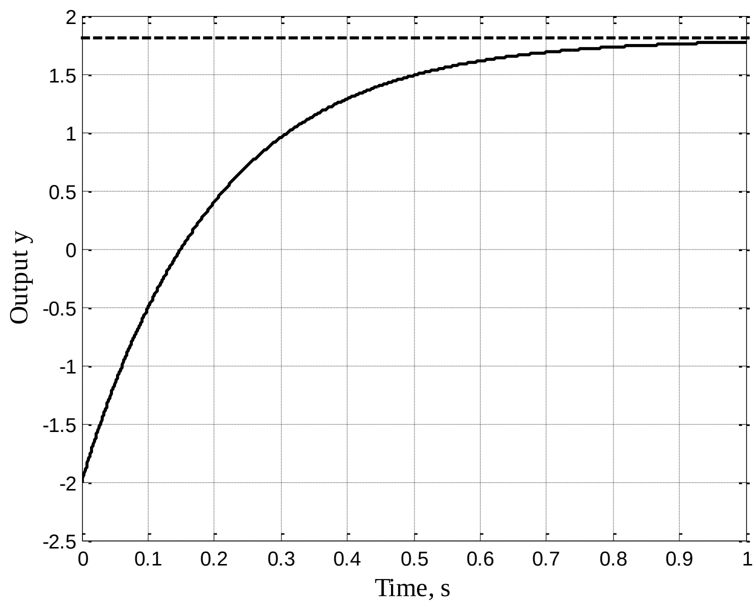
## Chapter 7: Analytical Solution of Linear Dynamic Systems

**7.1** Re-write the first-order ODE in standard form (divide all terms by 10):

$$0.2\dot{y} + y = 0.3u$$

The single root is $r = -1/0.2 = -5$ and therefore the response is a stable exponential function. The time constant is $\tau = 0.2$ s and hence the system reaches steady state in $t_S = 4\tau = 0.8$ s. The steady-state value for a constant input $u = 6$ is $y_{SS} = (0.3)(6) = 1.8$. A *hand-drawn sketch* of the response would match the response plot (below) and include labels for the initial condition $y(0) = -2$, settling time $t_S = 0.8$ s, and steady-state response $y_{SS} = 1.8$.



**7.2** a) First, we obtain the roots of the characteristic equation (below):

$$2r^2 + 12r + 68 = 0$$

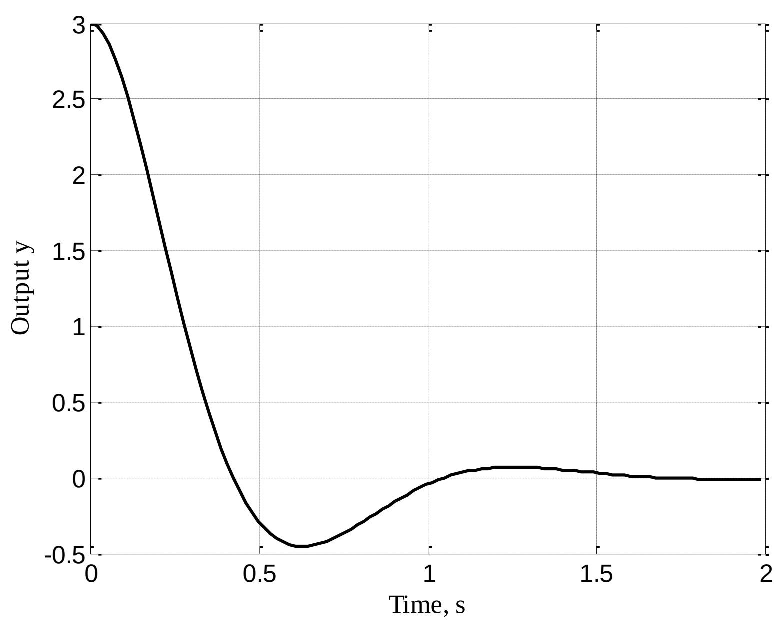
The two roots are complex: $r_{1,2} = -3 \pm j5$. Hence, the homogeneous response **does** exhibit oscillations. Another way to show this is to re-write the I/O equation in the standard form for a second-order system:

$$\ddot{y} + 6\dot{y} + 34y = 0 \quad \text{or} \quad \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 \quad \text{Hence } \zeta = 0.514 < 1 \text{ (underdamped)}$$

b) For an underdamped second-order system the settling time is approximately

$$t_S = \frac{4}{\zeta\omega_n} = \frac{4}{0.514\sqrt{34}} = 1.3333 \text{ s}$$

c) Because the second-order system is underdamped, the homogeneous response will exhibit decaying oscillations at frequency $\omega_d$ = 5 rad/s (or, period = 1.257 s). The homogeneous response will begin at the initial condition $y(0)$ = 3 with "zero slope" and exhibit a damped sinusoidal response that decays to zero in about 1.3333 s (a bit longer than one period). A sketch would match the plot:



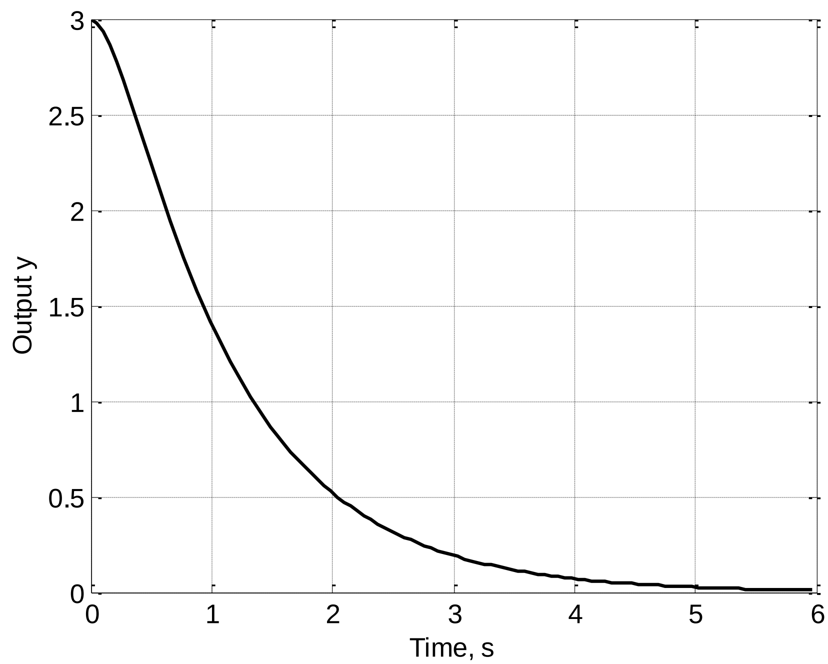
**7.3** a) First, we obtain the roots of the characteristic equation (below):

$$4r^2 + 22r + 18 = 4(r+1)(r+4.5) = 0$$

The two roots are real and negative: $r_1 = -1$ and $r_2 = -4.5$ and hence the homogeneous response is comprised of two decaying exponential functions (no oscillations).

b) The homogeneous response has the form $y_H(t) = c_1e^{-t} + c_2e^{-4.5t}$ and the *slowest* exponential mode "dies out" at time $t_S = 4$ s. Hence the settling time for the system is 4 s.

c) The homogeneous response will begin at the initial condition $y(0) = 3$ with "zero slope" and then exponentially decay to zero in about 4 s. A sketch would match the plot below.



**7.4** In all cases the denominator polynomial (characteristic equation) will have the form

$$(s+r_1)(s+r_2)=0$$

a) Roots $r_1=-2.5$ and $r_2=-0.2$: $(s+2.5)(s+0.2)=s^2+2.7s+0.5$ (denominator)

Hence the transfer function has the form $G(s)=\dfrac{c}{s^2+2.7s+0.5}$

Since the DC gain of this transfer function is $G(s=0)=c/0.5=0.5$ we obtain $c$ = 0.25 and the transfer function is

$$\boxed{G(s)=\frac{0.25}{s^2+2.7s+0.5}}$$

b) Roots $r_1=-3$ and $r_2=-10$: $(s+3)(s+10)=s^2+13s+30$ (denominator)

Since the DC gain of this transfer function is 6 we obtain

$$\boxed{G(s)=\frac{180}{s^2+13s+30}}$$

Note: $G(s=0)=180/30=6$

c) Roots $r_{1,2}=-2\pm j4$: $(s+2+j4)(s+2-j4)=s^2+4s+20$ (denominator)

Since the DC gain of this transfer function is 125 we obtain

$$\boxed{G(s)=\frac{2500}{s^2+4s+20}}$$

Note: $G(s=0)=2500/20=125$

d) Roots $r_{1,2}=-0.4\pm j1.6$: $(s+0.4+j1.6)(s+0.4-j1.6)=s^2+0.8s+2.72$

Since the DC gain of this transfer function is 0.02 we obtain

$$\boxed{G(s)=\frac{0.0544}{s^2+0.8s+2.72}}$$

Note: $G(s{=}0)=\dfrac{0.0544}{2.72}=0.02$

**7.5** a) We can obtain the steady-state response to a constant input by using the DC gain:

$$G(s) = \frac{84}{(3s^2 + 21s + 36)(s^2 + 2s + 9)} \;\rightarrow\; G(s = 0) = \frac{84}{(36)(9)} = 0.2593$$

Therefore the steady-state output is $y_{SS}$ = (4)(0.2593) = **1.0370**

b) The four characteristic roots are determined by setting the denominator to zero

$$(3s^2 + 21s + 36)(s^2 + 2s + 9) = 0 \quad \text{or} \quad 3(s+3)(s+4)(s^2 + 2s + 9) = 0$$

Hence the four roots (poles) are $r_1 = -3$, $r_2 = -4$, and $r_{3,4} = -1 \pm j2.8284$

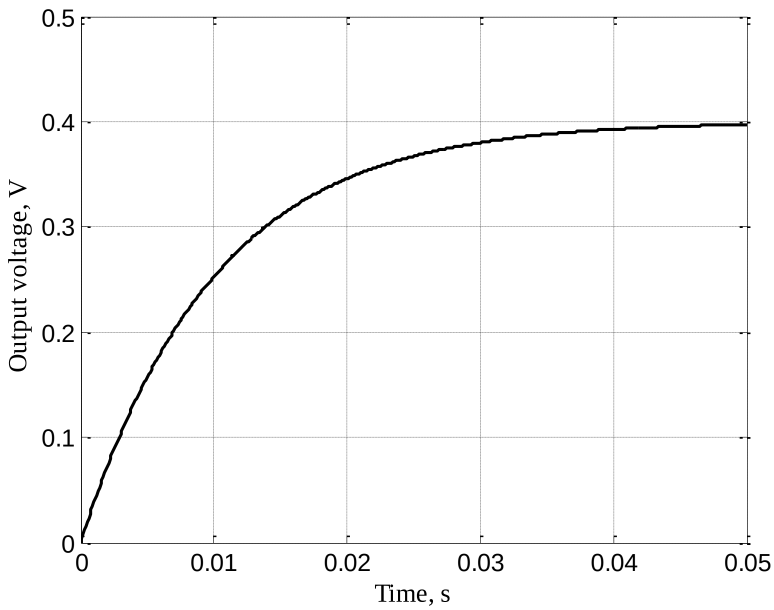
The three exponential decay modes for the transient response are $e^{-3t}$, $e^{-4t}$ (for the two real roots) and $e^{-t}$ (for the two complex roots). Since the ***slowest*** exponential mode is $e^{-t}$, the **overall** settling time is $\boldsymbol{t_S}$ **= 4 s** since $e^{-4}$ is "small."

