Chapter 9: Frequency Response Analysis

9.1 To find the sinusoidal transfer function $G(j\omega)$, replace s with $j\omega$

$$G(j\omega) = \frac{4}{0.2j\omega + 1}$$

or

$$\boxed{G(j\omega) = \frac{4}{1 + j0.2\omega}}$$

9.2 To find the sinusoidal transfer function $G(j\omega)$, replace s with $j\omega$

$$G(j\omega) = \frac{3j\omega + 1}{2(j\omega)^2 + 6j\omega + 40}$$

Substituting $j^2 = -1$ we obtain

$$\boxed{G(j\omega) = \frac{1 + j3\omega}{40 - 2\omega^2 + j6\omega}}$$

9.3 First, we determine the sinusoidal transfer function of the given I/O equation:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3}{2s+10} \quad \rightarrow \text{ the sinusoidal transfer function is } G(j\omega) = \frac{3}{10 + j2\omega}$$

We know from Eq. (9-17) that the frequency response has the form

$$y_{ss}(t) = |G(j\omega)| U_0 \sin(\omega t + \phi) \qquad \text{where } \omega = 4 \text{ rad/s, } U_0 = 18$$

We use the sine function because the input is a sine function. Finally, we need the magnitude and phase angle of the sinusoidal transfer function $G(j\omega)$ at input frequency $\omega = 4$ rad/s

Magnitude: $|G(j4)| = \dfrac{\sqrt{3^2 + 0^2}}{\sqrt{10^2 + 8^2}} = \dfrac{3}{\sqrt{164}} = 0.2343$

Phase angle: $\phi = \arg[3] - \arg[10 + j8] = \tan^{-1}\left(\dfrac{0}{3}\right) - \tan^{-1}\left(\dfrac{8}{10}\right) = -38.66$ deg or -0.6747 rad

Because the input magnitude is $U_0 = 18$, the frequency response is

$$\boxed{y_{ss}(t) = 4.2167 \sin(4t - 0.6747)}$$

9.4 The sinusoidal transfer function of the given I/O equation:

$$G(s) = \frac{0.8s + 2}{4s^2 + s + 30} \quad \rightarrow \text{ the sinusoidal transfer function is } G(j\omega) = \frac{2 + j0.8\omega}{30 - 4\omega^2 + j\omega}$$

We know from Eq. (9-18) that the frequency response has the form

$$y_{ss}(t) = |G(j\omega)| U_0 \cos(\omega t + \phi) \quad \text{where } \omega = 5 \text{ rad/s, } U_0 = 0.3$$

We use the cosine function because the input is a cosine function. We need the magnitude and phase angle of the sinusoidal transfer function $G(j\omega)$ at input frequency $\omega = 5$ rad/s

Magnitude: $|G(j5)| = \dfrac{\sqrt{2^2 + 4^2}}{\sqrt{(-70)^2 + 5^2}} = \dfrac{\sqrt{20}}{\sqrt{4925}} = 0.0637$

Phase angle: $\phi = \dfrac{\arg[2 + j4]}{\arg[-70 + j5]} = \tan^{-1}\left(\dfrac{4}{2}\right) - \tan^{-1}\left(\dfrac{5}{-70}\right) = 63.43 - 175.91 = -112.48$ deg

Because the input magnitude is $U_0 = 0.3$, the frequency response is

$$\boxed{y_{ss}(t) = 0.0191\cos(5t - 1.9631)}$$

9.5 The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s with $j\omega$

$$G(s) = \frac{2s+3}{s^2+8s} \quad \rightarrow \quad G(j\omega) = \frac{2j\omega+3}{(j\omega)^2+8j\omega} = \frac{3+j2\omega}{-\omega^2+j8\omega}$$

Magnitude: $$\left|G(j2)\right| = \frac{\sqrt{3^2+4^2}}{\sqrt{(-4)^2+16^2}} = \frac{\sqrt{25}}{\sqrt{272}} = \boxed{0.3032}$$

Phase angle: $$\phi = \frac{\arg[3+j4]}{\arg[-4+j16]} = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{16}{-4}\right) = 53.13 - 104.04 = \boxed{-50.91 \text{ deg}}$$

9.6 The sinusoidal transfer function is

$$G(j\omega)=\frac{0.5j\omega+0.4}{(j\omega)^2+6j\omega+8} \quad \text{or} \quad G(j\omega)=\frac{0.4+j0.5\omega}{8-\omega^2+j6\omega}$$

We know from Eq. (9-17) that the frequency response has the form

$$y_{ss}(t)=\left|G(j\omega)\right|U_0\sin(\omega t+\phi) \qquad \text{where } \omega = 5 \text{ rad/s, } U_0 = 2.7$$

We use the sine function because the input is a sine function. We need the magnitude and phase angle of the sinusoidal transfer function $G(j\omega)$ at input frequency $\omega = 5$ rad/s

Magnitude: $\left|G(j5)\right|=\dfrac{\sqrt{0.4^2+2.5^2}}{\sqrt{(-17)^2+30^2}}=\dfrac{\sqrt{6.41}}{\sqrt{1189}}=0.0734$

Phase angle: $\phi=\dfrac{\arg[0.4+j2.5]}{\arg[-17+j30]}=\tan^{-1}\left(\dfrac{2.5}{0.4}\right)-\tan^{-1}\left(\dfrac{30}{-17}\right)=80.91-119.54=-38.63$ deg

Because the input magnitude is $U_0 = 2.7$, the frequency response is

$$\boxed{y_{ss}(t)=0.1982\sin(5t-0.6742)}$$

9.7 The sinusoidal transfer function is

$$G(j\omega) = \frac{2.4}{0.6(j\omega)^2 + 8j\omega + 36} \quad \text{or} \quad G(j\omega) = \frac{2.4}{36 - 0.6\omega^2 + j8\omega}$$

We know from Eq. (9-18) that the frequency response has the form

$$y_{ss}(t) = |G(j\omega)| U_0 \cos(\omega t + \phi) \qquad \text{where } \omega = 20 \text{ rad/s, } U_0 = 30.2$$

We use the cosine function because the input is a cosine function. We need the magnitude and phase angle of the sinusoidal transfer function $G(j\omega)$ at input frequency $\omega = 20$ rad/s

Magnitude: $|G(j20)| = \dfrac{\sqrt{2.4^2 + 0^2}}{\sqrt{(-204)^2 + 160^2}} = \dfrac{2.4}{\sqrt{67{,}216}} = 0.00926$

Phase angle: $\phi = \dfrac{\arg[2.4]}{\arg[-204 + j160]} = \tan^{-1}\left(\dfrac{0}{2.4}\right) - \tan^{-1}\left(\dfrac{160}{-204}\right) =$ –141.89 deg (–2.477 rad)

Because the input magnitude is $U_0 = 30.2$, the frequency response is

$$\boxed{y_{ss}(t) = 0.2796 \cos(20t - 2.4765)}$$

9.8 The sinusoidal transfer function is

$$G(j\omega) = \frac{2j\omega + 6}{(j\omega)^3 + 2j\omega + 26} \quad \text{or} \quad G(j\omega) = \frac{6 + j2\omega}{26 + j(2\omega - \omega^3)}$$

We know from Eq. (9-17) that the frequency response has the form

$$y_{ss}(t) = |G(j\omega)| U_0 \sin(\omega t + \phi) \qquad \text{where } \omega = 3 \text{ rad/s, } U_0 = 0.6$$

We use sine because the input is a sine function. We need the magnitude and phase angle:

Magnitude: $|G(j3)| = \dfrac{\sqrt{6^2 + 6^2}}{\sqrt{26^2 + (-21)^2}} = \dfrac{\sqrt{72}}{\sqrt{1117}} = 0.2539$

Phase angle: $\phi = \dfrac{\arg[6 + j6]}{\arg[26 - j21]} = \tan^{-1}\left(\dfrac{6}{6}\right) - \tan^{-1}\left(\dfrac{-21}{26}\right) = 45 - -38.93 = 83.93 \text{ deg}$

Because the input magnitude is $U_0 = 0.6$, the frequency response is

$$\boxed{y_{ss}(t) = 0.1523\sin(3t + 1.4648)}$$

9.9 a) We begin with the sinusoidal transfer function

$$G(j\omega)=\frac{0.6}{0.25j\omega+1} \quad \text{or} \quad G(j\omega)=\frac{0.6}{1+j0.25\omega}$$

The magnitude in decibels (dB) is computed by taking $20\log_{10}$ of $G(j\omega)$

$$\left|G(j\omega)\right|_{\text{dB}}=20\log_{10}\left|G(j\omega)\right|=20\log_{10}(0.6)-20\log_{10}\sqrt{1^2+(0.25\omega)^2} \qquad \text{(A)}$$

For very low frequencies we have $\omega\to 0$ so Eq. (A) becomes

$$\left|G(j\omega)\right|_{\text{dB}}\approx 20\log_{10}(0.6)-20\log_{10}(1)= \mathbf{20\log_{10}(0.6)\ dB}$$

b) For very high frequencies we have $\omega\to\infty$ and Eq. (A) becomes

$$\left|G(j\omega)\right|_{\text{dB}}\approx 20\log_{10}(0.6)-20\log_{10}(0.25\omega)$$

or, $\left|G(j\omega)\right|_{\text{dB}}=20\log_{10}(0.6)-20\log_{10}(0.25)-20\log_{10}(\omega)$

Re-write the previous equation as $\left|G(j\omega)\right|_{\text{dB}}=20\log_{10}(2.4)-20\log_{10}(\omega)$ (B)

