

Gene genealogies in diploid populations evolving according to sweepstakes reproduction

— approximating $\mathbb{E}[R_i(n)]$ for the time-changed $\Omega\text{-}\delta_0\text{-Beta}(\gamma, 2 - \alpha, \alpha)$ -coalescent

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Let $\#A$ denote the number of elements in a finite set A . For a given coalescent $\{\xi^n\}$ write $L_i(n) \equiv \int_0^{\tau(n)} \#\{\xi \in \xi^n(t) : \#\xi = i\} dt$ and $L(n) \equiv \int_0^{\tau(n)} \#\xi^n(t) dt$ where $\tau(n) \equiv \inf\{t \geq 0 : \#\xi^n(t) = 1\}$. Then $L(n) = L_1(n) + \dots + L_{n-1}(n)$. Write $R_i(n) \equiv L_i(n)/L(n)$ for $i = 1, 2, \dots, n-1$. With this C++ simulation code we approximate $\mathbb{E}[R_i(n)]$ when $\{\xi^n(G)\} \equiv \{\xi^n(G(t)) : t \geq 0\}$ is the time-changed continuous-time $\Omega\text{-}\delta_0\text{-Beta}(\gamma, 2 - \alpha, \alpha)$ coalescent with $0 < \alpha < 2$ and time-change function $G(t)$.

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²compiled @ 2:18pm on Monday 20th October, 2025
CTANGLE 4.12.1 (TeX Live 2025/Debian)

g++ (Debian 15.2.0-4) 15.2.0
kernel 6.16.12+deb14-amd64 GNU/Linux
GNU bash, version 5.3.3(1)-release (x86_64-pc-linux-gnu)
GSL 2.8
CWEAVE 4.12.1 (TeX Live 2025/Debian)
L^AT_EX This is LuaHBTeX, Version 1.22.0 (TeX Live 2025/Debian) Development id: 7673
written using GNU Emacs 30.1

1 Copyright

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2 compilation and output

This CWEB [KL94] document (the `.w` file) can be compiled with `cweave` to generate a `.tex` file, and with `ctangle` to generate a `.c` [KR88] C++ code file.

Use the shell tool `spix` on the script appearing before the preamble (the lines starting with `%$`); simply

```
spix /path/to/the/sourcefile
```

where `sourcefile` is the `.w` file

One may also copy the script into a file and run `parallel` [Tan11] :

```
parallel --gnu -j1 :::: /path/to/scriptfile
```

3 intro

Write $\mathbb{N} \equiv \{1, 2, \dots\}$, $[n] = \{1, 2, \dots, n\}$, $]n] \equiv \{2, 3, \dots, n\}$ for all $n \in \mathbb{N}$. The Ω - δ_0 -Beta($\gamma, 2 - \alpha, \alpha$)-coalescent is a continuous-time Ξ -coalescent with transition rates

$$\begin{aligned} \lambda_{n; k_1, \dots, k_r; s} &= \mathbb{1}_{\{r=1, k_1=2\}} \frac{C_\kappa}{C_{\alpha, \gamma}} + \\ &+ \frac{c\alpha}{C_{\alpha, \gamma} m^\alpha} \sum_{\ell=0}^{s \wedge (4-r)} \binom{s}{\ell} \frac{(4)_{r+\ell}}{4^{k+\ell}} \int_0^1 \mathbb{1}_{\{0 < t \leq \gamma\}} t^{k+\ell-\alpha-1} (1-t)^{n+\alpha-k-\ell-1} dt \end{aligned} \quad (1)$$

where $0 < \gamma \leq 1$, $B(p, a, b) \equiv \int_0^1 \mathbb{1}_{\{0 < t \leq p\}} t^{a-1} (1-t)^{b-1} dt$, and

$$\begin{aligned} C_\kappa &= \frac{2}{4m^2} \left(\mathbb{1}_{\{\kappa=2\}} + \mathbb{1}_{\{\kappa>2\}} \frac{c_\kappa}{2^\kappa (\kappa-2)(\kappa-1)} \right) \\ C_{\kappa, \alpha, \gamma} &= C_\kappa + \frac{c\alpha}{4m^\alpha} B(\gamma, 2 - \alpha, \alpha) \\ m &= \mathbb{1}_{\{\kappa=2\}} \left(2 \frac{\pi^2}{3} - 3 \right) + \mathbb{1}_{\{\kappa>2\}} \left(1 + 2^{\kappa-1} \frac{\kappa}{\kappa-1} \right) \end{aligned}$$

In (1) $n \geq 2$, $k_1, \dots, k_r \in]n]$ and $2 \leq \sum_i k_i \leq n$ for all $r \in [4]$, $s \equiv n - \sum_i k_i$.