**7.6** The mathematical model of the RC circuit is $RC\dot{e}_O + e_O = e_{\text{in}}(t)$

Therefore the transfer function is

$$G(s) = \frac{E_O(s)}{E_{\text{in}}(s)} = \frac{1}{RCs+1} = \frac{1}{0.01s+1} \quad \text{(using } RC = 0.01 \text{ s)}$$

Clearly the DC gain is 1 and the time constant is $\tau = RC = 0.01$ s. Since the input is a constant 0.4 V, the steady-state output voltage is also 0.4 V. The output voltage will show an exponential rise from its initial value (zero) to 0.4 V and reach steady state in settling time $t_S = 4\tau = 0.04$ s. A sketch should show $e_O(0) = 0$, $e_O(\infty) = 0.4$ V, and settling time (i.e., match the plot below).

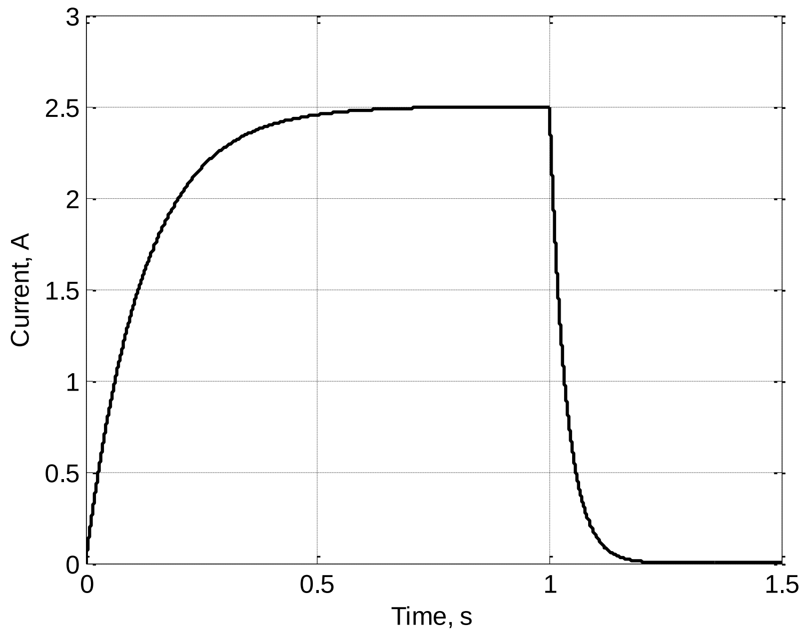


**7.7** The two mathematical models of the electrical system are

$$L\dot{I}_L + R_1 I_L = e_{\text{in}}(t) \ \ \text{(Switch 1)} \quad \text{and} \quad L\dot{I}_L + (R_1 + R_2)I_L = 0 \ \ \text{(Switch 2)}$$

First, consider the response for time $0 \le t \le 1$ (Switch 1) when $e_{\text{in}}(t)$ = 4 V. The time constant of the "Switch 1" model is $\tau_1 = L / R_1$ = 0.2/1.6 = 0.125 s. Hence the settling time is $4\tau_1$ = 0.5 s which is less than the switching time (1 s), so the current reaches a steady-state value. The steady-state current for the "Switch 1" model is $I_{L_{ss}} = e_{\text{in}} / R_1$ = 4/1.6 = 2.5 A.

When time $t$ > 1 s, we have the "Switch 2" model with time constant $\tau_2 = L/(R_1 + R_2)$ = 0.0357 s. Hence the settling time is $4\tau_2$ = 0.1429 s (relative to the switch time $t$ = 1), and the current $I_L$ decays to zero and reaches zero at approximately time $t$ = 1.1429 s. A sketch will match the plot below.



**7.8** The characteristic equation is the denominator polynomial of $G(s)$ set to zero:

$$2s^2 + 3s + 24 = 0 \quad \text{or} \quad s^2 + 1.5s + 12 = 0$$

The roots (poles) are $s_{1,2} = -0.75 \pm j3.3819$. Because the roots (or poles) are complex, the transient response **does** exhibit oscillations.

Method 2: the standard second-order form is $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. Hence $\omega_n = \sqrt{12} = 3.464$ rad/s and $\zeta = 0.217 < 1$ (underdamped) and the transient response shows oscillations.

**7.9** a) The characteristic equation is the denominator polynomial of $G(s)$ set to zero

$$0.15s^2 + 1.4s + 128 = 0 \quad \text{or} \quad s^2 + 9.33s + 853.33 = 0$$

The roots (poles) are $s_{1,2} = -4.6667 \pm j28.8367$. Because the roots (or poles) are complex, the transient response **does** exhibit oscillations.

Note that the standard second-order form is $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. Hence the undamped natural frequency is $\omega_n = \sqrt{853.33} = 29.2119$ rad/s and $\zeta = 0.1598 < 1$ (underdamped).

b) The oscillation frequency is the *damped* frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 28.8367$ rad/s which is the same as the imaginary part of the roots. The frequency in hertz is $\omega_d/2\pi$ = **4.59 Hz.**

c) The period of oscillation is $2\pi/\omega_d$ = 0.2178 s (= 1/4.59 Hz). The settling time for the underdamped second-order system is $t_S = 4/(\zeta\omega_n)$ = 0.8571 s. Hence the transient response has approximately 0.8571/0.2178 = 3.9 (~ 4) cycles before reaching steady state.

**7.10** The damping ratio $\zeta$ is the magnitude of the real part of the root divided by the distance from the root to the origin (see Fig. 7.17). If the complex root is $s = -a \pm jb$ then we have

$$\zeta = \frac{a}{\sqrt{a^2 + b^2}}$$

1) Roots are $s = -3 \pm j2$. Hence $\zeta = 3/\sqrt{3^2 + 2^2}$ = **0.8321**

2) Roots are $s = -2 \pm j3$. Hence $\zeta = 2/\sqrt{2^2 + 3^2}$ = **0.5547**

3) Roots are $s = -3 \pm j3$. Hence $\zeta = 3/\sqrt{3^2 + 3^2}$ = **0.7071**

→ **root-pair (1)** has the greatest damping ratio.

**7.11** We will use the log-decrement method. The peak values and peak times of the impulse response are estimated from the impulse-response plot:

Peak 1: $x_1$ = 0.026 m, $t_1$ = 0.1 s Peak 2: $x_2$ = 0.012 m, $t_1$ = 0.5 s

Hence the period is approximately $T_{\text{period}}$ = 0.5 – 0.1 = 0.4 s. The log decrement is

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{0.026}{0.012} = 0.7732$$

The damping ratio is approximately

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \mathbf{0.1221}$$

The damped frequency is $\omega_d$ = $2\pi/T_{\text{period}}$ = 15.708 rad/s. Because damped frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2}$ , the *undamped* natural frequency can be determined from $\omega_d$ and $\zeta$ to be

$$\omega_n = \omega_d / \sqrt{1-\zeta^2} = \mathbf{15.8264\ rad/s}$$

**7.12** a) The general solution form for the homogeneous response of an underdamped system with two complex roots at $r_{1,2} = \alpha \pm j\beta$ is

$$y_H(t) = ce^{\alpha t}\cos(\beta t + \phi) \qquad \text{(A)}$$

We see that the **real** part of the root will be the exponent of the exponential function while the **imaginary** part will be the frequency of the sinusoidal function. Because we have two pairs of complex roots we will have *two* damped sinusoidal functions in the form of Eq. (A). Using the two root pairs given in the problem, the homogeneous response has the form

$$\boxed{y_H(t) = c_1 e^{-1.2t}\cos(6.3t + \phi_1) + c_2 e^{-0.4t}\cos(4.2t + \phi_2)}$$

The unknown constants are $c_1$, $c_2$, $\phi_1$, and $\phi_2$. We could have used sine instead of cosine.

b) The two exponential functions $e^{-1.2t}$ and $e^{-0.4t}$ determine the decay rate for the two damped sinusoidal functions. The "slowest" decaying exponential function is $e^{-0.4t}$ which requires about 10 s to approximately reach zero since $e^{-4} = 0.0183$ is "small." Hence the settling time is **10 s.**

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**7.13** Because the free response is $x(t) = 6\cos 50t$ , we note that the *undamped* natural frequency is $\omega_n = 50$ rad/s.

The mechanical model of an unforced mass-spring system is $m\ddot{x} + kx = 0$. The standard I/O equation form of a second-order system with **no** damping ($\zeta = 0$) is $\ddot{x} + \omega_n^2 x = 0$. Hence the undamped natural frequency of the mass-spring system is

$$\omega_n = \sqrt{\frac{k}{m}} = 50 \text{ rad/s}$$

Since mass is $m = 0.2$ kg, the spring constant is ***k* = 500 N/m**

**7.14** The I/O equation is $2\ddot{y} + 8\dot{y} + 6y = 3u$

a) The characteristic equation can be written from inspection:

$$2r^2 + 8r + 6 = 0 \quad \text{or} \quad 2(r+1)(r+3) = 0$$

The two characteristic roots are $r_1 = -1$ and $r_2 = -3$.

b) The following transfer function can be derived from the given I/O equation:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3}{2s^2 + 8s + 6}$$

The poles of $G(s)$ are determined by setting the denominator polynomial to zero:

$$2s^2 + 8s + 6 = 0$$

Therefore the two poles of $G(s)$ are $s_1 = -1$ and $s_2 = -3$, which are identical to the roots in (a).

c) We can obtain a SSR for the states $x_1 = y$ and $x_2 = \dot{y}$. The resulting state equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} u$$

The eigenvalues are computed from the determinant $|\lambda \mathbf{I} - \mathbf{A}| = 0$

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 3 & \lambda + 4 \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

The eigenvalues are $\lambda_1 = -1, \lambda_2 = -3$ which are identical to the roots and poles in (a) and (b).

d) Because the two roots are real and negative the zero-input response consists of two decaying exponential functions

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}$$

The response starts at $y(0) = 5$ with "zero slope" due to $\dot{y}(0) = 0$ and eventually decays to zero. The slowest exponential decay mode is $e^{-t}$ which takes about 4 s to reach zero.