Equation (B) is a straight line plotted on a log-log scale where the x-axis is $\log_{10}(\omega)$. Hence the slope of the asymptote for $\omega\to\infty$ is **–20 dB/decade.**

c) The flat (low-frequency) asymptote is $20\log_{10}(0.6)$. The sloping high-frequency asymptote is $20\log_{10}(2.4)-20\log_{10}(\omega)$. Set these two asymptotes equal to determine the intersection.

$$20\log_{10}(0.6)=20\log_{10}(2.4)-20\log_{10}(\omega) \quad \text{or} \quad 20\log_{10}(\omega)=20\log_{10}(2.4)-20\log_{10}(0.6)$$

Re-write as $20\log_{10}(\omega)=20\log_{10}(2.4/0.6)$

Therefore the two asymptotes intersect at corner frequency ω_c = 2.4/0.6 **= 4 rad/s**

9.10 The transfer function for the RL system is

$$G(s)=\frac{1}{Ls+R}=\frac{1}{0.02s+1.5} \qquad \text{(using } L = 0.02 \text{ H and } R = 1.5\ \Omega\text{)}$$

By definition the bandwidth is the frequency range $0 \le \omega \le \omega_B$ where the magnitude remains within 3 dB from its DC gain. To compute magnitude we need the sinusoidal transfer function

$$G(j\omega)=\frac{1}{0.02j\omega+1.5} \qquad \text{or} \qquad G(j\omega)=\frac{1}{1.5+j0.02\omega} \qquad \text{(A)}$$

The magnitude of the DC gain (in dB) is $20\log_{10}(1/1.5)$ = –3.5218 dB. Hence we want to find the cutoff frequency ω_B where the magnitude is –6.5218 dB. Therefore the absolute value of the sinusoidal transfer function at the cutoff frequency is $10^{-6.5218/20}$ = 0.47196.

The magnitude of $G(j\omega)$ is determined by using Eq. (A)

$$\left|G(j\omega)\right|=\frac{\sqrt{1^2+0^2}}{\sqrt{1.5^2+(0.02\omega)^2}}=\frac{1}{\sqrt{2.25+0.0004\omega^2}} \qquad \text{(B)}$$

Setting Eq. (B) to 0.47196 and solving for ω yields the cutoff frequency ω_B = 74.822 rad/s. The cutoff frequency in Hz is determined by dividing rad/s by 2π.

Bandwidth: $0 \le \omega \le$ 11.91 Hz

9.11 The mathematical model of the mechanical system is $m\ddot{x}+b\dot{x}+kx=kx_{\text{in}}(t)$

$$\rightarrow \text{ the transfer function is } G(s)=\frac{X(s)}{X_{\text{in}}(s)}=\frac{k}{ms^2+bs+k}$$

Using m = 2 kg, b = 20 N-s/m and k = 500 N/m the sinusoidal transfer function becomes

$$G(j\omega)=\frac{500}{500-2\omega^2+j20\omega}$$

We know from Eq. (9-17) that the frequency response has the form

$$x_{ss}(t)=\left|G(j\omega)\right|U_0\sin(\omega t+\phi) \qquad \text{where } \omega = 50 \text{ rad/s, } U_0 = 0.04$$

We use sine because the input is a sine function. We need the magnitude and phase angle:

Magnitude: $\left|G(j50)\right|=\dfrac{\sqrt{500^2+0^2}}{\sqrt{(-4500)^2+1000^2}}=\dfrac{500}{\sqrt{2.125(10^7)}}=0.1085$

Phase angle: $\phi=\dfrac{\arg[500+j0]}{\arg[-4500+j1000]}=\tan^{-1}\left(\dfrac{0}{500}\right)-\tan^{-1}\left(\dfrac{1000}{-4500}\right)=\ -167.47 \text{ deg}$

Because the input magnitude is U_0 = 0.04, the frequency response is

$$\boxed{x_{ss}(t)=0.0043\sin(50t-2.9229) \quad \text{m}}$$

9.12 The DC gain of the transfer function in Problem 9.11 is $k/k = 1$, or 0 dB. At the cutoff frequency (bandwidth) the magnitude is –3 dB or $10^{-3/20} = 0.708$. The magnitude of the sinusoidal transfer function in Problem 9.11 is set equal to 0.708

$$|G(j\omega_B)| = \frac{\sqrt{500^2 + 0^2}}{\sqrt{(500 - 2\omega_B^2)^2 + (20\omega_B)^2}} = \frac{500}{\sqrt{4\omega_B^4 - 1600\omega_B^2 + 250{,}000}} = 0.708$$

Solving for bandwidth we obtain $\boldsymbol{\omega_B}$ **= 22.80 rad/s**

The resonant frequency for a second-order system is $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$

For the given system we have $\omega_n = \sqrt{k/m} = 15.8114$ rad/s and $\zeta = b/2\sqrt{km} = 0.3162$

Hence, the resonant frequency is $\boldsymbol{\omega_r}$ **= 14.1421 rad/s**

The maximum transmissibility occurs at the resonant frequency; therefore we evaluate the magnitude of the sinusoidal transfer function at $\omega = \omega_r = 14.1421$ rad/s

$$|G(j\omega_r)| = \frac{500}{\sqrt{4\omega_r^4 - 1600\omega_r^2 + 250{,}000}} = 1.6667$$ → **max TR = 1.6667**

9.13 The mathematical model of the mechanical system is $m\ddot{x} + (b_1 + b_2)\dot{x} + kx = f_a(t)$

The transfer function and sinusoidal transfer function are

$$G(s) = \frac{X(s)}{F_a(s)} = \frac{1}{0.6s^2 + 3.4s + 80} \quad \text{and} \quad G(j\omega) = \frac{1}{80 - 0.6\omega^2 + j3.4\omega}$$

a) We know from Eq. (9-17) that the frequency response has the form

$$x_{ss}(t) = |G(j\omega)| U_0 \sin(\omega t + \phi) \qquad \text{where } \omega = 8 \text{ rad/s, } U_0 = 2$$

We use sine because the input is a sine function. We need the magnitude and phase angle:

Magnitude: $|G(j8)| = \dfrac{\sqrt{1^2 + 0^2}}{\sqrt{41.6^2 + 27.2^2}} = \dfrac{1}{\sqrt{2470.4}} = 0.0201$

Phase angle: $\phi = \dfrac{\arg[1 + j0]}{\arg[41.6 + j27.2]} = \tan^{-1}\left(\dfrac{0}{1}\right) - \tan^{-1}\left(\dfrac{27.2}{41.6}\right) =$ –33.1785 deg

Because the input magnitude is $U_0 = 2$, the frequency response is

$$\boxed{x_{ss}(t) = 0.0402 \sin(8t - 0.5791) \quad \text{m}}$$

b) The **largest** output amplitude will occur when $\omega = \omega_r$ (resonant frequency). The resonant frequency is $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$.

For the system we have $\omega_n = \sqrt{k/m} = 11.5470$ rad/s and $\zeta = (b_1 + b_2)/2\sqrt{km} = 0.2454$

The resonant frequency is **ω_r = 10.8295 rad/s.**

9.14 The transfer function and sinusoidal transfer function from Problem 9.2 are

$$G(s) = \frac{3s+1}{2s^2+6s+40} \quad \text{and} \quad G(j\omega) = \frac{1+j3\omega}{40-2\omega^2+j6\omega}$$

The MATLAB commands to compute the magnitude and phase angle with $\omega = 2$ rad/s are

```
>> w = 2;                              % Input frequency, rad/s
>> numG = 1 + j*3*w;                   % Numerator of G(jω) (complex)
>> denG = 40 - 2*w^2 + j*6*w;          % Denominator of G(jω) (complex)
>> magG = abs(numG)/abs(denG);         % Compute the magnitude (absolute value)
>> phiG = angle(numG)-angle(denG);     % Compute the phase angle (in radians)
```

Executing the above commands yields **`magG` = 0.1780** and **`phiG` = 1.0469 rad**

We can verify this solution using MATLAB's `bode` command as follows:

```
>> sysG = tf([3 1],[2 6 40]);          % Define transfer function G(s)
>> w = 2;                              % Input frequency, rad/s
>> [mag,phase] = bode(sysG,w);         % Magnitude and phase at ω = 2 rad/s
```

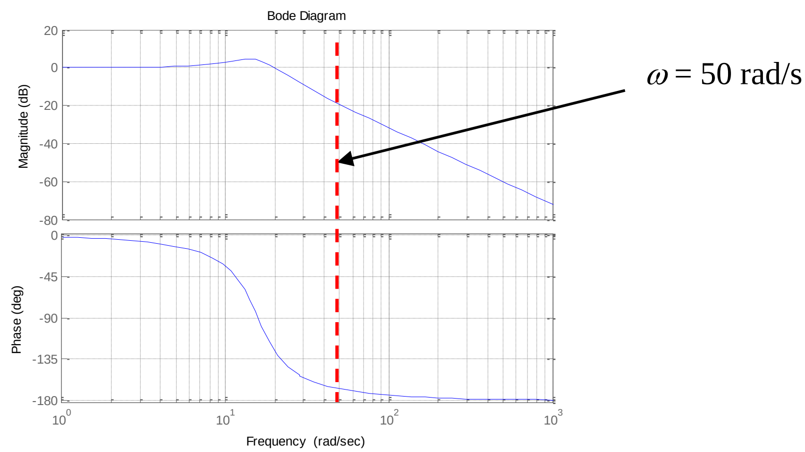
Executing the above commands yields **`mag` = 0.1780** and **`phase` = 59.982 deg (= 1.0469 rad)**

