Multiple Emitter Location and Signal Parameter Estimation

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Abstract—**Processing the signals received on an array of sensors for the location of the emitter is of great enough interest to have been treated under many special case assumptions. The general problem considers sensors with arbitrary locations and arbitrary directional characteristics (gain/phase/polarization) in a noise/interference environment of arbitrary covariance matrix. This report is concerned first with the multiple emitter aspect of this problem and second with the generality of solution. A description is given of the multiple signal classification (MUSIC) algorithm, which provides asymptotically unbiased estimates of 1) number of incident wavefronts present; 2) directions of arrival (DOA) (or emitter locations); 3) strengths and cross correlations among the incident waveforms; 4) noise/interference strength. Examples and comparisons with methods based on maximum likelihood (ML) and maximum entropy (ME), as well as conventional beamforming are included. An example of its use as a multiple frequency estimator operating on time series is included.**

INTRODUCTION

THE TERM MULTIPLE signal classification (MUSIC) is used to describe experimental and theoretical techniques involved in determining the parameters of multiple wavefronts arriving at an antenna array from measurements made on the signals received at the array elements.

The general problem considers antennas with arbitrary locations and arbitrary directional characteristics (gain/phase/polarization) in a noise/interference environment of arbitrary covariance matrix. The multiple signal classification approach is described; it can be implemented as an algorithm to provide asymptotically unbiased estimates of

1) number of signals;
2) directions of arrival (DOA);
3) strengths and cross correlations among the directional waveforms;
4) polarizations;
5) strength of noise/interference.

These techniques are very general and of wide application. Special cases of MUSIC are

1) conventional interferometry;
2) monopulse direction finding (DF), i.e., using multiple colocated antennas;
3) multiple frequency estimation.

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Editor's Note—This paper was first published in the *Proceedings of the RADC Spectrum Estimation Workshop,* held in October 1979 at Griffiss Air Force Base, NY. The document was limited in circulation, but the author's "MUSIC" algorithm is a principal candidate in the field of spectral estimation and rather widely referenced in the literature. Therefore, it seemed appropriate to reprint it in this special issue.

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THE DATA MODEL

The waveforms received at the M array elements are linear combinations of the D incident wavefronts and noise. Thus, the multiple signal classification approach begins with the following model for characterizing the received M vector X as in

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_M \end{bmatrix} = \begin{bmatrix} a(\theta_1) & a(\theta_2) & \cdots & a(\theta_D) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_D \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix}$$

or

$$X = AF + W. \tag{1}$$

The incident signals are represented in amplitude and phase at some arbitrary reference point (for instance the origin of the coordinate system) by the complex quantities $F_1, F_2, \cdots, F_D$. The noise, whether "sensed" along with the signals or generated internal to the instrumentation, appears as the complex vector W.

The elements of X and A are also complex in general. The a_{ij} are known functions of the signal arrival angles and the array element locations. That is, a_{ij} depends on the ith array element, its position relative to the origin of the coordinate system, and its response to a signal incident from the direction of the jth signal. The jth column of A is a "mode" vector $a(\theta_j)$ of responses to the direction of arrival θ_j of the jth signal. Knowing the mode vector $a(\theta_1)$ is tantamount to knowing θ_1 (unless $a(\theta_1) = a(\theta_2)$ with $\theta_1 \neq \theta_2$, an unresolvable situation, a type I ambiguity).

In geometrical language, the measured X vector can be visualized as a vector in M dimensional space. The directional mode vectors $a(\theta_j) = a_{ij}$ for $i = 1, 2, \cdots, M$, i.e., the columns of A, can also be so visualized. Equation (1) states that X is a particular linear combination of the mode vectors; the elements of F are the coefficients of the combination. Note that the X vector is confined to the range space of A. That is, if A has two columns, the range space is no more than a two-dimensional subspace within the M space and X necessarily lies in the subspace. Also note that $a(\theta)$, the continuum of all possible mode vectors, lies within the M space but is quite nonlinear. For help in visualizing this, see Fig. 1. For example, in an azimuth-only direction finding system, θ will consist of a single parameter. In an azimuth/elevation/

