Laboratory 5: Rigid and Non-Rigid Structure from Motion

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Abstract

Nothing for now.

1 Inputs:

The input is a Matrix that one explain below:

$$Input = \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_f \end{pmatrix} \tag{1}$$

Where I_k is a Matrix $2 \times p$, that represent a image in the frame k, such as $p = numberoftrackpoints$ and $f = numberofframes$

$$I_k = \begin{pmatrix} x_1 & x_2 & \cdots & x_p \\ y_1 & y_2 & \cdots & y_p \end{pmatrix} \tag{2}$$

2 Outputs:

We want to obtain a representation in the 3D plane of the points tracked in the 2D images.

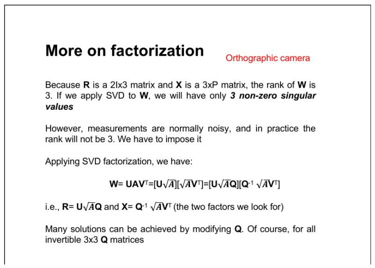
$$Output = \begin{pmatrix} x_1 & x_2 & \cdots & x_p \\ y_1 & y_2 & \cdots & y_p \\ z_1 & z_2 & \cdots & z_p \end{pmatrix} \tag{3}$$

Where (x, y, z) represents the 3D coordinates in the $\Re^3$ space.

3 First Part (Rigid structure from motion by factorization)

In this task, I must complete the orthogonal factorization function (**rigidfactorization ortho.m**). Also, we run the code with the provided example (a synthetic sequence with an ogre is provided to you) and report the 3D reconstruction error.

In this task, factorization for a rigid body must be implemented, as shown in the following slide figure 1:



More on factorization Orthographic camera

Because **R** is a 2Ix3 matrix and **X** is a 3xP matrix, the rank of **W** is 3. If we apply SVD to **W**, we will have only ***3 non-zero singular values***

However, measurements are normally noisy, and in practice the rank will not be 3. We have to impose it

Applying SVD factorization, we have:

$$\mathbf{W}= \mathbf{U}\mathbf{A}\mathbf{V}^T=[\mathbf{U}\sqrt{A}][\sqrt{A}\mathbf{V}^T]=[\mathbf{U}\sqrt{A}\mathbf{Q}][\mathbf{Q}^{-1}\sqrt{A}\mathbf{V}^T]$$

i.e., $\mathbf{R}= \mathbf{U}\sqrt{A}\mathbf{Q}$ and $\mathbf{X}= \mathbf{Q}^{-1}\sqrt{A}\mathbf{V}^T$ (the two factors we look for)

Many solutions can be achieved by modifying **Q**. Of course, for all invertible 3x3 **Q** matrices

Figure 1: Factoring slide

3.1 Code:

Now we introduce the code, we will explain each line number, but first I want to mention that the value of Zc is not the input matrix, but this is the input matrix minus the average of all the points in each frame. This way we no longer have to consider translation in the calculations.
Therefore the average matrix would be the following (with a dimension of $2f \times p$):

$$Mean = \begin{pmatrix} Mean_1 & Mean_1 & \cdots Mean_1 \\ Mean_2 & Mean_2 & \cdots Mean_2 \\ \vdots & \vdots & \ddots & \vdots \\ Mean_f & Mean_f & \cdots Mean_f \end{pmatrix} \tag{4}$$

where

$$Media_k = \begin{pmatrix} \frac{x_1+x_2+\cdots+x_p}{p} \\ \frac{y_1+y_2+\cdots+y_p}{p} \end{pmatrix} \tag{5}$$

for any k between 1 and f. Therefore from 4 we have:

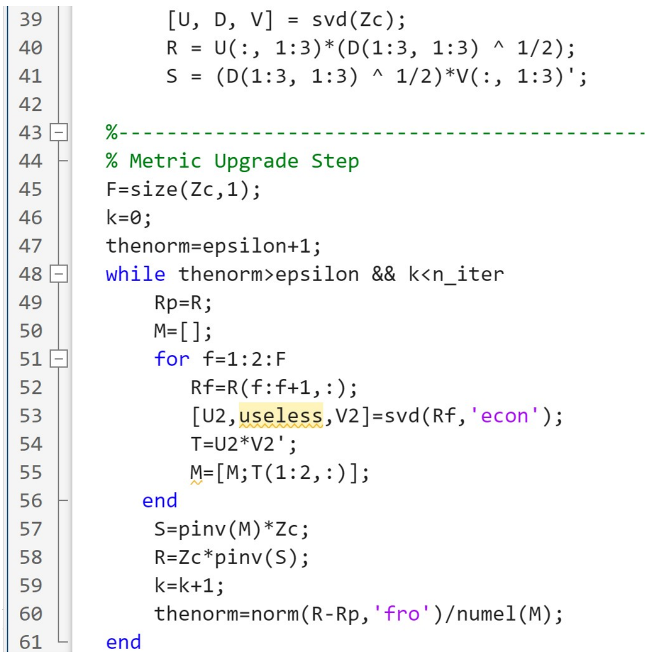
$$Zc = Input - Mean \tag{6}$$

Having said this, we continue with the application of the code figure 2.

In the first line, we calculate the factorization called Singular value decomposition(SVD).

In the second line, we approximate $R = U(:, 1 : 3)D(1 : 3, 1 : 3)^{1/2}$ and $S = D(1 : 3, 1 : 3)^{1/2}V(:, 1 : 3)'$ (where R represents rotation and S represents shape) in such a way that we only take the first three columns of U and, the first three columns of V and, the first three rows and the first three columns of D. This is because, being a rigid body, the range of R and S is 3.

In the "while loop" we try to reduce $\|Zc - RU\|$ to the threshold (in the code is epsilon). The idea is to use the rotation condition ($R_i R_i' = I$ where I is identity matrix) which is used to calculate T in the code and thus be able to calculate the new matrices S and R.



```
     [U, D, V] = svd(Zc);
     R = U(:, 1:3)*(D(1:3, 1:3) ^ 1/2);
     S = (D(1:3, 1:3) ^ 1/2)*V(:, 1:3)';

%-------------------------------------------
% Metric Upgrade Step
F=size(Zc,1);
k=0;
thenorm=epsilon+1;
while thenorm>epsilon && k<n_iter
    Rp=R;
    M=[];
    for f=1:2:F
       Rf=R(f:f+1,:);
       [U2,useless,V2]=svd(Rf,'econ');
       T=U2*V2';
       M=[M;T(1:2,:)];
   end
    S=pinv(M)*Zc;
    R=Zc*pinv(S);
    k=k+1;
    thenorm=norm(R-Rp,'fro')/numel(M);
end
```

Figure 2: source code

3.2 Examples:

Here we will show how to reconstruct the face of an Ogre with the given frames and using the following input parameters:

- threholder: 10^{-7}
- maximum number of iterations: 200

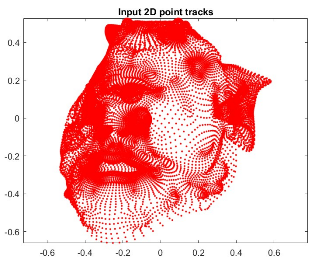


Figure 3: the original face of the Ogre.

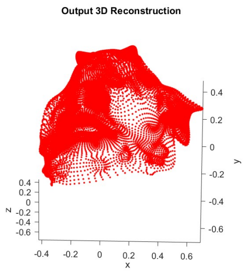


Figure 4: The Ogre's face reconstructed with the algorithm.

As can be seen in Figure 5, the reconstruction of the ogre's face is quite good, the error between the real ogre and the reconstructed one is quite low Figure 6.

