Modeling and Control of Manipulators

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Chapter 1

Introduction

Robotic manipulators, with their versatility and precision, have found their place in a wide range of applications across various industries and sectors. Here, we explore the diverse realms where robotic manipulators have made a significant impact:

- Manufacturing and Industrial Automation: Robotic arms are deployed in factories for tasks like welding, painting, and assembling. They enhance productivity, reduce error rates, and ensure consistent quality, revolutionizing manufacturing.
- Healthcare and Medical Robotics: Surgical robots assist surgeons in delicate procedures, reducing invasiveness and enabling remote surgeries.
- Space Exploration: Robotic arms on spacecraft and rovers collect samples, deploy instruments, and perform repairs in space and on other celestial bodies.
- Underwater Exploration: Robotic arms are routinely used in Remotely Operated Vehicles, to perform maintenance and repair operations in deep-water sites, where the presence of human divers is just not possible.
- Agriculture and Agribotics: Autonomous drones with manipulators perform planting, harvesting, and crop monitoring, increasing yields and sustainability in farming.
- Elderly assistance: There is an increasing trend in the adoption of human-like robots for the assistance of elderly people in day to day living, or just to keep them company.
- Entertainment and Creative Industries: Animatronic characters and precision-controlled camera rigs enhance entertainment experiences in theme parks and the film industry.
- Research and Development: Robotic manipulators are essential tools for testing prototypes, exploring materials, and developing new technologies.

In each of these applications, robotic manipulators contribute to increased efficiency, precision, and safety, pushing the boundaries of what technology can achieve.

1.1 Physics of a Manipulator

A manipulator is a mechanical device or robotic system designed to perform precise and controlled movements, often resembling the dexterity and range of motion of the human arm. In its essence, a manipulator is a serial connection of rigid bodies via rotational or translational objects, which allow the structure to move, even if such movements are limited. The rigid bodies are called *links* and the connection elements are called *joints*. Figure 1.1 shows the sketch of a typical 6 degrees of freedom (DOF) manipulator used in industrial settings. The DOF of a mechanical system are the number of independent parameters that define its configuration. In particular, the robot is composed by three 1-DOF rotational joints, and a 3-DOF ball wrist. For modeling purposes, the wrist can be seen as three 1-DOF rotational joints, whose origin is coinciding in the same point.

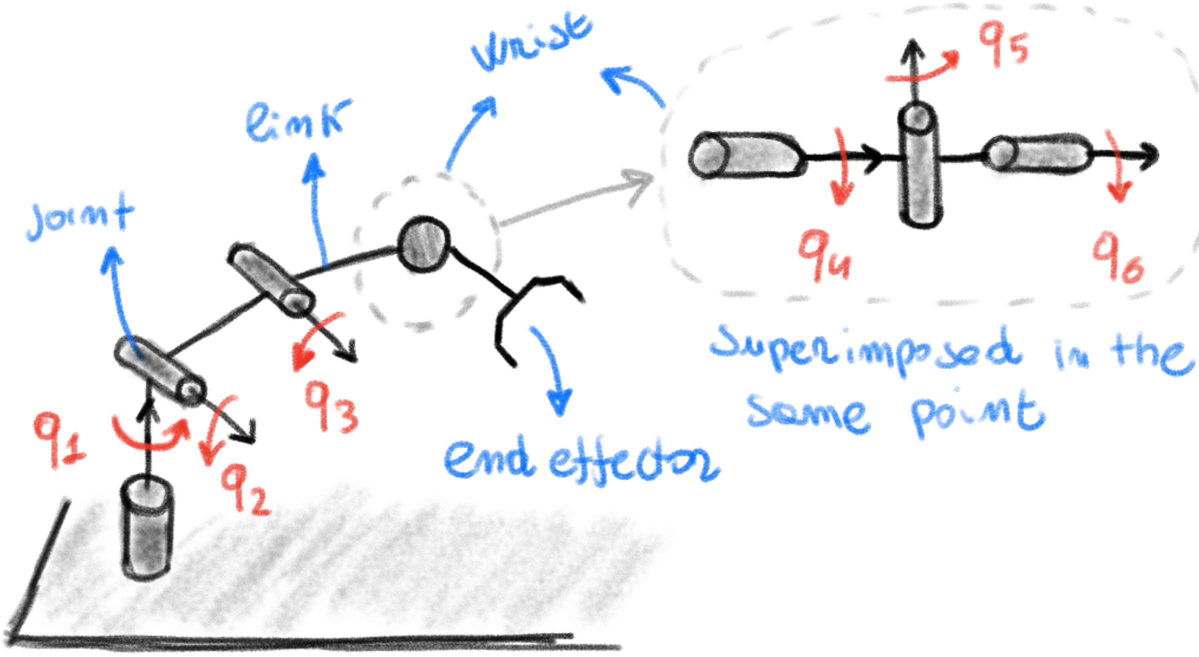

Figure 1.1: A typical 6 degrees of freedom manipulator

At a certain instant, the configuration of the robot is established by the values assigned to the various joints and mathematically is represented by the *joint configuration vector*

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, \tag{1.1}$$

where n is the number of joints and q_i represents the angle of rotation (for rotational joints) or the distance (for translational joints) with respect to a zero position initially established.

The example reported in the figure shows the structure of a manipulator with 6-DOF. There can be manipulators with less or more DOF, although that is the prevalent one employed in the industries. Indeed, having a structure with 6-DOF is the necessary condition to be able to locate the terminal body of the robot, often called the end-effector, in all possible orientations within a certain region of the 3 dimensional space. This region is called the *task workspace* of the robot.

As a comparison, the human arm has 7 DOF: 3 in the wrist, 1 in the elbow, and 3 in the shoulder. Having more than 6-DOF is useful to manage to robot configuration: the configuration can be changed without changing the position and orientation of the end-effector. Therefore, the 7-DOF configuration is *redundant* with respect the objective of keeping the end-effector in a given position and orientation.

The configuration of the robot changes over time. This depends on the rate of change of its joints, and is represented by the joint velocity vector

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}. \tag{1.2}$$

In turn, the joint velocity vector changes in time due to the second derivative of the joint values, i.e.

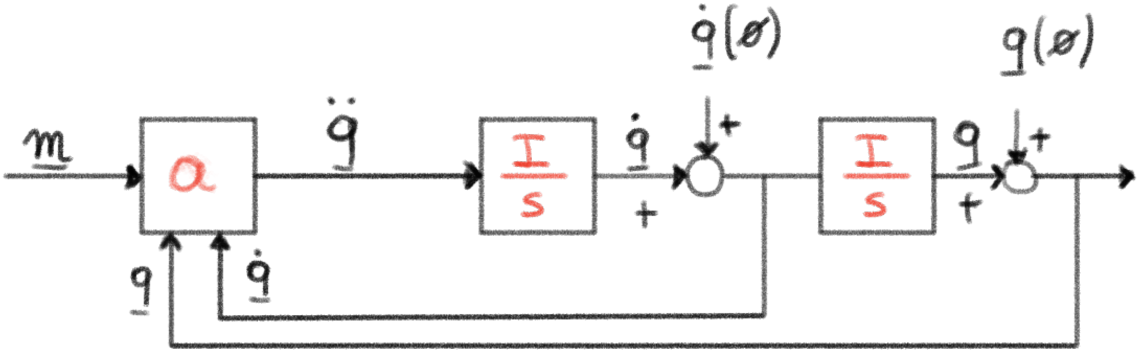

Figure 1.2: A closed loop scheme representing the dynamic model of a manipulator. Block "a" corresponds to equation (1.6).

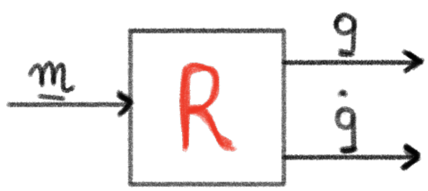

Figure 1.3: A compact representation of equation (1.6) and the integration of joint acceleration to obtain joint velocity and position vectors.

due to the joint acceleration vector

$$\ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix}. \tag{1.3}$$

There is no need to consider further derivatives, since the robot is an articulated mechanical system and its motion is originated from the torques and forces provided by electric motors. These values are organized in the so called join torque vector

$$\boldsymbol{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}. \tag{1.4}$$

The link between the torque provided by the motors, and the resulting joint acceleration is given by the dynamics of the manipulator. Mathematically, the dynamics can be modelled as

$$\boldsymbol{A}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{B}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{m}, \tag{1.5}$$

where $\boldsymbol{A}(\boldsymbol{q}) \in \mathbb{R}^{n \times n}$ is the Inertia matrix, $\boldsymbol{B}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{n \times n}$ is the Coriolis' effect matrix, and $\boldsymbol{c}(\boldsymbol{q})$ is a vector that keeps into account gravitational effects. In general, external forces different from gravitational forces can also be added on the side of $\boldsymbol{m}$ (for example, a force generated by a contact with the end-effector). Since $\boldsymbol{A}(\boldsymbol{q})$ is always invertible, the acceleration can be obtained as

$$\ddot{\boldsymbol{q}} = \boldsymbol{A}(\boldsymbol{q})^{-1}(\boldsymbol{m} - \boldsymbol{B}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{c}(\boldsymbol{q}). \tag{1.6}$$

The previous relationship will be more compactly represented as in Fig. 1.3. Clearly, if one generates $\boldsymbol{m}$ in some way, the robot will move, but the movement will not be predictable. As we want the robot to do a given task, we need to find out how to generate the torque vector appropriately.

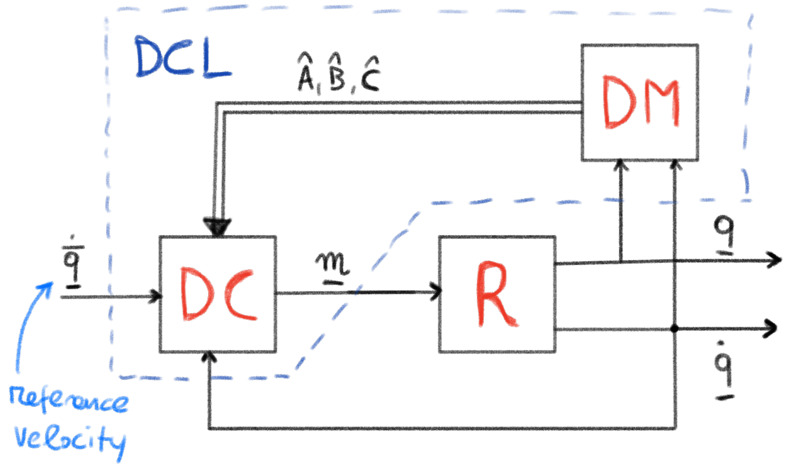

Figure 1.4: A block representation of the Dynamic Control Layer. The DM block provides the estimated matrices to the Dynamic Controller (DC) block, which generates the torque vector in input to the Robot.

1.2 A General Approach to Robot Control

We will consider a backstepping approach to the control of the manipulator. We will do so by organizing the functionalities and algorithms into an hierarchy of two layers of control.

1.2.1 Dynamic Control Layer

The first layer, called Dynamic Control Layer (DCL), is directly connected to the robot. It is in charge of making the robot commandable through joint velocity reference signals, instead of torques. To do so, it is necessary to design a controller that receives desired joint velocities as input, together with the feedback velocities, and generates a joint torque vector capable of driving the actual joint velocities toward the desired ones.

Different types of controllers can be build. For example, a feedback linearization controller, a sliding mode controller, or a simpler proportional integrative derivative (PID) controller. Let us consider a feedback linearization controller. To build such a controller, let us consider first a dynamic model functional block, which is an algorithm capable of estimating the matrices $\hat{\boldsymbol{A}}(\boldsymbol{q})$, $\hat{\boldsymbol{B}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, and $\hat{\boldsymbol{c}}(\boldsymbol{q})$ from the values of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. There are fast algorithms that provide the numerical values of these matrices and vector as output, with timings suitable for a real time implementation. On the basis of the availability of such matrices, a controller that can best approximate the desired velocity can be constructed. The resulting control scheme can be seen in 1.4. The controller is implemented in the Dynamic Controller (DC) block. It takes as reference the desired joint velocities $\dot{\bar{\boldsymbol{q}}} \in \mathbb{R}^n$, and as feedback the actual velocities $\dot{\boldsymbol{q}}$. Furthermore, it exploits the Dynamic Model block to have the estimated matrices of the dynamic model of the robot. The output is the joint torque vector $\boldsymbol{m}$, which is fed to the actuators of the robot.

If the performance of the controller are sufficiently good, and the references are slowly varying (i.e., they are inside the bandwidth of the controller), then the above representation can be approximated with the one shown in Fig. 1.4.

1.2.2 Kinematic Control Layer

On top of the DCL lies the Kinematic Control Layer (KCL). This layer is in charge of providing the desired reference velocity $\dot{\bar{\boldsymbol{q}}}$ to the underlying DCL block. This reference should be constructed so that the robot moves as desired in the Cartesian space.

To this aim, it is first necessary to understand how the arm is positioned in the Cartesian space. Intuitively, the positioning of each link is a function of the joint position vector $\boldsymbol{q}$, as the position and orientation of a link will depend on how the previous joints are rotated (or translated). To better

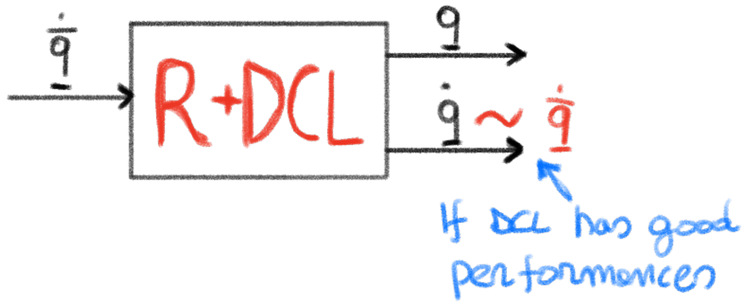

Figure 1.5: A block representation of the Dynamic Control Layer together with the Robot, forming a sort of Kinematic system. The output velocity can be approximated with the input desired velocity, if the performance of the DCL are sufficiently good.

understand the position and orientation of the links in the Cartesian space, let us attach a *frame* to each link, including to the base and to the end-effector. A frame is defined by three unitary vectors, orthogonal to each other, originating in the same point, which is called the origin of the frame.