9.15 The following MATLAB commands create the Bode diagram for the transfer function $G(s)$ for Problem 9.11

```
>> sysG = tf(500,[2 20 500]);          % define transfer function G(s)
>> bode(sysG)                          % create and plot Bode diagram
```

Because the system input is $x_{\text{in}}(t) = 0.04\sin 50t$ (m) we estimate the magnitude and phase angle using the Bode diagram for frequency $\omega = 50$ rad/s: Magnitude = –20 dB, Phase = –165 deg (or, –2.88 rad). The absolute-value magnitude is $10^{-20/20} = 0.1$. The frequency response is

$$\boxed{x_{ss}(t) = 0.004\sin(50t - 2.88) \quad \text{m}}$$



A more accurate calculation of the frequency response can be obtained by using MATLAB's `bode` command with input $\omega = 50$ rad/s:

```
>> sysG = tf(500,[2 20 500]);          % define transfer function G(s)
>> w = 50;                             % define input frequency ω = 50 rad/s
>> [mag,phase] = bode(sysG,w)          % compute magnitude and phase
```

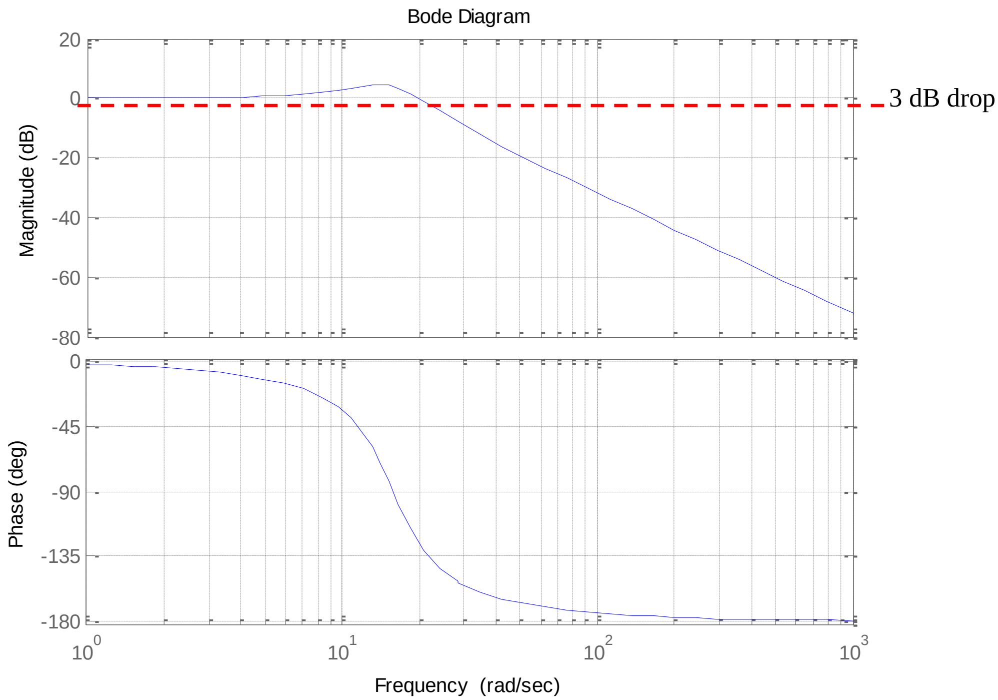
The result is `mag` = 0.1085 and `phase` = –167.47 deg (–2.9229 rad). Hence a more accurate frequency response is

$$\boxed{x_{ss}(t) = 0.0043\sin(50t - 2.9229) \quad \text{m}}$$

9.16 The bode diagram is computed using the MATLAB commands and system $G(s)$

```
>> sysG = tf(500,[2 20 500]);        % define transfer function G(s)
>> bode(sysG)                        % create and plot Bode diagram
```

We estimate the cutoff frequency where we observe the 3 dB drop from the DC gain (0 dB) to be approximately ω_B = **23 rad/s (bandwidth).** The peak magnitude occurs at magnitude of roughly 4.5 dB at a frequency of 15 rad/s. Hence the **resonant** frequency is about ω_r **= 15 rad/s** and the **peak transmissibility is about $10^{4.5/20}$ = 1.68.**

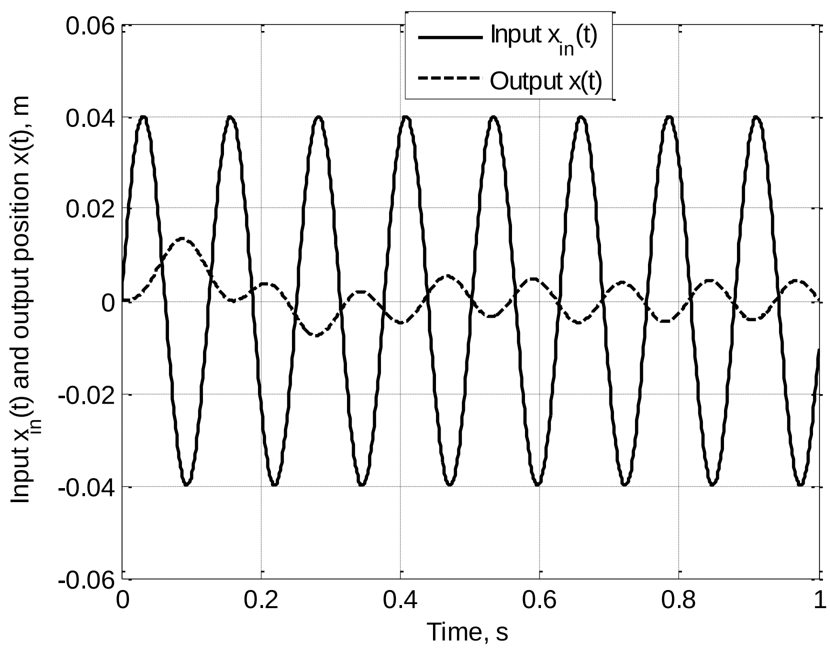


9.17 The MATLAB commands to simulate the frequency response are:

```
>> sysG = tf(500,[2 20 500]);          % define transfer function G(s)
>> t = 0:0.001:1;                      % time vector
>> w = 50;                             % input frequency
>> x_in = 0.04*sin(w.*t);              % input displacement xin(t)
>> [x,t] = lsim(sysG,x_in,t);          % Simulate frequency response x(t)
>> plot(t,x_in,t,x)                    % Plot input and output
```

The plot (below) shows the frequency response. The solid line is the input $x_{\text{in}}(t) = 0.04\sin 50t$ (m) and the dashed line is the position of the mass $x(t)$ (in m). We see that at steady state the response $x(t)$ has an amplitude of about 0.004 m and a period equal to the input (i.e., input frequency $\omega = 50$ rad/s). The "peaks" of the response $x(t)$ are nearly aligned with the "valleys" of the input $x_{\text{in}}(t)$ and hence the output is nearly (but not quite) 180 deg out-of-phase with the input (the phase is about –165 deg). Therefore an estimate of the frequency-response equation is

$$\boxed{x_{ss}(t) = 0.004\sin(50t - 2.9) \quad \text{m}}$$



9.18 The transfer function of the mechanical system in Problem 9.13 is

$$G(s) = \frac{X(s)}{F_a(s)} = \frac{1}{ms^2 + (b_1 + b_2)s + k} = \frac{1}{0.6s^2 + 3.4s + 80}$$

We can verify the frequency response results in Problem 9.13 using MATLAB's `bode` command:

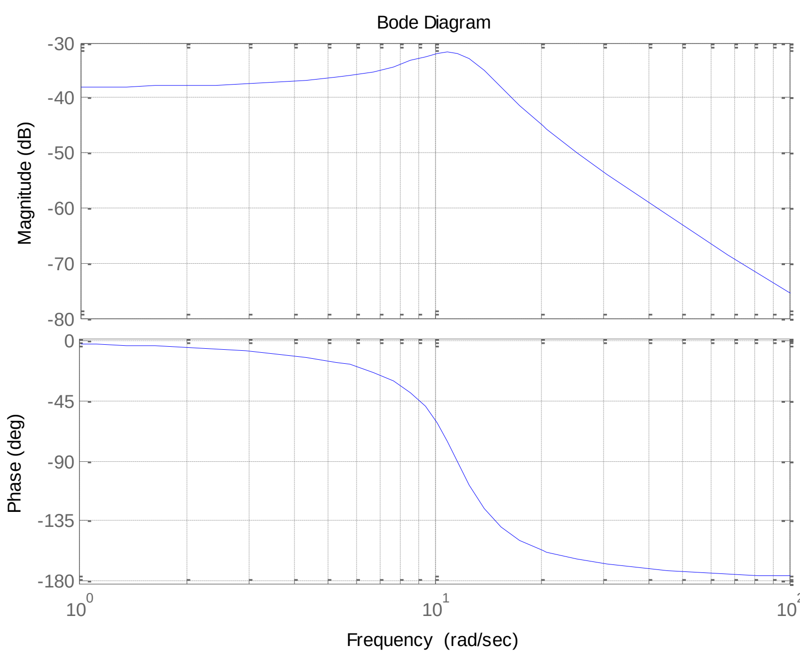
a) The input force is $f_a(t) = 2\sin 8t$ N. The magnitude and phase of the sinusoidal transfer function at $\omega = 8$ rad/s can be computed using the MATLAB commands below:

```
>> sysG = tf(1,[0.6 3.4 80]);          % define transfer function G(s)
>> w = 8;                              % define input frequency ω = 8 rad/s
>> [mag,phase] = bode(sysG,w)          % compute magnitude and phase
```