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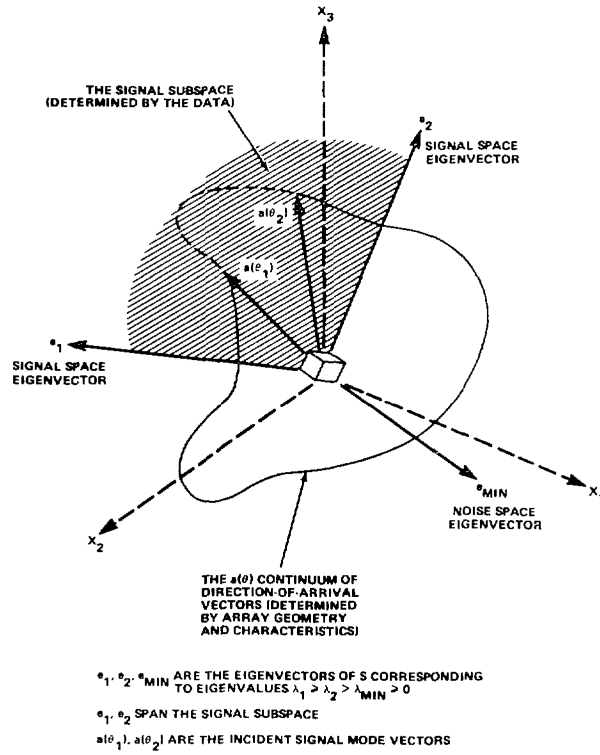

Fig. 1. Geometric portrayal for three-antenna case.

range system, θ will be replaced by θ, ϕ, r for example. In any case, $a(\theta)$ is a vector continuum such as a "snake" (azimuth only) or a "sheet" (Az/El) twisting and winding through the M space. (In practice, the procedure by which the $a(\theta)$ continuum is measured or otherwise established corresponds to calibrating the array.)

In these geometrical terms (see Fig. 1), the problem of solving for the directions of arrival of multiple incident wavefronts consists of locating the intersections of the $a(\theta)$ continuum with the range space of A. The range space of A is, of course, obtained from the measured data. The means of obtaining the range space and, necessarily, its dimensionality (the number D of incident signals) follows.

The S Matrix

The $M \times M$ covariance matrix of the X vector is

$$S \triangleq \overline{XX^*} = A\overline{FF^*}A^* + \overline{WW^*}$$

or

$$S = APA^* + \lambda S_0 \tag{2}$$

under the basic assumption that the incident signals and the noise are uncorrelated. Note that the incident waveforms represented by the elements of F may be uncorrelated (the $D \times D$ matrix $P \triangleq \overline{FF^*}$ is diagonal) or may contain completely correlated pairs (P is singular). In general, P will be "merely" positive definite which reflects the arbitrary degrees of pair-wise correlations occurring between the incident waveforms.

When the number of incident wavefronts D is less than the number of array elements M, then APA^* is singular; it has a rank less than M. Therefore

$$|APA^*| = |S - \lambda S_0| = 0. \tag{3}$$

This equation is only satisfied with λ equal to one of the eigenvalues of S in the metric of S_0. But, for A full rank and P positive definite, APA^* must be nonnegative definite. Therefore λ can only be the minimum eigenvalue $\lambda_{\min}$. Therefore, any measured $S = \overline{XX^*}$ matrix can be written

$$S = APA^* + \lambda_{\min} S_0, \qquad \lambda_{\min} \geq 0 \tag{4}$$

where $\lambda_{\min}$ is the smallest solution to $|S - \lambda S_0| = 0$. Note the special case wherein the elements of the noise vector W are mean zero, variance σ^2, in which case, $\lambda_{\min} S_0 = \sigma^2 I$.

Calculating a Solution

The rank of APA^* is D and can be determined directly from the eigenvalues of S in the metric of S_0. That is, in the complete set of eigenvalues of S in the metric of S_0, $\lambda_{\min}$ will not always be simple. In fact, it occurs repeated $N = M - D$ times. This is true because the eigenvalues of S and those of $S - \lambda_{\min} S_0 = APA^*$ differ by $\lambda_{\min}$ in all cases. Since the minimum eigenvalue of APA^* is zero (being singular), $\lambda_{\min}$ must occur repeated N times. Therefore, the number of incident signals estimator is

$$\hat{D} = M - \hat{N} \tag{5}$$

where $\hat{N}$ = the multiplicity of $\lambda_{\min}(S, S_0)$ and $\lambda_{\min}(S, S_0)$ is read "$\lambda_{\min}$ of S in the metric of S_0." (In practice, one can expect that the multiple $\lambda_{\min}$ will occur in a cluster rather than all precisely equal. The "spread" on this cluster decreases as more data is processed.)