**7.15** First, we determine the poles of the third-order transfer function $G_1(s)$

$$s^3 + 20.2s^2 + 124s + 24 = 0$$

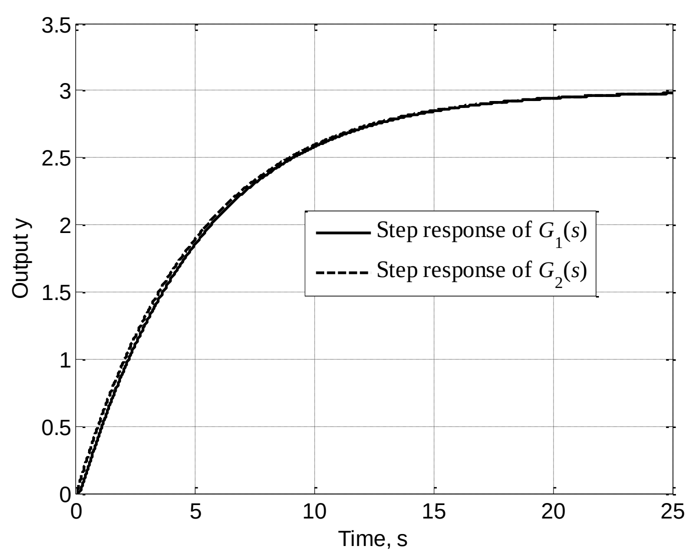
The three poles are $s_1 = -0.2$ and $s_2 = -10 \pm j4.4721$. The two underdamped poles show very high damping ($\zeta$ = 0.913) and a relatively short settling time $t_S$ = 0.4 s. The single real pole ($s_1 = -0.2$) exhibits an exponential decay $e^{-0.2t}$ which takes 20 s to die out to zero. Hence the damped sinusoidal (second-order) component of the transient response of $G_1(s)$ is barely noticeable and the first-order component $e^{-0.2t}$ dominates the transient response. It is easy to see that the single pole of the first-order transfer function $G_2(s)$ is also $s = -0.2$ and hence its transient response is also $e^{-0.2t}$, which matches the dominant pole of $G_1(s)$.

Finally, note that the DC gain of the third-order system is $G_1(s = 0) = 72/24 = 3$ which is also the DC gain of the first-order system, i.e., $G_2(s = 0) = 0.6/0.2 = 3$. So both transfer functions show the **same** steady-state response to a constant input. For these reasons the first-order system $G_2(s)$ is an excellent low-order approximation of the third-order system $G_1(s)$.

We can obtain the third-order and first-order step responses using the following MATLAB commands:

```
>> sysG1 = tf(72,[1 20.2 124 24]);        % define transfer function G1(s)
>> sysG2 = tf(0.6,[1 0.2]);               % define transfer function G2(s)
>> t = 0:0.01:25;                         % define time vector
>> [y1,t] = step(sysG1,t) ;               % unit-step response of G1(s)
>> [y2,t] = step(sysG2,t) ;               % unit-step response of G2(s)
>> plot(t,y1,t,y2)                        % plot both unit-step responses
```

The plot is below. The two unit-step responses are almost identical, and hence $G_2(s)$ is an excellent low-order approximate model for system $G_1(s)$.



**7.16** a) The eigenvalues are computed from the determinant $|\lambda\mathbf{I}-\mathbf{A}|=0$

$$|\lambda\mathbf{I}-\mathbf{A}|=\begin{vmatrix}\lambda+0.2 & 0.6\\ -2 & \lambda+4\end{vmatrix}=\lambda^2+4.2\lambda+2=0$$

The eigenvalues are $\lambda_1=-0.5476$ and $\lambda_2=-3.6524$.

b) The eigenvalues can be computed using the simple MATLAB commands; matches part (a)

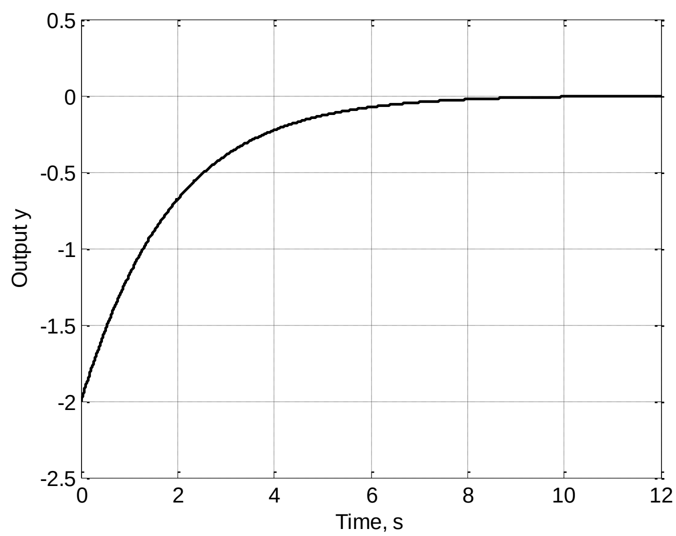
```
>> A = [-0.2 -0.6 ; 2 -4];             % define the state matrix A
>> eig(A)
```

c) Because the two roots are real and negative, the free (zero-input) response will consist of two decaying exponential functions, or $y(t)=c_1e^{-0.5476t}+c_2e^{-3.6524t}$. The "slow" exponential mode is $e^{-0.5476t}$ which takes about 7.3 s to decay to zero.

d) MATLAB commands are probably the easiest method: use the command `initial`

```
>> A = [-0.2 -0.6 ; 2 -4];             % define the state matrix A
>> B = [0 ; 1.5 ];                     % define the input matrix B
>> C = [1 0];                          % define the input matrix C
>> D = 0;                              % define the direct-link matrix D
>> sys = ss(A,B,C,D);                  % define sys as the SSR
>> x0 = [-2 ; -1];                     % define the initial state vector
>> t = 0:0.01:12;                      % define the time vector
>> [y,t]=initial(sys,x0,t);            % obtain the free response to the initial conditions
>> plot(t,y)                           % plot the free response
```

The plot (below) verifies the free response described in part (c).



**7.17** a) The MATLAB commands define state matrix **A** and determine its eigenvalues:

```
>> A = [0 1 0 ; 0 0 1 ; -12 -20 -9];        % define the state matrix A
>> eig(A)
```

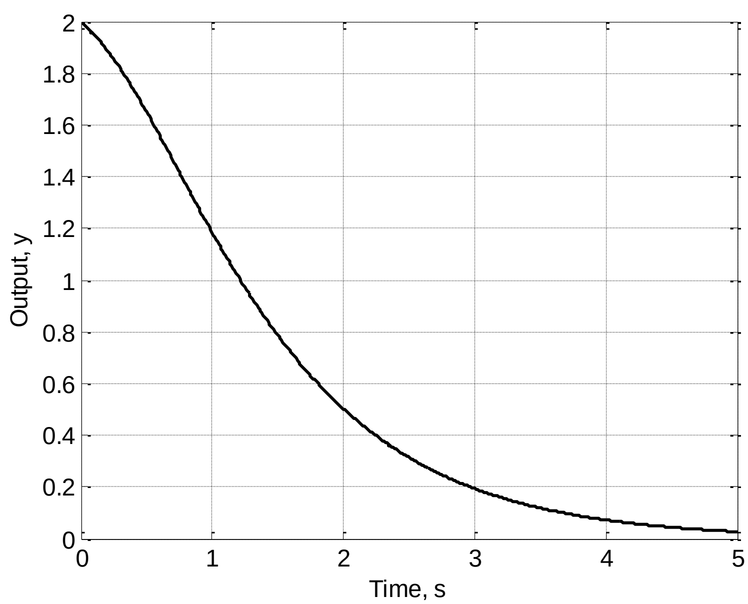
The three eigenvalues are –1, –2, and –6.

b) Because the three eigenvalues (roots) are real and negative, the free (zero-input) response will consist of three decaying exponential functions, or $y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-6t}$. The "slow" exponential mode is $e^{-t}$ which takes about 4 s to decay to zero.

c) MATLAB commands are probably the easiest method: use the command `initial`

```
>> A = [0 1 0 ; 0 0 1 ; -12 -20 -9];        % define the state matrix A
>> B = [0 ; 0 ; 0.4 ];                      % define the input matrix B
>> C = [1 0 0];                             % define the input matrix C
>> D = 0;                                   % define the direct-link matrix D
>> sys = ss(A,B,C,D);                       % define sys as the SSR
>> x0 = [2 ; -0.5 ; 0];                     % define the initial state vector
>> t = 0:0.01:5;                            % define the time vector
>> [y,t]=initial(sys,x0,t);                 % obtain the free response
>> plot(t,y)                                % plot the free response
```

The plot below verifies the free response described in part (b).



**7.18** a) The MATLAB commands define state matrix **A** and determine its eigenvalues:

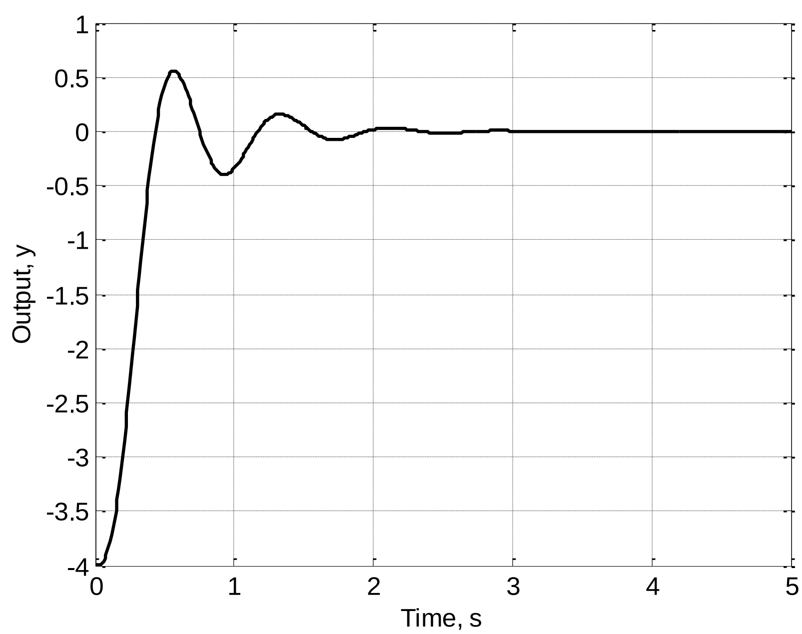
```
>> A = [0 1 0 ; 0 0 1 ; -340 -88 -9];        % define the state matrix A
>> eig(A)
```

The three eigenvalues are –5, –2 + $j$8, and –2 – $j$8.

b) Because the three eigenvalues (roots) have negative real parts the free response will eventually decay to zero. The free response will be the sum of one decaying exponential function and a damped sinusoidal function, or $y(t) = c_1 e^{-5t} + c_2 e^{-2t}\left(\sin 8t + \phi\right)$. The "slow" exponential mode is $e^{-2t}$ which takes about 2 s to decay to zero. The free response will oscillate at frequency 8 rad/s.

c) MATLAB commands are probably the easiest method. The plot verifies part (b).

```
>> A = [0 1 0 ; 0 0 1 ; -340 -88 -9];        % define the state matrix A
>> B = [0 ; 0 ; 2];                          % define the input matrix B
>> C = [1 0 0];                              % define the input matrix C
>> D = 0;                                    % define the direct-link matrix D
>> sys = ss(A,B,C,D);                        % define sys as the SSR
>> x0 = [-4 ; 0 ; 0];                        % define the initial state vector
>> t = 0:0.01:5;                             % define the time vector
>> [y,t]=initial(sys,x0,t);                  % obtain the free response
>> plot(t,y)                                 % plot the free response
```



**7.19** The mathematical model of the mass-damper system is $m\ddot{x} + b\dot{x} = f_a(t)$

a) Using velocity $v(t) = \dot{x}(t)$ as the dynamic variable the model becomes $m\dot{v} + bv = f_a(t)$. Because the force magnitude is 5 N and the very short pulse duration is 0.02 s, the input can be represented by an ideal impulse $0.1\delta(t)$ where (5 N)(0.02s) = 0.1 N-s is the weight or strength of the impulse. Substituting the numerical values for *m* and *b* the system model becomes

$$0.5\dot{v} + 3v = f_a(t) \qquad \text{or, in standard form:} \quad 0.1667\dot{v} + v = 0.3333 f_a(t)$$