The result is `mag` = 0.0201 and `phase` = –33.1786 deg (or –0.5791 rad). The amplitude of the output is (2 N)(0.0201) = 0.0402. The frequency response is

$$\boxed{x_{ss}(t) = 0.0402\sin(8t - 0.5791) \quad \text{m}}$$ (matches Problem 9.13a)

b) We can create the Bode diagram (below) using the MATLAB command `bode(sysG)`



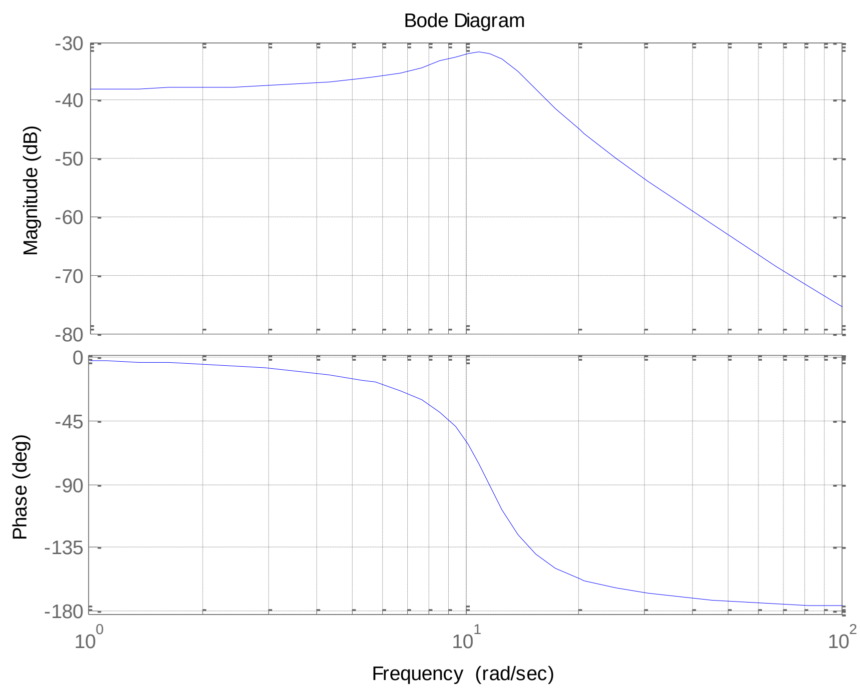
The peak magnitude occurs at a frequency in the range between 10-12 rad/s. We can use MATLAB's `bode` command as in part (a) to compute magnitudes at frequencies between 10-12 rad/s to determine the peak magnitude. The resonant frequency is approximately $\boldsymbol{\omega_r}$ **= 10.8 rad/s** which verifies the computation in Problem 9.13b.

9.19 The transfer function (from Problem 9.13) relating displacement x to the input force f_a is

$$G_1(s) = \frac{X(s)}{F_a(s)} = \frac{1}{ms^2 + (b_1 + b_2)s + k} = \frac{1}{0.6s^2 + 3.4s + 80}$$

The following MATLAB command creates the first Bode diagram (below):

```
>> sysG1 = tf(1,[0.6 3.4 80]);          % define transfer function G1(s)
>> bode(sysG1)                          % create the Bode diagram
```



Bode diagram for transfer function $G_1(s) = X(s)/F_a(s)$

The second desired transfer function is $G_2(s) = F_T(s)/F_a(s)$ where f_T is the force transmitted to the base. The force transmitted to the fixed base depends on the displacement and velocity of the mass as well as damper b_2 and spring k (see Fig. P9.13):

$$f_T = b_2\dot{x} + kx$$

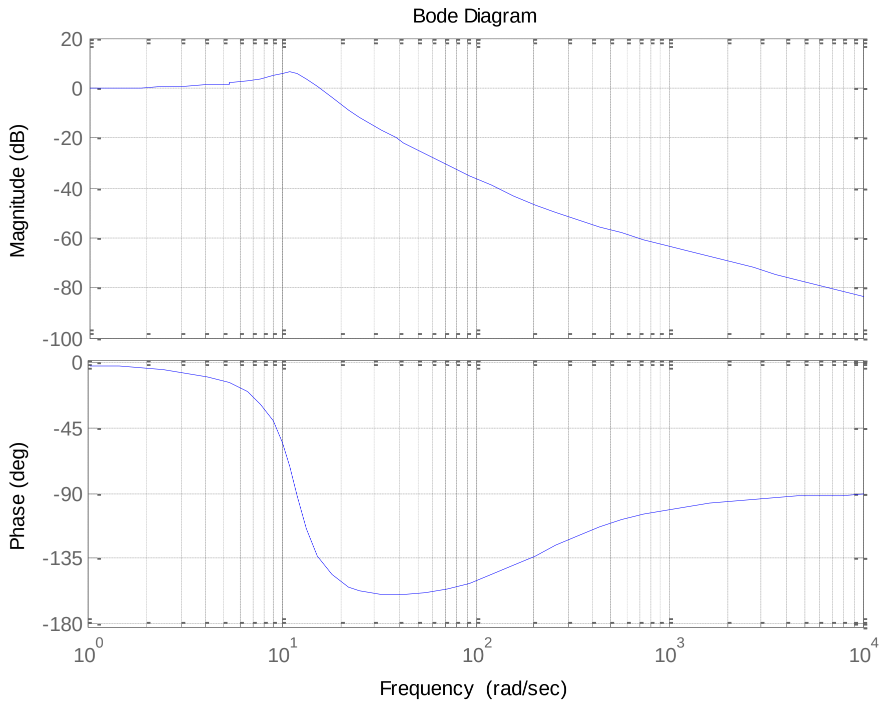
The transfer function relating transmitted force f_T to displacement x is $\dfrac{F_T(s)}{X(s)} = b_2 s + k$

Finally, the desired transfer function $G_2(s) = F_T(s)/F_a(s)$ can be obtained by multiplying $F_T(s)/X(s)$ and $G_1(s) = X(s)/F_a(s)$ as shown below

$$G_2(s) = \frac{F_T(s)}{F_a(s)} = \frac{F_T(s)}{X(s)}\frac{X(s)}{F_a(s)} = \frac{b_2 s + k}{ms^2 + (b_1 + b_2)s + k} = \frac{0.4s + 80}{0.6s^2 + 3.4s + 80}$$

The following MATLAB command creates the second Bode diagram (below):

```
>> sysG2 = tf([0.4 80],[0.6 3.4 80]);     % define transfer function G2(s)
>> bode(sysG2)                             % create the Bode diagram
```



Bode diagram for transfer function $G_2(s) = F_T(s)/F_a(s)$

From the two Bode (magnitude) diagrams we see that the resonant frequency is about 10.8 rad/s for <u>both</u> systems $G_1(s)$ and $G_2(s)$. The MATLAB command can compute the bandwidth

```
>> wB1 = bandwidth(sysG1)                  % compute bandwidth ωB for G1(s)
>> wB2 = bandwidth(sysG2)                  % compute bandwidth ωB for G2(s)
```

Both bandwidths are 17.2 rad/s.

For very low-frequency inputs, the Bode diagram for system $G_1(s)$ shows a magnitude of about –38 dB (or, absolute value of 0.0126). Hence the steady-state displacement for a (low-frequency) 2-N sinusoidal force is (2 N)(0.0126) = 0.025 m, which is the static deflection for a 2-N step input; i.e., f_a / k = (2 N)/(80 N/m) = 0.025 m.

For very low-frequency inputs, the Bode diagram for system $G_2(s)$ shows a magnitude of 0 dB (or, absolute value of 1). Hence the steady-state transmitted force for a (low-frequency) 2-N sinusoidal force is also 2 N, which makes sense because at very low frequencies the mass is nearly in static equilibrium (almost no motion) and the spring force kx balances the input force f_a.

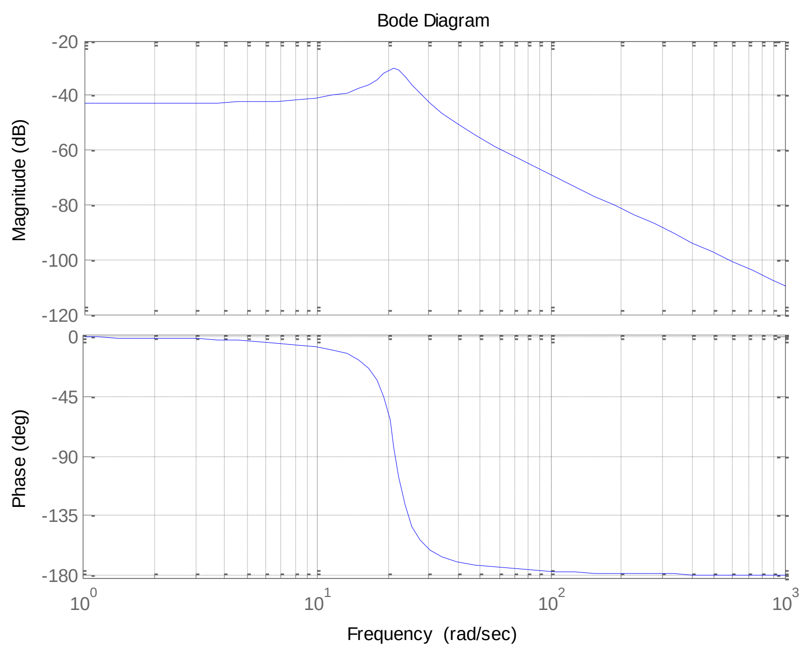
9.20 The mathematical model of the mechanical system is $m\ddot{z} + b\dot{z} + (k_1 + k_2)z = f_a(t)$

a) Hence, the transfer function (substituting the numerical values for m, b, k_1, and k_2) is

$$G(s) = \frac{Z(s)}{F_a(s)} = \frac{1}{0.3s^2 + 1.5s + 140} = \frac{3.3333}{s^2 + 5s + 466.6667}$$