The Signal and Noise Subspaces

The M eigenvectors of S in the metric of S_0 must satisfy $Se_i = \lambda_i S_0 e_i$, $i = 1, 2, \cdots, M$. Since $S = APA^* + \lambda_{\min} S_0$, we have $APA^* e_i = (\lambda_i - \lambda_{\min}) S_0 e_i$. Clearly, for each of the λ_i that is equal to $\lambda_{\min}$—there are N—we must have $APA^* e_i = 0$ or $A^* e_i = 0$. That is, the eigenvectors associated with $\lambda_{\min}(S, S_0)$ are orthogonal to the space spanned by the columns of A; the incident signal mode vectors!

Thus we may justifiably refer to the N dimensional subspace spanned by the N noise eigenvectors as the noise subspace and the D dimensional subspace spanned by the incident signal mode vectors as the signal subspace; they are disjoint.

The Algorithm

We now have the means to solve for the incident signal mode vectors. If E_N is defined to be the $M \times N$ matrix whose columns are the N noise eigenvectors, and the ordinary Euclidean distance (squared) from a vector Y to the signal subspace is $d^2 = Y^* E_N E_N^* Y$, we can plot $1/d^2$ for points along the $a(\theta)$ continuum as a function of θ. That is,

$$P_{MU}(\theta) = \frac{1}{a^*(\theta) E_N E_N^* a(\theta)}. \tag{6}$$

(However, the $a(\theta)$ continuum may intersect the D dimensional signal subspace more than D times; another unresolvable situation occurring only for the case of multiple incident signals—a type II ambiguity.) It is clear from the expression that MUSIC is asymptotically unbiased even for multiple incident wavefronts because S is asymptotically perfectly measured so that E_N is also. $a(\theta)$ does not depend on the data.

Once the directions of arrival of the D incident signals have been found, the A matrix becomes available and may be used to compute the parameters of the incident signals. The solution for the P matrix is direct[1] and can be expressed in terms of $(S - \lambda_{\min} S_0)$ and A. That is, since $APA^* = S - \lambda_{\min} S_0$,

$$P = (A^*A)^{-1} A^* (S - \lambda_{\min} S_0) A (A^*A)^{-1}. \quad (7)$$

Including Polarization

Consider a signal arriving from a specific direction θ_0. Assume that the array is not diverse in polarization; i.e., all elements are identically polarized, say, vertically. Certainly the DF system will be most sensitive to vertically polarized energy, completely insensitive to horizontal and partially sensitive to arbitrarily polarized energy. The array is only sensitive to the vertically polarized *component* of the arriving energy.

For a general or polarizationally diverse array, the mode vector corresponding to the direction θ_0 depends on the signal polarization. A vertically polarized signal will induce one mode vector and horizontal another, and right-hand circular (RHC) still another.

Recall that signal polarization can be completely characterized by a single complex number q. We can "observe" how the mode vector changes as the polarization parameter q for the emitter changes at the specific direction θ_0. It can be proven that as q changes through all possible polarizations, the mode vector sweeps out a *two-dimensional* "polarization subspace." Thus, only two independent mode vectors spanning the polarization subspace for the direction θ_0 are needed to represent any emitter polarization q at direction θ_0. The practical embodiment of this is that only the mode vectors of two emitter polarizations need be calculated or kept in store for direction θ_0 in order to solve for emitter polarizations where only one was needed to solve for DOA in a system with an array that was not polarizationally diverse.

These arguments lead to an equation similar to (6) for $P(\theta)$ but including the effects of polarization diversity among the array elements.

$$P_{MU}(\theta) = \frac{1}{\lambda_{\min}\left(\begin{bmatrix} a_x^*(\theta) \\ a_y^*(\theta) \end{bmatrix} E_N E_N^* \begin{bmatrix} a_x(\theta) & a_y(\theta) \end{bmatrix} \right)} \quad (8)$$

where $a_x(\theta)$ and $a_y(\theta)$ are the two continua corresponding to, for example, separately taken x and y linear incident wavefront polarizations. The eigenvector corresponding to $\lambda_{\min}$ in (8) provides the polarization parameter q since it is of the form $[1 \;\; q]^T$.