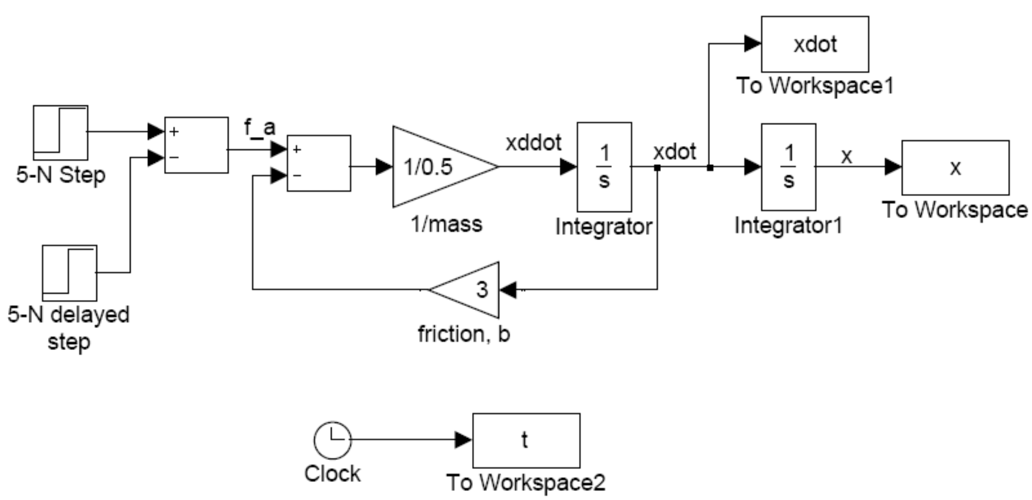
Hence, the time constant is $\tau$ = 0.1667 s and the initial magnitude of the impulse response is (0.1)(0.3333)/0.1667 = 0.2 m/s = $v$(0+). The impulse response is an exponential function $e^{-t/\tau} = e^{-6t}$. The impulse response for velocity is $v(t) = 0.2e^{-6t}$ m/s.

b) We can integrate the velocity response in part (a) to obtain the position response:

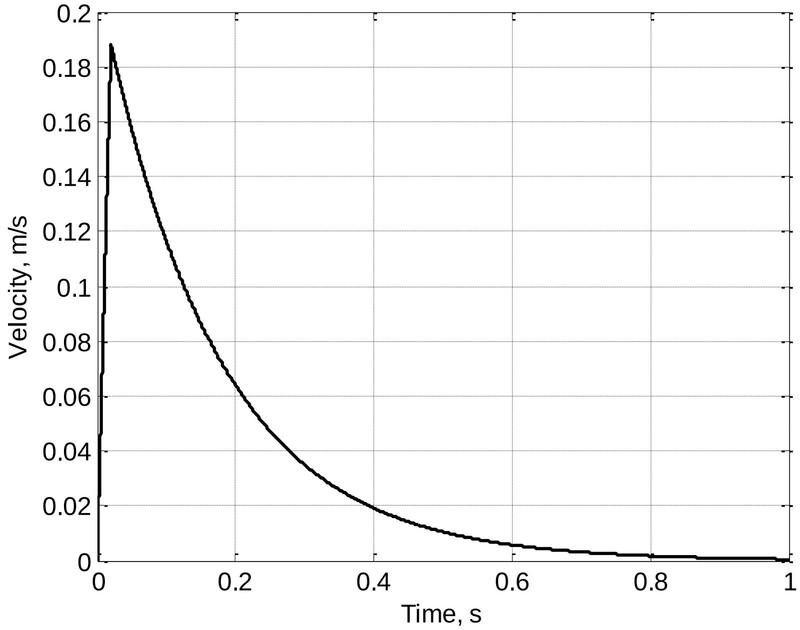
$$x(t) = x(0) + \int_0^t v(\lambda)d\lambda = \int_0^t 0.2e^{-6\lambda}d\lambda$$

Note $x(0)$ = 0 and $\lambda$ is the dummy integration variable. The solution is $x(t) = 0.0333(1 - e^{-6t})$ m

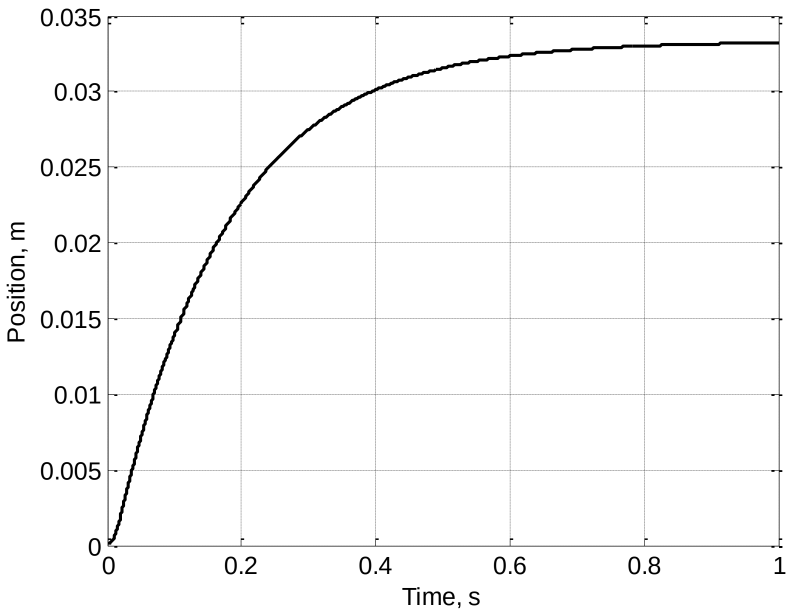
c) The Simulink model (below) will solve the mathematical model $m\ddot{x} + b\dot{x} = f_a(t)$ by integrating acceleration twice. Both integrators have zero initial conditions. Input force is a very short-duration pulse with a magnitude of 5 N and is created by adding two step functions. The fixed simulation step size is $10^{-4}$ s.




Plots of velocity $v(t)$ and position $x(t)$ follow (next page). Note that velocity starts at rest and exhibits a very sharp jump to nearly 0.2 m/s due to the short-duration pulse input. The simulation nearly matches the analytic solution $v(t) = 0.2e^{-6t}$ m/s in part (a) which was computed using an ideal impulse. Position $x(t)$ shows an exponential rise (from zero) to the computed steady-state of 0.0333 m.




**Prob. 7.19: velocity vs. time**




**Prob. 7.19: position vs. time**

**7.20** The system transfer function is

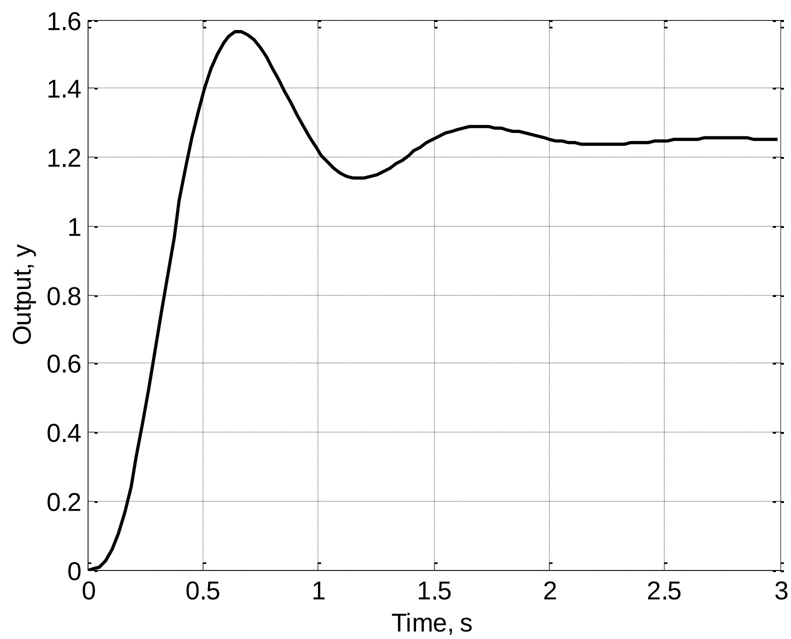
$$G(s) = \frac{800}{2s^3 + 24s^2 + 144s + 640}$$

The characteristic roots are the poles of the transfer function, i.e., $2s^3 + 24s^2 + 144s + 640 = 0$. Using MATLAB, the roots are

```
>> denG = [2 24 144 640];                % denominator of G(s)
>> roots(denG)
```

The three roots are $-8$, $-2 + j6$, and $-2 - j6$. Because the two of the roots are complex the transient response will exhibit oscillations with frequency 6 rad/s (period = $2\pi/6$ = 1.047 s). The real root will contribute an exponential mode $e^{-8t}$ which dies out in 0.5 s. The complex roots will contribute a damped sinusoidal mode with an exponential envelop of $e^{-2t}$ (dies out in 2 s). Because the DC gain is $G(s = 0) = 800/640 = 1.25$ and the input is a unit step, the steady-state response is $y_{SS} = 1.25$. MATLAB's `step` command can be used to obtain a numerical solution:

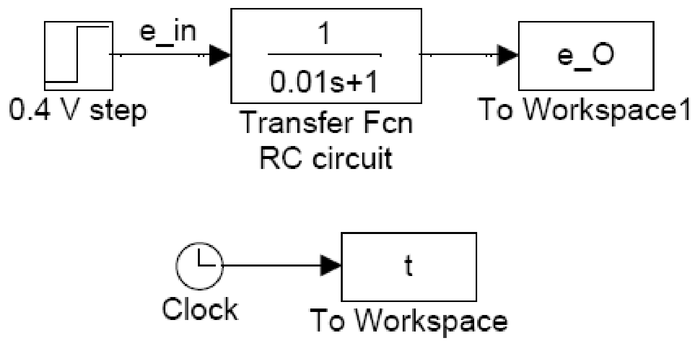
```
>> sysG = tf(800,[2 24 144 640]);        % define transfer function G(s)
>> t = 0:0.002:3;                        % define time vector
>> [y,t] = step(sysG);                   % step response
>> plot(t,y)                             % plot y(t)
```

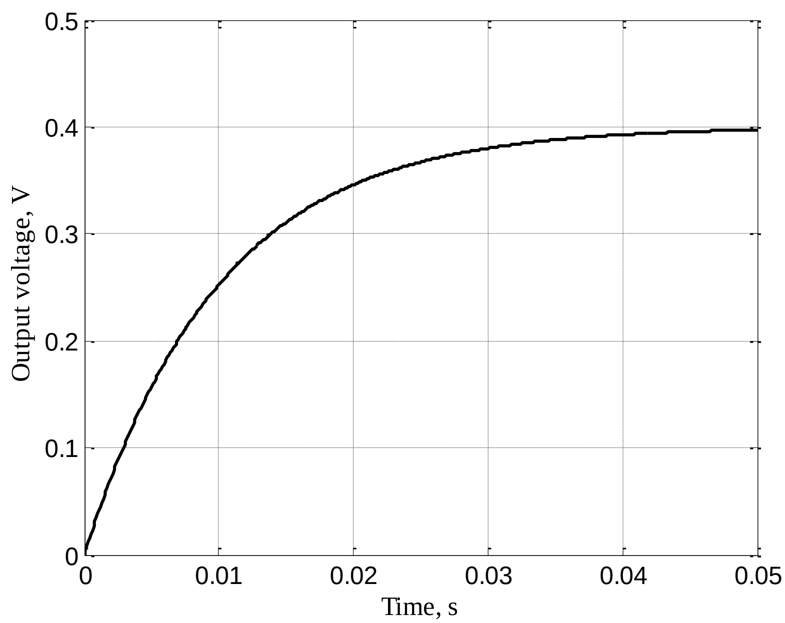


**7.21** The system transfer function for the RC circuit is

$$G(s) = \frac{E_O(s)}{E_{\text{in}}(s)} = \frac{1}{RCs+1} = \frac{1}{0.01s+1}$$

The Simulink model simulates the step response to a 0.4-V input. The output voltage response $e_O(t)$ matches the sketch in Problem 7.6.