Here we consider a time-changed version of the Ω - δ_0 -Beta($\gamma, 2 - \alpha, \alpha$)-coalescent corresponding to exponential population growth where the time-change function $G(t) = \int_0^t e^{\rho s} ds$ for some fixed $\rho \geq 0$.

Let $\#A$ denote the number of elements in a finite set A . For a given coalescent $\{\xi^n\}$ write $L_i(n) \equiv \int_0^{\tau(n)} \# \{\xi \in \xi^n(t) : \#\xi = i\} dt$ and $L(n) \equiv \int_0^{\tau(n)} \#\xi^n(t) dt$ where $\tau(n) \equiv \inf \{t \geq 0 : \#\xi^n(t) = 1\}$. Then $L(n) = L_1(n) + \dots + L_{n-1}(n)$. Write $R_i(n) \equiv L_i(n)/L(n)$ for $i = 1, 2, \dots, n-1$. With this C++ code we use simulations to approximate $\mathbb{E}[R_i(n)]$

The code follows in § 4.1–§ 4.19, we conclude in § 5. Comments within the code in **this font and color**

4 code

4.1 includes

the included libraries

5 ⟨ includes 5 ⟩ ≡

```
#include <iostream>
#include <iomanip>
#include <fstream>
#include <vector>
#include <numeric>
#include <random>
#include <functional>
#include <memory>
#include <utility>
#include <algorithm>
#include <ctime>
#include <cstdlib>
#include <cmath>
#include <list>
#include <string>
#include <fstream>
#include <chrono>
#include <unordered_set>
#include <forward_list>
#include <assert.h>
#include <math.h>
#include <fenv.h>
#include <unistd.h>
#include <limits>
#include <gsl/gsl_rng.h>
#include <gsl/gsl_randist.h>
#include <gsl/gsl_sf.h>
#include <boost/math/special_functions/beta.hpp>
#include <boost/math/special_functions/factorials.hpp>
```

This code is used in chunk 23.

4.2 constants

the global constants

```
6 <constants 6> ≡      /*
Model 0: 1 ≤ α < 2 : epsilon = cN**(α - 2)(1(κ > 2) + (κ = 2)log N); /*      */
Model 1 : α = 1 : epsilon = constant
0 < α < 1 : epsilon = constant * N * (cutoff)**(alpha - 1) /*
const double dalpha = 0.01;
const long double ALPHA = static_cast<long double>(dalpha);
const long double KAPPA = 2.0L;
const long double RHO = 0.0L;
const double dgamma = 0.1;
const long double C_C = 1.0L;      /*
when KAPPA > 2: const long double MM = 1 + 2κ * κ/(κ - 1); /*      */
when KAPPA = 2: /*
const long double MM = (2.0L * M_PII * M_PII/3.0L) - 3.0L;
const long double CA = ((KAPPA + 2.0L) + (KAPPA * KAPPA))/2.0L;
const long double CB = powl(2.0L, KAPPA) * ((KAPPA - 2.0L) * (KAPPA - 1.0L));
const long double CKAPPA = 2.0L * (KAPPA > 2.0L ? CA/CB : 1.0L)/(MM * MM);
const long double GAMMA = static_cast<long double>(dgamma);
const long double BETA = static_cast<long double>(gsl_sf_beta(2. - dalpha,
dalpha) * (dgamma < 1. ? gsl_sf_beta_inc(2. - dalpha, dalpha, dgamma) : 1.));
const long double CKAG = (CKAPPA + ((ALPHA * C_C * powl(2.0L, ALPHA)) * BETA / powl(MM,
ALPHA)/4.0L));      /*
SAMPLE_SIZE is number of diploid individuals sampled */
const unsigned int SAMPLE_SIZE = 100;
const int EXPERIMENTS = 2500;
```

This code is used in chunk 23.

4.3 random number generators

the random number generators

```
7 <rngs 7> ≡
    std::random_device randomseed;      /* 
        Standard mersenne twister random number engine */
    std::mt19937_64 rng(randomseed());
    gsl_rng *rngtype;
    static void setup_rng(const unsigned long int s)
    {
        const gsl_rng_type*T;
        gsl_rng_env_setup();
        T = gsl_rng_default;
        rngtype = gsl_rng_alloc(T);
        gsl_rng_set(rngtype, s);
    }
```

This code is used in chunk 23.