4 Second Part (Non-Rigid structure from motion by non-linear optimization and assuming a low-rank shape model)

In this task we must perform a nonlinear optimization for a non-rigid body, assuming a low rank. This code used Levenberg-Marquardt algorithm to optimize the non-linear equation that it is shown to next:

$$\underset{R^i,B_k,l_i^k,t^i}{\arg\min} \sum_{i=1}^{i} \|\widehat{W}^i - R^i \sum_{k=1}^{k} l_i^k B_k - t^i\|^2 + \gamma \sum_{i=1}^{i-1} \|L^i - L^{i+1}\|^2 + \phi \sum_{i=1}^{i-1} \|R^i - R^{i+1}\|^2 \tag{7}$$

Where: B_k is a row of bases matrix. R^i is a rotacon for a point in column i of X the 3D shape vector. l_i are the constants such that $\sum_{k=1}^{k} l_i^k B_k = X$. t^i is the translation. L^i is the vector of all constants in frame i. The norm is that of Frobenius.

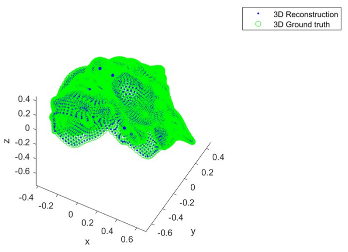
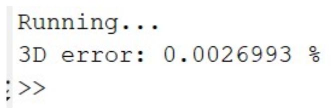


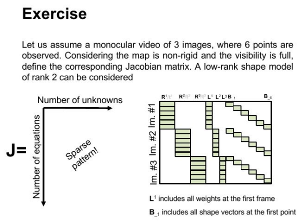
Figure 5: Comparison between the 3D representation of the Ogre's face against the one reconstructed by the algorithm.



```
Running...
3D error: 0.0026993 %
>>
```

Figure 6: Error between the ogre's face and the reconstructed face.

In order to calculate the Levenberg-Marquardt optimization we need to calculate the Jacobian form of the equation 7, we do this as shown in the following slide Figure 7



Exercise

Let us assume a monocular video of 3 images, where 6 points are observed. Considering the map is non-rigid and the visibility is full, define the corresponding Jacobian matrix. A low-rank shape model of rank 2 can be considered

$\mathbf{L}^1$ includes all weights at the first frame

$\mathbf{B}_{\_1}$ includes all shape vectors at the first point

Figure 7: Jacobian form

4.1 Code:

Now we will show how this Jacobian form is calculated in the code.

So first we will define the dimensions of Jacobian matrix. For this code we are using quaternions for the rotations, therefore R^i is 2×4.

Then the dimension of Jacobian matrix is defined in the following way

- number of rows of $Dim(J^{rows}) = 2\sum_{i=1}^{f}(numberofpointsvisibleinframei)$

- number of columns of $Dim(J_{columns}) = 4f + 2f + kf + 3kp$

Where f is the number of frames, p is the total number of points tracked and k is the total number of bases.

Explaining row sizes:

Green: Represents the visible points per frame and when we multiply it by two this is because the rotations are in quaternions and the quaternions are 2x4 and then they are two vectors per each points and that is why it is multiplied by two.

Explaining column sizes:

Red: Represents the value of the columns of the rotation matrix (2×4 quaternions) and we have a rotation matrix R^i for each frame i, therefore $4f$ where f is the number of frames.

Blue: Represents the translation matrix t^i, whose dimension is 1×2, for each frame i, therefore $2f$ where f is the number of frames.

Orange: Represents the matrix L_i whose dimension is $1 \times k$ for each frame i, therefore kf where f is the number of frames.

Yellow: Represents the matrix B_b whose dimension is $3 \times p$ for each base b, therefore $3pk$ where k is the number of bases.