The mutual positioning and orientation of a frame with respect to another one is represented by a 4×4 matrix called *Transformation Matrix*. If one wants to indicate the positioning of the end-effector frame $\langle e \rangle$ with respect to the base frame $\langle 0 \rangle$, then one would write ${}^0_e\boldsymbol{T}(\boldsymbol{q})$.

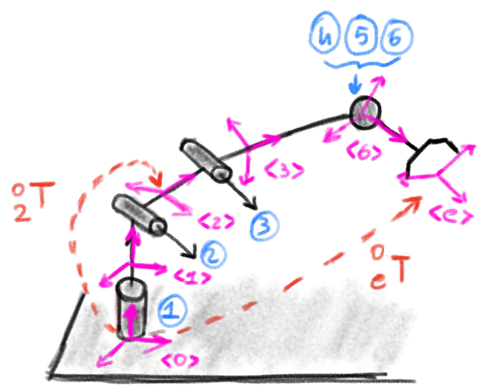

Figure 1.6: .

Of course, a transformation matrix can be computed for any couple of frames. Typically, it is of interest to know the positioning of the robot frames with respect to the base frame, hence from the knowledge of the joint position vector $\boldsymbol{q}$ one can compute ${}^0_1\boldsymbol{T}(\boldsymbol{q}); {}^0_2\boldsymbol{T}(\boldsymbol{q}), \ldots, {}^0_n\boldsymbol{T}(\boldsymbol{q}), {}^0_e\boldsymbol{T}(\boldsymbol{q})$.

The knowledge of all these transformation matrices is the minimal amount of information to know the position and orientation of the different links of the robot. Hence, we can imagine to have a *Geometric Model* block (GM) that, knowing the physics of the robot and the current posture through the joint position vector $\boldsymbol{q}$, produces as output the transformation matrices mentioned above.

Now that we have a mean to compute the position and orientation of all the links in real-time, it is necessary to understand how the joints should move so that the end-effector reaches the desired position and orientation. In other words, it is necessary to understand how to make ${}^0_e\boldsymbol{T}(\boldsymbol{q})$ equal to a value that corresponds to the desired position and orientation.

Given the dependency of ${}^0_e\boldsymbol{T}$ on $\boldsymbol{q}$, it is necessary to act upon the joint velocity vector. As we are considering a backstepping approach, we need to generate an appropriate desired joint velocity vector $\dot{\bar{\boldsymbol{q}}}$.

With reference to Fig. 1.7, let us consider a generic link $\langle i \rangle$. Such a frame changes its position and orientation over time, as the joints move. Therefore, it is characterized by a certain Cartesian velocity. Let us label such a velocity as $\dot{\boldsymbol{x}}_i \in \mathbb{R}^6$. This 6 dimensional vector is composed by two 3 dimensional vectors defined in the Cartesian space, nameley the angular velocity vector $\boldsymbol{\omega}_i$ and $\boldsymbol{v}_i$. These geometric vectors assume different values if we consider them on one frame or another one. To differentiate between these different instances of the same geometric vector, we use the notation ${}^0\boldsymbol{\omega}_i$, which considers the geometric vector projected into frame $\langle 0 \rangle$.

The vector $\dot{\boldsymbol{x}}_i$ is linearly related with the joint velocity vector through a posture-dependent matrix called the *Jacobian* matrix, leading to the following equation

$$\dot{\boldsymbol{x}}_i = \boldsymbol{J}_{i/0}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{1.7}$$

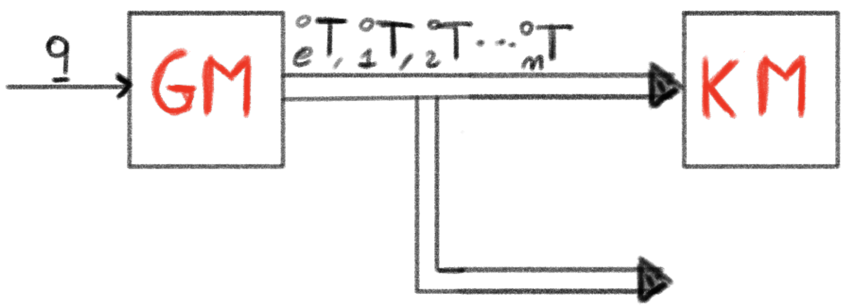

Figure 1.8: A block representation of.

where $\boldsymbol{J}_{i/0}(\boldsymbol{q}) \in \mathbb{R}^{6\times n}$ is the Jacobian matrix of frame $\langle i\rangle$ with respect to frame $\langle 0\rangle$. Given the serial nature of the manipulator, for a given frame $\langle i\rangle$, the only part of $\boldsymbol{q}$ that influence its velocity are the joints before and including joint i itself.

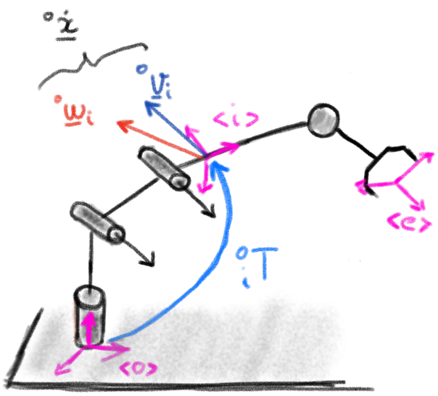

Figure 1.7: .

Hence, the previous relationship could also be written as

$$\dot{\boldsymbol{x}}_i = \boldsymbol{J}_{i/0}(q_1, \dots, q_{i-1}) \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_i \end{bmatrix}. \tag{1.8}$$

To evaluate this matrix for all the frames of the robot, it is important to highlight that the dependency of the Jacobian matrix on $\boldsymbol{q}$ is related to the position and orientation of all the frames preceding $\langle i\rangle$. Hence, it could be rewritten in terms of the transformation matrices of the preceding joints, as

$$\boldsymbol{J}_{i/0}(\boldsymbol{q}) = \boldsymbol{J}_{i/0}\left({}^0_1\boldsymbol{T}(\boldsymbol{q}), {}^0_2\boldsymbol{T}(\boldsymbol{q}), \dots, {}^{0}_{i-1}\boldsymbol{T}(\boldsymbol{q})\right). \tag{1.9}$$

Therefore, it possible to construct a Kinematic Model functional block that takes as input the output of the Geometric Model block (the transformation matrices), and computes in real-time all the *basic Jacobians* of the manipulator as depicted in Fig. 1.8. The dependency on $\boldsymbol{q}$ is implicit in the dependency on the transformation matrices.

Let us now more formally consider the objective of the KCL, by tackling the most common case in robotics. This corresponds to the sketch drawn in Fig. 1.9, where the end-effector frame $\langle e\rangle$ needs to be driven to a desired *goal* position represented by the goal frame $\langle g\rangle$.

First, let us immediately note that the end-effector frame, like any other frame, can only be driven to a point that belongs to its workspace. Indeed, the movement of the frame is constrained by the manipulator's structure. However, if it were free from constraints, then we could easily find a velocity in the Cartesian space to move it towards $\langle g\rangle$, both in position and orientation. Hence, let us consider a functional block called *Cartesian Controller* (CC), which on the basis of the feedback position of the end-effector, represented by the transformation matrix ${}^0_e\boldsymbol{T}(\boldsymbol{q})$, and on the desired goal position ${}^0_g\boldsymbol{T}(\boldsymbol{q})$, generates the desired generalized velocity $\bar{\dot{\boldsymbol{x}}}_e$ for the end-effector.

However, the frame is not free from constraints, but as it is attached to a manipulator's part, hence its movements depend on the motion of all the joints of the system. Therefore, this ideal Cartesian velocity must be attained with a specific motion of the joints, if possible. So a new fundamental block must be introduced to perform this mapping from the Cartesian velocity space to the joint velocity space. Such a block is called the *Inverse Kinematics* block and takes as input the desired Cartesian velocity $\bar{\dot{\boldsymbol{x}}}_e$ and the Jacobian $\boldsymbol{J}_{e/0}(\boldsymbol{q})$ of the end-effector. The block provides the desired joint velocities $\bar{\dot{\boldsymbol{q}}}$ that the DCL must track. These velocities, when applied, provide the *best approximation* to the desired Cartesian velocity. As an example, is the manipulator is constrained to move on planar surface,

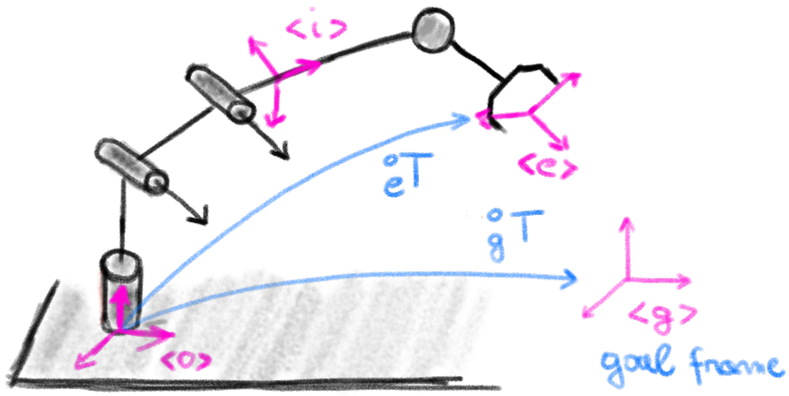

Figure 1.9: A block representation of.

and the goal is to catch an object flying in 3D space, then clearly the desired velocity $\dot{\boldsymbol{x}}_e$ will, in general, be outside of the plane and thus unattainable. The best approximation in this case will be reached by providing, in the Cartesian space, the projection of such desired velocity on the plane.

The overall control scheme can be seen in Fig. 1.10. The geometric model (GM), exploiting the joint position feedback, provides the transformation matrices to both the Kinematic Model (KM) and the Cartesian Controller. The Cartesian Controller, knowing the desired (goal) position for the end-effector represented by ${}^0_e\boldsymbol{T}(\boldsymbol{q})$, computes the desired Cartesian velocity $\dot{\boldsymbol{x}}_e$. This velocity is given as input to the Inverse Kinematics (IK) block, which finds the desired joint velocity vector $\dot{\boldsymbol{q}}$ that provides the best approximation. This velocity is the final output of the overall KCL and is given as input to the DCL, in particular to the Dynamic Controller (DC) block. In this particular example, the DC employs a feedback linearization approach, and hence receives the estimated dynamic parameters from the Dynamic Model (DM) block. On such a basis, it computes the joint wrench vector $\boldsymbol{m}$, which is finally given to the actuators. The dynamics of the system will make the joints move and the sensors will provide the new feedback joint velocities and positions.

In the rest of these notes, we will present the fundamentals underlying each of the KCL blocks. The details on the dynamic level are provided in a separate course.

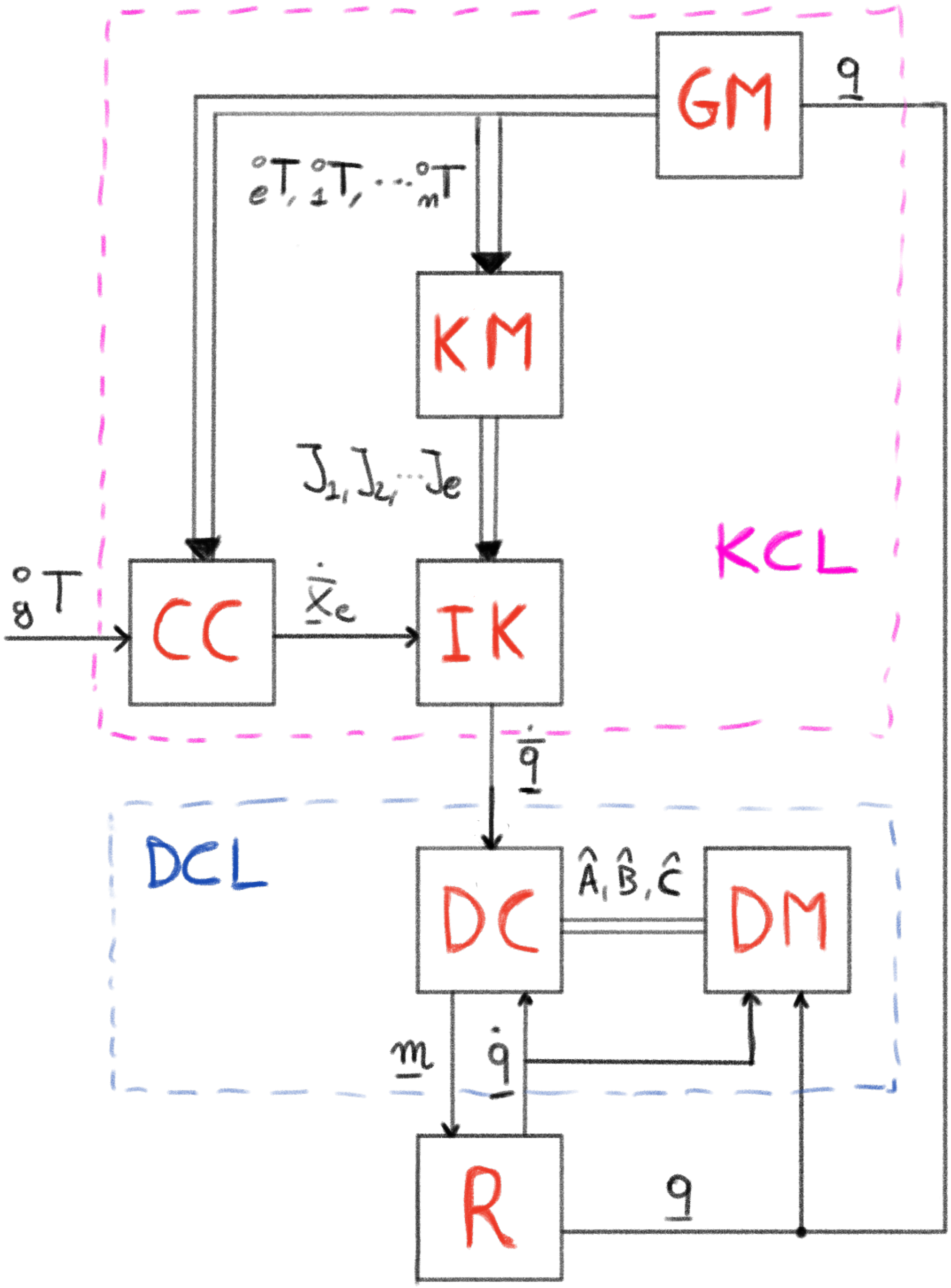

Figure 1.10: A block representation of.

