Econometrics II

The Autodistributed Lag (ADL) Model

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Outline: Dynamic Models for Stationary Time Series

1. Autoregressive Distributed Lag (ADL) Models
2. Dynamic Multipliers and Long-Run Solution
3. The Error Correction Model

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Autoregressive Distributed Lag (ADL) Models

- We want to model the interrelationships between variables.
  We are interested in the conditional mean, and assume linearity, e.g.,

$$E(y_t \mid y_1, \ldots, y_{t-1}, x_1, \ldots, x_{t-1}, x_t) = \delta + \theta y_{t-1} + \phi_0 x_t + \phi_1 x_{t-1}.$$

- This can also be written as linear regression

$$y_t = \delta + \theta y_{t-1} + \phi_0 x_t + \phi_1 x_{t-1} + \epsilon_t, \quad \text{with} \quad E(\epsilon_t \mid y_{t-1}, x_t, x_{t-1}) = 0, \quad (*)$$

  which is called the autoregressive distributed lag (ADL) model.

- y_t is stationary if x_t is and if the autoregressive polynomial is invertible, $|\theta| < 1$.
  Standard results for estimation and inference apply.
  $E(\epsilon_t \mid x_t) = 0$ excludes contemporaneous feedback from y_t to x_t.

- We are interested in the dynamic inter-relations:

$$\frac{\partial y_t}{\partial x_t}, \quad \frac{\partial y_{t+1}}{\partial x_t}, \quad \frac{\partial y_{t+2}}{\partial x_t}, \quad \ldots$$

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Dynamic Multipliers

- From the equations

$$
\begin{array}{lllllllllll}
y_t & = & \delta & + & \theta y_{t-1} & + & \phi_0 x_t & + & \phi_1 x_{t-1} & + & \epsilon_t \\
y_{t+1} & = & \delta & + & \theta y_t & + & \phi_0 x_{t+1} & + & \phi_1 x_t & + & \epsilon_{t+1} \\
y_{t+2} & = & \delta & + & \theta y_{t+1} & + & \phi_0 x_{t+2} & + & \phi_1 x_{t+1} & + & \epsilon_{t+2}
\end{array}
$$

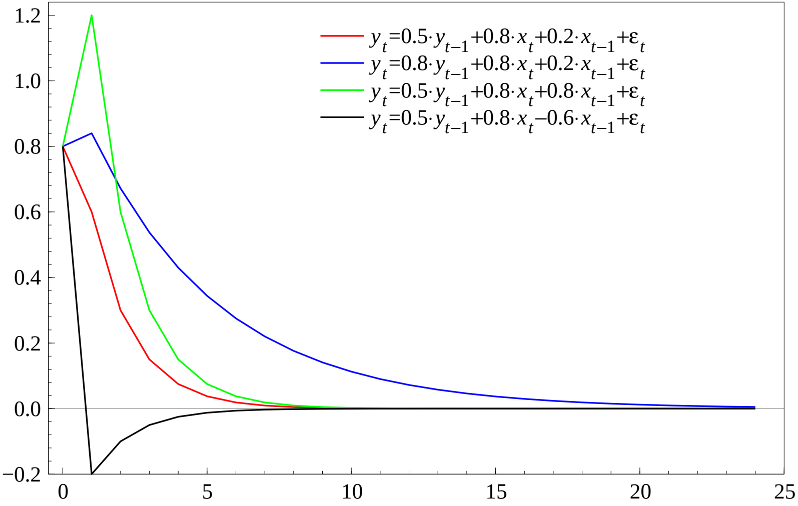
we can find the dynamic multipliers as the derivatives:

$$
\begin{array}{lllllll}
\frac{\partial y_t}{\partial x_t} & = & & & & & \phi_0 \\
\frac{\partial y_{t+1}}{\partial x_t} & = & \frac{\partial y_t}{\partial x_{t-1}} & = & \theta \frac{\partial y_t}{\partial x_t} + \phi_1 & = & \theta\phi_0 + \phi_1 \\
\frac{\partial y_{t+2}}{\partial x_t} & = & \frac{\partial y_t}{\partial x_{t-2}} & = & \theta \frac{\partial y_{t+1}}{\partial x_t} & = & \theta\left(\theta\phi_0 + \phi_1\right) \\
\frac{\partial y_{t+3}}{\partial x_t} & = & \frac{\partial y_t}{\partial x_{t-3}} & = & \theta \frac{\partial y_{t+2}}{\partial x_t} & = & \theta^2\left(\theta\phi_0 + \phi_1\right) \\
 & \vdots & & & & & \\
\frac{\partial y_{t+k}}{\partial x_t} & = & \frac{\partial y_t}{\partial x_{t-k}} & = & & & \theta^{k-1}\left(\theta\phi_0 + \phi_1\right).
\end{array}
$$

Due to stationarity, $|\theta| < 1$, shocks have transitory effects:

$$
\frac{\partial y_{t+k}}{\partial x_t} \to 0 \quad \text{as} \quad k \to \infty.
$$

Examples of Dynamic Multipliers



Long-Run Solution and Long-Run Multiplier

- Now we define the steady state as $y_t = y_{t-1}$, $x_t = x_{t-1}$, and $\epsilon_t = 0$. We obtain

$$\begin{aligned} y_t &= \delta + \theta y_t + \phi_0 x_t + \phi_1 x_t \\ y_t &= \frac{\delta}{1-\theta} + \frac{\phi_0 + \phi_1}{1-\theta} x_t, \end{aligned}$$

  which is known as the long-run solution.