Well then we need to define the shape of our Jacobian matrix by putting 1 where there are values and leaving 0 in the spaces that do not exist, as shown in the following code.

```
for i=1:n_frames
        for j=1:nnz(vij(i,:))
            J(2*j-1+computed_points:2*j+
               computed_points,6*j-5:6*j)=ones(2,6);
            J(2*j-1+computed_points:2*j+
               computed_points,K*j-K+1+6*n_frames:K*j
               +6*n_frames)=ones(2,K);
            J(2*j-1+computed_points:2*j+
               computed_points,3*K*j-3*K+1+(6+K)*
               n_frames:3*K*j+(6+K)*n_frames)=ones
               (2,3*K);
        end
        computed_points=computed_points+2*nnz(vij(i,:)
            );
    end
```

In the first for loop each frame is iterated, in the second for loop only the points visible in that frame are iterated.

```
J(2*j-1+computed_points:2*j+computed_points,6*j-5:6*j)
    =ones(2,6);
```

Here, the Jacobian is calculated for the rotation R part (in quaternions) and the translation part t, having this

$$\begin{pmatrix} R_{2\times 4} & | & t_{2\times 2} \end{pmatrix}_{2\times 6} \tag{8}$$

we can add a matrix of ones of 2×6
Now, the Jacobian is calculated for the constants L^i part, where $L^i = [l_1^i, l_2^i, \cdots, l_k^i]$ such that i is a frame.

```
J(2*j-1+computed_points:2*j+computed_points,K*j-K+1+6*
    n_frames:K*j+6*n_frames)=ones(2,K);
```

and for that, in the code is added a matrix of ones from $2 \times k$.

Now, the Jacobian is calculated for the bases B_i part, where

$$B_i = \begin{pmatrix} [b_1^i]_{3\times 1} & [b_2^i]_{3\times 1} & \cdots & [b_k^i]_{3\times 1} \end{pmatrix}_{3\times k} \tag{9}$$

```
J(2*j-1+computed_points:2*j+computed_points,3*K*j-3*K
    +1+(6+K)*n_frames:3*K*j+(6+K)*n_frames)=ones(2,3*K)
    ;
```

and for that, in the code is added a matrix of ones from $2 \times 3k$.

Now we calculate the Jacobian part of the priors, so the dimension of the Jacobian for the priors comes from the following expression

$$\underset{R^i, B_k, l_i^k, t^i}{\arg\min} \, \gamma \sum_{i=1}^{i-1} \|L^i - L^{i+1}\|^2 + \phi \sum_{i=1}^{i-1} \|R^i - R^{i+1}\|^2 \tag{10}$$

so from this equation 10 and the orange color of the equation 7 we deduce that the Jacobian dimension for the prior L would be

$$Dim(J_{rows}) = f - 1 Dim(J_{columns}) = kf \tag{11}$$

and the blue and red color of the equation 7 also we can deduce the Jacobian dimension for the prior R as

$$Dim(J_{rows}) = f - 1 Dim(J_{columns}) = 6f \tag{12}$$

Now we will show how we fill the indices that have a value in the Jacobian Matrix of the L prior with 1 and otherwise hold with zero.

```
if (priors.coeff_prior == 1)
     % prior terms on L
     L=spalloc(n_frames-1,(K+6)*n_frames + K*3*
         n_points,2*K*(n_frames-1));
     for i=1:n_frames-1
        L(i,K*i-K+1+6*n_frames:K*i+K+6*n_frames)=ones
            (1,2*K);
     end
     if K==1
         J(2*nnz(vij)+1:size(J,1), :)=L;
     else
         J(2*nnz(vij)+1:2*nnz(vij)+n_frames-1, :)=L;
     end
 end
```

We do the following, we take the L_i and L_{i+1} and join them as shown in the equation 13

$$\big([L_i]_{1\times k} \quad | \quad [L_{i+1}]_{1\times k}\big)_{1\times 2k} \tag{13}$$

and because of this in the code we add a matrix of ones $1 \times 2K$.
For the Jacobian Matrix of the R prior is the similar way, in this case we joint R^i and R^{i+1} like show in the equation 14

$$\big([R^i]_{1\times 6} \quad | \quad [R_{i+1}]_{1\times 6}\big)_{1\times 12} \tag{14}$$

and because of this in the code we add a matrix of ones 1×12.
Joining the Jacobian and the Jacobian of the priors we have the following graph figure 8.