Chapter 2

Geometric Fundamentals

The present chapter will review all basic geometric concepts and related algorithmic tools needed for later managing geometric problems regarding any type of robotic structure.

For sure, many of the concepts and formalisms presented in the initial sections are well known to the reader, possibly from previous courses. Even in this case, the reading of the chapter is recommended, at least for getting acquainted with the used notations.

2.1 World Coordinate System and Related Objects

A so-called *world coordinate system*, otherwise termed as the *world frame* $\langle 0 \rangle$, is represented by the *ordered triple* of *orthogonal* and *oriented axes* $x - y - z$, corresponding to a so-called right frame (i.e. with the orientation of the z axis obtained via the application of the "right-hand-rule" from x to y) that *ideal observers* have *agreed* to use for parameterizing the surrounding space.

The concept of "ideal observer" corresponds to an ideal entity capable of monitoring the evolution of the involved geometric or physical phenomena in whatever needed details, and capable of recording them with no storage limits, as well as capable of instantaneously communicating with other ideal observers via infinite capacity and speed channels. In the practice, instead, "non-ideal (real) observers" typically correspond to hardware/software subsystem organizations that, possibly distributed within the environment, perform measurements, data acquisition, data processing, data logging, data transmission, etc., but with obvious practical limitations with respect to the ideal ones.

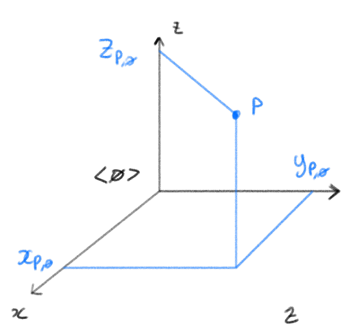

Figure 2.1: Point P and its components on frame $\langle 0 \rangle$.

With that in mind, as indicated in Fig. 2.1, the position of a point P (we shall always use capital letters for indicating points in space) with respect to the agreed frame $\langle 0 \rangle$ is termed as *point P projected on frame* $\langle 0 \rangle$ and is denoted as ${}^0P$, and is represented by the algebraic-vector of its ordered components, respectively on the $x - y - z$ axes of $\langle 0 \rangle$, that is

$$ {}^0P \triangleq \begin{bmatrix} x_{P,0} \\ y_{P,0} \\ z_{P,0} \end{bmatrix} , \tag{2.1} $$

where the indexes indicating the point P and the frame $\langle 0 \rangle$, in correspondence of each component of ${}^0P$ itself, only serve for recalling how such components are defined, and will be omitted in the rest of the notes, whenever the context is clearly understood. Moreover, let us trivially note that the position ${}^0O_0$ of the origin of frame $\langle 0 \rangle$, with respect to frame $\langle 0 \rangle$ itself, corresponds to the null position (i.e. the null algebraic vector).

Within such a parameterized space, other than points, we also have the so-called *applied projected vectors* on frame $\langle 0 \rangle$, simply defined and denoted as

$$ {}^0\left[P - Q\right] \triangleq \left[{}^0P - {}^0Q\right] , \tag{2.2} $$

corresponding to the algebraic vector of the components on $\langle 0 \rangle$ of the oriented segment that starting from point Q reaches point P.

Accordingly with its definition, an applied projected vector cannot be parallel shifted. Indeed, if we were instead parallel shifting it into another ${}^{0}[P' - Q']$, then we should consider it as a *different* vector. Finally, let us note how a projected point ${}^{0}P$ can be seen as an applied projected vector on frame $\langle 0 \rangle$, with starting point the origin of frame $\langle 0 \rangle$:

$$ {}^{0}P = {}^{0}[P - O_0] \,. \tag{2.3} $$

An important extension is the concept of *projected vector* on frame $\langle 0 \rangle$ denoted as

$$ {}^{0}(P - Q) \triangleq \left({}^{0}P - {}^{0}Q \right), \tag{2.4} $$

which now includes the *entire class* of applied projected vectors on frame $\langle 0 \rangle$, which are each one a parallel shift of a given seminal one ${}^{0}[P - Q]$. The use of round brackets, in lieu of squared ones, underlines this difference. Moreover, let us introduce a more compact vector notation:

$$ {}^{0}\boldsymbol{v}_{P,Q} \triangleq \left({}^{0}P - {}^{0}Q \right), \tag{2.5} $$

where the indication of the seminal points is maintained. Whenever the context is clear, the projected vector will be indicated simply as ${}^{0}\boldsymbol{v}$. Moreover, accordingly with the so-called *Grassman rule*, note that a position point ${}^{0}P$ can also be represented as

$$ {}^{0}P = {}^{0}Q + {}^{0}\boldsymbol{v}, \tag{2.6} $$

i.e. as a position point translated toward the desired position, accordingly with the translation provided by a given projected vector.

Finally, within the space parameterized by $\langle 0 \rangle$, a special ordered triple of projected vectors can be further introduced as

$$ \begin{aligned} {}^{0}\boldsymbol{i}_0 &\triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\top} \\ {}^{0}\boldsymbol{j}_0 &\triangleq \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\top} \\ {}^{0}\boldsymbol{k}_0 &\triangleq \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\top} \end{aligned} \tag{2.7} $$

corresponding to the well known so-called projected unit vectors of the x-y-z axes of frame $\langle 0 \rangle$, projected on frame $\langle 0 \rangle$ itself.

2.2 Other Coordinate Systems

Once the ideal observers have agreed about the world coordinate system $\langle 0 \rangle$, they can also agree on the use of other coordinated systems, of which at least one of them is positioned with respect to the originally agreed frame $\langle 0 \rangle$; and each one of the others still positioned with respect to $\langle 0 \rangle$ (directly or indirectly through other frames).

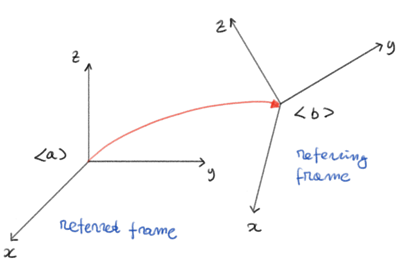

Figure 2.2: Point P and its components on frame $\langle 0 \rangle$.

Fig. 2.2 is an example of a graph indicating how a possible positioning of different frames $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$ has been progressively performed by starting from the seminal one $\langle 0 \rangle$.

Within such a graph, the tips of the arrows joining one frame with another are used for indicating a corresponding positioned frame, otherwise termed as a *referring frame*, while the tails indicate the corresponding *referred frame*, i.e. the one with respect to which the frame at the tip has been positioned. By proceeding in this way a so-called tree of frames (a connected graph) can consequently be constructed, as agreed by all the observers, termed as the *agreed universe of frames*. Let us note how, when positioning a referring frame $\langle b \rangle$ with respect to a referred one $\langle a \rangle$, then the point of view of an observer fixed with the referred frame $\langle a \rangle$ (i.e. just looking to where actually is located with respect to him) is de-facto implicitly adopted.

To describe the positioning of a generic frame $\langle b \rangle$ with respect to another one $\langle a \rangle$, the following set of information data is needed:

$$ \left\{ {}^{a}O_b ; \left({}^{a}\boldsymbol{i}_b, {}^{a}\boldsymbol{j}_b, {}^{a}\boldsymbol{k}_b \right) \right\}, \tag{2.8} $$

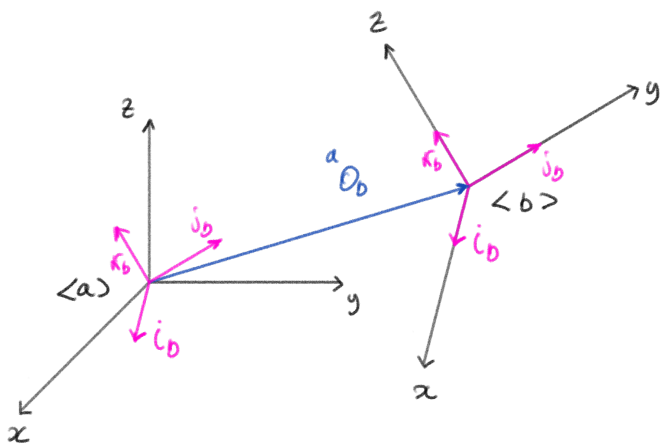

Figure 2.3: Point P and its components on frame $\langle 0 \rangle$.

composed by the where the origin O_b of frame $\langle b \rangle$ is located w.r.t. $\langle a \rangle$, i.e. the projection of O_b on the referred frame $\langle a \rangle$, indicated as ${}^aO_b$; and the ordered triple of unit vectors of $\langle b \rangle$, each one represented with components on frame $\langle a \rangle$. The second set of information serves for establishing which is the *orientation* of frame $\langle b \rangle$ with respect to frame $\langle a \rangle$.

2.2.1 Orientation Matrices

Let us now consider two frames $\langle a \rangle$, $\langle b \rangle$, and let us further consider a generic geometric vector $\boldsymbol{v}$, whose projection ${}^b\boldsymbol{v}$ is it also known. Let us now perform a change of projection coordinate for vector $\boldsymbol{v}$, i.e. let us devise a formula for deducing its projection ${}^a\boldsymbol{v}$ on frame $\langle a \rangle$. To this aim, let us formerly *express* the geometric vector $\boldsymbol{v}$ as the linear combination of the unit vectors of frame $\langle b \rangle$:

$$\boldsymbol{v} = \boldsymbol{i}_b x_b + \boldsymbol{j}_b y_b + \boldsymbol{k}_b z_b. \tag{2.9}$$

After projecting on frame $\langle a \rangle$ one obtains

$${}^a\boldsymbol{v} = {}^a\boldsymbol{i}_b x_b + {}^a\boldsymbol{j}_b y_b + {}^a\boldsymbol{k}_b z_b, \tag{2.10}$$

which rearranged in matrix form becomes

$${}^a\boldsymbol{v} = \begin{bmatrix} {}^a\boldsymbol{i}_b & {}^a\boldsymbol{j}_b & {}^a\boldsymbol{k}_b \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}. \tag{2.11}$$

Then, the 3×3 matrix

$${}^a_b\boldsymbol{R} \triangleq \begin{bmatrix} {}^a\boldsymbol{i}_b & {}^a\boldsymbol{j}_b & {}^a\boldsymbol{k}_b \end{bmatrix}. \tag{2.12}$$

is called *orientation matrix* or *rotation matrix*, and it allows to express the formula for the change of coordinates of a projected vector, i.e.

$${}^a\boldsymbol{v} = {}^a_b\boldsymbol{R}\,{}^b\boldsymbol{v}, \tag{2.13}$$

2.2.1.1 Properties of Orientation Matrices

A first noteworthy property of ${}^a_b\boldsymbol{R}$ is that, calling r_{ij} its generic entry,

$$r_{ij} \in [-1, 1] \;\; \forall \;\; i, j \in \{1, 2, 3\}. \tag{2.14}$$

Moreover, given the orthonormal nature of the basis vectors that compose an orientation matrix, it turns out orientation matrices are orthonormal, hence they satisfy

$${}^a_b\boldsymbol{R}\,{}^a_b\boldsymbol{R}^\top = \boldsymbol{I}_{3\times 3} \tag{2.15}$$

$${}^a_b\boldsymbol{R}^\top\,{}^a_b\boldsymbol{R} = \boldsymbol{I}_{3\times 3} \tag{2.16}$$

implying that

$$ {}^a_b\boldsymbol{R}^\top = {}^a_b\boldsymbol{R}^{-1}. \tag{2.17}$$

Moreover, consider again (2.13) and multiply by the inverse of the rotation matrix

$$ {}^b\boldsymbol{v} = {}^a_b\boldsymbol{R}^{-1}\,{}^a\boldsymbol{v} = {}^a_b\boldsymbol{R}^\top\,{}^a\boldsymbol{v}. \tag{2.18}$$

However, if the change of coordinate problem had been casted with the roles of frames $\langle a\rangle$ and $\langle b\rangle$ swapped, we would have obtained

$$ {}^b\boldsymbol{v} = {}^b_a\boldsymbol{R}\,{}^a\boldsymbol{v}, \tag{2.19}$$

then, due to the arbitrariness of $\boldsymbol{v}$, it follows that we must necessarily have that

$$ {}^a_b\boldsymbol{R}^\top = {}^b_a\boldsymbol{R}, \tag{2.20}$$

which represents the simple formula for knowing the orientation of a first frame with respect to a second one, once the orientation of the second one is known with respect to the first. In particular, the above equation implies the following inner structure of ${}^a_b\boldsymbol{R}$

$$ {}^a_b\boldsymbol{R} = \begin{bmatrix} {}^a\boldsymbol{i}_b & {}^a\boldsymbol{j}_b & {}^a\boldsymbol{k}_b \end{bmatrix} = \begin{bmatrix} {}^b\boldsymbol{i}_a^\top \\ {}^b\boldsymbol{j}_a^\top \\ {}^b\boldsymbol{k}_a^\top \end{bmatrix}, \tag{2.21}$$

which provides a very easy mean for immediately looking at the mutual orientations between two given frames.