**7.22** The 1-DOF mechanical system transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{0.25}{s^2 + 2s + 9}$$

a) Because the system is underdamped (note the poles are $-1 \pm j2.8284$) we can compute the step-response performance metrics using Table 7-4 (summarized below)

Undamped natural frequency $\omega_n = \sqrt{9} = 3$ rad/s, damping ratio $\zeta = 2/(2\omega_n) = 0.3333$.

Peak time $t_p = \dfrac{\pi}{\omega_n \sqrt{1-\zeta^2}}$ = 1.1107 s

Steady-state response = (step input in N)(DC gain) = (4 N)(0.25/9) = 0.1111 m

Maximum overshoot $M_{OS} = e^{-\zeta\pi / \sqrt{1-\zeta^2}}$ = 0.3293 (or 32.9%)
→ peak value = (1.3293)(0.1111 m) = 0.1477 m

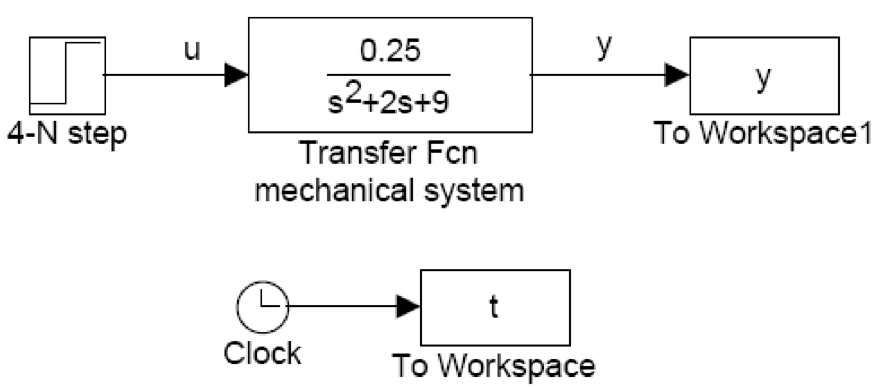
Settling time $t_S = \dfrac{4}{\zeta\omega_n}$ = 4 s

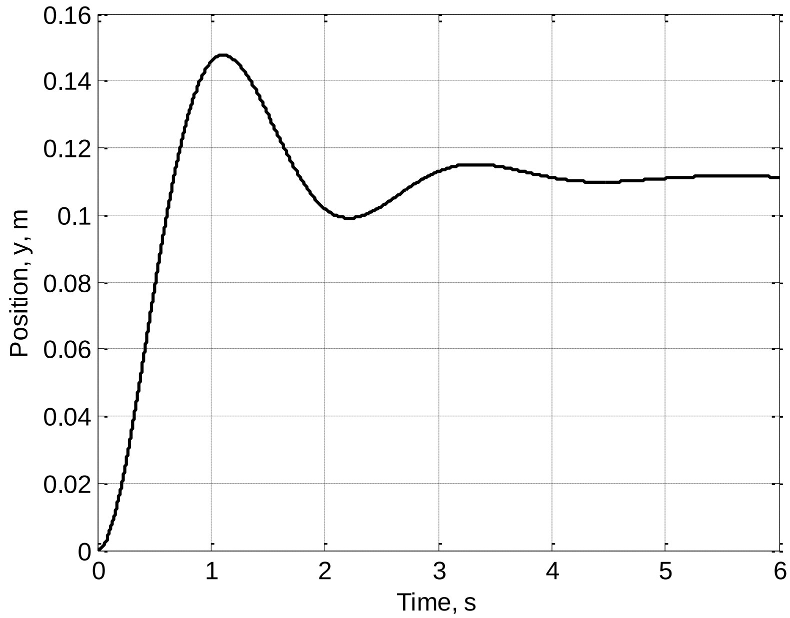
Period of oscillation $T_{\text{period}} = \dfrac{2\pi}{\omega_n \sqrt{1-\zeta^2}}$ = 2.2214 s

Number of cycles to S-S $N_{\text{cycles}} = \dfrac{2\sqrt{1-\zeta^2}}{\pi\zeta}$ = 1.8 cycles

An accurate sketch would show these response values - see simulation plot in part (b).

b) The below Simulink model will produce the step response - see plot of $y(t)$ on the next page.






**Prob. 7.22: position vs. time**

**7.23** The I/O equation of the 1-DOF mechanical system can be derived from the transfer function $G(s)$ in Problem 7.22:

$$\ddot{y} + 2\dot{y} + 9y = 0.25u$$

a) The system is underdamped (note the roots are $-1 \pm j2.8284$) and we can use the second-order underdamped metrics from Table 7-4 (see Problem 7.22)

Undamped natural frequency $\omega_n = \sqrt{9} = 3$ rad/s, damping ratio $\zeta = 2/(2\omega_n) = 0.3333$.

Peak time $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ = 1.1107 s

Steady-state response is **zero** since input $u = 0$ and the damped sinusoid decays to zero.

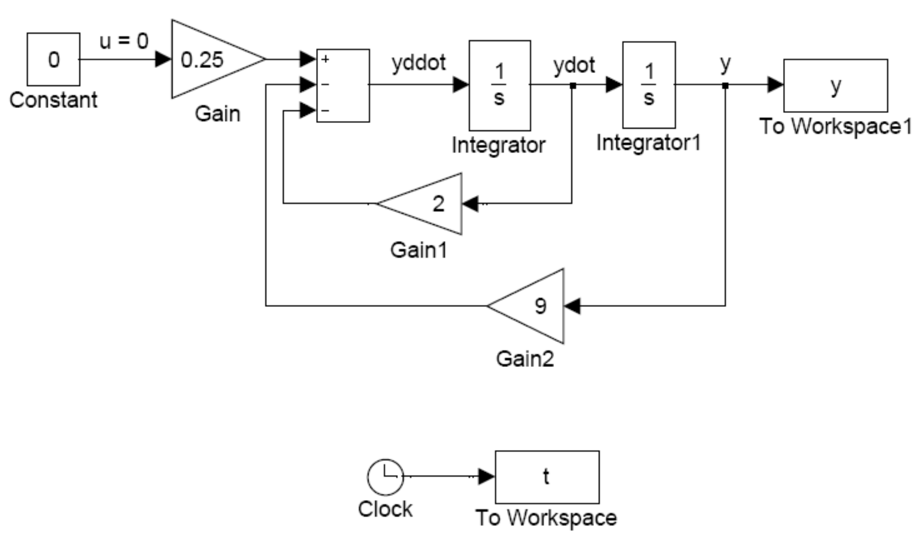
Settling time $t_S = \dfrac{4}{\zeta\omega_n}$ = 4 s

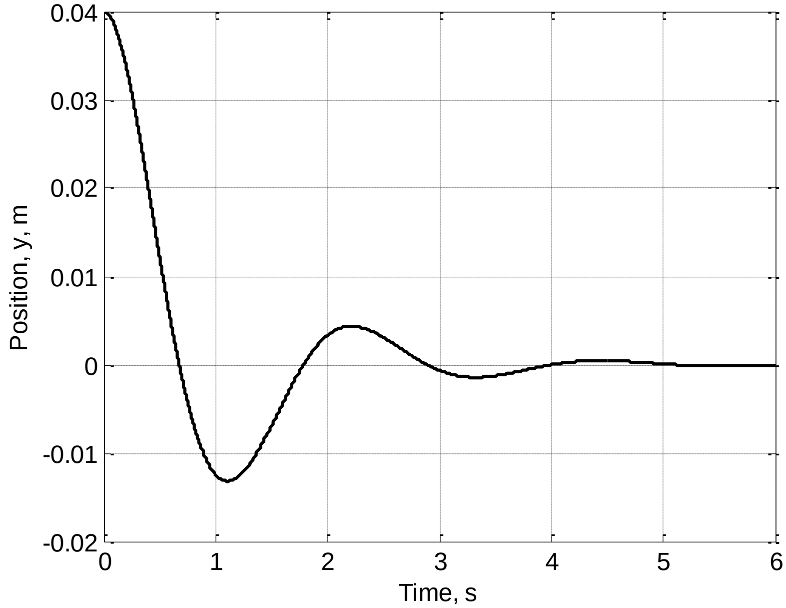
Period of oscillation $T_{\text{period}} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$ = 2.2214 s

Number of cycles to S-S $N_{\text{cycles}} = \dfrac{2\sqrt{1-\zeta^2}}{\pi\zeta}$ = 1.8 cycles

An accurate sketch would show these response values - see simulation plot in part (b) on the next page.

b) We cannot use transfer functions since the system has **non-zero initial conditions**. The below Simulink model uses integrator blocks and will produce the free response. Note that the input is zero and the integrators must have initial conditions $y(0) = 0.04$ m and $\dot{y}(0) = 0$.






