Computational Logic
L7.X.1 PL Exercises

Adolfo Villafiorita
Fausto Giunchiglia

UNIVERSITY OF TRENTO

Outline

- Informal 2 Formal
  - Rules of the thumb
  - Examples
- Reasoning
  - Truth Tables
  - Tableau

Formalizing NL

- It is the case that P:

  P

- It is not the case that P:

  $\neg P$

- P and Q. P but Q. Although P, Q:

  $(P \wedge Q)$

- P or Q. Either P or Q:

  $(P \vee Q)$

- P if and only if Q:

  $(P \leftrightarrow Q)$

- If P, then Q:

  $(P \rightarrow Q)$

- P if Q:

  $(Q \rightarrow P)$

- Neither P nor Q:

  $\neg (P \vee Q)$

  $\neg P \wedge \neg Q$

- It is not the case that both P and Q:

  $\neg (P \wedge Q)$

- Both not P and not Q:

  $(\neg P \wedge \neg Q)$

Formalizing NL

- It is the case that P:

  P

- It is not the case that P:

  not P

- P and Q. P but Q. Although P, Q:

  (P and Q)

- P or Q. Either P or Q:

  (P or Q)

- P if and only if Q:

  (P iff Q)

- If P, then Q:

  (P imp Q)

- P if Q:

  (Q imp P)

- Neither P nor Q:

  not (P or Q)
  not P and not Q

- It is not the case that both P and Q:

  not (P and Q)

- Both not P and not Q:

  (not P and not Q)

Meaning of Implication

"The meaning of the implication operator (→) may appear unintuitive, since we must get used to the notion that "falsehood implies everything." We should not confuse (→) with causation. That is, p→q may be true, yet p does not "cause"q in any sense. For example, let p be "it is raining," and q be "Sue takes her umbrella." We might assert that p→q is true. It might even appear that the rain is what caused Sue to take her umbrella. However, it could also be true that Sue is the sort of person who doesn't believe weather forecasts and prefers to carry an umbrella at all times"
(http://infolab.stanford.edu/~ullman/focs/ch12.pdf)

Meaning of Implication

- Use implication to restrict conditions: “In the situations in which … then …”
- Probably simpler to grasp if you reason with the negation (It is not the case that the consequence is false and the premiss is true)

Skip to the next slide, unless…

- Another difficulty might come from “unless”
- Consider the following:
  - Carla will go to the party (C) unless Stephen goes (S)
  - How will you formalize it?

… solution

- Consider the following:

S imp not C

not S imp C

- What happens if Stephen does not go to the party?

Formalizing NL

Let us consider a propositional language where ph means "Paola is happy", pp means "Paola paints a picture", and rh means "Renzo is happy".

Formalize the following sentences:

- "if Paola is happy and paints a picture then Renzo is not happy”
- "if Paola is happy, then she paints a picture”
- "Paola is happy only if she paints a picture”

Formalizing NL

Let us consider a propositional language where ph means "Paola is happy", pp means "Paola paints a picture", and rh means "Renzo is happy".

Formalize the following sentences:

- "if Paola is happy and paints a picture then Renzo is not happy”
  $(ph \land pp) \rightarrow \neg\, rh$
- "if Paola is happy, then she paints a picture”
  $ph \rightarrow pp$
- "Paola is happy only if she paints a picture”
  $\neg\, (ph \land \neg\, pp)$ or, equivalently, $ph \rightarrow pp$

Formalizing NL

- "If Davide comes to the party then Bruno and Carlo come too"
- "Carlo comes to the party only if Angelo and Bruno do not come"
- "If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"
- "Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"
- "A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"
- "Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes”

Formalizing NL

- "If Davide comes to the party then Bruno and Carlo come too"
  $D \rightarrow (B \wedge C)$
- "Carlo comes to the party only if Angelo and Bruno do not come"

  $C \rightarrow (\neg A \wedge \neg B)$
- "If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"

  $D \rightarrow (\neg C \rightarrow A)$
- "Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"

  $(C \rightarrow \neg D) \wedge (D \rightarrow \neg B)$
- "A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"

  $A \rightarrow (\neg B \wedge \neg C \rightarrow D)$
- "Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes”

  $(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \rightarrow (D \rightarrow C))$

Valid or not?