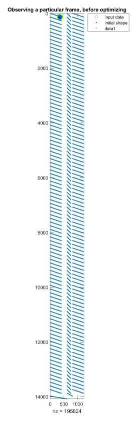


Figure 8: Jacobian result

4.2 Examples:

Here we will show how to reconstruct the human face with the given frames and using the following input parameters:

- low-rank K(it is the number of bases): 2
- $camera_prior$(it mean that the R prior is active): 1
- $coeff_prior$(it mean that the L prior is active): 1
- number of optimizer iterations: 50

In Figure 9 we can see the comparison of the input points of a frame in green against the estimated points of that frame in blue.

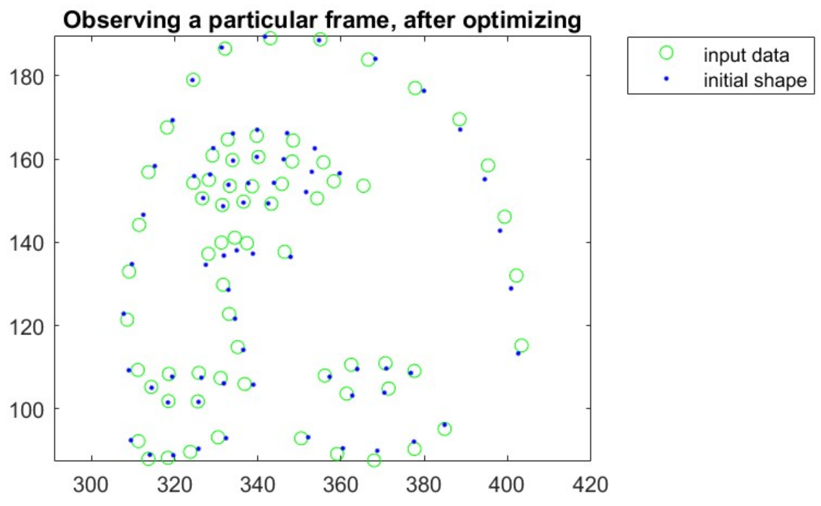


Figure 9: Input points compared to optimized points

In Figure 10 we can see the estimated 3D points for a frame.
In Figure 11 we see the calculation of each of the 50 optimizations, this method has a high cost in performance, but a good result is achieved as shown by the low error at the end.

