First Order Systems

Systems of ODE's

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(y_1, \ldots, y_n) \\ f_2(y_1, \ldots, y_n) \\ \vdots \\ f_n(y_1, \ldots, y_n) \end{pmatrix}$$

Example: linear systems

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ & & & \\ a_{n1} & & & a_{nn} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Linear Systems

$$\frac{d\mathsf{y}}{dt} = \mathsf{A}\mathsf{y} \qquad \mathsf{y}(t=0) = \mathsf{y}_0$$

$$\mathsf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & & \\ & & & \\ a_{n1} & & & a_{nn} \end{pmatrix}$$

Solving Linear Systems

Recall - eigenvalues and eigenvectors

The pair $\lambda, \mathbf{u}$ is an eigenvalue/eigenvector pair if

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}$$

In General an $n \times n$ matrix has n eigenvalue/eigenvector pairs.

Let $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$

The we write the system as

$$\mathbf{A}\mathbf{U} = \mathbf{U}\Lambda$$

or, known as the Schur decomposition

$$\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^{-1}$$

Computing eigenvalues

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u} \quad \implies \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = 0 \quad \implies \quad det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

Example find the eigenvalues of

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Using eigenvalues to solve ODE's

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \qquad \mathbf{y}(t=0) = \mathbf{y}_0$$

$$\frac{d\mathbf{y}}{dt} = \mathbf{U}\Lambda\mathbf{U}^{-1}\mathbf{y} \qquad \mathbf{y}(t=0) = \mathbf{y}_0$$

Multiply both sides with $\mathbf{U}^{-1}$

$$\frac{d\mathbf{U}^{-1}\mathbf{y}}{dt} = \Lambda\mathbf{U}^{-1}\mathbf{y} \qquad \mathbf{U}^{-1}\mathbf{y}(t=0) = \mathbf{U}^{-1}\mathbf{y}_0$$

define a variable $\mathbf{z} = \mathbf{U}^{-1}\mathbf{y}$

$$\frac{d\mathbf{z}}{dt} = \Lambda\mathbf{z} \qquad \mathbf{z}(t=0) = \mathbf{U}^{-1}\mathbf{y}_0 = \mathbf{z}_0$$

Using eigenvalues to solve ODE's

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

The system decoupled

$$\frac{dz_i}{dt} = \lambda_i z_i$$

The solution of the system depends on the eigenvalues.

Using eigenvalues to solve ODE's

Suppose that λ_i is an eigenvalue of A and u_i is an associated eigenvector. Then we have that

$$x_i(t) = \exp(\lambda_i t) u_i$$

is a solution of the system

$$\frac{dx_i}{dt} = \lambda_i \exp(\lambda_i t) u_i \qquad Ax(t) = \exp(\lambda_i t) A u_i = \lambda_i \exp(\lambda_i t) u_i$$

The fundamental matrix of solutions is therefore

$$X(t) = [x_1(t), \ldots, x_n(t)] = [u_1 \exp(\lambda_1 t), \ldots, u_n \exp(\lambda_n t)]$$

And the general solution is

$$y(t) = X(t)c$$

where c is a vector that depends on initial conditions.

Recipe for solving linear systems of ODE's

Given

$$\frac{dy}{dt} = Ay \quad y(0) = y_0$$

1. Find the eigenvalues/vectors of A
2. Form the fundamental solution
   $X(t) = [u_1 \exp(\lambda_1 t), \ldots, u_n \exp(\lambda_n t)]$
3. Solve for c given y_0

$$X(0)c = y_0$$

Recipe for solving linear systems of ODE's

Example: $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvec/value pairs

$$\left(1, \frac{1}{\sqrt{2}}[-1, 1], \right), \left(3, \frac{1}{\sqrt{2}}[1, 1], \right)$$

Fundamental solution

$$X(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\exp(t) & \exp(3t) \\ \exp(t) & \exp(3t) \end{pmatrix}$$

To find the solution solve the system

$$X(0)c = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Recipe for solving linear systems of ODE's - complex case

Recall - eigenvalues/vectors are complex conjugates of each other

Example: $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvec/value pairs

$$\left(2 + i, \frac{1}{\sqrt{2}}[1, i], \right), \left(2 - i, \frac{1}{\sqrt{2}}[1, -i], \right)$$

or

$$\lambda = \lambda_r \pm i\lambda_i \qquad u = u_r \pm iu_i$$

Could solve complex systen but can also show that the fundamental solution is

$$X(t) = [u_r \exp(\lambda_r t)\cos(\lambda_t), u_i \exp(\lambda_r t)\sin(\lambda_i t)]$$

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp(2t)\cos(t) & \exp(2t)\sin(t) \\ \exp(2t)\cos(t) & -\exp(2t)\sin(t) \end{pmatrix}$$