Validity $\supset$ Satisfiability $\supset$ not Unsatisfiability

Unsatisfiability $\supset$ not Satisfiable $\supset$ not Valid

if A is	then $\neg$ A is
Valid	Unsatisfiable
Satisfiable	not Valid
not Valid	Satisfiable
Unsatisfiable	Valid

Properties of Entailment

- Deduction Theorem:

  $\Theta \cup \{\varphi\} \models \psi$ iff $\Theta \models \varphi \to \psi$

- Contraposition Theorem

  $\Theta \cup \{\varphi\} \models \neg\psi$ iff $\Theta \cup \{\psi\} \models \neg\varphi$

- Contradiction Theorem

  $\Theta \cup \{\varphi\}$ is unsatisfiable iff $\Theta \models \neg\varphi$

List the models for the propositions

- List the models for the following formulas:
  1. $\boxed{A \wedge \neg B}$
  2. $(A \wedge B) \vee (B \wedge C)$
  3. $(A \vee B) \rightarrow C$
  4. $(\neg A \quad B) \quad C$

A	B	$A \wedge \neg B$
T	T	F
T	F	T
F	T	F
F	F	F

MODEL →

Truth Tables and Validity (1)

Use the truth tables method to determine whether $(p \to q) \lor (p \to \neg q)$ is valid.

p	q	$p \to q$	$\neg q$	$p \to \neg q$	$(p \to q) \lor (p \to \neg q)$
T	T	T	F	F	**T**
T	F	F	T	T	**T**
F	T	T	F	T	**T**
F	F	T	T	T	**T**

The formula is valid since it is satisfied by every interpretation.

Truth Tables and Validity (2)

Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with *F*) is valid.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	F
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

There exists an interpretation satisfying *F*, thus *F* is satisfiable.

Problem formalization (2)

Brown, Jones, and Smith are suspected of a crime. They testify as follows:

- Brown: “Jones is guilty and Smith is innocent”.
- Jones: “If Brown is guilty then so is Smith”.
- Smith: “I'm innocent, but at least one of the others is guilty”.

Formalization:

1. Choose an appropriate propositional representation for the sentences: Let B, J, and S be the statements “Brown is guilty”, “Jones is guilty”, and “Smith is guilty”, respectively.
2. Express the sentences (depositions) of each suspect as a PL formula. The three statements can be expressed as $J \land \neg S$, $B \rightarrow S$, and $\neg S \land (B \lor J)$

Problem formalization (2)

1. Write a truth table for the three testimonies.

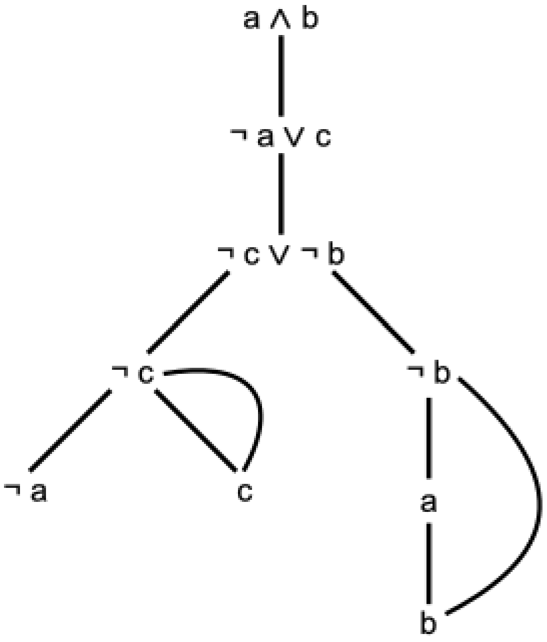
	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

Problem formalization (2)

- Use the truth table to answer the following questions:
  a) Are the three testimonies satisfiable?
  [Yes, assignment (6) makes them all true]
  b) The testimony of one of the suspects follows from that of another. Which from which?
  [$J \land \neg S \models \neg S \land (B \lor J)$]
  c) Assuming that everybody is innocent, who committed perjury?
  [Everybody is innocent corresponds to assignment (8), and in this case the statements of Brown and Smith are false.]
  d) Assuming that all testimonies are true, who is innocent and who is guilty?
  [Assuming that all testimonies are true corresponds to assignment (6). In this case Jones is guilty and the others are innocents.]

