Computational Methods for Geological Engineers

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First Order Systems

Systems of ODE's

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(y_1,\dots,y_n) \\ f_2(y_1,\dots,y_n) \\ \vdots \\ f_n(y_1,\dots,y_n) \end{pmatrix}$$

Example: linear systems

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & & \\ a_{n1} & & & a_{nn} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Linear Systems

$$\frac{d\mathsf{y}}{dt} = \mathsf{A}\mathsf{y} \qquad \mathsf{y}(t=0) = \mathsf{y}_0$$

$$\mathsf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & & \\ a_{n1} & & & a_{nn} \end{pmatrix}$$

Recall - eigenvalues and eigenvectors

The pair $\lambda, \mathbf{u}$ is an eigenvalue/eigenvector pair if

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

In General an $n \times n$ matrix has n eigenvalue/eigenvector pairs.

Let $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$

The we write the system as

$$\mathbf{A}\mathbf{U} = \mathbf{U}\Lambda$$

or, known as the Schur decomposition

$$\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^{-1}$$

Computing eigenvalues

$$\mathsf{A}\mathsf{u} = \lambda \mathsf{u} \implies (\mathsf{A} - \lambda \mathsf{I})\mathsf{u} = 0 \implies det(\mathsf{A} - \lambda \mathsf{I}) = 0$$

Example find the eigenvalues of

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Using eigenvalues to solve ODE's

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \qquad \mathbf{y}(t=0) = \mathbf{y}_0$$

$$\frac{d\mathbf{y}}{dt} = \mathbf{U}\Lambda\mathbf{U}^{-1}\mathbf{y} \qquad \mathbf{y}(t=0) = \mathbf{y}_0$$

Multiply both sides with $\mathbf{U}^{-1}$

$$\frac{d\mathbf{U}^{-1}\mathbf{y}}{dt} = \Lambda\mathbf{U}^{-1}\mathbf{y} \qquad \mathbf{U}^{-1}\mathbf{y}(t=0) = \mathbf{U}^{-1}\mathbf{y}_0$$

define a variable $\mathbf{z} = \mathbf{U}^{-1}\mathbf{y}$

$$\frac{d\mathbf{z}}{dt} = \Lambda\mathbf{z} \qquad \mathbf{z}(t=0) = \mathbf{U}^{-1}\mathbf{y}_0 = \mathbf{z}_0$$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

The system decoupled

$$\frac{dz_i}{dt} = \lambda_i z_i$$

The solution of the system depends on the eigenvalues.

Using eigenvalues to solve ODE's

Suppose that λ_i is an eigenvalue of A and u_i is an associated eigenvector. Then we have that

$$x_i(t) = \exp(\lambda_i t) u_i$$

is a solution of the system

$$\frac{dx_i}{dt} = \lambda_i \exp(\lambda_i t) u_i \qquad Ax(t) = \exp(\lambda_i t) A u_i = \lambda_i \exp(\lambda_i t) u_i$$

The fundamental matrix of solutions is therefore

$$X(t) = [x_1(t), \ldots, x_n(t)] = [u_1 \exp(\lambda_1 t), \ldots, u_n \exp(\lambda_n t)]$$

And the general solution is

$$y(t) = X(t)c$$

where c is a vector that depends on initial conditions.

Recipe for solving linear systems of ODE's

Given

$$\frac{dy}{dt} = Ay \quad y(0) = y_0$$

1. Find the eigenvalues/vectors of A
2. Form the fundamental solution
   $X(t) = [u_1 \exp(\lambda_1 t), \ldots, u_n \exp(\lambda_n t)]$
3. Solve for c given y_0

$$X(0)c = y_0$$

Recipe for solving linear systems of ODE's

Example: $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvec/value pairs

$$\left(1, \frac{1}{\sqrt{2}}[-1, 1],\right), \left(3, \frac{1}{\sqrt{2}}[1, 1],\right)$$

Fundamental solution

$$X(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\exp(t) & \exp(3t) \\ \exp(t) & \exp(3t) \end{pmatrix}$$

To find the solution solve the system

$$X(0)c = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Recipe for solving linear systems of ODE's - complex case

Recall - eigenvalues/vectors are complex conjugates of each other

Example: $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvec/value pairs

$$\left(2+i, \frac{1}{\sqrt{2}}[1, i], \right), \left(2-i, \frac{1}{\sqrt{2}}[1, -i], \right)$$

or

$$\lambda = \lambda_r \pm i\lambda_i \qquad u = u_r \pm iu_i$$

Could solve complex systen but can also show that the fundamental solution is

$$X(t) = [u_r \exp(\lambda_r t)\cos(\lambda_t), u_i \exp(\lambda_r t)\sin(\lambda_i t)]$$

