1-Modeling

1.4-Set theory

UNIVERSITÀ DI TRENTO

DataScientia
Unitas per Varietatem

1.4-Set Theory

- Inverse relation
- Relation properties
- Equivalence relation
- Order relation

1.4.2-Relations

Lecture index

Relations

Definition 1 (Relation) A relation R from the set A to the set B is a subset of the Cartesian product of A and B: $R \subseteq A \times B$.
If $(x, y) \in R$, then we will write xRy and we say 'x is R-related to y'.

Proposition A binary relation on a set A is a subset $R \subseteq A \times A$.

Given a relation R from A to B:

- the domain of R is the set $Dom(R) = \{a \in A | \text{there exists a } b \in B, aRb\}$

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Given a relation R from A to B:

- the domain of R is the set $Dom(R) = \{a \in A | \text{there exists a } b \in B, aRb\}$
- the co-domain of R is the set $Cod(R) = \{b \in B | \text{there exists an } a \in A, aRb\}$

Relations

Example 1 Given $A = \{1, 2, 3, 4\}$, $B = \{a, b, d, e, r, t\}$ and aRb iff in the Italian name of a there is the letter b, then $R = \{(2, d), (2, e), (3, e), (3, r), (3, t)\ , (4, a), (4, r), (4, t)\}$

Example 2 Given $A = \{3, 5, 7\}$, $B = \{2, 4, 6, 8, 10, 12\}$ and aRb iff a iff a is a divisor of b, then $R = \{(3, 6), (3, 12), (5, 10)\}$

Inverse relation

Definition 2 (Inverse relation) Let R be a relation from A to B. The inverse relation of R is the relation $R^{-1} \subseteq B \times A$ where

$$R^{-1} = \{(b, a) | (a, b) \in R\}$$

Relation Properties

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- **transitive** iff aRb and bRc imply aRc for all $a, b, c \in A$
- **anti-symmetric** iff aRb and bRa imply $a = b$ for all $a, b \in A$

Equivalence relation

Definition 3 (Equivalence relation) Let R be a binary relation on a set A. R is an equivalence relation iff it satisfies all the following properties:

- reflexive
- symmetric
- transitive

Remark
An equivalence relation is usually denoted with $\sim$ or $\equiv$

Set partition

Definition 4 (Partition of a set) Let A be a set, a partition of A is a family F of non-empty subsets of A so that:

- the subsets are pairwise disjoint
- the union of all subsets is the set A

Remark
Each element of A belongs to exactly one subset in F

Equivalence class

Definition 5 (Equivalence class) Let A be a set and $\equiv$ an equivalence relation on A, given an x $\in$ A we define equivalence class X the set of elements x' $\in$ A s.t. x' $\equiv$ x, formally:

$$X = \{x' | x' \equiv x\}$$

Remark

Any element x is sufficient to obtain the equivalence class X, which is denoted also with [x].

x $\equiv$ x' implies [x]=[x']=X

Quotient set

Definition 6 (Quotient set) We define quotient set of A with respect to an equivalence relation $\equiv$ as the set of equivalence classes defined by $\equiv$on A, and denote it with $A/\equiv$

Theorem 1 Given an equivalence relation $\equiv$ on A, the equivalence classes defined by $\equiv$ on A are a partition of A. Similarly, given a partition on A, the relation R defined as xRx' iff x and x' belong to the same subset, is an equivalence relation on A.

Equivalence class (example)

Example 3 (Parallelism relation) Two straight lines in a plane are parallel if they do not have any point in common or if they coincide. The parallelism relation $||$ is an equivalence relation since it is:

- reflexive: $r||r$
- symmetric: $r||s$ implies $s||r$
- transitive $r||s$ and $s||t$ imply $r||t$

We can thus obtain a partition in equivalence classes: intuitively, each class represent a direction in the plane.

Order relation

Definition 7 (Order) Let A be a set and R be a binary relation on A. R is an order (partial), usually denoted with $\leq$, if it satisfies the following properties:

- reflexive $a \leq a$
- anti-symmetric $a \leq b$ and $b \leq a$ imply $a = b$
- transitive $a \leq b$ and $b \leq c$ imply $a \leq c$

If the relation holds for all $a, b \in A$ then it is a total order.

A relation is a strict order, denoted with <, if it satisfies the following properties:

- transitive $a < b$ and $b < c$ imply $a < c$
- for all $a, b \in A$ either $a < b$ or $b < a$ or $a = b$

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