The following MATLAB command creates the Bode diagram using the transfer function

```
>> sysG = tf(1,[0.3 1.5 140]);          % define transfer function G(s)
>> bode(sysG)                           % create the Bode diagram
```



We can create the same Bode diagram using the state-space representation (SSR) for the system, where the state variables are $x_1 = z$ and $x_2 = \dot{z}$; the input is $u = f_a(t)$, and the output is $y = x_1 = z$. The SSR is

State equation: $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -466.67 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 3.3333 \end{bmatrix}u$ Output equation: $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$

The following MATLAB command creates the Bode diagram using the SSR

```
>> A = [ 0 1 ; -466.67 -5];        % define state matrix A
>> B = [ 0 ; 3.3333 ];             % define input matrix B
>> C = [ 1 0 ];                    % define output matrix C
>> D = 0;                          % define direct-link term D
>> sys = ss(A,B,C,D);              % define sys as SSR
>> bode(sys)                       % create the Bode diagram from SSR
```

The Bode diagram is identical to the diagram above produced using the transfer function.

b) The MATLAB command bandwidth can be used as shown below:

```
>> wB = bandwidth(sysG)                    % compute bandwidth ωB for G(s)
```

The bandwidth is **33.24 rad/s.**

The magnitude Bode diagram shows that a peak occurs at a resonant frequency of about 21 rad/s. Repeated trials of the MATLAB command `[mag,phase] = bode(sysG,w)` shows that the resonant frequency is about $\boldsymbol{\omega_r}$ **= 21.3 rad/s.**

9.21 a) The transfer function is $G(s) = \dfrac{1}{2s^2 + 32}$; therefore the sinusoidal transfer function is

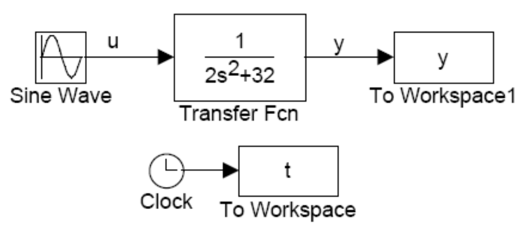
$$G(j\omega) = \frac{1}{2(j\omega)^2 + 32} = \frac{1}{32 - 2\omega^2}$$

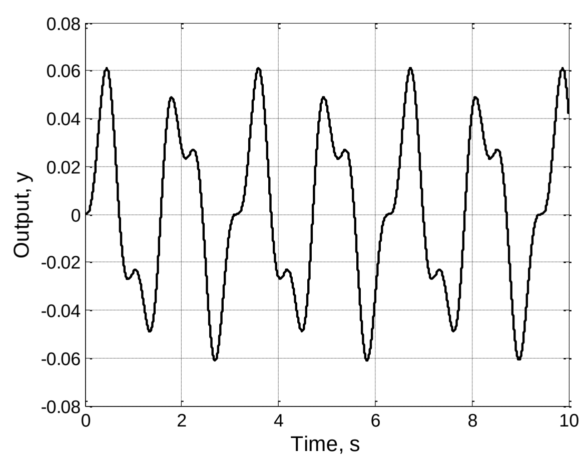
For the input frequency ω = 10 rad/s, the magnitude and phase are

Magnitude: $|G(j10)| = \dfrac{\sqrt{1^2 + 0^2}}{\sqrt{(32-200)^2 + 0^2}} = \dfrac{1}{168}$ = **0.00595**

Phase angle: $\phi = \dfrac{\arg[1 + j0]}{\arg[-168 + j0]} = \tan^{-1}\left(\dfrac{0}{1}\right) - \tan^{-1}\left(\dfrac{0}{-168}\right) =$ **180 deg**

b) The Simulink model (below) simulates the response to the sinusoidal input $u(t) = 3\sin 10t$. The output $y(t)$ is plotted below.





c) The simulation response plot above does not match the frequency response Eq. (9-17) because this system is *undamped* and therefore the transient response does not die out. Note that the system's undamped natural frequency is $\omega_n = \sqrt{32/2} = 4$ rad/s and hence the response $y(t)$ contains two sinusoidal functions with two frequencies: 4 rad/s (undamped natural frequency) and 10 rad/s (input frequency).

9.22 The system transfer function is

$$G(s) = \frac{576}{16s^2 + 38.4s + 576}$$

The two magnitude Bode plots show the same peak value, high-frequency slope, and shape; only the resonant frequency is different. We can compute the resonant frequency from knowledge of the undamped natural frequency ω_n and damping ratio ζ. First, divide $G(s)$ by 16:

$$G(s) = \frac{36}{s^2 + 2.4s + 36}$$

Therefore $\omega_n = \sqrt{36} = 6$ rad/s, and damping ratio $\zeta = 2.4/(2\omega_n) = 0.2$.

The resonant frequency is $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ = **5.755 rad/s** which matches **System 1.**

9.23 a) We can represent the complete system by using a state-space representation (SSR), where we can select the states as $x_1 = z$ (position), $x_2 = \dot{z}$ (velocity), and $x_3 = I$ (current). The input is $u = e_{in}(t)$ and the desired output is position $y = x_1 = z$. The complete SSR is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -51{,}667 & -91.667 & 76.667 \\ 0 & -230 & -400 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

The following MATLAB commands compute the frequency response for $\omega = 150$ rad/s

```
>> A = [0 1 0; -51667 -91.667 76.667; 0 -230 -400];   % define state matrix A
>> B = [ 0 ; 0 ; 100];                                 % define input matrix B
>> C = [ 1 0 0 ];                                      % define output matrix C
>> D = 0;                                              % define direct-link term D
>> sys = ss(A,B,C,D);                                  % define sys as SSR
>> w = 150;                                            % define input frequency ω
>> [mag,phase] = bode(sys,w)                           % compute magnitude and phase
```

The MATLAB result is `mag` = 0.000486 and `phase` = –52.508 deg (–0.9164 rad). The amplitude of the output is (8 V)(0.000486) = 0.00389 m (3.89 mm). Hence the frequency response is

$$\boxed{z_{ss}(t) = 0.00389 \sin(150t - 0.9164) \text{ m}}$$

b) The MATLAB command (below) determines the bandwidth $\omega_B = 318.33$ rad/s

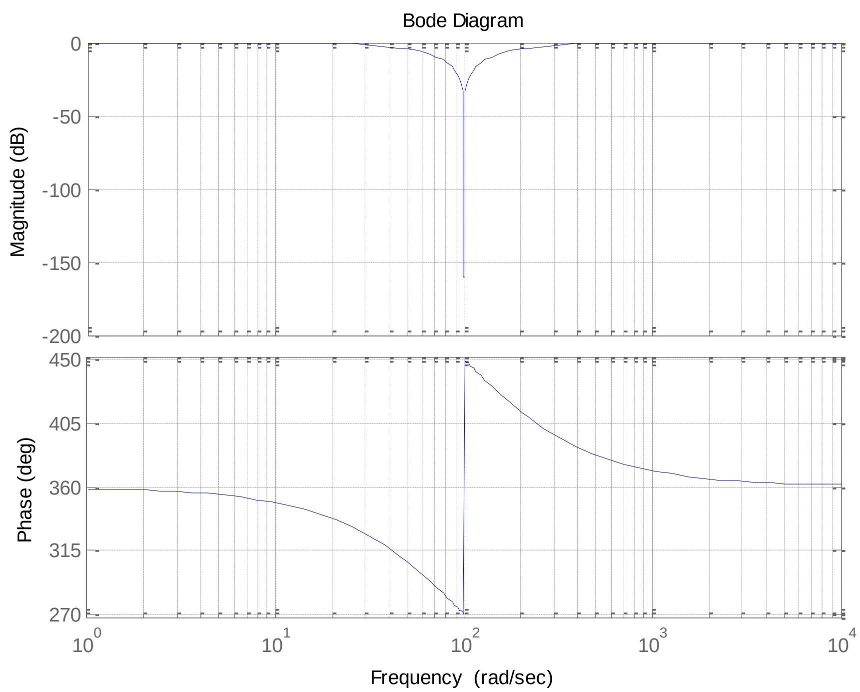
```
>> wB = bandwidth(sys)                                 % compute bandwidth ωB in rad/s
```

9.24 The notch filter transfer function is

$$G(s) = \frac{LCs^2 + 1}{LCs^2 + RCs + 1} = \frac{E_O(s)}{E_{\text{in}}(s)}$$

a) The following MATLAB commands create the Bode diagram (using values for L, C, and R)

```
>> L = 0.005;                              % define inductance L
>> C = 0.02;                               % define capacitance C
>> R = 1;                                  % define resistance R
>> sysG = tf([L*C 0 1],[L*C R*C 1])        % define transfer function G(s)
>> bode(sysG)                              % create the Bode diagram
```