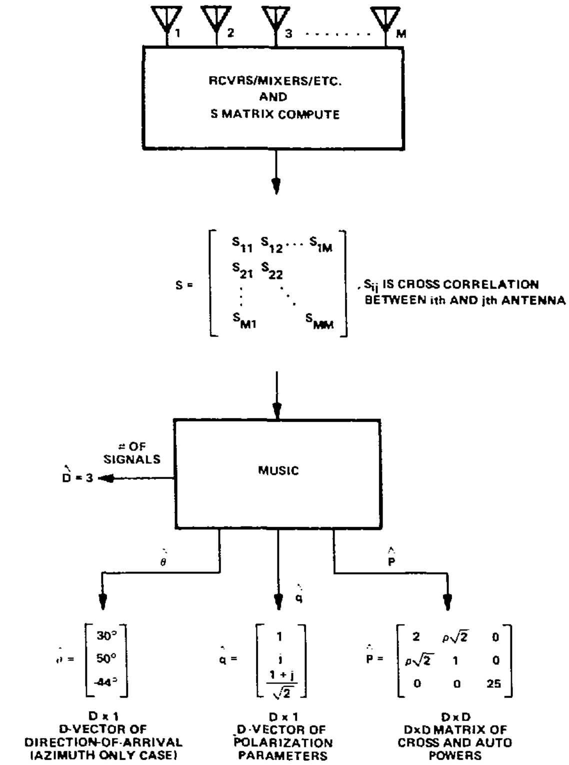

Fig. 2. Block diagram for multiple signal classification.

The Algorithm

In summary, the steps of the algorithm (see Fig. 2) are:

Step 0: collect data, form S;
Step 1: calculate eigenstructure of S in metric of S_0;
Step 2: decide number of signals D; (5);
Step 3: evaluate $P_{MU}(\theta)$ versus θ; (6) or (8);
Step 4: pick D peaks of $P_{MU}(\theta)$;
Step 5: calculate remaining parameters; (7).

The above steps have been implemented in several forms to verify and evaluate the principles and basic performance. Field tests have been conducted using actual receivers, arrays, and multiple transmitters. The results of these tests have demonstrated the potential of this approach for handling multiple signals in practical situations. Performance results are being prepared for presentation in another paper.

Comparison With Other Methods

In comparing MUSIC with ordinary beamforming (BF), maximum likelihood (ML), and maximum entropy (ME), the following expressions were used. See Figs. 3 and 4.

$$P_{\mathrm{BF}}(\theta) = a^*(\theta) S a(\theta)$$

$$P_{\mathrm{ML}}(\theta) = \frac{1}{a^*(\theta) S^{-1} a(\theta)}$$

$$P_{\mathrm{ME}}(\theta) = \frac{1}{a^*(\theta) c c^* a(\theta)}$$

[1] (added in reprint) Equation (7) is true if S_0, the noise covariance matrix, is the identity matrix. In general, although there are many estimators of P, the least squares estimate based on $X = AF + W$ with $WW^* = \lambda_{\min} S_0$ requires *whitening* which leads to

$$P = (A^* S_0^{-1} A)^{-1} A^* S_0^{-1} (S - \lambda_{\min} S_0) S_0^{-1} A (A^* S_0^{-1} A)^{-1}. \quad (7)$$

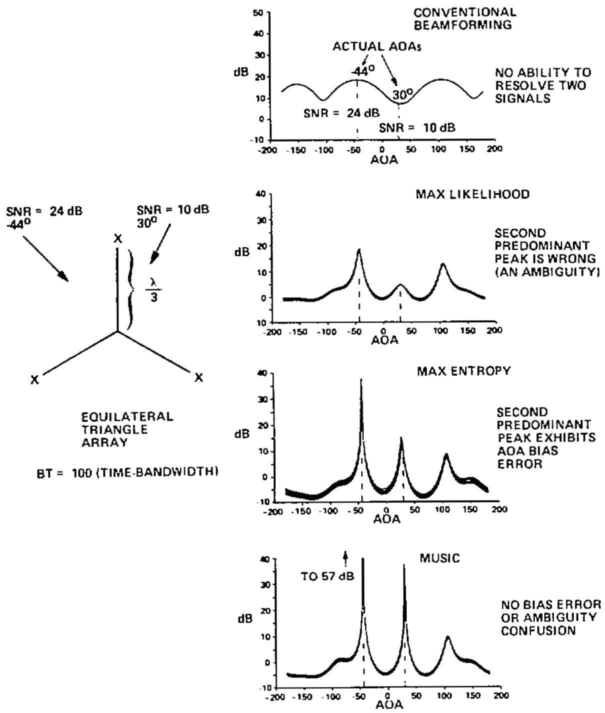

Fig. 3. Example of azimuth-only DF performance.