- Now consider a permanent shift in x_t so that $E(x_t)$ changes. The effect in $E(y_t)$ is

$$\frac{\partial E(y_t)}{\partial E(x_t)} = \frac{\phi_0 + \phi_1}{1-\theta} = \frac{\phi(1)}{\theta(1)},$$

  known as the long-run multiplier.

- Note that this is also the accumulated short-run effects:

$$\begin{aligned} \frac{\partial y_t}{\partial x_t} + \frac{\partial y_{t+1}}{\partial x_t} + \frac{\partial y_{t+2}}{\partial x_t} + \ldots &= \phi_0 + (\theta\phi_0 + \phi_1) + \theta(\theta\phi_0 + \phi_1) + \theta^2(\theta\phi_0 + \phi_1) + \ldots \\ &= \phi_0\left(1 + \theta + \theta^2 + \ldots\right) + \phi_1\left(1 + \theta + \theta^2 + \ldots\right) \\ &= \frac{\phi_0 + \phi_1}{1-\theta}. \end{aligned}$$

General Case

- Consider the general case ADL(p,q):

$$\theta(L)y_t = \delta + \phi(L)x_t + \epsilon_t,$$

where

$$\begin{aligned}\theta(L) &= 1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_p L^p \\ \phi(L) &= \phi_0 + \phi_1 L + \phi_2 L^2 + \ldots + \phi_q L^q.\end{aligned}$$

- The solution is given by

$$y_t = \theta^{-1}(L)\delta + \theta^{-1}(L)\phi(L)x_t,$$

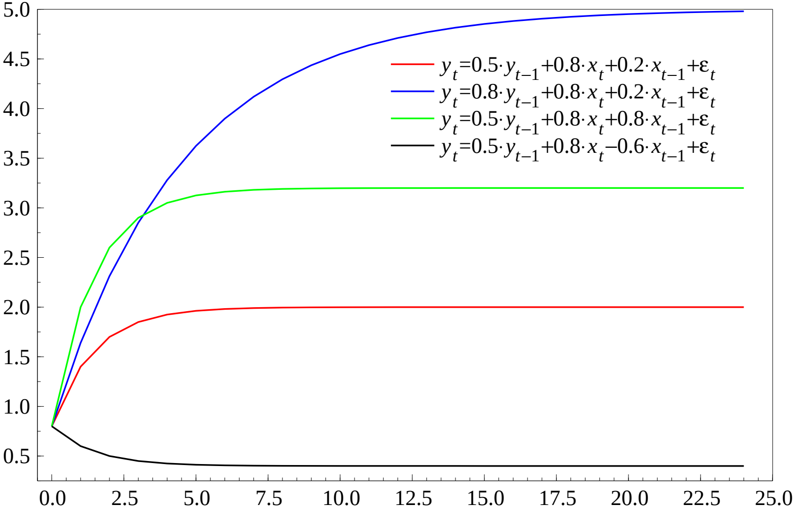
with long-run solution

$$y_t = \frac{\delta}{\theta(1)} + \frac{\phi(1)}{\theta(1)}x_t.$$

The long-run multiplier is

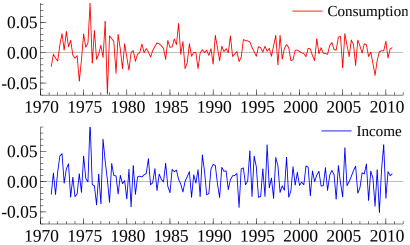
$$\frac{\phi(1)}{\theta(1)} = \frac{\phi_0 + \phi_1 + \phi_2 + \ldots + \phi_q}{1 - \theta_1 - \theta_2 - \ldots - \theta_p}.$$

