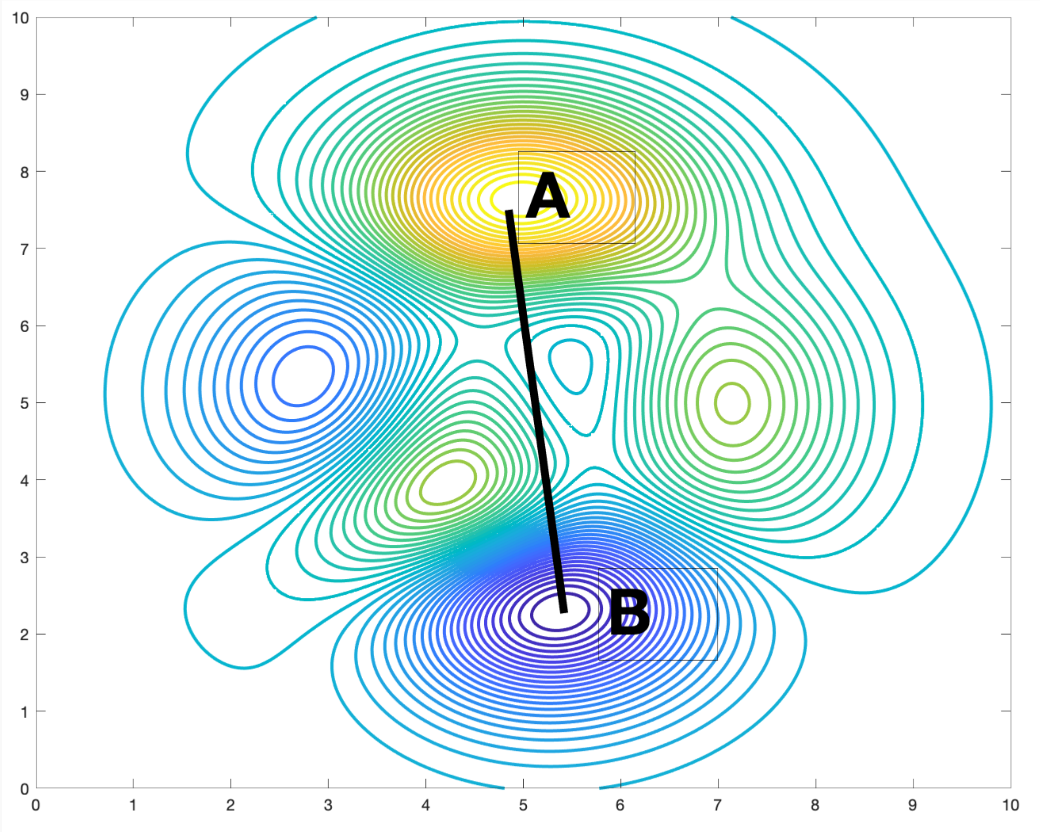
Finite Difference

Goal: Approximate the derivative

Why

- To solve ODE's on the computer (to come)
- To understand data
- For design

Example: Path design



What is the slope of the proposed path from A to B?

Is it too steep?

Finite Difference

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + ...$$

"Solve" for $f'(x)$

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{1}{2}f''(x)h + \frac{1}{6}f'''(x)h^2 + ...$$

Think small h, the leading error behaves like h

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \mathcal{O}(h)$$

Similarly (to be derived at home)

$$\frac{f(x) - f(x-h)}{h} = f'(x) + \mathcal{O}(h)$$

Higher Order Finite Difference

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \dots$$

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= f'(x) + \frac{1}{2}f''(x)h + \frac{1}{6}f'''(x)h^2 + \dots \\
\frac{f(x) - f(x-h)}{h} &= f'(x) - \frac{1}{2}f''(x)h + \frac{1}{6}f'''(x)h^2 + \dots
\end{aligned}$$

Add them together to obtain

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{1}{6}f'''(x)h^2 + \dots$$

Or equivalent

$$\frac{f(x+h) - f(x)}{h} = f'(x + h/2) + \mathcal{O}(h^2)$$

Finite Difference on a mesh

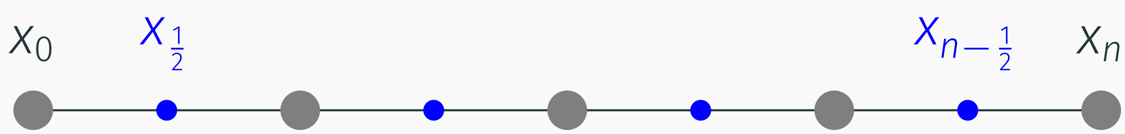
Components

- A mesh $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$
- The functions on the mesh $\mathbf{f} = [f(x_1), \ldots, f(x_n)]^\top$
- Approximate the derivative at x_j

$$f'(x_j) + \mathcal{O}(h) = \frac{f(x_j + h) - f(x_j)}{h} = \frac{f(x_{j+1}) - f(x_j)}{h}$$

- Approximate the derivative at $x_{j+\frac{1}{2}}$

$$f'(x_{j+\frac{1}{2}}) + \mathcal{O}(h^2) = \frac{f(x_j + h) - f(x_j)}{h} = \frac{f(x_{j+1}) - f(x_j)}{h}$$



The forward Euler Method

Numerically solve the ODE

$$y' = f(t, y)$$

Use forward difference to obtain

$$\frac{y(t+h) - y(t)}{h} = f(t, y)$$

or

$$y(t+h) = y(t) + hf(t, y)$$

Indexing

$$\mathbf{y} = [y(t=0), y(t=h), y(t=2h), ..., y(t=Nh)] = [\mathbf{y}_0, \ldots, \mathbf{y}_N]$$

we write

$$\mathbf{y}_{j+1} = \mathbf{y}_j + hf(t_j, \mathbf{y}_j)$$

The forward Euler Method

How to pick h?

- Solve the problem on two (nested) grids and compare the results
- Mesh size H

$$\mathbf{y}_{coarse} = [y(0), y(H), y(2H), ..., y(NH)]$$

- Do it again for $h = H/2$ and obtain

$$\mathbf{y}_{fine} = [y(0), y(h), y(2h), ..., y(2Nh)]$$

- Compare

$$\mathbf{r} = [y_{coarse}(H) - y_{fine}(2h), y_{coarse}(2H) - y_{fine}(4h) ...., y_{coarse}(NH) - y_{fine}(2Nh)]$$

- If error small enough stop.