**Prob. 7.23: position vs. time**

---

**7.24** The mathematical model of the frictionless mechanical system is

$$J\ddot{\theta} + k\theta = T_{\text{in}}(t)$$

a) Using the numerical values for $J$ and $k$ the transfer function is

$$G(s) = \frac{\Theta(s)}{T_{\text{in}}(s)} = \frac{1}{0.2s^2 + 100}$$

Hence the undamped natural frequency is $\omega_n = \sqrt{100/0.2}$ = 22.3607 rad/s. The damping ratio $\zeta$ is zero since there is no friction. Hence, the homogeneous (or natural) response will be an undamped harmonic (sinusoidal) function with the form

$$\theta_H(t) = c_1 \sin 22.3607t + c_2 \cos 22.3607t \text{ rad}$$

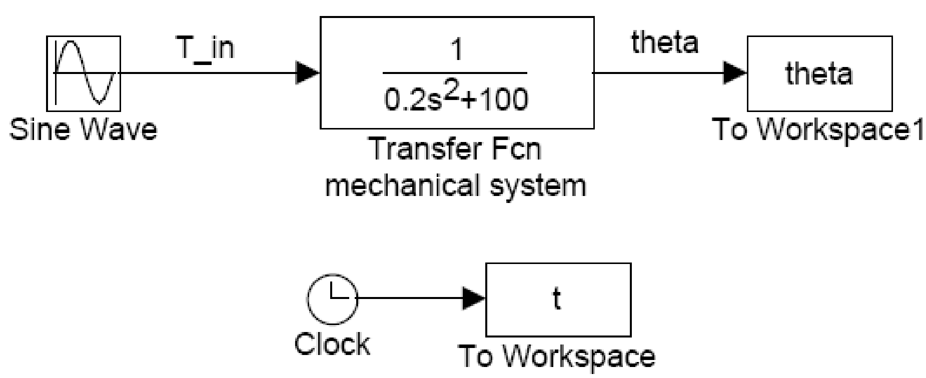
The forced response or particular solution (due to the sinusoidal input torque) will be a sinusoidal function with frequency 3 rad/s (same as the input) with unknown coefficients:

$$\theta_P(t) = c_3 \sin 3t + c_4 \cos 3t \text{ rad}$$

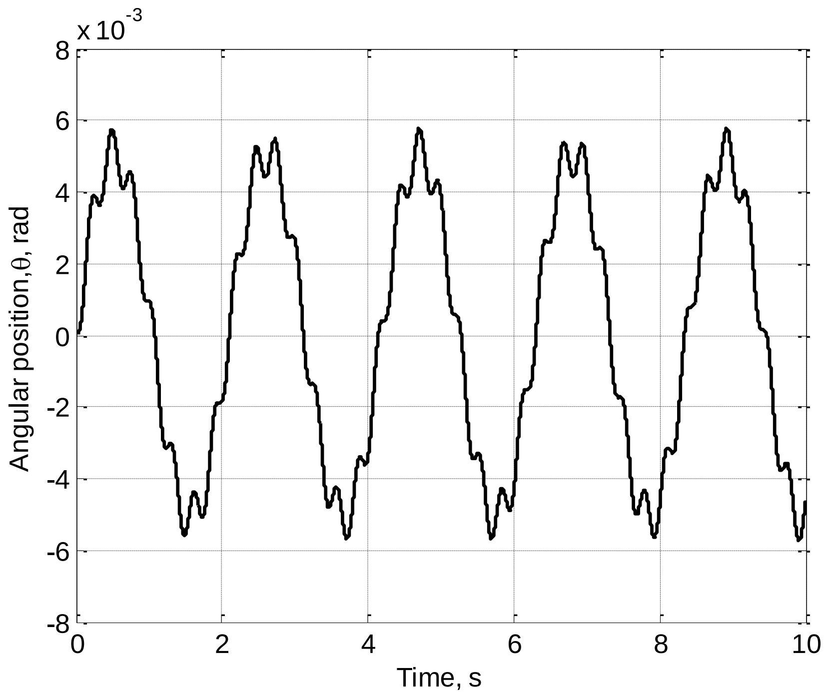
Hence the total solution is the sum of sinusoidal function at <u>two</u> frequencies, $\omega_n$ and 3 rad/s.

$$\theta(t) = c_1 \sin 22.3607t + c_2 \cos 22.3607t + c_3 \sin 3t + c_4 \cos 3t \text{ rad}$$

b) The Simulink model (below) will produce the desired system response.




The complete response $\theta(t)$ is shown on the next page. Note that the response is a **summation** of two harmonic (sinusoidal) functions: the first harmonic function has a frequency of 3 rad/s (period = 2.1 s) due to the torque input and the second harmonic function has frequency 22.36 rad/s (period = 0.28 s) due to $k/J$. This dual-frequency harmonic function matches the result in part (a).




**Prob. 7.24: angular position vs. time**

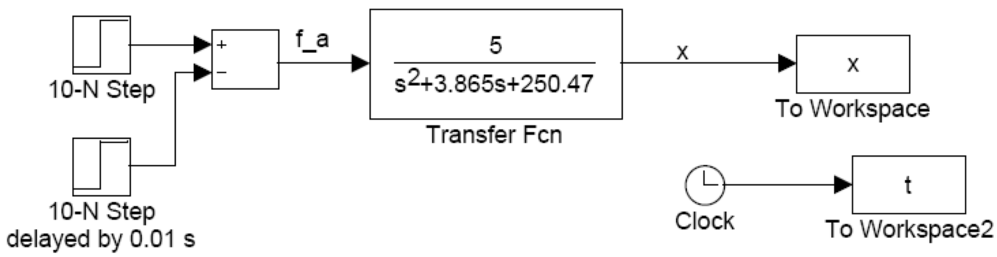
**7.25** The I/O equation of a linear second-order mechanical system is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f_a(t)}{m}$$

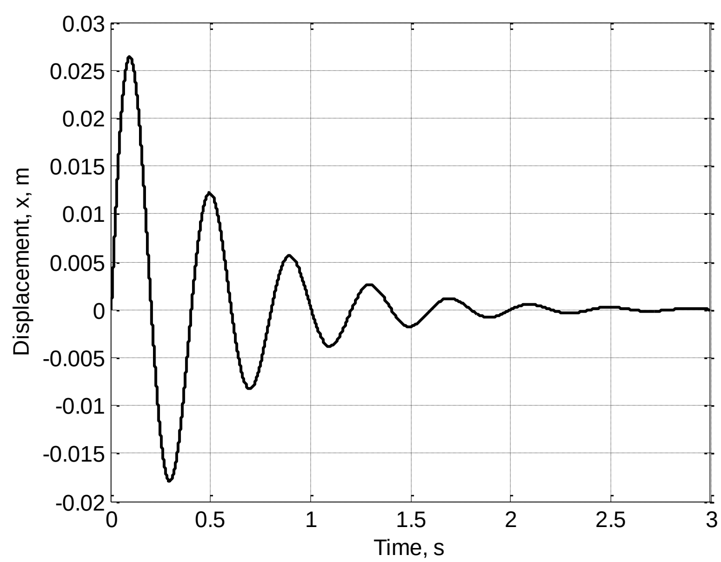
The numerical values determined in Problem 7.11 are $\zeta$ = 0.1221 (damping ratio) and $\omega_n$ = 15.8264 rad/s (undamped natural frequency), and mass $m$ = 0.2 kg. Hence the transfer function is

$$G(s) = \frac{X(s)}{F_a(s)} = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{5}{s^2 + 3.865s + 250.47}$$

The Simulink model below shows that two step functions are added to create a pulse input. One step input is delayed by 0.01 s.




The magnitude of the pulse input was varied until the peak response of $x(t)$ nearly matched 0.026 m (the peak response from the plot in Problem 7.11). A 10-N pulse input with pulse width of 0.01 s produced the following response (below) which nearly matches the plot in Problem 7.11. Because the pulse magnitude is 10 N and pulse width is 0.01 s, the strength of the impulse input is (10 N)(0.01 s) = **0.1 N-s.**



**7.26** The mathematical model of the thermal system is (see Example 4.7)

$$R_{\mathrm{EQ}} C\dot{T} + T = R_{\mathrm{EQ}} q_{\mathrm{BH}} + T_a$$

The equivalent thermal resistance of all surfaces is $\dfrac{1}{R_{\mathrm{EQ}}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_6} = \sum_{i=1}^{6} \dfrac{1}{R_i}$

Or, $R_{EQ}$ = 0.018014 deg C-s/J. Hence the time constant of the first-order system is $\tau = R_{\mathrm{EQ}}C$ = 8,700.8 s. The settling time is $t_S = 4\tau$ = 34,803 s = 9.67 hrs. At steady-state we have $\dot{T} = 0$ so the first-order model becomes

$$T(\infty) = R_{\mathrm{EQ}} q_{\mathrm{BH}} + T_a$$ = (0.018014 deg C-s/J)(1000 J/s) + 10 deg C = **28.01 deg C.**

The settling time and steady-state temperature match the numerical results from Problem 6.25. A sketch of $T(t)$ would be an exponential rise from 10 deg C to 28 deg C with settling time of 9.7 hrs (see solution of Problem 6.25).

---

**7.27** The mathematical model of the LC tuner circuit (see Problem 3.25) is

$$LC\ddot{e}_C + e_C = 0 \quad \text{or} \quad \ddot{e}_C + 1.6667(10^7)e_C = 0 \qquad \text{using } L = 3(10^{-3}) \text{ H and } C = 20(10^{-6}) \text{ F}$$

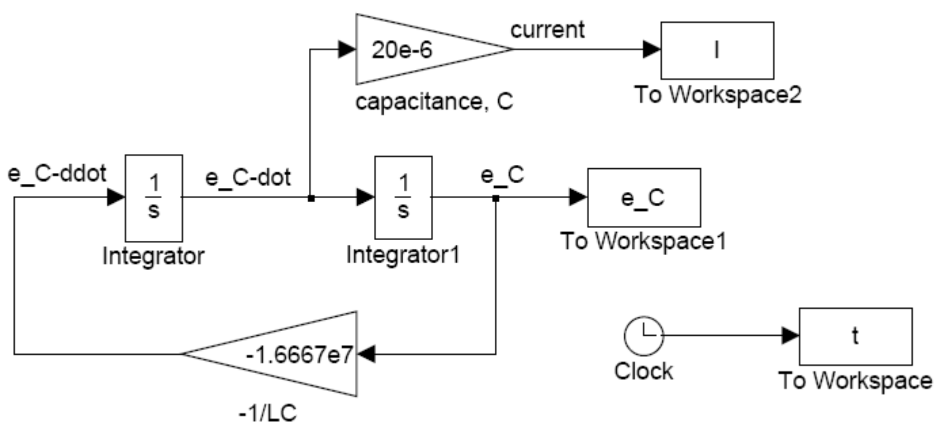
Therefore the second-order system has zero damping ($\zeta = 0$) and undamped natural frequency $\omega_n = \sqrt{1/LC} = 4{,}082.5$ rad/s. The voltage response $e_C(t)$ is an undamped sinusoidal function with initial voltage $e_C(0) = 2.5$ V and frequency 4082.5 rad/s (hence, use the cosine function):

Capacitor voltage: $e_C(t) = 2.5\cos 4082.5t$ V

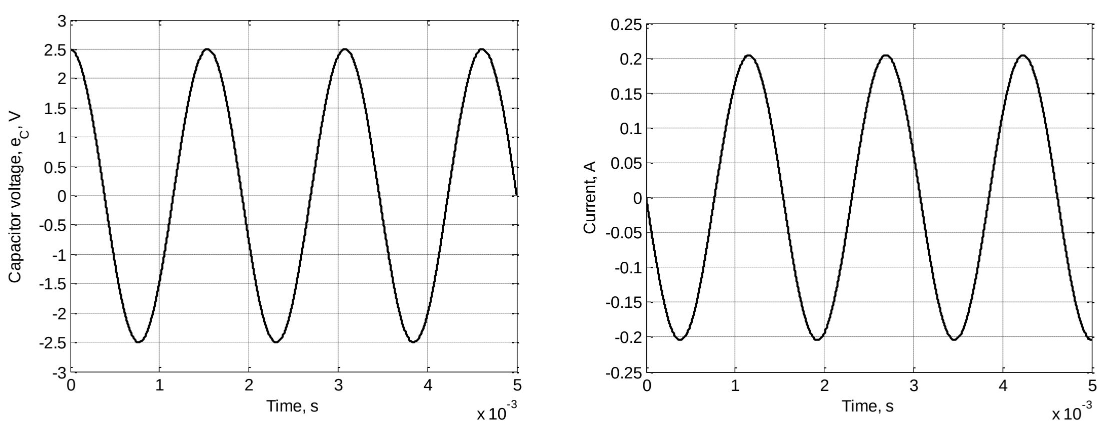
The current is $I(t) = C\dot{e}_C$ = (20E-6 F)(2.5 V)(–4082.5)sin4082.5$t$ = $-0.2041\sin 4082.5t$ A.

