Particle Flow

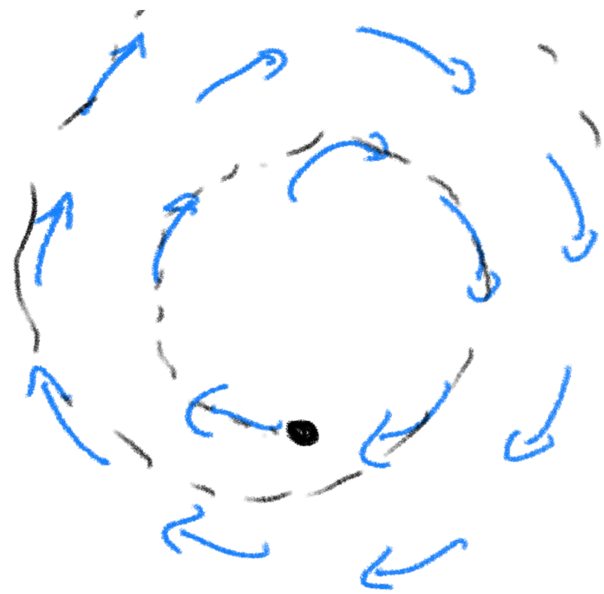
x – position vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad u = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{pmatrix}$$

velocity field

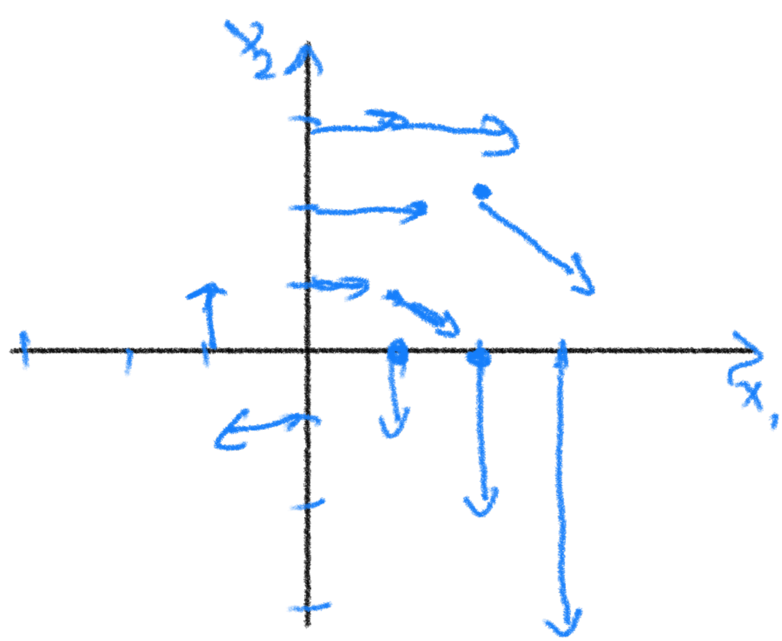
$$\frac{dx}{dt} = u(x)$$

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1(x_1 \ x_2) \\ u_2(x_1 \ x_2) \end{pmatrix} \\[2ex] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{t=0} = \begin{pmatrix} x_1(t=0) \\ x_2(t=0) \end{pmatrix} \end{cases}$$



Example

$$u_1 = x_2 \qquad u_2 = -x_1$$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\frac{d}{dt}x_1 = x_2 \qquad t=0$$

$$\frac{d}{dt}x_2 = -x_1$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$i \qquad -i$

$\begin{pmatrix} i \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -i \end{pmatrix}$

$$X(t) = \left[\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$X(t)\,c = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \end{pmatrix} \quad \begin{pmatrix} 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} c_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$c_1 = 2$$

$$x(t) = 2\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)$$

$$2\begin{pmatrix} \cos(t) \\ \sin(t \end{pmatrix}$$

Example 2

$$\frac{dx}{dt} = u(x)$$

$$u(x) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{pmatrix}$$

$$u_1 = x_1^2 + x_2^2$$

$u_2 = -x_1 x_2$

$$\frac{dx}{dt} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leftarrow \begin{matrix} \text{const} \\ \text{do} \end{matrix}$$

Numerical Solution

$$\begin{cases} \dfrac{dx}{dt} = u(x) \\ x(0) = x_0 \end{cases}$$

$x \in R^n$

$u : R^n \to R^n$

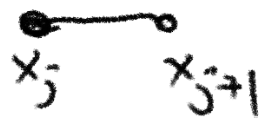
1) Forward Euler

$$\frac{dx}{dt} \simeq \frac{x_{j+1} - x_j}{\Delta t}$$

$$\frac{x_{j+1} - x_j}{\Delta t} = u(x_j)$$

Δt

$$x_{j+1} = x_j + \Delta t \, u(x_j)$$



2) Midpoint

$$\frac{dx}{dt} = \frac{x_{j+1} - x_{j-1}}{2\Delta t}$$

$$\frac{x_{j+1} - x_{j-1}}{2\Delta t} = u(x_j)$$

$$x_{j+1} = x_{j-1} + 2\Delta t \, u(x_j)$$