Recalling that for any two square non singular matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size:

$$ \det(\boldsymbol{AB}) = \det\boldsymbol{A}\,\det\boldsymbol{B} \tag{2.22}$$

(Binet Theorem), and that

$$ \det\boldsymbol{A} = \det\boldsymbol{A}^\top, \tag{2.23}$$

it follows that for any $3D$ matrix $\boldsymbol{M}$ for which $\boldsymbol{M}\,\boldsymbol{M}^\top = \boldsymbol{I}_{3\times3}$ it must also be

$$ \det\boldsymbol{M} = \pm 1. \tag{2.24}$$

It follows that any $3D$ rotation matrix $\boldsymbol{R}$ must have $|\det\boldsymbol{R}| = 1$: actually, given that a rigid rotation must preserve relative orientations (i.e. a right hand side orthonormal frame must be transformed in another right hand side orthonormal frame) it follows that

$$ \det\boldsymbol{R} = +1. \tag{2.25}$$

2.2.1.2 Group Properties of Rotation Matrices in $\Re^n$

Following the same line of reasoning described before for $3D$ rotations, one can define a rotation matrix in $\Re^n$ as a matrix $\boldsymbol{R}\in\Re^{n\times n}$ such that: $\boldsymbol{R}^\top\boldsymbol{R} = I_{n\times n}$ (assuring *isometry* or preservation of the euclidean norm) and $\det\boldsymbol{R} = +1$ (assuring the preservation of relative direction). Indeed, the set of all possible $n-$dimensional matrices having such properties can be shown to form a group in $\Re^n$. In particular, define

$$ SO(n) = \left\{\boldsymbol{R}\in\Re^{n\times n} : \boldsymbol{R}^\top\boldsymbol{R} = \boldsymbol{I}_{n\times n},\ \det\boldsymbol{R} = +1\right\} \tag{2.26}$$

that is called the special orthogonal group where the term special refers to the fact that $\det\boldsymbol{R} = +1$ rather than $\det\boldsymbol{R} = \pm1$. The fact that $SO(n)$ is named a group refers to the fact that its elements satisfy all the group axioms under the operation of matrix multiplication [**?**]:

- Closure: given any two elements $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of $SO(n)$ also $\boldsymbol{R}_1\boldsymbol{R}_2$ belongs to $SO(n)$.
- Identity: there exists an element $\boldsymbol{I}$ of $SO(n)$ such that $\boldsymbol{IR}\in SO(n)$ for any $\boldsymbol{R}\in SO(n)$.
- Inverse: for any $\boldsymbol{R}\in SO(n)$ there exists $\boldsymbol{R}^{-1}\in SO(n)$ such that $\boldsymbol{RR}^{-1} = \boldsymbol{I}$.
- Associativity: given any three elements $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3$ in $SO(n)$ it follows that $\boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3) = (\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3$.

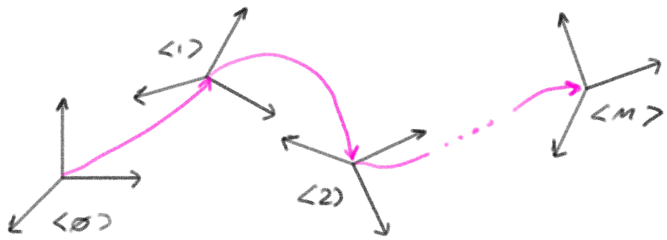

Figure 2.4: A chain of frames. By recursively going backwards the chain, each time applying the change of projection of a vector, one obtains the closure property of $SO(n)$ matrices.

The above results can be proven by direct calculation and are left as exercise. The closure property is particularly interesting in $3D$ applications as it reveals how rotations may be combined. To this aim, let us consider a chain of $n+1$ frames, as depicted in Fig. 2.4, for which we only evidence their cascaded orientation matrices ${}^{i-1}_{i}\boldsymbol{R}, i = 1, \ldots, n$. We want to deduce the orientation matrix ${}^0_n\boldsymbol{R}$ from the last frame with respect to the first one. To this aim, let us extend the problem considered in the previous section, and let us assume to know the projection ${}^n\boldsymbol{v}$ of a generic vector $\boldsymbol{v}$ on the last frame, and then let us evaluate its projection ${}^0\boldsymbol{v}$ on the first one. Repeating backwardly the change of coordinate formula developed in the previous section, one obtains:

$$ {}^0\boldsymbol{v} = \left({}^0_1\boldsymbol{R}{}^1_2\boldsymbol{R}\cdots{}^{i-1}_{i}\boldsymbol{R}\cdots{}^{n-1}_{n}\boldsymbol{R}\right){}^n\boldsymbol{v}, \tag{2.27} $$

that, accordingly with the arbitrariness of $\boldsymbol{v}$, allows stating that

$$ {}^0_n\boldsymbol{R} = {}^0_1\boldsymbol{R}{}^1_2\boldsymbol{R}\cdots{}^{i-1}_{i}\boldsymbol{R}\cdots{}^{n-1}_{n}\boldsymbol{R}. \tag{2.28} $$

2.3 Transformation Matrices

In the previous sections we have focused on the representation of the mutual orientation between two frames, neglecting the position information represented by ${}^aO_b$. Let us now introduce a compact notation that takes into account both the positioning information and the orientation one in a 4×4 matrix

$$ {}^a_b\boldsymbol{T} \triangleq \begin{bmatrix} {}^a_b\boldsymbol{R} & {}^aO_b \\ \boldsymbol{0}_{3\times 1} & 1 \end{bmatrix}, \tag{2.29} $$

which is termed *Transformation Matrix* (sometimes also called *Homogeneous Matrix*) of frame $\langle b\rangle$ with respect to frame $\langle a\rangle$. The reasons why the above matrix is constructed to be 4×4, with the first three elements of the fourth row always zero, and the last one always unitary, will be clarified in the sequel.

Let us consider a frame $\langle b\rangle$, positioned with respect to $\langle a\rangle$ within an agreed universe, via its transformation matrix ${}^a_b\boldsymbol{T}$. Let us further assume to know the position ${}^bP$ of a generic point P with respect to frame $\langle b\rangle$ and let us compute its position ${}^aP$ with respect to frame $\langle a\rangle$.

The above stated problem can be trivially solved, by observing that it will necessarily be

$$ {}^aP = {}^aO_b + {}^a_b\boldsymbol{R}{}^bP \tag{2.30} $$

as changing the coordinates of projected point from one frame to another one always requires its reference origin to be changed accordingly, as Fig. **??** shows.

Let us now represent projected points via the so called *homogeneous coordinates*, i.e. via the following 4×1 algebraic vector

$$ {}^b\bar{P} = \begin{bmatrix} {}^bP \\ 1 \end{bmatrix}, \tag{2.31} $$

where the scalar value 1 is always added as the fourth component. Then, exploiting this representation, the change of coordinates formula for a point can be more compactly rewritten as

$$ {}^b\bar{P} = {}^b_a\boldsymbol{T}{}^a\bar{P}. \tag{2.32} $$

Following similar steps as done for orientation matrices, it is possible to show that

$$ {}_b^a\boldsymbol{T}^{-1} = {}_a^b\boldsymbol{T}. \tag{2.33}$$

Moreover, the inversion of a transformation matrix can be found as:

$$ {}_b^a\boldsymbol{T}^{-1} = \begin{bmatrix} {}_b^a\boldsymbol{R}^\top & -{}_b^a\boldsymbol{R}^{\top\,a}O_b \\ \boldsymbol{0}_{3\times 1} & 1 \end{bmatrix}. \tag{2.34}$$

2.3.1 Evaluation of Transformation Matrices within Trees of Frames

Let us again consider a chain of $n+1$ frames, as depicted in Fig. **??**. Following the same idea exploited for orientation matrices, consider as known the position of a point P with respect to the last frame, i.e. ${}^n\bar{P}$. Then, its position w.r.t. the first frame $\langle 0\rangle$ can be found by repeatedly using the change of coordinates formula

$$ {}^0\bar{P} = \left({}_1^0\boldsymbol{T}{}_2^1\boldsymbol{T}\cdots{}_i^{i-1}\boldsymbol{T}\cdots{}_n^{n-1}\boldsymbol{T}\right){}^n\bar{P}, \tag{2.35}$$

that, accordingly with the arbitrariness of P, allows stating that

$$ {}_n^0\boldsymbol{T} = {}_1^0\boldsymbol{T}{}_2^1\boldsymbol{T}\cdots{}_i^{i-1}\boldsymbol{T}\cdots{}_n^{n-1}\boldsymbol{T}. \tag{2.36}$$

2.4 Free from Coordinate Notation

In the previous sections, we have shown how to change the projection of points, vectors and applied vectors to a different frame within an agreed universe of frames. It was shown how a frame can be described via different equivalent positioning with respect to the other frames. This makes the original agreed world frame $\langle 0\rangle$ actually loosing its character of privilege, which was only apparently assigned to it.

Therefore, let us introduce the so-called *free from coordinate* notation, for indicating points, vectors and applied vectors without mentioning any projection frame since, in force of existing transformation equivalence, it can be any one among the agreed ones.

More precisely, in terms of free from coordinate notation, points, vectors, and applied vectors, will be more simply denoted as

- P - Capital letters for points;
- $\boldsymbol{v} = (P - Q)$ - bold lower case letters, or round parentheses difference of points, for vectors;
- $[P - Q]$ - square bracketed difference of points, or a vector-point couple for applied vectors;

where the introduced notation identifies, for each item, the entire class of the associated projections, which transform each other via coordinate changes.

It must be explicitly noted how, with the use of such a notation, the corresponding items result into *symbolic* items, each one of them generally termed as a *geometric-vector*, or a *geometric-point*. Instead, their projected equivalent forms are *algebraic-vectors* (i.e. 3×1 matrices, or 4×1 in case of homogeneous coordinates).

Finally note how even the notation for denoting frames, say for instance a generic frame $\langle a\rangle$, can be it also interpreted as a free-from-coordinates notation, where the explicit indication of its referring frame is avoided, since it could actually be any one of the other frames within the agreed universe.

2.5 Universes of Frames and Reference Systems

The present subsection is more of conceptual nature than technical. It will fix some further concepts (mainly the concept of Reference System) that will permeate the contents of all the successive chapters.

We can start the discussion by noticing how the agreed universe of frames (i.e. coordinate systems) as well as the Euclidean space (i.e. the whole set of points parameterized by each frame of the agreed universe) has been de-facto presented as if they were "static entities" not subjected to any change after their establishment. Of course, this is definitely not true, since if we now explicitly consider the time, and also assume the frames of a universe maintaining their "individuality along time", then we certainly have situations where at least some of the universe frames are changing their transformation matrices along time, with respect to at least one of the other frames of the universe itself.

So, if we have a frame $\langle\alpha\rangle$ of the universe exhibiting such a behavior, at a first glance we might consequently say that is therefore "moving" inside the universe, with respect to each one of the frames with respect to which it is changing its transformation matrices.

However, if we conversely consider one or more of such other frames (i.e. those with respect to which has been at a first glance considered "moving") we could say that each one of them are moving with respect to $\langle\alpha\rangle$. This in total accordance with the trivial well known fact that the concept of motion is a *relative* one.

As the consequence of the motion relativity, when at least a frame of a universe changes in time its transformation matrix with respect to other frames, we can only say that it is the configuration of the entire universe which is actually changing. The interpretation for such changes is let to the "adopted point of view", i.e. the frame from which such changes are observed.

For a given frame $\langle\alpha\rangle$, we can consequently adopt the free-from-coordinates notation $\langle\alpha\rangle(t)$ for representing it within the time varying context induced by its supporting entire class of now time varying transformation matrices ${}^{i}_{\alpha}\boldsymbol{T}$, $i = 1, \ldots, n$, with respect to all other frames of the universe. If we also consider points characterized by their own "individuality" along time, we can use the free-from-coordinate notation $P(t)$ to denote them. A point that does not change its position with respect to a frame $\langle\alpha\rangle(t)$ is a pointed "attached" to $\langle\alpha\rangle(t)$; nevertheless, it can still be a point moving within the universe, if and only if $\langle\alpha\rangle(t)$ is moving in the previously specified relative sense.

Motions within the universe are also allowed for "individual vectors" (applied or not), as the results of the motion assigned to their identifying points.

At this point, let us consider the set of individual points which are not varying their position with respect to $\langle\alpha\rangle(t)$ itself, and let us term such a rigid space as the *Reference System* established by frame $\langle\alpha\rangle(t)$, to be for instance denote as $\Sigma_\alpha(t)$.

Let us note that any other frame $\langle b\rangle(t)$ exhibiting a constant position and orientation with respect to $\langle\alpha\rangle(t)$, shares with it the same rigid space. Hence, a reference system can also be intended as established by the entire class of additional frames that maintain constant position and orientation with respect to the given one.