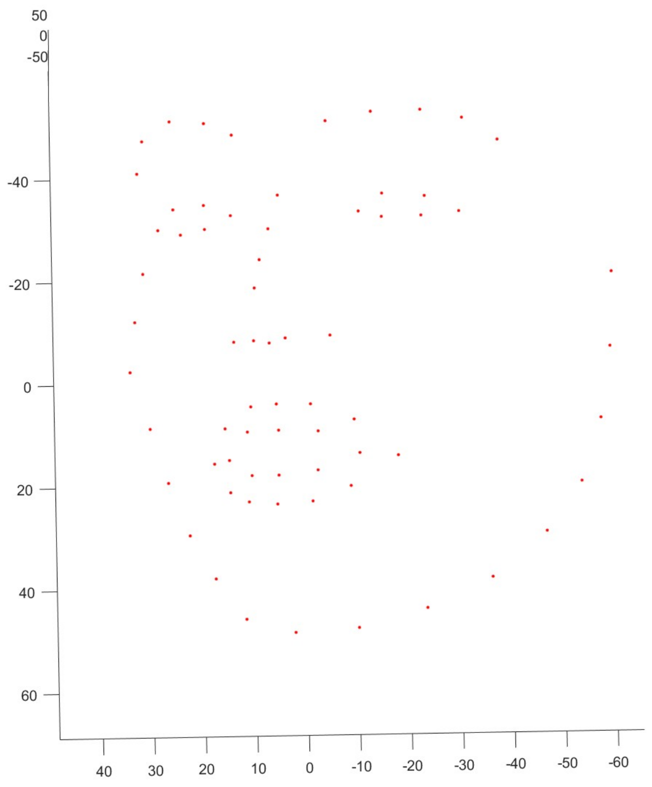
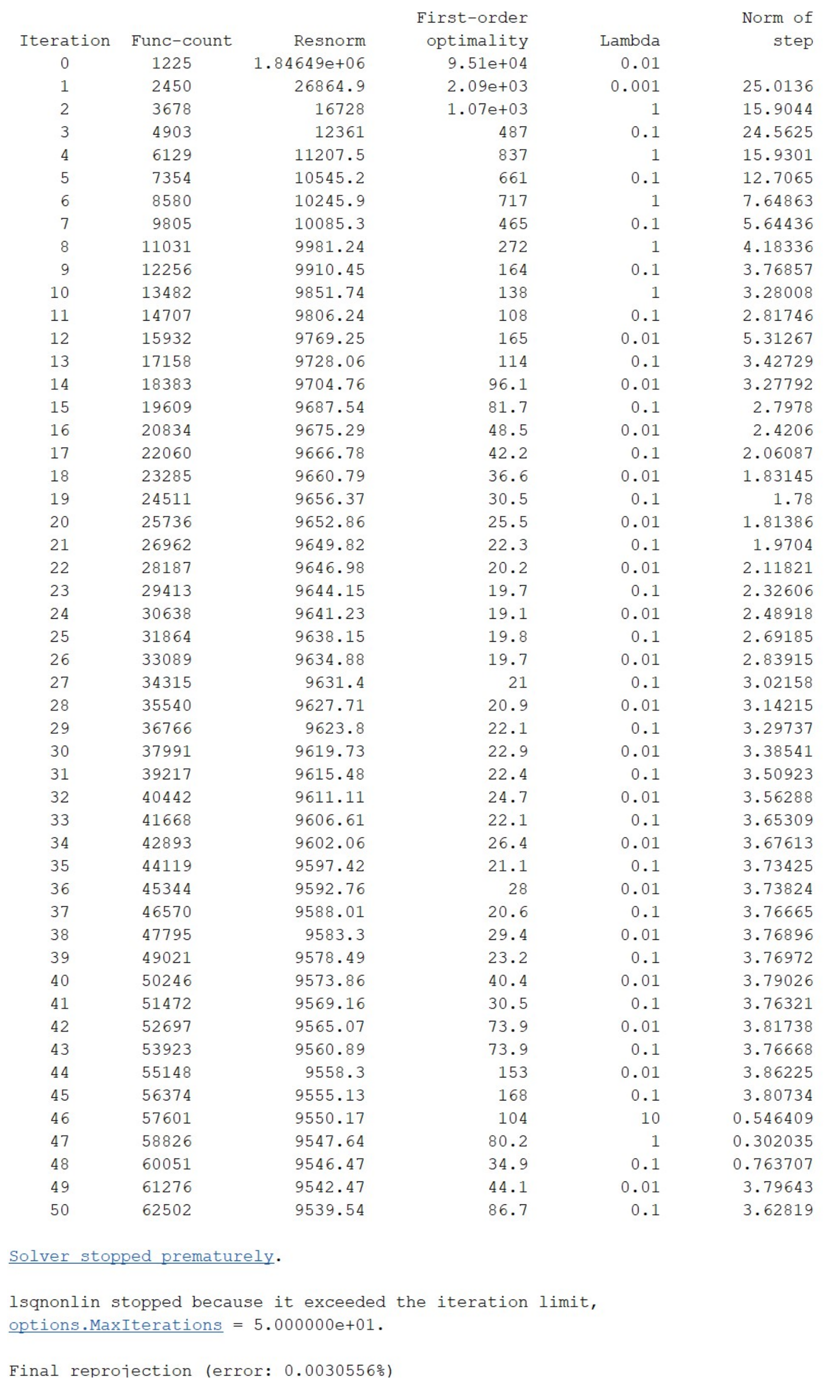