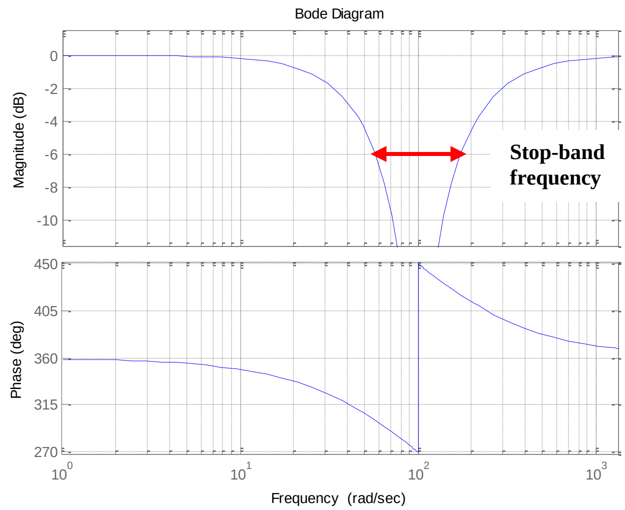
Bode diagram for L = 0.005 H, C = 0.02 F, and R = 1 Ω

Note that the magnitude plot has a DC gain of 0 dB at all frequencies except near ω = 100 rad/s. The large negative-dB "notch" at ω = 100 rad/s means that there is essentially zero output when the input frequency is near the "notch frequency" $\omega_N = \sqrt{1/(LC)} = \sqrt{10^4}$ = 100 rad/s.

b) When the output amplitude is reduced by one-half the corresponding change in decibels is

$$20\log_{10}(0.5) \cong -6\,\text{dB}$$

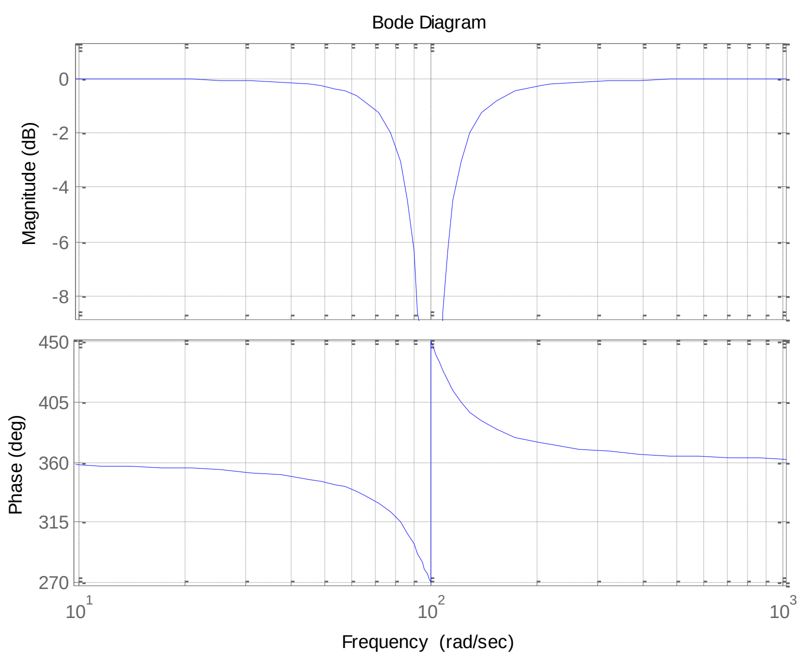
Hence, we can use the magnitude Bode diagram to estimate the frequency where the magnitude drops from 0 dB (low frequencies) to –6 dB. A "zoomed-in" view around the notch in the magnitude plot is shown on the next page.



Bode diagram for *L* = 0.005 H, *C* = 0.02 F, and *R* = 1 Ω (zoomed-in view)

Hence the "stop band" frequency range where the magnitude is less than –6 dB is roughly **between 60 and 140 rad/s** (i.e., symmetric about the notch frequency ω_N = 100 rad/s).

c) Executing the previous MATLAB commands with R = 0.2 Ω creates the Bode diagram shown below. Note that the "notch" is sharper and has a narrower stop-band frequency range. The stop-band frequency range (i.e., magnitude less than –6 dB) is **between 90 and 110 rad/s.**



Bode diagram for *L* = 0.005 H, *C* = 0.02 F, and *R* = 0.2 Ω (zoomed-in view)

9.25 a) Since the input force is $f_{\text{in}}(t) = 2\sin 50t$ N, we simply read the Bode diagram at input frequency $\omega = 50$ rad/s: Magnitude of $G(j50) = -16$ dB and phase of $G(j50) = -135$ deg (or -2.3562 rad). The absolute-value magnitude is $10^{-16/20} = 0.1585$. Hence, the amplitude of the frequency response is (2 N)(0.1585) = 0.317 N (amplitude of the transmitted force). The complete frequency response of the transmitted force is

$$\boxed{f_T(t) = 0.317\sin(50t - 2.3562) \ \ \text{N}}$$

b) Transmissibility of 1.75 is equal to an amplitude ratio (magnitude) in decibels of

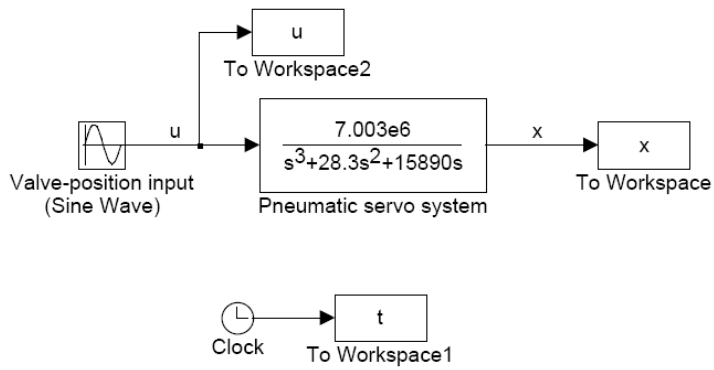
$$20\log_{10}(1.75) = 4.86 \ \text{dB}$$

Hence, the magnitude Bode plot cannot exceed 4.86 dB so that TR $\leq$ 1.75. We see from the Bode diagram that the magnitude exceeds roughly 5 dB for the frequency range $10 < \omega < 20$ rad/s and this is the frequency range that must be avoided in order to keep transmissibility less than 1.75.

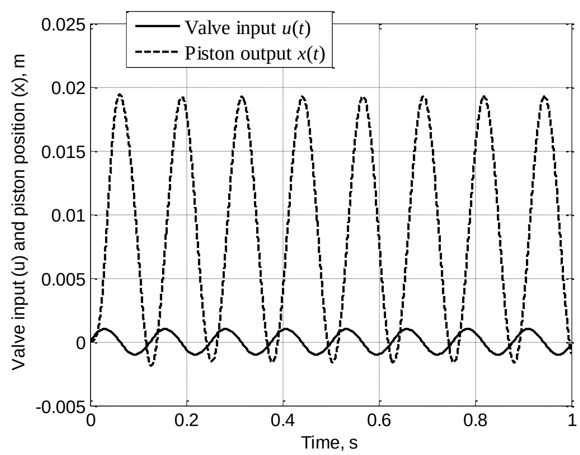
9.26 The linearized transfer function for the pneumatic system is

$$G(s) = \frac{7.003(10^6)}{s(s^2 + 28.3s + 15890)} = \frac{X(s)}{U(s)}$$

a) The Simulink model has a sinusoidal valve input $u(t) = 0.001\sin 50t$ m.



A plot of piston position $x(t)$ and input $u(t)$ is below:



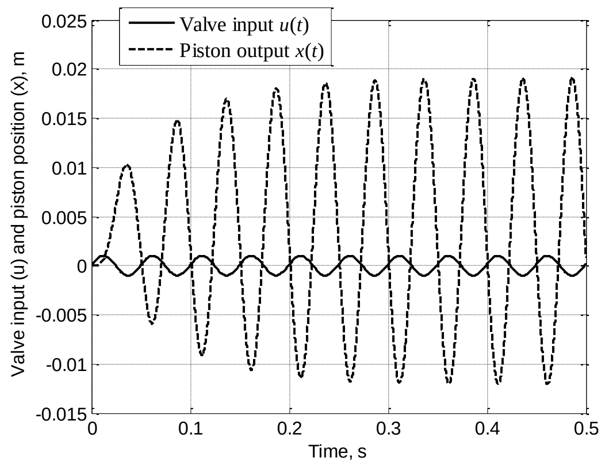
8-Hz input *u*(*t*) and system response *x*(*t*)

The response plot $x(t)$ does not oscillate about zero and is therefore different from the "standard" frequency-response equation (9-17):

$$x_{ss}(t) = |G(j\omega)|U_0 \sin(\omega t + \phi)$$

The difference can be explained by observing the poles of the system transfer function. The standard frequency-response equation (9-17) was derived by assuming that all poles of $G(s)$ have negative real parts; that is, the corresponding transient response eventually decays to zero. The pneumatic servo transfer function $G(s)$ has a pole at the origin ($s = 0$) and hence part of the natural response is a ***constant*** that does not decay to zero. This constant is an "offset" that is added to the sinusoidal frequency response, and hence the frequency response does not oscillate about zero.