Sketches would show that the capacitor voltage oscillates between $\pm 2.5$ V at frequency 4082.5 rad/s (period = 0.0015 s) and the current oscillates between $\pm 0.2041$ A at the same frequency.

The Simulink model using the integrator-block method is below.




Plots of capacitor voltage $e_C(t)$ and current $I(t)$ from the numerical simulation are below. These plots verify the sketches of the cosine and sine functions for $e_C(t)$ and $I(t)$



**7.28** The mathematical model of the LC tuner circuit (see Problem 3.25) is $LC\ddot{e}_C + e_C = 0$

Let the state variables be $x_1 = e_C$ and $x_2 = I$.

The first state equation is $\dot{x}_1 = \dot{e}_C = \dfrac{1}{C} I = \dfrac{1}{C} x_2$

The second state equation is $\dot{x}_2 = \dot{I} = C\ddot{e}_C = \dfrac{-1}{L} e_C = \dfrac{-1}{L} x_1$

The matrix-vector state equation is

$$\boxed{\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1/C \\ -1/L & 0 \end{bmatrix} \mathbf{x}}$$

State equation

The eigenvalues are

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -1/C \\ 1/L & \lambda \end{bmatrix} = \lambda^2 + \frac{1}{LC} = 0$$

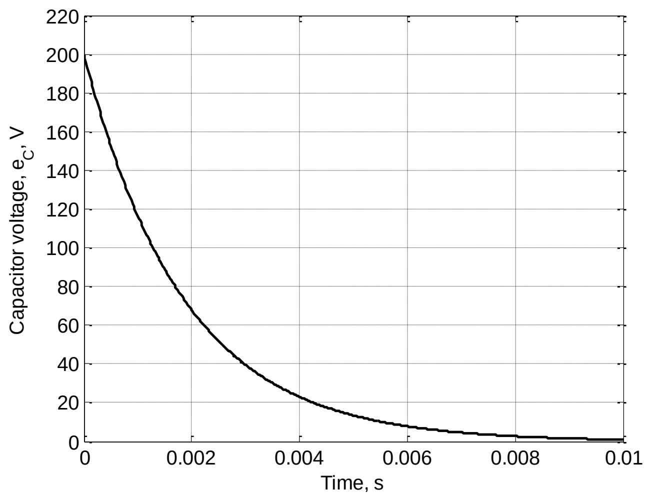
The eigenvalues are $\lambda = \pm j\sqrt{1/LC} = \pm\, j4082.5$

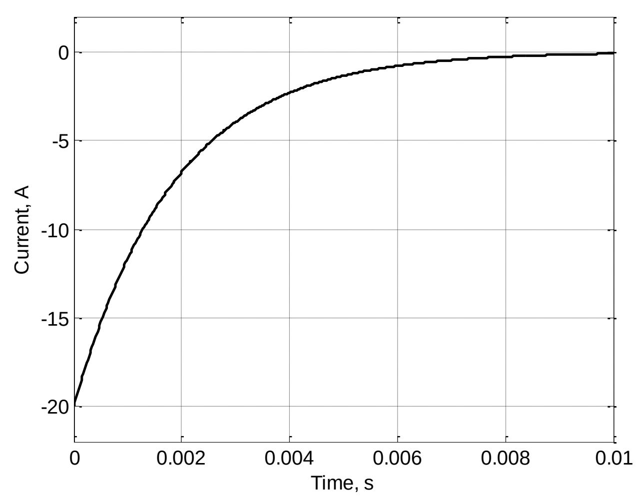
Note that in Problem 7.27 the capacitor voltage and current of the LC tuner circuit are $e_C(t) = 2.5\cos 4082.5t$ V and $I(t) = -0.2041\sin 4082.5t$ A. Both are undamped harmonic functions with a frequency of 4082.5 rad/s. Because the two eigenvalues are imaginary numbers the homogeneous response is an undamped sinusoidal function with a frequency equal to the imaginary number ($\omega$ = 4082.5 rad/s in this case).

**7.29** The mathematical model of the RC circuit (see Problem 3.26) is $RC\dot{e}_C + e_C = 0$

Using the values $R$ = 10 Ω and $C$ = 185(10$^{-6}$) F, the time constant is $\tau = RC$ = 0.00185 s. Hence the settling time is $t_S = 4\tau$ = 0.0074 s. Clearly at steady state $\dot{e}_C = 0$ and therefore $e_C(\infty) = 0$. A sketch of the capacitor voltage $e_C(t)$ would be an exponential decay from $e_C(0)$ = 200 V to zero at approximately 0.0074 s. The mathematical solution is $e_C(t) = 200e^{-t/\tau} = 200e^{-540.54t}$ V.

The current is $I(t) = C\dot{e}_C$ which is the product of the slope (time derivative) of $e_C(t)$ and capacitance $C$. The derivative is $\dot{e}_C(t) = -1.0811(10^5)e^{-540.54t}$ so $I(t) = C\dot{e}_C = -20e^{-540.54t}$ A. A sketch of the current $I(t)$ would be an exponential rise from –10 A to zero at about $t_S$ = 0.0074 s. Sketches would match the plots below.



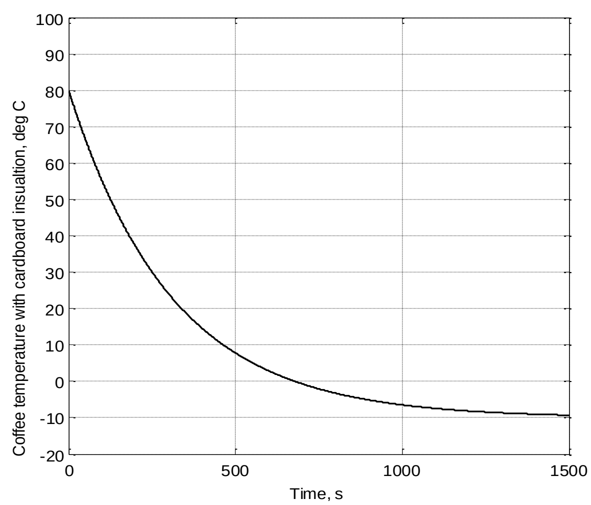



**7.30** a) The thermal model of the coffee cup is $RC\dot{T} + T = T_a$

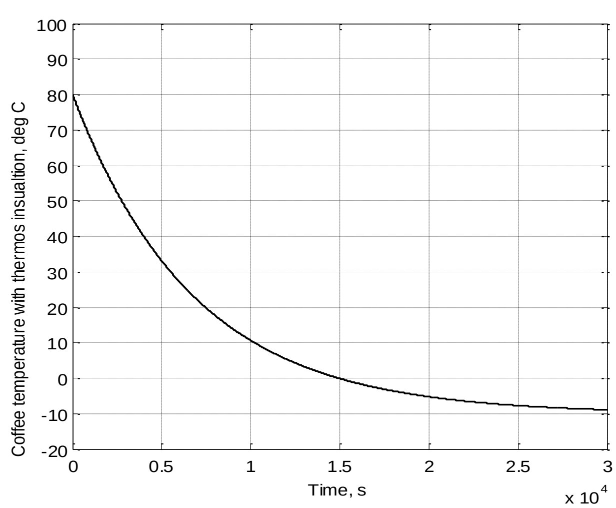
Using $R$ = 0.25 deg C-s/J (cardboard), $C$ = 1237 J/deg C, and $T_a$ = –10 deg C the model becomes

$$309.25\dot{T} + T = -10 \quad \text{(deg C)}$$

Clearly the time constant is $\tau = RC$ = 309.25 s and therefore the settling time is $t_S = 4\tau$ = 1237 s. The steady-state temperature is $T_{ss} = T_a$ = –10 deg C (ambient temperature). Therefore the temperature response $T(t)$ starts at 80 deg C and exponentially decays to –10 deg C in 1237 s (20.6 min). A hand-sketch would match the below plot.

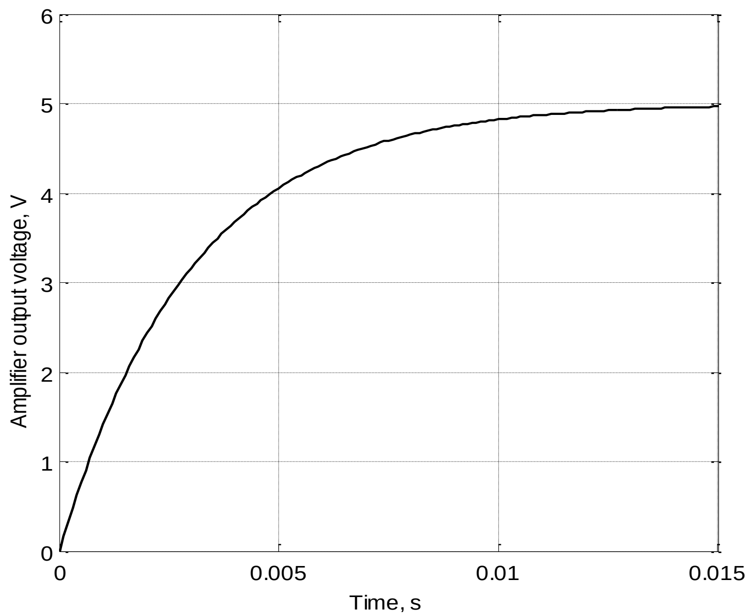



b) Now, $R$ = 5.5 deg C-s/J (thermos). The new time constant is $\tau = RC$ = 6803.5 s and therefore the settling time is $t_S = 4\tau$ = 27,214 s (7.6 hr). The initial and final temperatures are the same; only the settling time has changed. A hand-sketch would match the plot below.