Intrinsic to the definition of reference system (or rigid space) is the possibility of enlarging the original universe of frames with the infinite continuous sets of frames belonging, each one of them, to the reference systems of the original ones. This possibility makes the universe becoming continuous also with respect to frames, as it was already as regard points, vectors and applied vectors. Furthermore, the originally agreed *discrete set* of frames actually lose its character of privilege. Indeed, any collection of frames within the different reference systems could be used for constructing (or in a sense reconstructing) the same universe.

Finally, let us explicitly note how there exists a general tendency to consider the three distinct terms "frames" (i.e. "coordinate systems"), "reference systems/rigid spaces" and "ideal observers", as synonymous one of the other, notwithstanding the fact that they are slightly differently originated concepts. For instance, an observer should be more properly assigned to a reference system rather than to a specific frame, i.e. it should be delocalized within the assigned reference system. However, its attachment to a specific frame is in any case accepted, within the obvious understanding of considering it equivalent to any other attached to a frame of the same reference system (i.e. they all are de-facto the same observer).

2.6 Vector Operations

Let us now introduce how vector operations can be performed. A vector can be *expressed* on a frame, say $\langle a\rangle$, yielding

$$\boldsymbol{v} = x_a \boldsymbol{i}_a + y_a \boldsymbol{j}_a + z_a \boldsymbol{v}_a, \tag{2.37}$$

which is a symbolic representation of $\boldsymbol{v}$, i.e. still a geometric one, not an algebraic one. If we instead consider the *projection* on frame $\langle a\rangle$ we mean the following algebraic vector:

$${}^{a}\boldsymbol{v} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}. \tag{2.38}$$

All vector operations must be projected on the same frame, and the result will also be on that frame. Consider the geometric expression

$$2\boldsymbol{v} = (x_a \boldsymbol{i}_a + y_a \boldsymbol{j}_a + z_a \boldsymbol{v}_a) + (x_b \boldsymbol{i}_b + y_b \boldsymbol{j}_b + z_b \boldsymbol{v}_b). \tag{2.39}$$

In order to compute it, we need to project it on a frame. Of course, we cannot compute ${}^a\boldsymbol{v} + {}^b\boldsymbol{v}$. Instead, the correct formulation would require the use of an orientation matrix, namely

$$2^a\boldsymbol{v} = {}^a\boldsymbol{v} + {}^a_b\boldsymbol{R}^b\boldsymbol{v} \tag{2.40}$$

2.6.1 Linear Combinations

In general, a linear combination of n vectors, named $\boldsymbol{v}_1, \dots \boldsymbol{v}_n$ can be written in geometric, free from coordinate notation as

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n \tag{2.41}$$

Once projected on a given frame, say $\langle 0 \rangle$, the above formula becomes

$${}^0\boldsymbol{v} = c_1{}^0\boldsymbol{v}_1 + c_2{}^0\boldsymbol{v}_2 + \cdots + {}^0\boldsymbol{v}_n \tag{2.42}$$

2.6.2 Scalar Product

In free from coordinate notation, the scalar product between two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ is defined as

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = \boldsymbol{v}_2 \bullet \boldsymbol{v}_1 = \|\boldsymbol{v}_1\|\|\boldsymbol{v}_2\| \cos\theta. \tag{2.43}$$

Note that the above equation is directly defined without require any concept of frame, but it is instead simply rooted in its geometrical meaning. Once projected on a frame, say $\langle a \rangle$, the above formula becomes

$${}^a(\boldsymbol{v}_1 \bullet \boldsymbol{v}_2) = {}^a\boldsymbol{v}_1^{\top}\,{}^a\boldsymbol{v}_2. \tag{2.44}$$

It is then interesting to define the following linear operator

$$(\boldsymbol{v}_1 \bullet) \tag{2.45}$$

as the 1×3 matrix that performs the scalar product between the input vector and $\boldsymbol{v}_1$. Once projected on a frame, the operator simply becomes the transpose of $\boldsymbol{v}_1$ computed on that frame.

2.6.3 Vector Product

In free from coordinate notation, the vector product (also called cross product) between two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ is defined as

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{cases} \|\boldsymbol{v}_1 \times \boldsymbol{v}_2\| = \|\boldsymbol{v}_1\|\|\boldsymbol{v}_2\| \sin\theta \\ \boldsymbol{v}_1 \times \boldsymbol{v}_2 \perp \boldsymbol{v}_1, \boldsymbol{v}_2 \end{cases}, \tag{2.46}$$

and where the direction of the resulting vector is established according to the right hand rule. Again, the above equation is directly defined without require any concept of frame. The vector product is anti-commutative, i.e.,

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = -(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \tag{2.47}$$

Once projected on a frame, say $\langle a \rangle$, the above formula becomes

$${}^a(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = {}^a[\boldsymbol{v}_1 \times]\,{}^a\boldsymbol{v}_2, \tag{2.48}$$

where

$${}^a[\boldsymbol{v}_1 \times] \triangleq \begin{bmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{bmatrix} \tag{2.49}$$

and where x_a, y_a, z_a are the components of $\boldsymbol{v}_1$ once projected on $\langle a \rangle$. Similarly to the scalar product, the linear operator

$$[\boldsymbol{v} \times] \tag{2.50}$$

is the 3×3 matrix that performs the vector product between the input vector and $\boldsymbol{v}_1$. The operator is also called the skew-symmetric operator, due to its anti-symmetric property.

2.7 Three-parameter Representation of Orientation Matrices

For the subsequent developments, it is now important to introduce other more compact ways of representing an orientation matrix ${}^a_b\boldsymbol{R}$ between two generic frames $\langle a\rangle$, $\langle b\rangle$.

To this aim, let us recall that orientation matrices are orthonormal, therefore their 9 elements must satisfy the following 6 constraints:

- Unimodularity. Its three columns (rows) are unit vectors, that is

$$\begin{aligned} {}^a\boldsymbol{i}_b^\top {}^a\boldsymbol{i}_b &= 1, \\ {}^a\boldsymbol{j}_b^\top {}^a\boldsymbol{j}_b &= 1, \\ {}^a\boldsymbol{k}_b^\top {}^a\boldsymbol{k}_b &= 1. \end{aligned} \tag{2.51}$$

- Orthogonality. Its three columns (rows) are orthogonal to each other, that is

$$\begin{aligned} {}^a\boldsymbol{i}_b^\top {}^a\boldsymbol{j}_b &= 0, \\ {}^a\boldsymbol{j}_b^\top {}^a\boldsymbol{k}_b &= 0, \\ {}^a\boldsymbol{k}_b^\top {}^a\boldsymbol{i}_b &= 0. \end{aligned} \tag{2.52}$$

Therefore, due to the existence of the 6 constraints shown above, it is possible to represent ${}^a_b\boldsymbol{R}$ with three parameters only, even if in a not unique manner.

2.7.1 Euler angles

Any orientation can be achieved by composing three elementary rotations in sequence. These can occur either about the axes of a fixed coordinate system (extrinsic rotations), or about the axes of a rotating coordinate system (intrinsic rotations) initially aligned with the fixed one. Usually the rotating frame is assumed attached to a rigid body. We have the followinig distinction:

- Proper Euler angles: x-z-x, y-z-y, ...
- Tait-Bryan angles: z-y-x, x-y-z, ...

The notation z-y'-x" denotes an intrinsic rotation, first around the axis z, then about the newly found axis y' and finally about the axis x". The sequence corresponds to the yaw-pitch-roll sequence, which is the most used.

$${}^a_b\boldsymbol{R} = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) \tag{2.53}$$

2.7.2 Exponential (Angle-Axis) Representation

A particularly interesting minimal representation of 3D rotation matrices is the so called *exponential* or *angle-axis* representation. To introduce such a parameterization, let us recall the definition of the exponential of a matrix operator. Given a real square matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ and $\theta \in \mathbb{R}$, the exponential matrix of $\theta\boldsymbol{M}$ belongs to $\mathbb{R}^{n\times n}$ and is defined as

$$e^{\theta\boldsymbol{M}} = \sum_{k=0}^{\infty} \frac{(\theta\boldsymbol{M})^k}{k!}. \tag{2.54}$$

Then, let us consider two frames $\langle\alpha\rangle$, $\langle\beta\rangle$, initially coinciding. Let us further consider an applied geometric unit vector $(\boldsymbol{v}, O_\alpha) = (\boldsymbol{v}, O_\beta)$ passing through the common origin of the two frames, whose initial projection on $\langle\alpha\rangle$ is the same of that on $\langle\beta\rangle$. Then let us consider that frame $\langle\beta\rangle$ (and consequently its associated reference system) is purely rotated around $\boldsymbol{v}$ of an angle θ, even negative, accordingly with the right-hand rule, as indicated in Fig. **??**.

So doing, we note that the axis-line defined by $(\boldsymbol{v}, O_\alpha) = (\boldsymbol{v}, O_\beta)$ remains common to both the reference systems of the two frames $\langle\alpha\rangle$, $\langle\beta\rangle$, (like the axis of the hinge of a door, which belongs to the door whatever is its rotation, as well as to the wall) we have that

$${}^\alpha\boldsymbol{v} = {}^\beta\boldsymbol{v} \triangleq {}^*\boldsymbol{v},\ \forall\theta, \tag{2.55}$$

where the introduced notation ${}^{*}\boldsymbol{v}$ recalls that the geometric vector $\boldsymbol{v}$ can be indifferently projected on frames $\langle\alpha\rangle$ or $\langle\beta\rangle$.

Then, the orientation matrix constructed in the above way is said to be represented by its equivalent angle-axis representation that admits the following equivalent analytical expressions (Rodrigues formula)

$$\boldsymbol{R}({}^{*}\boldsymbol{v},\theta) = e^{[{}^{*}\boldsymbol{v}\wedge]\theta} = \boldsymbol{I}_{3\times 3} + [{}^{*}\boldsymbol{v}\wedge]\sin\theta + [{}^{*}\boldsymbol{v}\wedge]^{2}\,(1-\cos\theta), \tag{2.56}$$

where the last expression is the direct consequence of the fact that the matrix exponent is actually a skew symmetric one.

2.7.3 Unit Vector Lemma

The angle-axis vector can be computed using the following "unit vector lemma". Given two frames $\langle a\rangle$, $\langle b\rangle$, the following equivalence holds:

$$\begin{cases} (\boldsymbol{i}_a \wedge \boldsymbol{i}_b) + (\boldsymbol{j}_a \wedge \boldsymbol{j}_b) + (\boldsymbol{k}_a \wedge \boldsymbol{k}_b) = 2\boldsymbol{v}\sin\theta \\ (\boldsymbol{i}_a \cdot \boldsymbol{i}_b) + (\boldsymbol{j}_a \cdot \boldsymbol{j}_b) + (\boldsymbol{k}_a \cdot \boldsymbol{k}_b) = 1 + 2\cos\theta \end{cases} \tag{2.57}$$

These equations allow us to compute the angle-axis vector $\boldsymbol{v},\theta$ from the knowledge of the rotation matrix ${}^{a}_{b}\boldsymbol{R}$.

Let us express (2.57) in a frame of our choice, say $\langle s\rangle$, obtaining

$$\begin{cases} [{}^{s}\boldsymbol{i}_a\wedge]{}^{s}\boldsymbol{i}_b + [{}^{s}\boldsymbol{j}_a\wedge]{}^{s}\boldsymbol{j}_b + [{}^{s}\boldsymbol{k}_a\wedge]{}^{s}\boldsymbol{k}_b = 2{}^{s}\boldsymbol{v}\sin\theta \\ ({}^{s}\boldsymbol{i}_a^{\top}\,{}^{s}\boldsymbol{i}_b) + ({}^{s}\boldsymbol{j}_a^{\top}\,{}^{s}\boldsymbol{j}_b) + ({}^{s}\boldsymbol{k}_a^{\top}\,{}^{s}\boldsymbol{k}_b) = 1 + 2\cos\theta \end{cases} \tag{2.58}$$

where we have defined the skew matrix as

$$[{}^{s}\boldsymbol{i}_a\wedge] = \begin{bmatrix} 0 & -{}^{s}\boldsymbol{i}_{a_z} & {}^{s}\boldsymbol{i}_{a_y} \\ {}^{s}\boldsymbol{i}_{a_z} & 0 & -{}^{s}\boldsymbol{i}_{a_x} \\ -{}^{s}\boldsymbol{i}_{a_y} & {}^{s}\boldsymbol{i}_{a_x} & 0 \end{bmatrix}. \tag{2.59}$$

Solving (2.58) gives us ${}^{s}\boldsymbol{v},\theta$.

Note that:

$${}^{s}_{a}\boldsymbol{R} = [{}^{s}\boldsymbol{i}_a|...] \qquad {}^{s}_{b}\boldsymbol{R} = [{}^{s}\boldsymbol{i}_b|...] \tag{2.60}$$

This means that ${}^{s}_{a}\boldsymbol{R}$ and ${}^{s}_{b}\boldsymbol{R}$ contain all the vectors needed in the projected versor lemma equation. From their knowledge it is possible to compute the angle-axis representation, which will be projected on the common frame $\langle s\rangle$.

Chapter 3

Kinematics Fundamentals

3.1 Time Derivative of Orientation and Transformation Matrices

Let us start by considering the rate of change of a generic transformation matrix between two frames, within a given universe. This analysis, and in particular the analysis regarding its orientation part, will allow us to introduce the important concept of angular velocity vector.

To these aims, let us start by considering two frames $\langle a \rangle$, $\langle b \rangle$, extracted from a given universe, which are assumed to be moving inside it, and in general moving one with respect to the other. This is described by the, time varying, transformation matrix ${}^a_b\boldsymbol{T}(t)$. In particular, let us adopt the point of view of an observer at all times fixed with frame $\langle a \rangle$ (i.e. an observer in the reference system of frame $\langle a \rangle$).