Tableau

- A method to systematically decompose a formula
- The decomposition process identifies constraints on the possible interpretation of the propositional letters of the formulas
- This allows to build interpretation functions and demonstrate validity (if the negated is unsatisfiable)



From Wikimedia Commons, the free media repository

Tableau Rules

$A \wedge B$ A B	$A \vee B$ $A \qquad B$	$A \rightarrow B$ $\neg A \qquad B$
$A \leftrightarrow B$ $A \wedge B \qquad \neg A \wedge \neg B$	$\neg\neg A$ A	$\neg(A \wedge B)$ $\neg A \qquad \neg B$
$\neg(A \vee B)$ $\neg A$ $\neg B$	$\neg(A \rightarrow B)$ A $\neg B$	$\neg(A \leftrightarrow B)$ $A \wedge \neg B \qquad \neg A \wedge B$

Tableau: Rules of the “thumb”

- All branches closed
  - Formula is unsatisfiable (no models)
- One branch open:
  - Formula is satisfiable (at least one model)
- All branches open:
  - **Formula is satisfiable**

- **Problem 1: Is ψ is satisfiable?**

  Approach:
  - build a tableau for ψ
  - find an open branch.
- **Problem 2: is ψ is valid?**

  Approach
  - Build a tableau for ¬ψ
  - Check all branches are closed

Tableaux

Q. Prove the following: $\vDash (q \vee p) \rightarrow (p \vee q)$

NOTE: "$\supset$" is a (widely) used alternative representation of "$\rightarrow$"

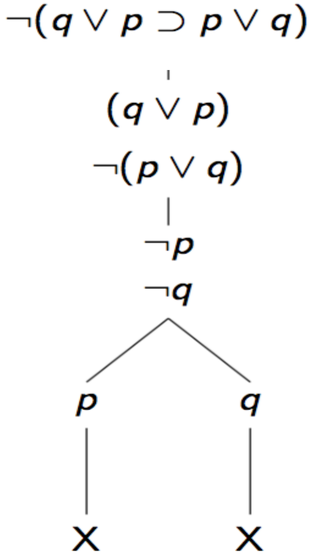
A. We demonstrate that the negation is not satisfiable (= there is no model satisfying the negated thesis)

All branches closed.

Rules of the thumb:

- "and": same node; "or" branches
- transform formulas so that truth is preserved
- In general, prefer "and"s over "or"s

Tableaux




```
1. not (q or p) imp (p or q)

1.1 (q or p)
    not (p or q)

1.1.1 not p
      not q

1.1.1.1 p

1.1.1.2 q

1.1.1.1.1 CLOSED

1.1.1.2.1 CLOSED
```

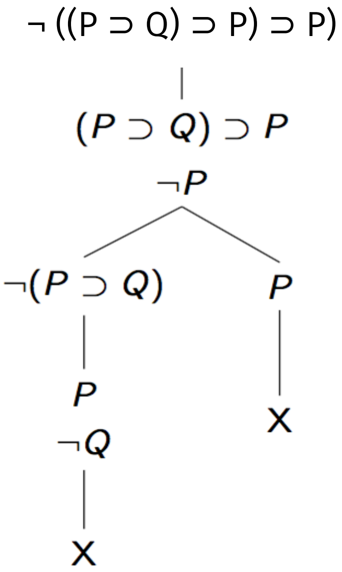
Tableaux

Q. Prove the following: $\models (P \rightarrow (Q \rightarrow P)) \rightarrow P$

A. We demonstrate that the negation is not satisfiable (= there is no model satisfying the negated thesis)

All branches closed.

Tableaux




```
1. not ((p imp q) imp p) imp p)

1.1 (p imp q) imp p
    not p

1.1.1 not (p imp q)

1.1.2 p

1.1.1.1 p
        not q

1.1.2.1 CLOSED

1.1.1.1.1 CLOSED
```

Exercises

- Are these entailments valid?