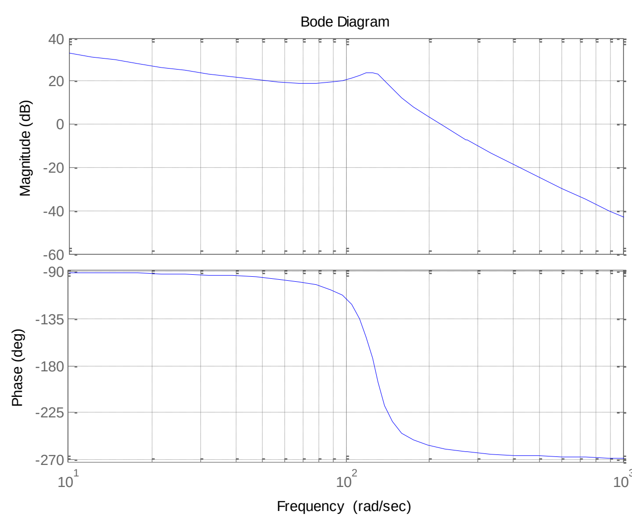
b) We can change the input frequency in the Simulink model to a 20-Hz input sinusoidal function $u(t) = 0.001\sin 126t$. The corresponding system response is below:



20-Hz input $u(t)$ and system response $x(t)$

The response plot shows that input $u(t)$ and output $x(t)$ are nearly 180 deg out of phase (i.e., the input "peaks" are aligned with the output "valleys."). The phase plot in the Bode diagram (produced by MATLAB) verifies the –180 deg phase at input frequency $\omega = 126$ rad/s. Note that the undamped natural frequency of $G(s)$ is $\omega_n = \sqrt{15{,}890} = 126.06$ rad/s which is equal to the input frequency.

```
>> sysG = tf(7.003e6,[1 28.3 15890 0])     % define transfer function G(s)
>> bode(sysG)                              % create the Bode diagram
```

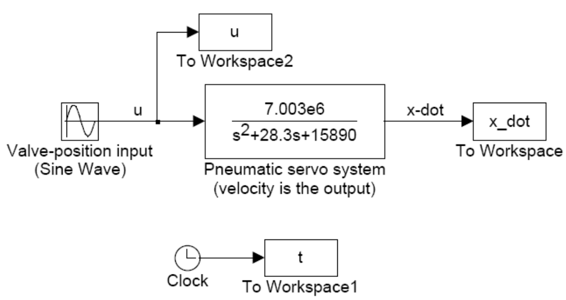


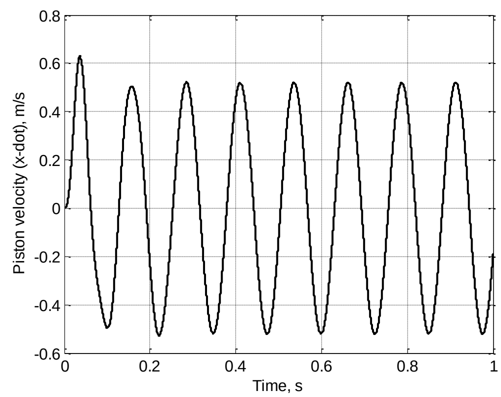
Bode diagram with piston position as the output

c) The transfer function with velocity as the output can be obtained by multiplying the position-output transfer function $G(s)$ by s

$$G_2(s) = \frac{7.003(10^6)}{s^2 + 28.3s + 15890} \qquad \text{(velocity is the output)}$$

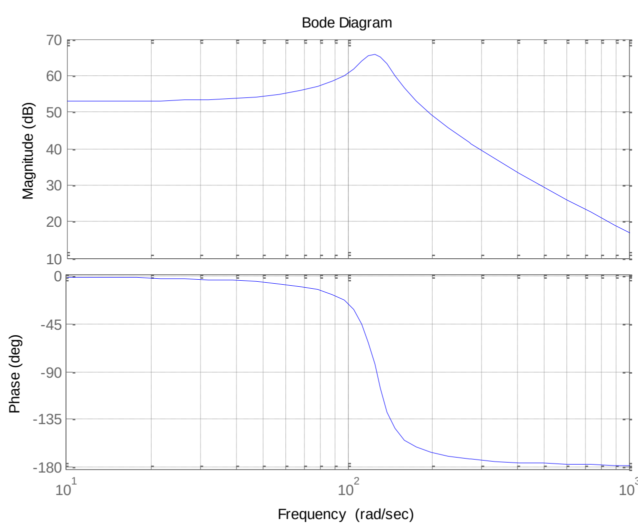
The Simulink diagram with the velocity-output transfer function is shown below. Note that the pneumatic servo transfer function is 2nd-order (no pole at $s = 0$). The velocity response is plotted below. Note that velocity oscillates about zero.





Velocity response to an 8-Hz valve input

The frequency response of the velocity can be computed from the corresponding Bode diagram of transfer function $G_2(s)$, shown below:



Bode diagram with piston velocity as the output

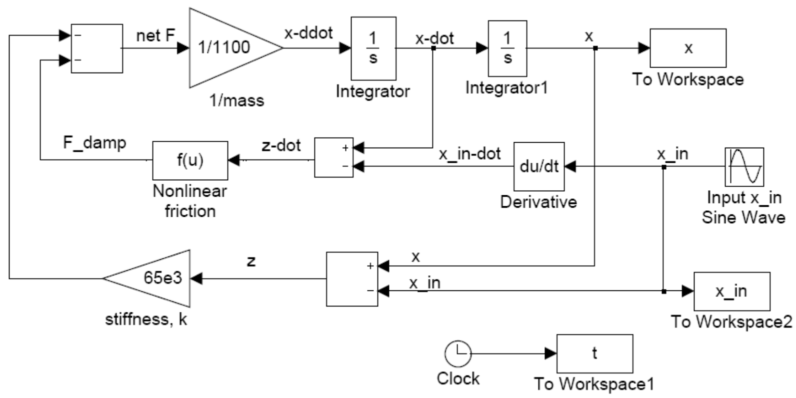
At input frequency $\omega = 50$ rad/s (~8 Hz) the magnitude is 54 dB (or 520.1) and the phase is –6 deg (or –0.105 rad). Hence the velocity frequency response is

$$\dot{x}_{ss}(t) = 0.520\sin(50t - 0.105) \quad \text{m/s}$$

9.27 The mathematical model of the 1-DOF nonlinear suspension system is

$$m\ddot{x} = -F_d(\dot{z}) - kz \qquad \text{or,} \qquad m\ddot{x} + \frac{4500}{\sqrt{\dot{z}^2 + v^2}}\dot{x} + kx = \frac{4500}{\sqrt{\dot{z}^2 + v^2}}\dot{x}_{\text{in}}(t) + kx_{\text{in}}(t)$$

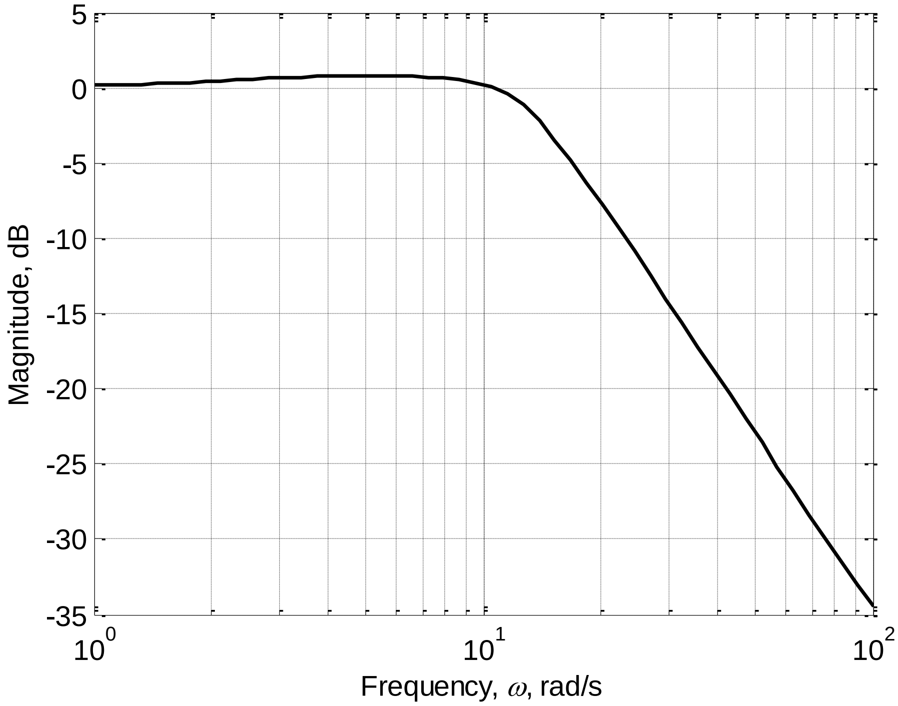
where $\dot{z} = \dot{x} - \dot{x}_{\text{in}}(t)$ is the relative velocity and v = 0.2 m/s. The Simulink model is below. Note that a user-defined function computes the nonlinear damping force given relative velocity $\dot{z} = \dot{x} - \dot{x}_{\text{in}}(t)$.



The following M-file “loops” through a range of input frequencies $0.1 < \omega < 100$ rad/s, executes the Simulink model, stores the steady-state amplitude ratio (x/x_{in}), and creates the magnitude Bode diagram (in dB).

```
%   Mfile for Problem 9.27

r2d = 180/pi;               %  rad-to-deg

%  input frequencies
Nw = 50;
w = logspace(0,2,Nw);       %  log-spaced frequencies from 0.1 to 100 rad/s

%  Loop for frequency
for i=1:Nw
    w_in = w(i);
    f_in_Hz = w_in/(2*pi);

    %  re-compute step size based on period
    Tperiod = 2*pi/w_in;                        %  period, sec
    t_stop = round(w_in*3)*Tperiod;
    dt = Tperiod/500;

    %  execute Simulink model
    sim P9_27

    Npts = numel(x);                            %  number of data pts
    N_half = round(Npts/2);
    amp_x_SS = max(x(N_half:Npts));
    amp_xin = max(x_in);
    magdB(i) = 20*log10(amp_x_SS/amp_xin);      %  magnitude, dB
end

%  Bode magnitude plot
semilogx(w,magdB)
grid
xlabel('Frequency, \it\omega\rm, rad/s')
ylabel('Magnitude, dB')
```

The Bode magnitude plot from running the M-file is shown below:



Bode magnitude plot of 1-DOF system with nonlinear damper

We see that the magnitude exhibits a DC gain of 0 dB, or unity transmissibility at low frequencies. The system has a small 1-dB resonant peak at about 5 rad/s and drops off at high frequencies. Therefore this nonlinear suspension system has good damping and attenuates high-frequency input.

