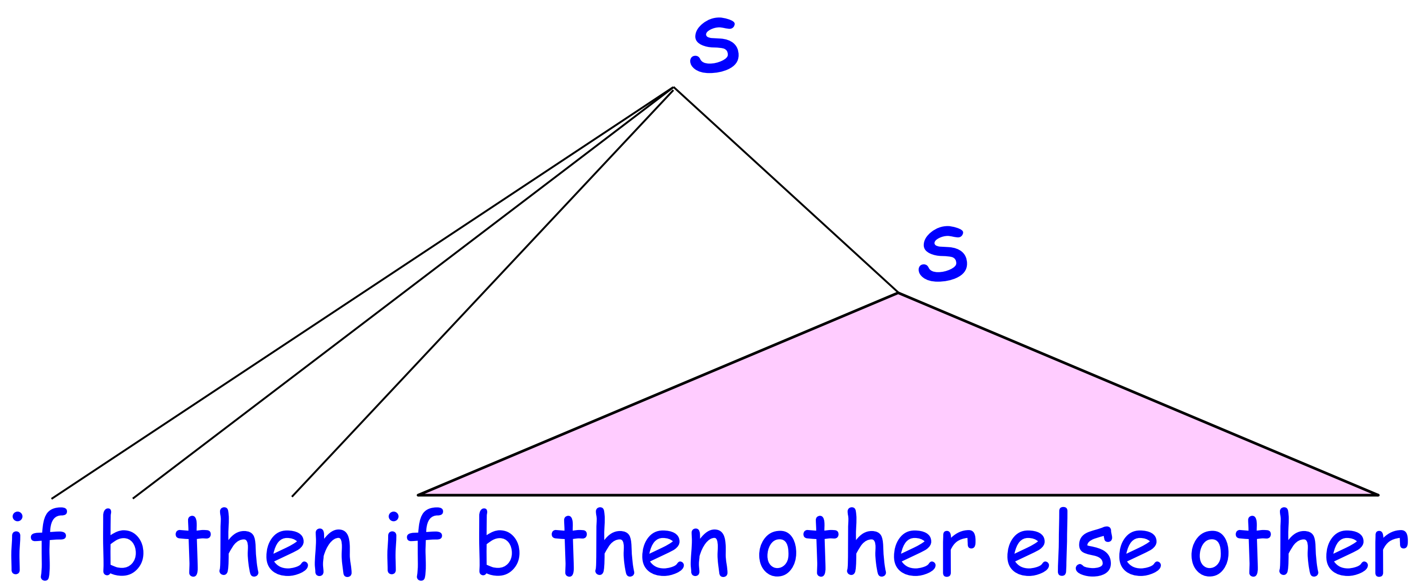
Dangling Else

$S \rightarrow$ if b then S | if b then S else S | other

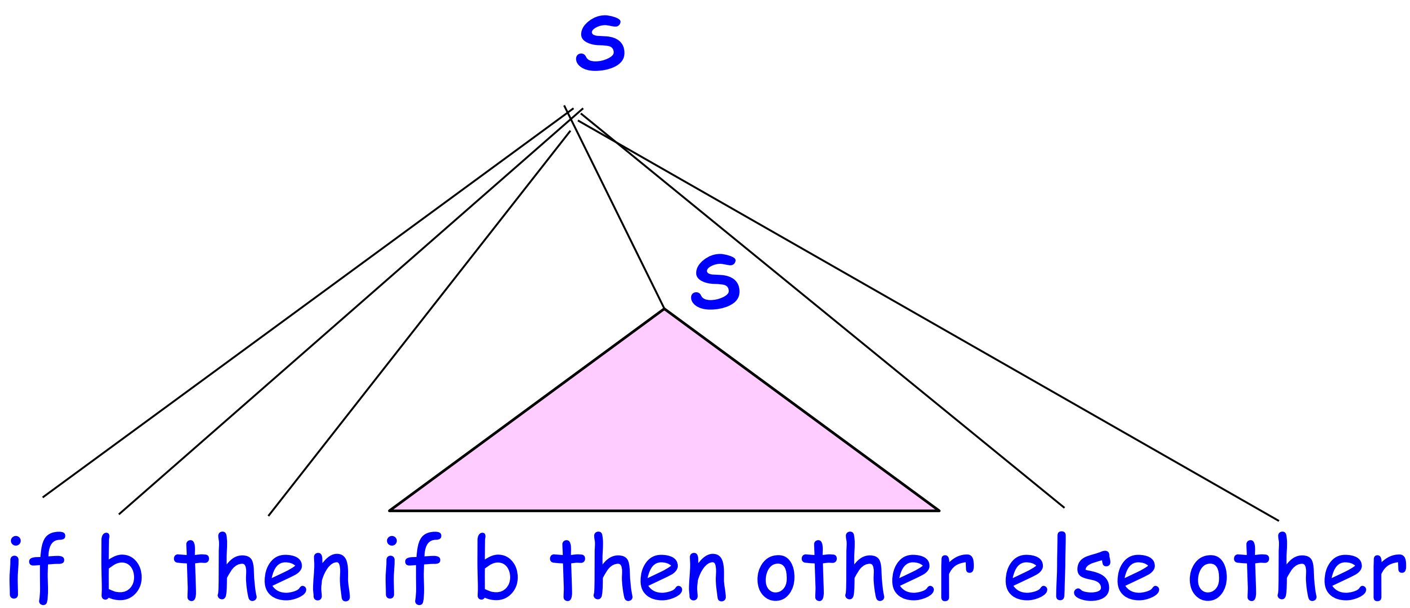
- Take
- w = if b then if b then other else other

- Which "then" matches "else"?

Dangling Else



Dangling Else



Dangling Else

$S \rightarrow$ if b then S | if b then S else S | other

- Ambiguous!

Observation

- Ambiguity is undecidable

- No algorithm can be designed to decide whether a grammar is ambiguous or not