4.4 compute e^x checking for under and overflow

```
compute  $e^x$  checking for under and overflow
1 long double veldi( const long double v )
2 {
3     feclearexcept(FE_ALL_EXCEPT);
4
5     const long double svar = expl( v ) ;
6
7     if( fetestexcept( FE_OVERFLOW ) != 0 ? LDBL_MAX :
8         fetestexcept(FE_UNDERFLOW) != 0 ? 0.0L : svar ) ;
9
10 }  

11 {  

12     <math>\langle e^x \text{ with checks} \rangle \equiv  

13     long double veldi(const long double v)  

14     {  

15         feclearexcept(FE_ALL_EXCEPT);  

16         const long double svar = expl(v);  

17         return (fetestexcept(FE_OVERFLOW) != 0 ? LDBL_MAX : (fetestexcept(FE_UNDERFLOW) != 0 ?  

18                 0.0L : svar));  

19     }  

20 }
```

This code is used in chunk 23.

4.5 number of collisions

get the factorial number of collisions $\prod_{j=2}^n (\sum_i \mathbb{1}_{\{k_i=j\}})!$

```
9 <numbercollisions 9> ≡
  static long double numbercollisions ( const std::vector< unsigned int > &__k ) { /*
    __k is in ascending order */
  assert ( std::all_of ( __k.cbegin(), __k.cend(), [] (const auto x)
  {
    return x > 1;
  }
  ) );
  double l
  {}
  ;
  switch ( __k.size() ) {
  case 1:
  {
    l = 1;
    break;
  }
  case 2:
  {
    assert( __k[0] ≤ __k[1]);
    l = ( __k[0] ≡ __k[1] ? 2 : 1);
    break;
  }
  case 3:
  {
    assert( __k[1] ≤ __k[2]);
    assert( __k[0] ≤ __k[1]);
    l = ( __k[0] ≡ __k[2] ? 6 : ( __k[0] ≡ __k[1] ? 2 : ( __k[1] ≡ __k[2] ? 2 : 1)));
    break;
  }
  case 4:
  {
    assert( __k[2] ≤ __k[3]);
    assert( __k[1] ≤ __k[2]);
    assert( __k[0] ≤ __k[1]);
    l = ( __k[0] ≡ __k[3] ? 24 : ( __k[0] ≡ __k[2] ? 6 : (( __k[0] ≡ __k[1] ? ( __k[2] ≡
      __k[3] ? 4 : 2) : ( __k[1] ≡ __k[3] ? 6 : ( __k[1] ≡ __k[2] ? 2 : ( __k[2] ≡ __k[3] ?
        2 : 1))))));
    break;
  }
  default: break;
}
assert(l > DBL_EPSILON);
return static_cast<long double>(l); }
```

This code is used in chunk 23.

4.6 multinomial constant

compute the log of the multinomial constant

$$\binom{m}{k_1 \dots k_r s} \frac{1}{\prod_{j=2}^n (\sum_i 1_{\{k_i=j\}})!}$$

```
10  <multinomial constant 10> ≡
long double multinomialconstant (const unsigned int m, const std::vector<unsigned
int >&v__k ) {
const long double s = lgammal(static_cast<long
double>(1 + m - std::accumulate(v__k.cbegin(), v__k.cend(), 0)));
const long double d = static_cast<long double> ( std::accumulate
(v__k.cbegin(), v__k.cend(), 0.0L, [] (long double a, const auto x)
{
    return a + lgammal(static_cast<long double>(x + 1));
}) );
/* numbercollisions § 4.5 */
return (lgammal(static_cast<long double>(m + 1)) - d - s - logl(numbercollisions(v__k)));
}
```

This code is used in chunk 23.

4.7 the (incomplete) beta function

return the logarithm of the (incomplete) beta function; when complete it is of course $\log \Gamma(a) + \log \Gamma(b) - \log \Gamma(a+b)$

11 ⟨ the beta function 11 ⟩ ≡

```
static long double betafunc(const double x, const double a, const double b)
{
    /* the GSL incomplete beta function is normalised by the complete beta function */
    /*
     * 0 < x <= 1 is the cutoff point */
    assert(x > 0.);
    assert(a > 0.);
    assert(b > 0.);      /*
        the standard way would be gsl_sf_beta_inc(a, b, x) * gsl_sf_beta(a, b) */

const long double f = static_cast<long double>((x < 1. ? gsl_sf_hyperg_2F1(a + b, 1,
    a + 1., x) : 1.));
assert(LDBL_EPSILON < f);      /*
    return 1_{\{x < 1\}}(\log f + a \log x + b \log(1-x) - \log a) + 1_{\{x = 1\}}(\log \Gamma(a) + \log \Gamma(b) - \log \Gamma(a+b))
*/
return (x < 1. ? (logl(f) + (static_cast<long double>((a * log(x)) + (b * log(1-x)) - log(a))) :
    (lgamma(static_cast<long double>(a)) + lgamma(static_cast<long
    double>(b)) - lgamma(static_cast<long double>(a + b))));
```

}

This code is used in chunk 23.