Then, the time evolution of ${}^a_b\boldsymbol{T}(t)$ is governed by its time derivative ${}^a_b\dot{\boldsymbol{T}}(t)$ taking the form

$$ {}^a_b\dot{\boldsymbol{T}}(t) = \begin{bmatrix} {}^a_b\dot{\boldsymbol{R}}(t) & {}^a\dot{O}_b(t) \\ \mathbf{0}_{3\times 1} & 0 \end{bmatrix}, \tag{3.1} $$

from which ${}^a_b\boldsymbol{T}(t)$ follows via time integration, from an initial condition ${}^a_b\boldsymbol{T}(0)$, of the following *separate* time integrations:

$$ {}^a_b\boldsymbol{R}(t) = \int_0^t {}^a_b\dot{\boldsymbol{R}}(t)dt + {}^a_b\boldsymbol{R}(0) \tag{3.2} $$

$$ {}^aO_b(t) = \int_0^t {}^a\dot{O}_b(t)dt + {}^aO_b(0). \tag{3.3} $$

The above equations imply and are implied by the following expressions termed *progressive-sums of infinitesimal contributions* associated to the corresponding time integrals:

$$ {}^a_b\boldsymbol{R}(t+dt) = {}^a_b\boldsymbol{R}(t) + {}^a_b\dot{\boldsymbol{R}}(t)dt + O(dt) \tag{3.4} $$

$$ {}^aO_b(t+dt) = {}^aO_b(t) + {}^a\dot{O}_b(t)dt + O(dt), \tag{3.5} $$

in which the notation $O(dt)$ means "infinitesimal terms of order greater than dt".

Let us remark how in (3.4) the sum of infinitesimal contributions ${}^a_b\dot{\boldsymbol{R}}(t)dt$ does not, in general, preserves the orthogonality of the resulting matrix. Indeed, let us consider the *exact* form of *progressive products of infinitesimal-rotation matrices*, for representing the evolution ${}^a_b\boldsymbol{R}(t)$ via a sequence of infinitesimal rotations:

$$ {}^a_b\boldsymbol{R}(t+dt) = {}^a_b\boldsymbol{R}(t)e^{\left[{}^b\boldsymbol{d\theta}_{b/a}\wedge\right]} = e^{\left[{}^a\boldsymbol{d\theta}_{b/a}\wedge\right]}{}^a_b\boldsymbol{R}(t), \tag{3.6} $$

where the infinitesimal rotation vector $\boldsymbol{d\theta}_{b/a}$ has been directly defined, without any relationship with any pre-existing finite one. Note that the lower-right index b/a, appearing in both projected vectors and related free-from-coordinate abstraction, has to be read as "frame $\langle b \rangle$ with respect to $\langle a \rangle$". This serves for recalling which is the referred frame ($\langle a \rangle$), with respect to which the referring one ($\langle b \rangle$) is currently infinitesimally changing its orientation.

Let us now represent the geometric vector $\boldsymbol{d\theta}_{b/a}$ via the following geometric differential form

$$ \boldsymbol{d\theta}_{b/a} \triangleq \boldsymbol{\omega}_{b/a}dt, \tag{3.7} $$

where the introduced geometric vector $\boldsymbol{\omega}_{b/a}$ has the following properties:

- its unit vector $\boldsymbol{u}_{b/a}$ identifies at time t the instantaneous axis around which frame $\langle b\rangle$ can be considered as instantaneously rotating with respect to $\langle a\rangle$;
- its norm $\dot{\alpha} \triangleq \boldsymbol{u}_{b/a} \cdot \boldsymbol{\omega}_{b/a}$ identifies the instantaneous rotation rate at time instant t.

Such a geometric vector is termed the *angular velocity vector* of frame $\langle b\rangle$ with respect to $\langle a\rangle$. Again, let us remark how the term "velocity" does not mean, in general, that the vector corresponds to the time-derivative of a pre-existing vector.

By formerly substituting the definition of angular velocity (3.7) in (3.6) and then in (3.4) one obtains

$$ {}^a_b\boldsymbol{R}(t)e^{\left[{}^b\boldsymbol{\omega}_{b/a}\wedge\right]dt} = e^{\left[{}^a\boldsymbol{\omega}_{b/a}\wedge\right]dt}\,{}^a_b\boldsymbol{R}(t) = {}^a_b\boldsymbol{R}(t) + {}^a_b\dot{\boldsymbol{R}}(t)dt + O(dt). \tag{3.8}$$

Rearranging some terms allow to find out

$$ {}^a_b\dot{\boldsymbol{R}}(t) = \frac{{}^a_b\boldsymbol{R}(t)\left(e^{\left[{}^b\boldsymbol{\omega}_{b/a}\wedge\right]dt} - \boldsymbol{I}_{3\times 3}\right)}{dt} - \frac{O(dt)}{dt} = \frac{\left(e^{\left[{}^a\boldsymbol{\omega}_{b/a}\wedge\right]dt} - \boldsymbol{I}_{3\times 3}\right){}^a_b\boldsymbol{R}(t)}{dt} - \frac{O(dt)}{dt} \tag{3.9}$$

that for $dt \to 0$ finally leads to the following (hint: represent the matrix-exponential as its power-series and then note how the ratios with dt of all terms of greater infinitesimal order, necessarily go to zero):

$$ {}^a_b\dot{\boldsymbol{R}}(t) = {}^a_b\boldsymbol{R}(t)\left[{}^b\boldsymbol{\omega}_{b/a}\wedge\right] = \left[{}^a\boldsymbol{\omega}_{b/a}\wedge\right]{}^a_b\boldsymbol{R}(t), \tag{3.10}$$

which are matrix differential equations termed (for historical reasons) *Strap-down equations* of the first and second type, respectively. Such equations establish the searched structural constraints to which ${}^a_b\dot{\boldsymbol{R}}(t)$ obeys and highlight that the *sole cause* for the time change of a rotation matrix, of a frame with respect to another one, is just the angular velocity vector of the referring frame with respect to the referred one.

3.2 Composition of Angular Velocity Vectors

Let us consider three frames $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$ for which we assume to know $\boldsymbol{\omega}_{b/a}, \boldsymbol{\omega}_{c/b}$. We wish to evaluate $\boldsymbol{\omega}_{c/a}$. In this situation we have that

$$ {}^a_c\boldsymbol{R}(t+dt) = {}^a_b\boldsymbol{R}(t+dt){}^b_c\boldsymbol{R}(t+dt) = {}^a_b\boldsymbol{R}(t)e^{\left[{}^b\boldsymbol{\omega}_{b/a}\wedge\right]dt}{}^b_c\boldsymbol{R}(t)e^{\left[{}^c\boldsymbol{\omega}_{c/b}\wedge\right]dt}. \tag{3.11}$$

Now, using the r.h.s equality in (3.6), let us transfer the infinitesimal orientation matrices to the right, i.e.

$$ {}^a_c\boldsymbol{R}(t+dt) = {}^a_b\boldsymbol{R}(t){}^b_c\boldsymbol{R}(t)e^{\left[{}^c\boldsymbol{\omega}_{b/a}\wedge\right]dt}e^{\left[{}^c\boldsymbol{\omega}_{c/b}\wedge\right]dt} = {}^a_c\boldsymbol{R}(t)e^{\left[{}^c\boldsymbol{\omega}_{b/a}\wedge\right]dt}e^{\left[{}^c\boldsymbol{\omega}_{c/b}\wedge\right]dt}. \tag{3.12}$$

From the previous equation, let us compute the associated incremental ratio at time t as

$$ \frac{{}^a_c\boldsymbol{R}(t+dt) - {}^a_c\boldsymbol{R}(t)}{dt} = \frac{{}^a_c\boldsymbol{R}(t)\left(e^{\left[{}^c\boldsymbol{\omega}_{b/a}\wedge\right]dt}e^{\left[{}^c\boldsymbol{\omega}_{c/b}\wedge\right]dt} - \boldsymbol{I}_{3\times 3}\right)}{dt}, \tag{3.13}$$

whose limit for $dt \to 0$ provides the following result (hint: represent the matrix exponential as series expansions; then imagine to execute the resulting countable products; and then, as usual, keep into account that all ratios by dt of infinitesimals of order greater than dt tend to zero for $dt \to 0$):

$$ {}^a_c\dot{\boldsymbol{R}}(t) = {}^a_c\boldsymbol{R}(t)\left[{}^c\left(\boldsymbol{\omega}_{b/a} + \boldsymbol{\omega}_{c/b}\right)\wedge\right], \tag{3.14}$$

where we recognize the strap-down equation of the first type for frame $\langle c\rangle$ with respect to frame $\langle a\rangle$. Consequently, we have that

$$ \boldsymbol{\omega}_{c/a} = \boldsymbol{\omega}_{b/a} + \boldsymbol{\omega}_{c/b}, \tag{3.15}$$

which is commonly known as the *law of composition (or addition) of angular velocity vectors*. Finally, if we consider $\langle c\rangle \equiv \langle a\rangle$ we immediately get the well know result

$$ \boldsymbol{\omega}_{a/b} = -\boldsymbol{\omega}_{b/a}. \tag{3.16}$$

As a direct consequence of the law of addition for angular velocity vectors, let us remark how the angular velocity vector $\boldsymbol{\omega}_{b/a}$ between two frames $\langle a\rangle$, $\langle b\rangle$, in its geometric form is actually *common* to that of any other couples of frames $\langle\bar{a}\rangle$, $\langle\bar{b}\rangle$ respectively belonging to their reference systems. Therefore, the notation $\boldsymbol{\omega}_{b/a}$ should be more properly read as the "angular velocity vector of the whole reference system associated to frame $\langle b\rangle$, with respect to the one associated to frame $\langle a\rangle$". This is because the observer is not only as associated with frame $\langle a\rangle$, but actually with the whole reference system established by $\langle a\rangle$ itself.

3.3 Time Derivative of Vectors

Within a given, generally time varying universe, consider a geometric vector $\boldsymbol{s}(t)$. Let us recall how such a geometric vector represents the *entire class* of its projected vectors within the given universe considered at time t. Furthermore, remember that a vector is considered "moving" inside the universe if it changes projection coordinates with respect to at least one of its frames.

Accordingly with this consideration, the way $\boldsymbol{s}(t)$ changes in time its coordinates depends on the projection frame. Therefore, let us start by considering the "time derivative of a *projected-vector* on a frame", i.e. the time derivative of the algebraic vector representing its projection on a given frame. To this aim, let us consider a frame $\langle a\rangle$, with an observer associated to it. For its point of view, the observer will see the time changes occurring on the projected vector ${}^a\boldsymbol{s}(t)$ as

$$ {}^a\dot{\boldsymbol{s}}(t) \triangleq \frac{d}{dt}{}^a\boldsymbol{s}(t) \triangleq D^a\boldsymbol{s}(t), \tag{3.17}$$

where the shorter notation ${}^a\dot{\boldsymbol{s}}(t)$ has to be intended under the implicit understanding of always performing *differentiation after projection*. Furthermore, accordingly with the way it is constructed (i.e. by differentiating each component of ${}^a\boldsymbol{s}(t)$), ${}^a\dot{\boldsymbol{s}}(t)$ consequently results into an algebraic vector directly defined on frame $\langle a\rangle$.

The previous equation (3.17) implies that ${}^a\boldsymbol{s}(t)$ can be reconstructed via the simple time-integration

$$ {}^a\boldsymbol{s}(t) = \int_0^t {}^a\dot{\boldsymbol{s}}(t)dt + {}^a\boldsymbol{s}(0) \tag{3.18}$$

$$ {}^a\boldsymbol{s}(t+dt) = {}^a\boldsymbol{s}(t) + {}^a\dot{\boldsymbol{s}}(t)dt + O(dt) \tag{3.19}$$

3.3.1 Change of Differentiation Frame

Another observer, located on frame $\langle b\rangle$, will see the projected vector ${}^b\boldsymbol{s}(t)$ vary accordingly with ${}^b\dot{\boldsymbol{s}}(t)$. We are interested in establishing a link between these two point of views. In general

$$ {}^b\dot{\boldsymbol{s}}(t) \neq {}^b_a\boldsymbol{R}\,{}^a\dot{\boldsymbol{s}}(t). \tag{3.20}$$

Indeed, applying the rotation matrix ${}^b_a\boldsymbol{R}$ to the time derivative ${}^a\dot{\boldsymbol{s}}(t)$ means only the projection of the time derivative ${}^a\dot{\boldsymbol{s}}(t)$ on another frame, i.e.

$$ {}^b({}^a\dot{\boldsymbol{s}}(t)) = {}^b_a\boldsymbol{R}\,{}^a\dot{\boldsymbol{s}}(t), \tag{3.21}$$

where the notation on the left hand side must be preserved due to the fact that, in general, ${}^b({}^a\dot{\boldsymbol{s}}(t)) \neq {}^b\dot{\boldsymbol{s}}(t)$.