  $\{P \lor Q, \neg P\} \models Q$

  $\{P \rightarrow Q, Q\} \models P$

  $\{P \rightarrow Q\} \models \neg(Q \rightarrow P)$

  $\{P \rightarrow (\neg H \rightarrow C), P \rightarrow \neg H\} \models P \rightarrow C$

  $\{J \lor Y, Y \rightarrow (\neg S \rightarrow C), \neg J \rightarrow S\} \models C$

Exercise

- You have a pet chameleon. You adore it, but you don't know it well. After doing some research in the library, here's what you have discovered:
  1. your chameleon can be in three moods: it can be happy, upset, or indignant;
  2. it can be in only one mood at any given time;
  3. in each mood, a chameleon takes on a specific color: in particular, when a chameleon is indignant or upset it turns purple;
  4. chameleon always become upset when they are hungry;
  5. chameleon always become indignant when they are busy eating and you suddenly start to pet them;
  6. if the food bowl is empty then you can be sure your chameleon is hungry;
  7. if the food bowl is not empty and a chameleon is hungry, it immediately engages itself in busy eating;
  8. when a chameleon has slept enough it becomes happy.
- Given the above knowledge:

  (a) Suppose you see that the food bowl is empty: show that it follows from the data above that your chameleon is not happy;

  (b) if your chameleon is not purple when you are petting it, is the food bowl empty? (show that your answer follows from the data above)
- Source: http://disi.unitn.it/~bernardi/Courses/Logic06/oct_27_pl.pdf

More Exercises

- Consider the following problem: are the following formulae valid Decide on it with the (analytic) tableau procedure.
  1. $(p \to q) \to (\neg q \to \neg p)$
  2. $(q \to p) \to p$
  3. $(\bot \land \neg\neg q) \lor q$

- Use the tableau procedure and: (1) prove that the following formulae are satisfiable; (2) build a model for each of them.
  1. $\neg(p \to (p \land q))$
  2. $(p \lor (p \land q))$
  3. $(\bot \land \neg\neg q) \lor q$
  4. $(p \lor q) \land (\neg p \lor \neg q)$

CNF

$\mathrm{CNF}(p)$	=	p if $p \in$ **PROP**
$\mathrm{CNF}(\neg p)$	=	$\neg p$ if $p \in$ **PROP**
$\mathrm{CNF}(\phi \supset \psi)$	=	$\mathrm{CNF}(\neg\phi) \otimes \mathrm{CNF}(\psi)$
$\mathrm{CNF}(\phi \wedge \psi)$	=	$\mathrm{CNF}(\phi) \wedge \mathrm{CNF}(\psi)$
$\mathrm{CNF}(\phi \vee \psi)$	=	$\mathrm{CNF}(\phi) \otimes \mathrm{CNF}(\psi)$
$\mathrm{CNF}(\phi \equiv \psi)$	=	$\mathrm{CNF}(\phi \supset \psi) \wedge \mathrm{CNF}(\psi \supset \phi)$
$\mathrm{CNF}(\neg\neg\phi)$	=	$\mathrm{CNF}(\phi)$
$\mathrm{CNF}(\neg(\phi \supset \psi))$	=	$\mathrm{CNF}(\phi) \wedge \mathrm{CNF}(\neg\psi)$
$\mathrm{CNF}(\neg(\phi \wedge \psi))$	=	$\mathrm{CNF}(\neg\phi) \otimes \mathrm{CNF}(\neg\psi)$
$\mathrm{CNF}(\neg(\phi \vee \psi))$	=	$\mathrm{CNF}(\neg\phi) \wedge \mathrm{CNF}(\neg\psi)$
$\mathrm{CNF}(\neg(\phi \equiv \psi))$	=	$\mathrm{CNF}(\phi \wedge \neg\psi) \otimes \mathrm{CNF}(\psi \wedge \neg\phi)$

where $(C1 \wedge \ldots \wedge Cn) \otimes (D1 \wedge \ldots \wedge Dm)$ is defined as:
$(C1 \vee D1) \wedge \ldots \wedge (C1 \vee Dm) \wedge \ldots \wedge (Cn \vee D1) \wedge \ldots \wedge (Cn \vee Dm)$

DPLL

- Apply DPLL to the following:
  - 1. $(p \supset q \supset r) \wedge p \wedge \neg q$
  - 2. $(p \wedge q) \vee \neg p \supset r$
  - 3. $(p \wedge r) \vee (\neg q \wedge p) \vee (\neg r \wedge \neg p)$