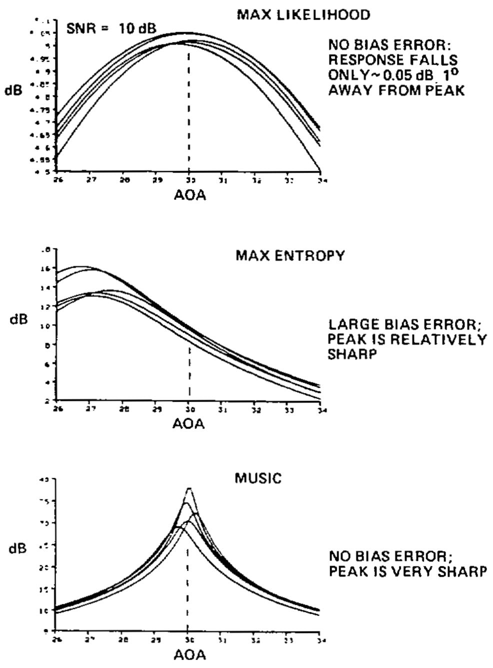

Fig. 4. Example of azimuth-only DF performance (scale expanded about weaker signal at 30°).

where c is a column of S^{-1}. The beamformer expression calculates for plotting the power one would measure at the output of a beamformer (summing the array element signals after inserting delays appropriate to steer or look in a specific direction) as a function of the direction.

$P_{\mathrm{ML}}(\theta)$ calculates the log likelihood function under the assumptions that X is a mean zero, multivariate Gaussian and that there is only a single incident wavefront present. For multiple incident wavefronts, $P_{\mathrm{ML}}(\theta)$ becomes

$$P_{\mathrm{ML}}(\theta) = \frac{1}{\lambda_{\min}(A_\theta^* S^{-1} A_\theta)}$$

which implies a D dimensional search (and plot!).

$P_{\mathrm{ME}}(\theta)$ is based on selecting one of the M array elements as a "reference" and attempting to find weights to be applied to the remaining $M - 1$ received signals to permit their sum with a MMSE fit to the reference. Since there are M possible references, there are M generally different $P_{\mathrm{ME}}(\theta)$ obtained from the M possible column selections from S^{-1}. In the comparison plots, a particular reference was consistently selected.

An example of the completely general MUSIC algorithm applied to a problem of steering a multiple feed parabolic dish antenna is shown in Fig. 5. sin x/x pencil beamshapes skewed slightly off boresight are assumed for the element patterns. Since the six antennas are essentially colocated, the DF capacity arises out of the antenna beam pattern diversity. The computer was used to simulate the "noisy" S matrix that would arise in practice for the conditions desired and then to subject it to the MUSIC algorithm. Fig. 5 shows how three directional signals are distinguished and their polarizations estimated even though two of the arriving signals are highly similar (90 percent correlated).

The application of MUSIC to the estimation of the frequencies of multiple sinusoids (arbitrary amplitudes and phases) for a very limited duration data sample is shown in Fig. 6. The figure suggests that, even though there was no actual noise included, the rounding of the data samples to six decimal digits has already destroyed a significant portion of the information present in the data needed to resolve the several frequencies.

Summary and Conclusion

As this paper was being prepared, the works of Gething [1] and Davies [2] were discovered, offering a part of the solution discussed here but in terms of simultaneous equations and special linear relationships without recourse to eigenstructure. However, the geometric significance of a vector space setting and the interpretation of the S matrix eigenstructure was missed. More recent work by Reddi [3] is also along the lines of the work presented here though limited to uniform, collinear arrays of omnidirectional elements and also without clear utilization of the entire noise subspace. Ziegenbein [4] applied the same basic concept to time series spectral analysis referring to it as a Karhunen–Loeve transform though treating aspects of it as "ad hoc." El-Behery and MacPhie [5] and Capon [6] treat the uniform collinear array of omnidirectional elements using the maximum likelihood method. Pisarenko [7] also treats time series and addresses only the case of a full complement of sinusoids; i.e., a one-dimensional noise subspace.