Figure 10: 3D reconstruction



```
                                          First-order                     Norm of
 Iteration  Func-count     Resnorm        optimality       Lambda           step
     0          1225    1.84649e+06        9.51e+04         0.01
     1          2450        26864.9        2.09e+03        0.001          25.0136
     2          3678          16728        1.07e+03            1          15.9044
     3          4903          12361             487          0.1          24.5625
     4          6129        11207.5             837            1          15.9301
     5          7354        10545.2             661          0.1          12.7065
     6          8580        10245.9             717            1          7.64863
     7          9805        10085.3             465          0.1          5.64436
     8         11031        9981.24             272            1          4.18336
     9         12256        9910.45             164          0.1          3.76857
    10         13482        9851.74             138            1          3.28008
    11         14707        9806.24             108          0.1          2.81746
    12         15932        9769.25             165         0.01          5.31267
    13         17158        9728.06             114          0.1          3.42729
    14         18383        9704.76            96.1         0.01          3.27792
    15         19609        9687.54            81.7          0.1           2.7978
    16         20834        9675.29            48.5         0.01           2.4206
    17         22060        9666.78            42.2          0.1          2.06087
    18         23285        9660.79            36.6         0.01          1.83145
    19         24511        9656.37            30.5          0.1             1.78
    20         25736        9652.86            25.5         0.01          1.81386
    21         26962        9649.82            22.3          0.1           1.9704
    22         28187        9646.98            20.2         0.01          2.11821
    23         29413        9644.15            19.7          0.1          2.32606
    24         30638        9641.23            19.1         0.01          2.48918
    25         31864        9638.15            19.8          0.1          2.69185
    26         33089        9634.88            19.7         0.01          2.83915
    27         34315         9631.4              21          0.1          3.02158
    28         35540        9627.71            20.9         0.01          3.14215
    29         36766         9623.8            22.1          0.1          3.29737
    30         37991        9619.73            22.9         0.01          3.38541
    31         39217        9615.48            22.4          0.1          3.50923
    32         40442        9611.11            24.7         0.01          3.56288
    33         41668        9606.61            22.1          0.1          3.65309
    34         42893        9602.06            26.4         0.01          3.67613
    35         44119        9597.42            21.1          0.1          3.73425
    36         45344        9592.76              28         0.01          3.73824
    37         46570        9588.01            20.6          0.1          3.76665
    38         47795         9583.3            29.4         0.01          3.76896
    39         49021        9578.49            23.2          0.1          3.76972
    40         50246        9573.86            40.4         0.01          3.79026
    41         51472        9569.16            30.5          0.1          3.76321
    42         52697        9565.07            73.9         0.01          3.81738
    43         53923        9560.89            73.9          0.1          3.76668
    44         55148         9558.3             153         0.01          3.86225
    45         56374        9555.13             168          0.1          3.80734
    46         57601        9550.17             104           10         0.546409
    47         58826        9547.64            80.2            1         0.302035
    48         60051        9546.47            34.9          0.1         0.763707
    49         61276        9542.47            44.1         0.01          3.79643
    50         62502        9539.54            86.7          0.1          3.62819

Solver stopped prematurely.

lsqnonlin stopped because it exceeded the iteration limit,
options.MaxIterations = 5.000000e+01.

Final reprojection (error: 0.0030556%)
```

Figure 11: Optimizer iterations