Examples of Long-Run Multipliers

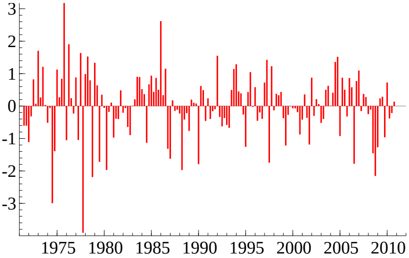


Empirical Example: Consumption and Income

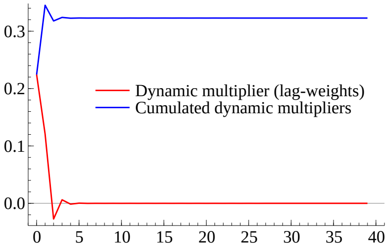
(a) Growth of Consumption and Income



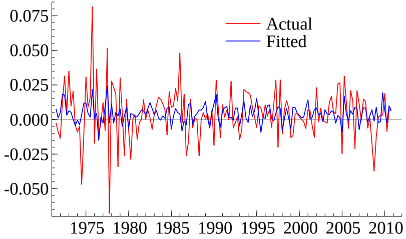
(b) Standardized Residuals



(c) Dynamic Multipliers



(d) Predicted Values from ADL Model



Empirical Example: Consumption and Income

We consider growth in consumption and income, $c_t = \Delta \log(\text{CONS}_t)$ and $y_t = \Delta \log(\text{INC}_t)$, for Denmark. An ADL(1,1) model yields the following:

```
      Modelling c by OLS: 1971(3) - 2010(4)

                  Coefficient   Std.Error  t-value  t-prob
c_1               -0.225873     0.07793    -2.90    0.0043
Constant           0.00300594   0.001429    2.10    0.0370
y                  0.224342     0.05854     3.83    0.0002
y_1                0.171446     0.05969     2.87    0.0047

sigma                0.0167947  RSS              0.0434375234
R^2                   0.123735  F(3,154) =    7.249 [0.000]**
Adj.R^2               0.106665  log-likelihood        423.531
no. of observations        158  no. of parameters           4
mean(c)             0.00415263  se(c)               0.0177691
```

Recall the main assumptions we have made...

Apart from outliers, the model is reasonably well specified:

```
AR 1-5 test:       F(5,149)   =   1.3762 [0.2364]
ARCH 1-4 test:     F(4,150)   =  0.66247 [0.6190]
Normality test:    Chi^2(2)   =   20.706 [0.0000]**
Hetero test:       F(6,151)   =   4.6249 [0.0002]**
RESET23 test:      F(2,152)   =   5.0161 [0.0078]**
```

The long-run solution is derived in PcGive:

```
Solved static long-run equation for c
             Coefficient  Std.Error  t-value  t-prob
Constant     0.00245208   0.001156   2.12     0.0355
y            0.322863     0.07508    4.30     0.0000
```

where, e.g.,

$$\beta = \frac{\phi_0 + \phi_1}{1 - \theta} = \frac{0.224342 + 0.171446}{1 + 0.225873} = 0.32286.$$

The standard errors are functions of the covariance matrix ($\delta-$method).

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The Error Correction Model

- An alternative formulation which incorporates the long-run solution is the ECM:

$$\begin{aligned} y_t &= \delta + \theta y_{t-1} + \phi_0 x_t + \phi_1 x_{t-1} + \epsilon_t \\ y_t - y_{t-1} &= \delta + (\theta - 1) y_{t-1} + \phi_0 x_t + \phi_1 x_{t-1} + \epsilon_t \\ y_t - y_{t-1} &= \delta + (\theta - 1) y_{t-1} + \phi_0 (x_t - x_{t-1}) + (\phi_0 + \phi_1) x_{t-1} + \epsilon_t \\ \Delta y_t &= \delta + \underbrace{(\theta - 1)}_{\gamma_1} y_{t-1} + \phi_0 \Delta x_t + \underbrace{(\phi_0 + \phi_1)}_{\gamma_2} x_{t-1} + \epsilon_t, \qquad (*) \end{aligned}$$

or alternatively

$$\Delta y_t = \phi_0 \Delta x_t \underbrace{-(1-\theta)}_{\gamma_1} \left(y_{t-1} - \underbrace{\frac{\delta}{1-\theta}}_{\alpha} - \underbrace{\frac{\phi_0 + \phi_1}{1-\theta}}_{\beta} x_{t-1} \right) + \epsilon_t. \qquad (**)$$

- So Δy_t reacts on deviations from the long-run solution

$$y_t - y_t^* = y_t - \alpha - \beta x_t.$$

It error-corrects or equilibrium-corrects. $y_t^* = \alpha + \beta x_t$ is the attractor.

Empirical Example: ECM for Consumption

We can estimate $(*)$ directly. Defining Δc_t and Δy_t we get:

	Coefficient	Std.Error	t-value	t-prob
Constant	0.00300594	0.001429	2.10	0.0370
dy	0.224342	0.05854	3.83	0.0002
c_1	-1.22587	0.07793	-15.70	0.0000
y_1	0.395788	0.09540	4.15	0.0001

which is the model

$$\widehat{\Delta c_t} = 0.224 \cdot \Delta y_t - 1.226 \cdot c_{t-1} + 0.396 \cdot y_{t-1} + 0.003 \qquad (*)$$

$$= 0.224 \cdot \Delta y_t - 1.226 \cdot \left(c_{t-1} - \underbrace{\frac{0.396}{1.226}}_{=0.323} \cdot y_{t-1} - \underbrace{\frac{0.003}{1.226}}_{=0.0025} \right). \qquad (**)$$

We could also estimate $(**)$, but that is a non-linear regression.