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp(2t)\cos(t) & \exp(2t)\sin(t) \\ \exp(2t)\cos(t) & -\exp(2t)\sin(t) \end{pmatrix}$$

Introduction

- We will study second-order differential equations of the form:

$$y'' + ay' + by = f(t)$$

- Discuss general and particular solutions
- Solve two examples step by step
- Physical Examples:
    - Harmonic Oscillators
    - RLC Circuits

Harmonic Oscillators: Derivation

Mass-Spring System:

- Consider a mass m attached to a spring with spring constant k.
- Newton’ s Second Law states that the net force acting on the mass equals its mass times acceleration:

$$m\frac{d^2y}{dt^2} = F$$

- Hooke’ s Law states that the restoring force of a spring is proportional to the displacement:

$$F = -ky$$

- Substituting into Newton’ s Second Law:

$$m\frac{d^2y}{dt^2} = -ky$$

- Dividing by m:

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

- This is a second-order homogeneous differential equation.

Harmonic Oscillators: Damped Case

Adding Damping:

- Suppose there is a damping force proportional to velocity: $F_d = -c\frac{dy}{dt}$
- Newton' s Second Law with damping:

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}$$

- Rearranging:

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$$

- This is a second-order linear differential equation with damping.

RLC Circuits: Derivation

Kirchhoff’ s Voltage Law (KVL):

- Consider a series RLC circuit with resistance R, inductance L, and capacitance C.
- Kirchhoff’ s Voltage Law states:

$$V_L + V_R + V_C = 0$$

- Using voltage-current relationships:
  - Inductor: $V_L = L\frac{d^2Q}{dt^2}$
  - Resistor: $V_R = R\frac{dQ}{dt}$
  - Capacitor: $V_C = \frac{Q}{C}$
- Substituting into KVL:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

- This is a second-order linear differential equation describing charge Q in the circuit.

RLC Circuits: Current-Based Formulation

- Since current is $I = \frac{dQ}{dt}$, differentiating the charge equation:

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$

- This is a second-order ODE for the current in the RLC circuit.
- The equation structure is analogous to damped harmonic motion.

Homogeneous Case: Characteristic Equation

Equation:

$$y'' + ay' + by = 0$$

- Assume a solution of the form $y = e^{rt}$
- Substituting into the equation gives the characteristic equation:

$$r^2 + ar + b = 0$$

- Solve for roots r_1, r_2 to determine the general solution

General Solution Cases

- Distinct real roots: r_1, r_2

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- Repeated root: $r_1 = r_2 = r$

$$y_h = (C_1 + C_2 t)e^{rt}$$

- Complex roots: $r = \alpha \pm i\beta$

$$y_h = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

Non-Homogeneous Case: Particular Solution

- General solution:

$$y = y_h + y_p$$

- Find a particular solution y_p using:
    - Method of undetermined coefficients (for polynomials, exponentials, sines/cosines)
    - Variation of parameters (more general approach)

Example: Homogeneous Equation

- Consider the equation:

$$y'' - 3y' + 2y = 0$$

- Characteristic equation:

$$r^2 - 3r + 2 = 0$$

- Factoring:

$$(r-1)(r-2) = 0 \Rightarrow r = 1, 2$$

- General solution:

$$y_h = C_1 e^t + C_2 e^{2t}$$

Example: Non-Homogeneous Equation

- Consider:

$$y'' - 3y' + 2y = e^t$$

- Solve the homogeneous part first:

$$y_h = C_1 e^t + C_2 e^{2t}$$

- Find a particular solution: Assume $y_p = Ae^t$
- Substitute into the equation:

$$Ae^t - 3Ae^t + 2Ae^t = e^t$$

- Solve for A:

$$(1 - 3 + 2)Ae^t = e^t \Rightarrow A = 1$$

- General solution:

$$y = C_1 e^t + C_2 e^{2t} + e^t$$

Example: RLC Circuit

- Consider $L = 1H$, $R = 2\Omega$, $C = 1F$.
- The equation for charge is:

$$Q'' + 2Q' + Q = 0$$

- Characteristic equation:

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$$

- General solution:

$$Q(t) = (C_1 + C_2 t)e^{-t}$$

Example: RLC Circuit with Forcing

- Consider $Q'' + 2Q' + Q = \cos t$
- Homogeneous solution:

$$Q_h = (C_1 + C_2 t)e^{-t}$$

- Particular solution: Assume $Q_p = A\cos t + B\sin t$
- Compute derivatives and substitute to solve for A, B
- General solution:

$$Q(t) = (C_1 + C_2 t)e^{-t} + A\cos t + B\sin t$$

Example: Complex Roots

- Consider the equation:

$$y'' + 2y' + 5y = 0$$

- Characteristic equation:

$$r^2 + 2r + 5 = 0$$

- Solving for r using the quadratic formula:

$$r = \frac{-2 \pm \sqrt{2^2 - 4(5)}}{2(1)}$$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$r = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

- General solution:

$$y = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$$

- Solution form: Exponential decay with oscillation.

Summary

- Derived second-order ODEs for harmonic oscillators and RLC circuits
- Identified three cases of homogeneous solutions
- Learned how to find a particular solution

Next steps: Try additional exercises!

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Introduction

- Many physical systems are modeled using second-order ODEs.
- Converting them into a system of first-order ODEs simplifies analysis.
- We explore stability, stationary points, and phase diagrams.

General Form of a Second-Order ODE

A second-order differential equation:

$$y'' + ay' + by = f(t)$$

Introducing variables:

$$x_1 = y, \quad x_2 = y'$$

Leads to the system:

$$\begin{cases} x_1' = x_2 \\ x_2' = -ax_2 - bx_1 + f(t) \end{cases}$$

Example: Simple Harmonic Oscillator

$$y'' + \omega^2 y = 0$$

Define:

$$x_1 = y, \quad x_2 = y'$$

Gives the system:

$$\begin{cases} x_1' = x_2 \\ x_2' = -\omega^2 x_1 \end{cases}$$

Stationary Points and Classification

- Stationary points: Solutions where $x_1' = 0$ and $x_2' = 0$.
- Linear system: $\mathbf{x}' = A\mathbf{x}$ where A is the coefficient matrix.
- Eigenvalues of A determine system behavior:
  - Real, distinct, opposite signs $\Rightarrow$ Saddle point.
  - Real, same sign $\Rightarrow$ Node (Stable/Unstable).
  - Complex $\Rightarrow$ Spiral (Stable/Unstable).

Example: Damped Harmonic Oscillator

$$y'' + 2\zeta\omega y' + \omega^2 y = 0$$

Converting:

$$\begin{cases} x_1' = x_2 \\ x_2' = -2\zeta\omega x_2 - \omega^2 x_1 \end{cases}$$

Eigenvalues classify behavior:

- $\zeta > 1$ (Overdamped - Two real roots).
- $\zeta = 1$ (Critically damped - Repeated root).
- $0 < \zeta < 1$ (Underdamped - Complex roots).

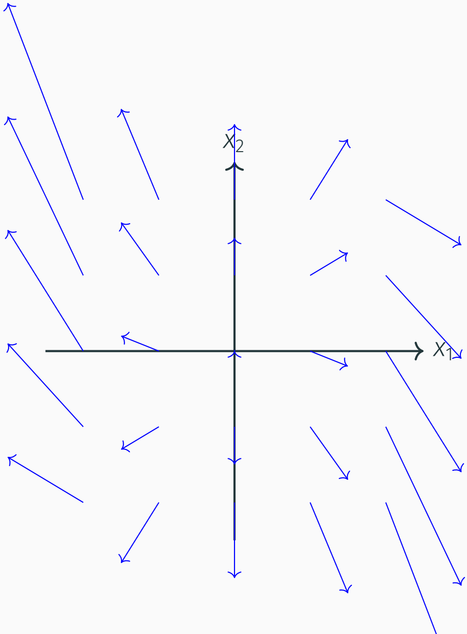
Example: Nonlinear System

Consider the system:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1 - x_1^3 \end{cases}$$

- Nonlinear term $-x_1^3$ makes phase portrait interesting.
- Stationary points: $x_1 = 0, x_2 = 0$.

Phase Diagram Example



Further Exercises

- Convert $y'' + y' + y = 0$ to first-order system.
- Classify stationary points for $\begin{cases} x_1' = x_2, x_2' = -x_1 - 0.5x_2 \end{cases}$.
- Sketch phase diagram for $x_2' = x_1 - x_1^3$.

Conclusion

- Second-order ODEs can be rewritten as first-order systems.
- Stationary points classify system behavior.
- Phase diagrams provide insight into stability.