Ordinary Differential Equations

ODE's - Ordinary differential equations

$$\frac{d\mathbf{y}}{dt} = f(t, \mathbf{y}; p)$$

or more specifically

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(y_1, \ldots, y_n; p) \\ \vdots \\ f_n(y_1, \ldots, y_n; p) \end{pmatrix}$$

Appear in many applications

- Particle flow
- Disease propagation
- Fake news detection
- Geochemistry
- ...

ODE's - Classification

- Linear first order $$\frac{dy}{dt} = Ay + f(t)$$
- Linear higher order $$\frac{d^2y}{dt^2} = Ay + \sin^2(t)$$
- Nonlinear first order $$\frac{dy}{dt} = y^2$$
- Nonlinear higher order $$\frac{d^2y}{dt^2} = y^2 + \cos(y)$$
- System, linear $$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
- System, nonlinear

$$\frac{d^2y}{dt^2} + y = \cos(3t)$$ is $$\frac{d^5y}{dt^5} + y^3 = 0$$

Separation of variables

We have a special case

$$\frac{dy}{dx} = f(x, y) = \frac{g(x)}{w(y)}$$

Then

$$w(y)dy = g(x)dx$$

$$\int^{y} w(y)dy = \int^{x} g(x)dx + C$$

Integrate and solve for $y(x)$

Examples

Exponential model

$$\frac{dy}{dx} = \lambda y$$

$$\frac{dy}{y} = \lambda \, dx$$

$$\lg y = \lambda x + C$$

$$y = e^{c+\lambda x} = \underbrace{e^{c}}_{a} e^{\lambda x} = a e^{\lambda x}$$

Examples

Logistic model

$$\frac{dy}{dx} = \lambda y \left(1 - \frac{y}{a}\right)$$

Hint -

$$\frac{1}{y(1-y/a)} = \frac{1}{y} + \frac{1}{a+y}$$

$$\frac{dy}{y(1-\frac{y}{a})} = \lambda dx$$

$$\frac{dy}{y} + \frac{dy}{a+y} = \lambda dx$$