9.28 The state equation of the ¼-car system (see Problem 2.30 and 6.23) is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1/m_1 & -b/m_1 & k_1/m_1 & b/m_1 \\ 0 & 0 & 0 & 1 \\ k_1/m_2 & b/m_2 & -(k_1+k_2)/m_2 & -b/m_2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_2/m_2 \end{bmatrix}u$$

The state variables are $x_1 = z_1$, $x_2 = \dot{z}_1$, $x_3 = z_2$, and $x_4 = \dot{z}_2$. The input is $u = z_{in}(t)$.

a) An impulsive input $z_{in}(t)$ (a road bump) will excite the natural frequencies of the system. The natural frequencies are associated with the imaginary parts of the characteristic roots, or poles, or eigenvalues. Because we have a SSR it is easiest to determine the eigenvalues of the 4×4 **A** matrix using the MATLAB commands:

```
>> m1 = 250;                                          % mass of ¼ car, kg
>> m2 = 30;                                           % mass of wheel/axle, kg
>> k1 = 1.6e4;                                        % suspension stiffness, N/m
>> b = 980;                                           % suspension friction, N-s/m
>> k2 = 1.6e5;                                        % tire stiffness, N/m
>> Arow1 = [ 0 1 0 0 ];                               % A (row 1)
>> Arow2 = [ -k1/m1 -b/m1 k1/m1 b/m1 ];               % A (row 2)
>> Arow3 = [ 0 0 0 1 ];                               % A (row 3)
>> Arow4 = [ k1/m2 b/m2 (-k1-k2)/m2 -b/m2 ];          % A (row 2)
>> A = [Arow1 ; Arow2 ; Arow3 ; Arow4];               % A matrix
>> lambda = eig(A)                                    % eigenvalues of the A matrix
```

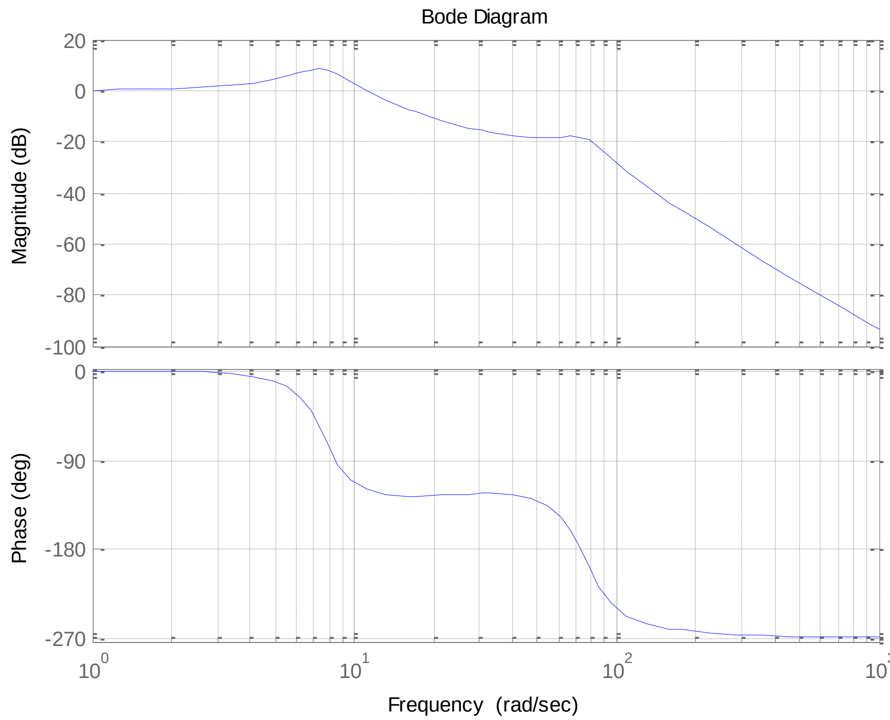
The four eigenvalues are complex: $\lambda_{1,2} = -16.6503 \pm j74.0592$ and $\lambda_{3,4} = -1.6430 \pm j7.5193$. Hence the vibration frequencies are the imaginary parts: **ω_1 = 7.519 rad/s and ω_2 = 74.059 rad/s**.

a) The system output is the ¼-car mass position: $y = z_1 = x_1$

$$y = [1 \quad 0 \quad 0 \quad 0]\mathbf{x} + [0]u$$

The following added MATLAB commands produce the Bode diagram (using the SSR)

```
>> B = [ 0 ; 0 ; 0 ; k2/m2 ];                         % B matrix
>> C = [ 1 0 0 0 ];                                   % C matrix (output, y = z1)
>> D = 0;                                             % D matrix (null)
>> sys = ss(A,B,C,D);                                 % create SSR system
>> bode(sys);                                         % draw Bode diagram
```



The magnitude plot of the Bode diagram shows two resonant peaks at approximately ω = 7.4 rad/s and ω = 70 rad/s, which are very close to the natural (damped) frequencies in part (a). The magnitudes at these resonant frequencies are about 8 dB (at ω = 7.4 rad/s) and –18 dB (at ω = 70 rad/s). Hence the two transmissibility values are $10^{8/20}$ = **2.512** and $10^{-18/20}$ = **0.126**.

9.29 The mathematical model of the 1-DOF mechanical system is

$$m\ddot{x} + b\dot{x} + kx = b\dot{x}_b(t) + kx_b(t)$$

The system's transfer function is

$$G(s) = \frac{X(s)}{X_b(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Because we are given the amplitude of the frequency response we must compute the magnitude of the sinusoidal transfer function:

$$G(j\omega) = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k} = \frac{k + jb\omega}{k - 1.4\omega^2 + jb\omega}$$

where we have substituted mass m = 1.4 kg. The magnitude of the sinusoidal transfer function for input frequency ω = 5 rad/s (first table entry) is

Magnitude: $|G(j5)| = \dfrac{\sqrt{k^2 + (5b)^2}}{\sqrt{(k - (1.4)(25))^2 + (5b)^2}} = \dfrac{0.5070}{0.5}$ = 1.0140

or, squaring both sides:

$$k^2 + 25b^2 = (1.0140)^2\left(k^2 - 70k + 1225 + 25b^2\right) \qquad \text{(A)}$$

Similarly we can compute the magnitude with input frequency ω = 30 rad/s (second table entry)

Magnitude: $|G(j30)| = \dfrac{\sqrt{k^2 + (30b)^2}}{\sqrt{(k - (1.4)(900))^2 + (30b)^2}} = \dfrac{0.7008}{0.5}$ = 1.4016

or

$$k^2 + 900b^2 = (1.4016)^2\left(k^2 - 2520k + 1{,}587{,}600 + 900b^2\right) \qquad \text{(B)}$$

While Equations (A) and (B) provide two equations with two unknowns (k and b) the equations are nonlinear. Perhaps a simpler method is to use matrix methods with a third equation using the third table entry, ω = 60 rad/s:

Magnitude: $|G(j60)| = \dfrac{\sqrt{k^2 + (60b)^2}}{\sqrt{(k - (1.4)(3600))^2 + (60b)^2}} = \dfrac{0.4975}{0.5}$ = 0.9950

or

$$k^2 + 3600b^2 = (0.9950)^2\left(k^2 - 10{,}080k + 25{,}401{,}600 + 3600b^2\right) \qquad \text{(C)}$$

Equations (A)-(C) provide three equations with three unknowns k^2, k, and b^2. A matrix formulation is

$$\begin{bmatrix} -0.0282 & 71.9737 & -0.7049 \\ -0.96448 & 4950.4961 & -868.0343 \\ 0.009975 & 9979.4520 & 35.9100 \end{bmatrix} \begin{bmatrix} k^2 \\ k \\ b^2 \end{bmatrix} = \begin{bmatrix} 1259.54 \\ 3.1188125(10^6) \\ 2.5148219(10^7) \end{bmatrix}$$

Multiplying the inverse of the 3×3 matrix by the right-hand side vector yields the solution:

$$\begin{bmatrix} k^2 \\ k \\ b^2 \end{bmatrix} = \begin{bmatrix} 6.2436(10^6) \\ 2500.3 \\ 3729.408 \end{bmatrix}$$

Hence the composite spring constant is k = 2500.3 N/m and the friction coefficient is b = 61.07 N-s/m. Note that there is some loss of numerical accuracy due to carrying finite digits for the very large numbers. As a check, we can use the MATLAB command `bode` as follows:

```
>> m = 1.4;                              % mass, kg
>> k = 2500.3;                           % stiffness, N/m
>> b = 61.07;                            % friction, N-s/m
>> sysG = tf([b k],[m b k]);             % create transfer function G(s)
>> [mag,phase] = bode(sysG,5);           % magnitude and phase for ω = 5 rad/s
```

The magnitude is `mag = 1.0140` which matches the first entry in the table. We can use these MATLAB commands to verify the other entries in the table for ω = 30, 60, and 100 rad/s.