Let us now formally derive the correct relationship between the two time derivatives. Let us consider the time derivative of the projected vector ${}^a\boldsymbol{s}(t)$, but this time let us express it as the result of the projection on frame $\langle a\rangle$ of ${}^b\boldsymbol{s}(t)$, that is:

$$ {}^a\dot{\boldsymbol{s}}(t) = D^a\boldsymbol{s}(t) = D\left({}^a_b\boldsymbol{R}\,{}^b\boldsymbol{s}(t)\right) = {}^a_b\boldsymbol{R}\left(D^b\boldsymbol{s}(t)\right) + {}^a_b\dot{\boldsymbol{R}}\,{}^b\boldsymbol{s}(t) = {}^a_b\boldsymbol{R}\,{}^b\dot{\boldsymbol{s}}(t) + {}^a_b\dot{\boldsymbol{R}}\,{}^b\boldsymbol{s}(t), \tag{3.22}$$

which, using the strap-down equation (3.10), leads to the following two equivalent expressions

$$ {}^a\dot{\boldsymbol{s}}(t) = \begin{cases} {}^a_b\boldsymbol{R}\,{}^b\dot{\boldsymbol{s}}(t) + {}^a_b\boldsymbol{R}\left[{}^b\boldsymbol{\omega}_{b/a}\wedge\right]{}^b\boldsymbol{s}(t) \\ {}^a_b\boldsymbol{R}\,{}^b\dot{\boldsymbol{s}}(t) + \left[{}^a\boldsymbol{\omega}_{b/a}\wedge\right]{}^a_b\boldsymbol{R}\,{}^b\boldsymbol{s}(t) \end{cases}. \tag{3.23}$$

These equations show how ${}^{a}\dot{\boldsymbol{s}}(t)$ generally differs from the mere projection on $\langle a\rangle$ of the time derivative ${}^{b}\dot{\boldsymbol{s}}(t)$ evaluated by $\langle b\rangle$ unless $\boldsymbol{\omega}_{b/a} \wedge \boldsymbol{s} = 0$. This would be the case when frames $\langle a\rangle$ and $\langle b\rangle$ exhibit *no-rotational motion* one with respect to the other at the current instant (i.e. $\boldsymbol{\omega}_{b/a} = 0$) or when $\boldsymbol{\omega}_{b/a}$ is completely aligned with $\boldsymbol{s}$.

The above two equivalent relationships can be expressed in a more compact notational way, once a free-from-coordinate notation is used for representing the arguments of the above appearing projected cross-product:

$$ {}^{a}\dot{\boldsymbol{s}} = {}^{a}\left({}^{b}\dot{\boldsymbol{s}}\right) + {}^{a}\left(\boldsymbol{\omega}_{b/a} \wedge \boldsymbol{s}\right) = {}^{a}\left({}^{b}\dot{\boldsymbol{s}}\right) - {}^{a}\left(\boldsymbol{s} \wedge \boldsymbol{\omega}_{b/a}\right) \tag{3.24} $$

which is called *law of change of differentiation frame in projected form.*

Finally, the above law can be expressed using only free from coordinate notations, via the definition of the following geometric vector

$$ D_a\boldsymbol{s} \triangleq \frac{d_a}{dt}\boldsymbol{s} \triangleq \boldsymbol{i}_a\dot{x}_a + \boldsymbol{j}_a\dot{y}_a + \boldsymbol{k}_a\dot{z}_a, \tag{3.25} $$

which is termed as *time-derivative of a geometric vector with respect to a frame.* By simple inspection, we have that

$$ {}^{a}D_a\boldsymbol{s} = D^{a}\boldsymbol{s} = {}^{a}\dot{\boldsymbol{s}}. \tag{3.26} $$

Consequently, we can write the *law of change of differentiation frame in geometric form*

$$ D_a\boldsymbol{s} = D_b\boldsymbol{s} + \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{s} = D_b\boldsymbol{s} - \boldsymbol{s} \wedge \boldsymbol{\omega}_{b/a} \tag{3.27} $$

for which the same considerations previously made for its projected form can obviously be repeated, without any change.

As a concluding remark, the time derivative $D_a\boldsymbol{s}$ of a geometric vector, with respect to a frame $\langle a\rangle$, is the same one of those evaluated with respect to any frame of its reference system. Therefore, the notation $D_a\boldsymbol{s}$ should be read as the "time derivative of geometric vector $\boldsymbol{s}$ with respect to the reference system associated to frame $\langle a\rangle$".

3.4 Time Derivative of Constant Module Vectors

Let us consider the particular case of a vector $\boldsymbol{s}$ exhibiting a constant non-zero module, that is such that $|\boldsymbol{s}| = \sigma > 0$ and $\dot{\sigma} = 0$, to be then differentiated with respect to a given frame $\langle a\rangle$. To this aim, let us instrumentally consider any frame $\langle h\rangle$ constraited to admit $\boldsymbol{s}$ as one of its fixed vector, i.e. ${}^{h}\dot{\boldsymbol{s}} = 0$.

For the law of change of differentiation frame we have

$$ D_a\boldsymbol{s} = \boldsymbol{w}_{h/a} \wedge \boldsymbol{s};\ \ \forall\langle h\rangle\ s.t.\ D_h\boldsymbol{s} = 0 \tag{3.28} $$

where with $\boldsymbol{w}_{h/a}$ we have denoted the angular velocity vector of *anyone* of the frames obeying to the indicated condition. Let us note how the time derivative of a non-zero constant-module vector is necessarily *orthogonal* to the vector itself. From the equation above, we can immediately see that class of angular velocity vectors $\boldsymbol{w}_{h/a}$ associated to a non-zero constant-module vector is represented by the following linear manifold

$$ \boldsymbol{w}_{h/a} \triangleq \boldsymbol{\omega}_{s/a} + \boldsymbol{n}z\ \forall z \in \mathbb{R}, \tag{3.29} $$

where $\boldsymbol{n} \triangleq \boldsymbol{s}/\sigma$ is the unit vector of $\boldsymbol{s}$ and where $\boldsymbol{\omega}_{s/a}$ corresponds to the unique angular velocity vector that, within the class, exhibits the minimum module. Such a minimum-module vector denotes the *angular-velocity-vector of a non-zero constant-module vector* $\boldsymbol{s}$ with respect to frame $\langle a\rangle$. It is used to write the minimal representation of the time derivative of a non-zero constant-module vector with respect to frame $\langle a\rangle$:

$$ D_a\boldsymbol{s} = \boldsymbol{\omega}_{s/a} \wedge \boldsymbol{s} \tag{3.30} $$

3.5 Time Derivative of Generic Vectors

Let us now consider a geometric vector $\boldsymbol{s}$, again assumed to be non-zero, but not necessarily with constant module:

$$ \boldsymbol{s} \triangleq \boldsymbol{n}\sigma;\ |\boldsymbol{n}| = 1;\ \sigma > 0 \tag{3.31} $$

Then the differentiation of $\boldsymbol{s}$ with respect to a generic frame $\langle a \rangle$ yields

$$D_a \boldsymbol{s} = D_a (\boldsymbol{n}\sigma) = \boldsymbol{n}\dot{\sigma} + \sigma D_a \boldsymbol{n} \tag{3.32}$$

where the first term is proportional to $\boldsymbol{n}$, hence is aligned to $\boldsymbol{s}$, and the second term is proportional to the time-derivative of the non-zero constant module unit-vector $\boldsymbol{n}$, hence is orthogonal to $\boldsymbol{s}$. For these reasons, let us term (3.32) as the *aligned/orthogonal representation (AO) of the first type* of the time derivative of a vector.

A representation equivalent to the above can also be obtained by starting directly from $D_a(\boldsymbol{s})$ and by decomposing it in the following way

$$D_a \boldsymbol{s} = \boldsymbol{n}(\boldsymbol{n}\bullet) D_a \boldsymbol{s} + [\boldsymbol{I}_{3\times 3} - \boldsymbol{n}(\boldsymbol{n}\bullet)] D_a \boldsymbol{s}, \tag{3.33}$$

which is termed as *AO representation of the second type* of the time derivative of a vector. By direct comparison between (3.32) and (3.33) we get

$$\begin{aligned} \dot{\sigma} &= \boldsymbol{n} \bullet D_a(\boldsymbol{s}), \\ \sigma D_a \boldsymbol{n} &= [\boldsymbol{I}_{3\times 3} - \boldsymbol{n}(\boldsymbol{n}\bullet)] D_a \boldsymbol{s}. \end{aligned} \tag{3.34}$$

Recalling that $\boldsymbol{n}$ is a unit vector (hence constant module) and the formula (3.30) for the derivative of a constant module vector, its derivative is also equal to

$$D_a \boldsymbol{n} = \boldsymbol{\omega}_{n/a} \wedge \boldsymbol{n}. \tag{3.35}$$

Therefore, expanding (3.32) with the latest result we have the so called *the AO representation of the third type*:

$$D_a \boldsymbol{s} = \boldsymbol{n}\dot{\sigma} + \boldsymbol{\omega}_{s/a} \wedge \boldsymbol{s}, \tag{3.36}$$

where $\boldsymbol{\omega}_{s/a} \triangleq \boldsymbol{\omega}_{n/a}$ is the angular velocity vector exhibited by $\boldsymbol{s}$ (and consequently by all vectors aligned with it, including $\boldsymbol{n}$) whenever considered with invariant module equal to the one at current time.

3.6 Time Derivative of Rotation Vectors

Within this subsection, the expressions of the time derivative of rotation vectors between couples of frames and between couples of unit vectors will be discussed. The final results will be formerly stated and commented, before providing their proof.

3.6.1 Time Derivative of Rotation Vectors for Couples of Frames

Consider two frames $\langle a \rangle$, $\langle b \rangle$ exhibiting the rotation matrix ${}^a_b\boldsymbol{R}$, and associated geometric (free from coordinate notation) rotation vector $\boldsymbol{\rho} \triangleq \boldsymbol{\rho}_{b/a}$. Let us assume frame $\langle b \rangle$ continuously changing its orientation with respect to $\langle a \rangle$ under the action of the relative angular velocity vector $\boldsymbol{\omega}_{b/a}$. We have already discussed the time derivative of ${}^a_b\boldsymbol{R}$ (see the strap down equations (3.10)). We are now interested in the time derivative of $\boldsymbol{\rho}$.

First of all, let us consider

$$\boldsymbol{\rho} = \boldsymbol{n}\theta;\ \boldsymbol{\rho} \neq 0;\ \theta \triangleq |\boldsymbol{\rho}|;\ \boldsymbol{n} \triangleq \boldsymbol{\rho}/\theta. \tag{3.37}$$

Then, following (3.32) let us write

$$D_a \boldsymbol{\rho} = \boldsymbol{n}\dot{\theta} + \theta D_a \boldsymbol{n}. \tag{3.38}$$

Using the fact that

$${}^a D_a \boldsymbol{\rho} = {}^b D_b \boldsymbol{\rho} \tag{3.39}$$

since by definition of rotation vector ${}^a\boldsymbol{\rho} = {}^b\boldsymbol{\rho}$, and hence it is consequently clear how two different observers respectively located in $\langle a \rangle$ and $\langle b \rangle$ will see such a projected common vector necessarily varying with the same modalities.

Then, it results

$$D_a\boldsymbol{\rho} = \boldsymbol{n}(\boldsymbol{n}\bullet)\boldsymbol{\omega}_{b/a} + \frac{\theta}{2}\left[\frac{\boldsymbol{I}_{3\times3}}{\tan(\theta/2)} - (\boldsymbol{n}\wedge)\right]\left[\boldsymbol{I}_{3\times3} - \boldsymbol{n}(\boldsymbol{n}\bullet)\right]\boldsymbol{\omega}_{b/a}. \tag{3.40}$$

If we group $\boldsymbol{\omega}_{b/a}$ we have

$$D_a\boldsymbol{\rho} = \left[\boldsymbol{n}(\boldsymbol{n}\bullet) + \boldsymbol{N}_a(\theta)\right]\boldsymbol{\omega}_{b/a};\ \boldsymbol{N}_a(\theta) \triangleq \frac{\theta}{2}\left[\frac{\boldsymbol{I}_{3\times3}}{\tan(\theta/2)} - (\boldsymbol{n}\wedge)\right]\left[\boldsymbol{I}_{3\times3} - \boldsymbol{n}(\boldsymbol{n}\bullet)\right], \tag{3.41}$$

where the linear operator $\boldsymbol{N}_a(\theta)$ transforms $\boldsymbol{\omega}_{b/a}$ into the sum of the indicated two vectors, both of them orthogonal to $\boldsymbol{\rho}$.