**7.31** a) The power amplifier is a first-order I/O system: $0.003\dot{e}_O + e_O = 25e_{\text{in}}(t)$

Clearly the time constant is $\tau$ = 0.003 s and therefore the settling time is $t_S = 4\tau$ = 0.012 s. The DC gain of the amplifier transfer function is 25 and therefore the steady-state voltage for a 0.2 V step input is 5 V. The voltage response $e_O(t)$ will exponentially reach 5 V in 0.012 s. A hand-sketch would match the below plot.




b) We can compute the steady-state spool-valve position from the individual DC gains of the three transfer functions: power amp DC gain = 25, solenoid DC gain = 1.6, and spool-valve DC gain = 1/1800 = 5.5556($10^{-4}$). The product of the three DC gains is 0.0222. Hence the steady-state valve position is the product of the total DC gain (0.0222) and magnitude of the step input voltage, $e_{\text{in}}(t)$ = 0.2 V, or, (0.0222)(0.2) = **0.0044 m = 4.4 mm.**

c) The I/O equation for the spool valve is $0.035\ddot{z} + 7\dot{z} + 1800z = f(t)$

Dividing by 0.035 we obtain the standard form for a second-order system

$$\ddot{z} + 200\dot{z} + 51{,}428.6z = 28.571f(t) \quad \text{or} \quad \ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = 28.571f(t)$$

The undamped natural frequency is $\omega_n$ = 226.7787 rad/s and the damping ratio is $\zeta = 200/(2\omega_n)$ = 0.441 and hence the spool valve step response is a damped sinusoidal that eventually settles at the steady-state value of 0.0044 m (4.4 mm) computed in part (b). The transient step-response characteristics can be computed using the performance equations in Table 7-4:

Peak time: $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ = 0.0154 s

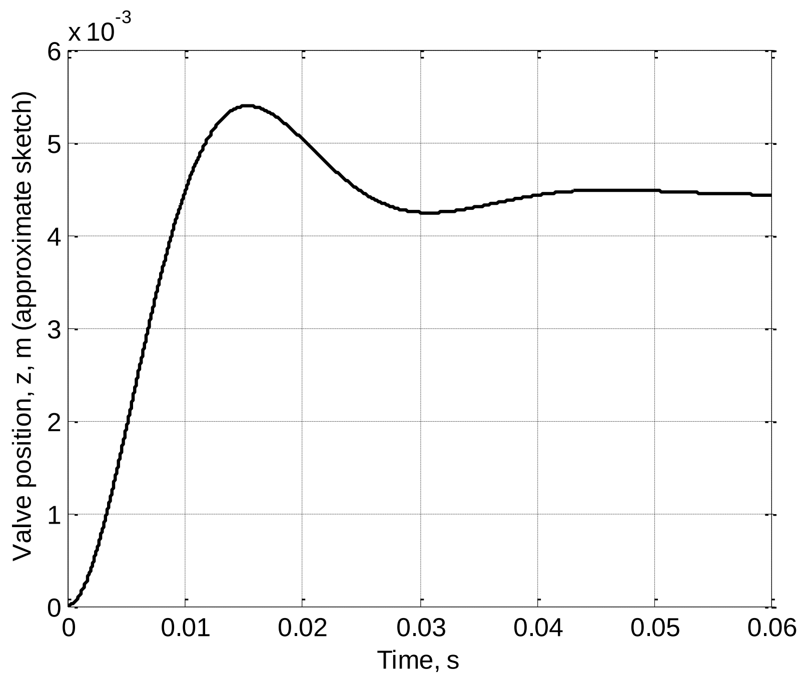
Maximum overshoot: $M_{OS} = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ = 0.2136, or $z_{\max}$ = (1.2136)(0.0044) = 0.0053 m

Settling time: $t_S = \dfrac{4}{\zeta\omega_n}$ = 0.04 s

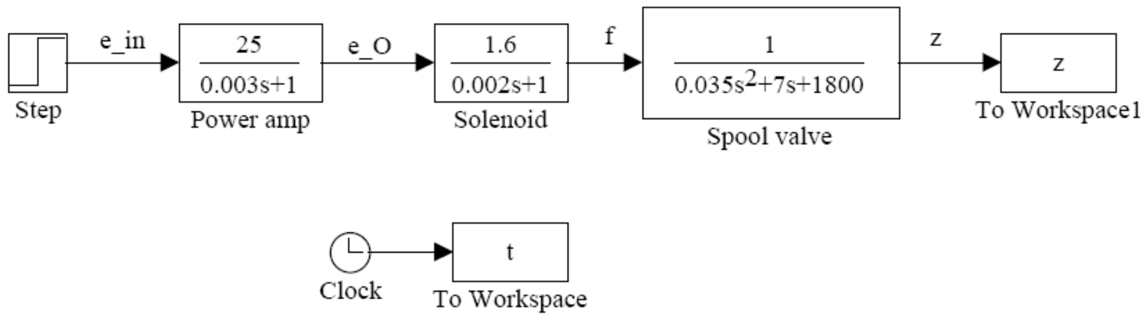
Period of oscillation: $T_{\text{period}} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$ = 0.0309 s

Number of cycles to steady state: $N_{\text{cycles}} = \dfrac{2\sqrt{1-\zeta^2}}{\pi\zeta}$ ~ 1.3 cycles

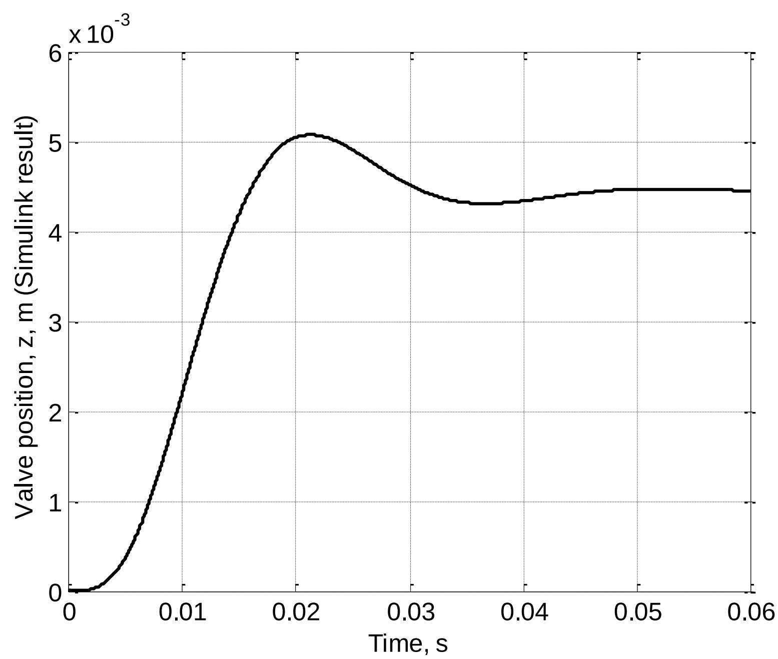
A *hand-sketch* would show (with labels) the above calculated values and match the below plot.




Finally, we can verify the approximate hand-sketch (i.e., step response of the 2nd-order spool valve model) by a numerical simulation of the *complete* system using Simulink (next page):




The step response of the complete Simulink model is below. Note that the steady-state value (0.0044 m) matches the approximate sketch. However, the addition of two first-order models (power amp and solenoid) has delayed the response. Note that the amplifier and solenoid have time constants of 0.003 and 0.002 s, respectively, which correspond to settling times around 0.012 and 0.008 s. Therefore, the second-order spool valve response is delayed (compare the peak times between the approximate sketch and the full simulation). The peak response of the full simulation is also diminished (damped) compared to the approximate sketch.



**7.32** The measured motor characteristics are a) steady-state $\omega_{ss}$ = 85 rad/s for a 2-V step voltage input, b) settling time $t_S$ = 0.6 s, and c) the speed response from zero to 85 rad/s is an exponential rise.

Because the response is exponential we can model the system using the "standard form" of a first-order transfer function:

$$G(s) = \frac{\omega(s)}{E_{\text{in}}(s)} = \frac{a}{\tau s + 1}$$

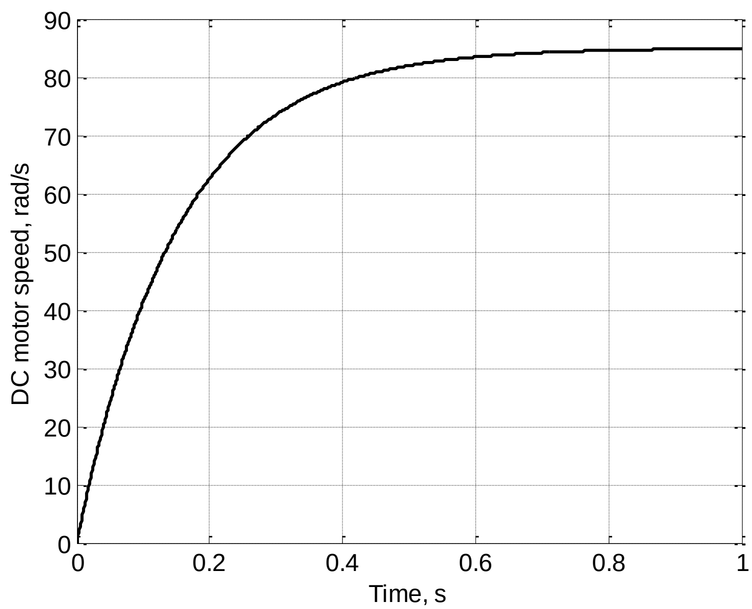
The time constant is $\tau = t_S/4 = 0.15$ s and the DC gain is $G(0) = a = (85 \text{ rad/s})/(2 \text{ V}) = 42.5$. Hence an appropriate transfer function model of the DC motor is

$$\boxed{G(s) = \frac{\omega(s)}{E_{\text{in}}(s)} = \frac{42.5}{0.15s + 1}}$$

The following MATLAB commands simulate the step response of this transfer function model.

```
>> sysG = tf(42.5,[ 0.15  1 ]);          % define transfer function G(s)
>> t = 0:0.001:1;                        % define time vector t
>> e_in = 2*ones(size(t));               % define 2-V step input e_in(t)
>> [w,t] = lsim(sysG,e_in,t);            % obtain response ω(t), output speed
```

The plot of the simulation results (below) verifies the experimental characteristics of the DC motor.



**7.33** The model of the mass-spring-damper system is $m\ddot{x} + b\dot{x} + kx = f_a$

The standard form of the second-order system is

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f_a \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}f_a$$

The mass is $m$ = 3 kg. Maximum overshoot for a step input only depends on damping ratio $\zeta$ (see Table 7-4). Substitute each value of $b$, $k$ and compute damping ratio:

Option A: $\zeta = \dfrac{b}{2\sqrt{mk}} = \dfrac{90}{2\sqrt{(3)(11{,}300)}} = 0.2444$

Option B: $\zeta = \dfrac{b}{2\sqrt{mk}} = \dfrac{88}{2\sqrt{(3)(8{,}700)}} = 0.2724$

Since **Option B** has the largest damping ratio it has the smallest percent overshoot to a step input.

**7.34** a) The measured output shows a classic underdamped response which we can model with a second-order transfer function:

$$G(s) = \frac{\Theta(s)}{T_{\text{in}}(s)} = \frac{a\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The steady-state output is 0.012 rad for a step input torque of 0.6 N-m. Hence the DC gain of the transfer function is $a$ = 0.012/0.6 = 0.02.

The peak response is 0.016 rad which is a 33.333% overshoot of the steady-state value of 0.012 rad. The maximum overshoot is

$$M_{OS} = e^{-\zeta\pi / \sqrt{1-\zeta^2}} = 0.333333 \quad \rightarrow \quad \text{damping ratio } \zeta = 0.3301$$

The period of the response is 1.1 s. Therefore the damped frequency is $\omega_d = 2\pi / 1.1$ = 5.712 rad/s. The undamped natural frequency is $\omega_n = \omega_d / \sqrt{1-\zeta^2}$ = 6.0512 rad/s. The second-order transfer function is

$$\boxed{G(s) = \frac{\Theta(s)}{T_{\text{in}}(s)} = \frac{0.73234}{s^2 + 3.995s + 36.617}}$$

b) The following MATLAB commands simulate the step response of this transfer function model.

```
>> sysG = tf(0.73234,[1 3.995 36.617]);    % define transfer function G(s)
>> t = 0:0.002:3;                          % define time vector t
>> T_in = 0.6*ones(size(t));               % define 0.6 N-m step input Tin(t)
>> [theta,t] = lsim(sysG,T_in,t);          % obtain response θ(t)
```

The simulation result (plotted below) shows a good match with Fig. P7.34; therefore the second-order transfer function $G(s)$ is an accurate representation of the mechanical system.