The approach presented here for multiple signal classification is very general and of wide application. The method is interpretable in terms of the geometry of complex M spaces wherein the eigenstructure of the measured S matrix plays the central role. MUSIC provides asymptotically unbiased esti-

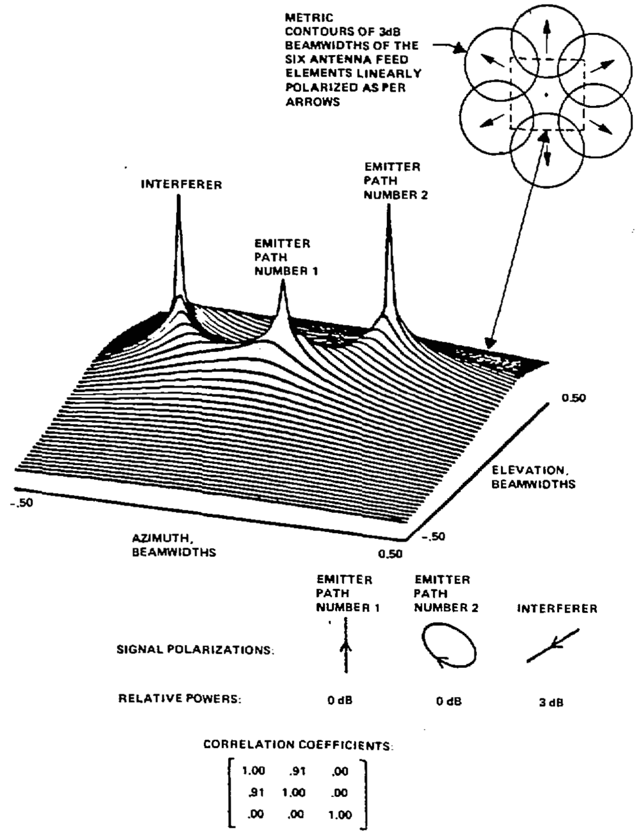

Fig. 5. MUSIC applied to a multiple feed, parabolic dish antenna system.

DATA: 16 COMPLEX TIME SAMPLES OF 6 CISOIDS

FREQ (Hz)	REL AMP (dB)
78.1	-62.33
134.1	-11.10
138.6	0.0
142.9	-11.10
152.9	-40.0
165.3	-26.02

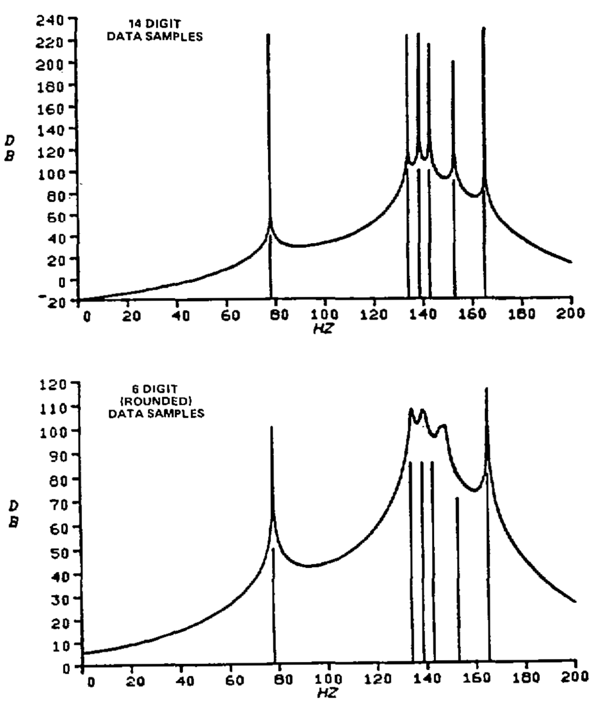

Fig. 6. Example of MUSIC used for frequency estimation.

mates of a general set of signal parameters approaching the Cramer–Rao accuracy bound. MUSIC models the data as the sum of point source emissions and noise rather than the convolution of an all pole transfer function driven by a white noise (i.e., autoregressive modeling, maximum entropy) or maximizing a probability under the assumption that the X vector is zero mean, Gaussian (maximum likelihood for Gaussian data). In geometric terms MUSIC minimizes the distance from the $a(\theta)$ continuum to the signal subspace whereas maximum likelihood minimizes a weighted combination all component distances.

No assumptions have been made about array geometry. The array elements may be arranged in a regular or irregular pattern and may differ or be identical in directional characteristics (amplitude/phase) provided their polarization characteristics are all identical. The extension to include general polarizationally diverse antenna arrays will be more completely described in a separate paper.

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