3.6.2 Time Derivative of Rotation Vectors for Couples of Vectors

Let us consider two unit vectors a, b and the rotation vector $\boldsymbol{\rho} = \boldsymbol{n}\theta$ between the two vectors. Our goal is to compute the time derivative of $\boldsymbol{\rho}$ with respect to an observer on a generic frame $\langle\alpha\rangle$, that is

$$D_\alpha\boldsymbol{\rho} = \boldsymbol{n}\dot{\theta} + \theta D_\alpha\boldsymbol{n}. \tag{3.42}$$

To this aim, let us first compute $\dot{\theta}$ recalling that

$$\boldsymbol{a}\wedge\boldsymbol{b} = \boldsymbol{n}\sin\theta \tag{3.43}$$
$$\boldsymbol{a}\bullet\boldsymbol{b} = \cos\theta \tag{3.44}$$

The, by differentiating the second equality (observer independent since it is a scalar quantity) one obtains

$$\boldsymbol{a}\bullet\left(\boldsymbol{w}_{b/\alpha}\wedge\boldsymbol{b}\right) + \boldsymbol{b}\bullet\left(\boldsymbol{w}_{a/\alpha}\wedge\boldsymbol{a}\right) = -\sin\theta\dot{\theta} \tag{3.45}$$

where we have maintained the point of view of an observer in $\langle\alpha\rangle$ and where $\boldsymbol{w}_{a/\alpha}, \boldsymbol{w}_{b/\alpha}$ are each one an element of the class of angular velocity vectors associated to $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively (see (3.29)). By applying the cyclic property of the mixed products among vectors, one obtains

$$(\boldsymbol{a}\wedge\boldsymbol{b})\bullet\left(\boldsymbol{w}_{b/\alpha} - \boldsymbol{w}_{a/\alpha}\right) \triangleq (\boldsymbol{a}\wedge\boldsymbol{b})\bullet\boldsymbol{w}_{b/a} = \sin\theta\dot{\theta} \tag{3.46}$$

where the angular velocity difference $\boldsymbol{w}_{a/\alpha} - \boldsymbol{w}_{b/\alpha}$ results into the relative angular velocity $\boldsymbol{w}_{b/a}$. Substituting (3.43) in the above equation, we get the searched relationship

$$\dot{\theta} = \boldsymbol{n}\bullet\boldsymbol{w}_{b/a} \tag{3.47}$$

Let us now focus on the evaluation of $D_\alpha\boldsymbol{n}$:

$$\begin{aligned} D_\alpha\boldsymbol{n} &= D_\alpha\left(\frac{1}{\sin\theta}\boldsymbol{a}\wedge\boldsymbol{b}\right) \\ &= D_\alpha\left(\frac{1}{\sin\theta}\right)(\boldsymbol{a}\wedge\boldsymbol{b}) + \frac{1}{\sin\theta}D_\alpha(\boldsymbol{a}\wedge\boldsymbol{b}) \\ &= -\frac{\cos\theta\dot{\theta}}{\sin^2\theta}(\boldsymbol{a}\wedge\boldsymbol{b}) + \frac{1}{\sin\theta}\left(D_\alpha(\boldsymbol{a})\wedge\boldsymbol{b} + \boldsymbol{a}\wedge D_\alpha(\boldsymbol{b})\right). \end{aligned} \tag{3.48}$$

To evaluate the rightmost term in the above equation, let us consider the derivative of the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$D_\alpha\boldsymbol{a} \triangleq \boldsymbol{w}_{a/\alpha}\wedge\boldsymbol{a} = \left[\left(\boldsymbol{I}_{3\times3} - \boldsymbol{n}(\boldsymbol{n}\bullet)\right)\boldsymbol{w}_{a/\alpha}\right]\wedge\boldsymbol{a} + \left(\boldsymbol{n}(\boldsymbol{n}\bullet)\boldsymbol{w}_{a/\alpha}\right)\wedge\boldsymbol{a} \tag{3.49}$$
$$D_\alpha\boldsymbol{b} \triangleq \boldsymbol{w}_{b/\alpha}\wedge\boldsymbol{b} = \left[\left(\boldsymbol{I}_{3\times3} - \boldsymbol{n}(\boldsymbol{n}\bullet)\right)\boldsymbol{w}_{b/\alpha}\right]\wedge\boldsymbol{b} + \left(\boldsymbol{n}(\boldsymbol{n}\bullet)\boldsymbol{w}_{b/\alpha}\right)\wedge\boldsymbol{b} \tag{3.50}$$

and let us substitute these expressions in the evaluation of $\frac{1}{\sin\theta}\left(D_\alpha(\boldsymbol{a})\wedge\boldsymbol{b} + \boldsymbol{a}\wedge D_\alpha(\boldsymbol{b})\right)$ obtaining:

$$\frac{1}{\sin\theta}\left(D_\alpha(\boldsymbol{a})\wedge\boldsymbol{b} + \boldsymbol{a}\wedge D_\alpha(\boldsymbol{b})\right) \triangleq \boldsymbol{t}_1 + \boldsymbol{t}_2 \tag{3.51}$$

where

$$
\begin{aligned}
\boldsymbol{t}_1 &\triangleq \frac{1}{\sin\theta}\left(\left\{\left[\left(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet)\right)\boldsymbol{w}_{a/\alpha}\right]\wedge\boldsymbol{a}\right\}\wedge\boldsymbol{b}+\boldsymbol{a}\wedge\left\{\left[\left(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet)\right)\boldsymbol{w}_{b/\alpha}\right]\wedge\boldsymbol{b}\right\}\right)\\
&= \frac{1}{\sin\theta}\left(\boldsymbol{b}\wedge\left\{\boldsymbol{a}\wedge\left[\left(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet)\right)\boldsymbol{w}_{a/\alpha}\right]\right\}-\boldsymbol{a}\wedge\left\{\boldsymbol{b}\wedge\left[\left(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet)\right)\boldsymbol{w}_{b/\alpha}\right]\right\}\right)\\
&= -\frac{1}{\sin\theta}\left(\boldsymbol{a}\wedge\{\boldsymbol{b}\wedge[(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet))]\}\boldsymbol{w}_{b/\alpha}-\boldsymbol{b}\wedge\{\boldsymbol{a}\wedge[(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet))]\}\boldsymbol{w}_{a/\alpha}\right)\\
&= -\frac{1}{\sin\theta}\left(\boldsymbol{a}\wedge\{\boldsymbol{b}\wedge[(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet))]\}\boldsymbol{w}_{b/\alpha}-\boldsymbol{b}\wedge\{\boldsymbol{a}\wedge[(\boldsymbol{I}_{3\times3}-\boldsymbol{n}(\boldsymbol{n}\bullet))]\}\boldsymbol{w}_{a/\alpha}\right)\\
&= \boldsymbol{N}(\theta)\boldsymbol{w}_{b/\alpha}-\boldsymbol{M}(\theta)\boldsymbol{w}_{a/\alpha}
\end{aligned}
\tag{3.52}
$$

with the obvious definitions of the operators $\boldsymbol{N}(\theta)$ and $\boldsymbol{M}(\theta)$. Concerning the second term we have

$$
\begin{aligned}
\boldsymbol{t}_2 &\triangleq \frac{1}{\sin\theta}\left\{\left[\left(\boldsymbol{n}(\boldsymbol{n}\bullet)\boldsymbol{w}_{a/\alpha}\right)\wedge\boldsymbol{a}\right]\wedge\boldsymbol{b}+\boldsymbol{a}\wedge\left[\left(\boldsymbol{n}(\boldsymbol{n}\bullet)\boldsymbol{w}_{b/\alpha}\right)\wedge\boldsymbol{b}\right]\right\}\\
&= \frac{1}{\sin\theta}\left\{(\boldsymbol{n}\bullet)\boldsymbol{w}_{a/\alpha}\left[(\boldsymbol{n}\wedge\boldsymbol{a})\wedge\boldsymbol{b}\right]+(\boldsymbol{n}\bullet)\boldsymbol{w}_{b/\alpha}\left[\boldsymbol{a}\wedge(\boldsymbol{n}\wedge\boldsymbol{b})\right]\right\}\\
&= \frac{1}{\sin^2\theta}\left\{(\boldsymbol{n}\bullet)\boldsymbol{w}_{a/\alpha}\left[((\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{a})\wedge\boldsymbol{b}\right]+(\boldsymbol{n}\bullet)\boldsymbol{w}_{b/\alpha}\left[\boldsymbol{a}\wedge((\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{b})\right]\right\}\\
&= \frac{1}{\sin^2\theta}\left\{-(\boldsymbol{n}\bullet)\boldsymbol{w}_{a/\alpha}(\boldsymbol{a}\bullet\boldsymbol{b})\left[\boldsymbol{a}\wedge\boldsymbol{b}\right]+(\boldsymbol{n}\bullet)\boldsymbol{w}_{b/\alpha}(\boldsymbol{a}\bullet\boldsymbol{b})\left[\boldsymbol{a}\wedge\boldsymbol{b}\right]\right\}\\
&= \frac{\cos\theta\left(\boldsymbol{n}\bullet(\boldsymbol{w}_{b/\alpha}-\boldsymbol{w}_{a/\alpha})\right)}{\sin^2\theta}(\boldsymbol{a}\wedge\boldsymbol{b})
\end{aligned}
\tag{3.53}
$$

Recalling the result of (3.47), it is evident that $\boldsymbol{t}_2$ cancels out with the term $-\frac{\cos\theta\dot{\theta}}{\sin^2\theta}(\boldsymbol{a}\wedge\boldsymbol{b})$ that was present in (3.48). This could not be otherwise, as $\boldsymbol{n}$ is a constant unit vector and its overall derivative cannot have terms directed along $\boldsymbol{n}$ itself. Hence, we have

$$D_\alpha\boldsymbol{n} = \boldsymbol{N}(\theta)\boldsymbol{w}_{b/\alpha}-\boldsymbol{M}(\theta)\boldsymbol{w}_{a/\alpha} \tag{3.54}$$

Hence

$$D_\alpha\boldsymbol{\rho} = \boldsymbol{n}\dot{\theta}+\theta D_\alpha\boldsymbol{n} = \boldsymbol{n}\boldsymbol{n}\bullet\boldsymbol{w}_{b/a}+\theta\boldsymbol{N}(\theta)\boldsymbol{w}_{b/\alpha}-\theta\boldsymbol{M}(\theta)\boldsymbol{w}_{a/\alpha}. \tag{3.55}$$

Note that, if we consider our observer on one of the two frames $\langle a\rangle$ or $\langle b\rangle$ the equation simplifies. For example, considering the derivative with respect to frame $\langle a\rangle$:

$$D_a\boldsymbol{\rho} = \boldsymbol{n}\boldsymbol{n}\bullet\boldsymbol{w}_{b/a}+\theta\boldsymbol{N}(\theta)\boldsymbol{w}_{b/a}. \tag{3.56}$$

In this particular case, we notice that the derivative of $\boldsymbol{\rho}$ is affected along $\boldsymbol{n}$ only by the projection of $\boldsymbol{w}_{b/a}$ along it, while in the plane orthogonal to $\boldsymbol{n}$ only by the orthogonal components of $\boldsymbol{w}_{b/a}$ to $\boldsymbol{n}$.

3.7 Time Derivative of Points

Within a given, generally time varying universe, now consider a geometric point $P(t)$. Let us remember how such a geometric point represents the entire class of its positions, or projections, on the frames of the given universe considered at time t and it is considered as "moving" if it changes its projection coordinates at least with respect to one of its frames. Consequently, here it does not make sense to speak about the time-derivative of a geometric point in an absolute sense, but only after having specified the frame with respect to which the time derivative has to be intended.

The time derivative of a geometric point $P(t)$ with respect to a frame $\langle a\rangle$ is defined as the following geometric vector

$$\boldsymbol{v}_{P/a} \triangleq D_a\boldsymbol{r}_{P,a} = \frac{d_a}{dt}\boldsymbol{r}_{P,a};\ \boldsymbol{r}_{P,a} \triangleq (P-O_a) \tag{3.57}$$

traditionally termed as the *linear velocity vector* of the point with respect to a frame.

First of all note, by inspection, that, once projected on the *same* reference/differentiation frame, vector ${}^a\boldsymbol{v}_{P/a}$ yields

$${}^a\boldsymbol{v}_{P/a} = {}^a\dot{P} \tag{3.58}$$

which is the time derivative of the projected point ${}^aP$ that can be used to reconstruct the evolution of ${}^aP(t)$ via time integration, i.e.

$$\begin{aligned} {}^aP(t) &= \int_0^t {}^a\dot{P}dt + {}^aP(0) \\ {}^aP(t+dt) &= {}^aP(t) + {}^a\dot{P}dt + O(dt) \end{aligned} \tag{3.59}$$

3.7.1 Composition of Linear Velocity Vectors

Consider again a geometric point $P(t)$, where now two different observers are respectively located on the two frames $\langle a \rangle$, $\langle b \rangle$. Each observer will associated to the same point P the velocity vectors $\boldsymbol{v}_{P/a}$ and $\boldsymbol{v}_{P/b}$. Now, let us write $\boldsymbol{r}_{P,a}$ equivalently as

$$\boldsymbol{r}_{P,a} = \boldsymbol{r}_{P,b} + \boldsymbol{r}_{b,a} \tag{3.60}$$

then by differentiating we obtain

$$\boldsymbol{v}_{P/a} = D_a(\boldsymbol{r}_{b,a} + \boldsymbol{r}_{P,b}) = D_a\boldsymbol{r}_{b,a} + D_a\boldsymbol{r}_{P,b}. \tag{3.61}$$

The first term is the velocity of the origin of frame $\langle b \rangle$ with respect to $\langle a \rangle$, i.e. $\boldsymbol{v}_{b/a}$. Furthermore, let us apply the law of change of differentiation frame to $D_a\boldsymbol{r}_{P,b}$ obtaining

$$\boldsymbol{v}_{P/a} = \boldsymbol{v}_{b/a} + D_b\boldsymbol{r}_{P,b} + \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{r}_{P,b}, \tag{3.62}$$

in which we recognize that $D_b\boldsymbol{r}_{P,b} \triangleq \boldsymbol{v}_{P/b}$. Therefore, the *law of composition of linear velocity vectors* immediately follows as

$$\boldsymbol{v}_{P/a} = \boldsymbol{v}_{P/b} + \boldsymbol{v}_{b/a} + \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{r}_{P,b}. \tag{3.63}$$

3.7.2 Points Attached to Rigid Spaces

If the point P belongs to the rigid space of $\langle b \rangle$ then $\boldsymbol{v}_{P/b} \equiv 0$, therefore (3.63) becomes

$$\boldsymbol{v}_{P/a} = \boldsymbol{v}_{b/a} + \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{r}_{P,b}, \tag{3.64}$$

which is termed as the *law of composition of linear velocity vectors for points on a rigid space*, which is probably well know to the reader.

3.8 Time Derivative of Distance Vectors

Having defined the time derivative of points with respect to a frame (i.e. their linear velocity with respect to a frame) let us now tackle the problem of the time-derivative of a distance vector, i.e. of vectors which are really descending from a point difference. Therefore, let us consider the vector

$$\boldsymbol{s} = (P - Q). \tag{3.65}$$

Its time derivative with respect to a generic frame $\langle a \rangle$ is

$$D_a\boldsymbol{s} \triangleq \boldsymbol{v}_{s/a} = D_aP - D_aQ = \boldsymbol{v}_{P/a} - \boldsymbol{v}_{Q/a}. \tag{3.66}$$

Bibliography