4.8 read merger sizes

read in merger sizes summing to a given number; the merger sizes are available in the files

`Q_<m>_.txt`

where `m` is the given number

12 $\langle \text{readmergersizes } 12 \rangle \equiv$

```
static void readmergersizes (const unsigned int n, const unsigned int j, std::vector <
    unsigned int > &v__mergers ) {
    std::ifstream f("Q_" + std::to_string(n) + ".txt");
    std::string line { };
    ;
    v__mergers.clear();
    for (unsigned int i = 0; std::getline (f, line) & i < j; ++i) { if (i ≥ j - 1) {
        std::stringstream ss (line);
        v__mergers = std::vector < unsigned int > ( std::istream_iterator < unsigned int > (ss),
        {} );
        assert(v__mergers.size() > 0);
        assert ( std::all_of (v__mergers.cbegin(), v__mergers.cend(), [](const auto &x)
        {
            return x > 1;
        })
        );
        f.close();
    }
}
```

This code is used in chunk 23.

4.9 $\lambda_{n;k_1,\dots,k_r;s}$

compute $\lambda_{n;k_1,\dots,k_r;s} (1)$

13 $\langle \lambda_{n;k_1,\dots,k_r;s} 13 \rangle \equiv$

```

long double lambdanks (const unsigned int m, const std::vector < unsigned
int > &v__k) { /*  

    r is number of simultaneous mergers */  

const unsigned int r = v__k.size(); /*  

    k = k1 + ⋯ + kr */  

const unsigned int k = std::accumulate(v__k.cbegin(), v__k.cend(), 0);  

assert(k > 1);  

assert(k ≤ m);  

const unsigned int s = m - k; /*  

    log(x/y) = log Γ(x+1) - log Γ(y+1) - log Γ(x-y+1) */  

auto logchoose = [](const unsigned int x, const unsigned int y)  

{  

    return (lgamma(static_cast<long double>(x+1)) - lgamma(static_cast<long  

double>(y+1)) - lgamma(static_cast<long double>(x-y+1)));  

}  

;  

long double l = 0.0_L; /*  

    (4)p ≡ 4(4-1)⋯(4-p+1) and (4)0 ≡ 1 */  

auto ff = [](const unsigned int p)  

{  

    return (p > 0 ? static_cast<long double>(boost::math::falling_factorial(4.0,p)) : 1.0_L);  

}  

;  

for (std::size_t ell = 0; ell ≤ (s < 4 - r ? s : 4 - r); ++ell) {  

    l += veldi(logchoose(s, ell) + ff(static_cast<long double>(r + ell)) + betafunc(GAMMA,  

        static_cast<long double>(k + ell) - ALPHA,  

        static_cast<long double>(m - k - ell) + ALPHA) - (logl(4.0_L) * static_cast<long  

double>(k + ell)));  

}  

l *= (ALPHA * C_C) * powl(2.0_L, ALPHA);  

l /= (CKAG * powl(MM, ALPHA));  

l += (k < 3 ? CKAPPA / CKAG : 0.0_L);  

assert(l > 0.0_L);  

return (veldi(logl(l) + multinomialconstant(m, v__k))); }
```

This code is used in chunk 23.

4.10 rates for a given number of blocks

read in all the possible merger sizes when a given number of blocks and compute the rate (1)

14 \langle rates when given number of blocks 14 $\rangle \equiv$

```

void allrates_when_n_blocks (const unsigned int n, std::vector < long double > &v__lambdan, std::vector < long double > &v__rates) {
    std::vector < unsigned int > v__k
    {}
    ;
    v__k.clear(); std::string line {}
    ;
    std::ifstream f("Q_" + std::to_string(n) + ".txt");
    assert(f.is_open());
    std::stringstream ss
    {}
    ;
    long double l
    {}
    ;
    v__rates.clear();
    while ( std::getline (f, line) ) {
        ss = std::stringstream ( line );
        v__k = std::vector < unsigned int > ( std::istream_iterator < unsigned int > (ss),
        {} );
        assert(v__k.size() > 0);
        assert ( std::all_of (v__k.cbegin(), v__k.cend(), [](const auto &x)
        {
            return x > 1;
        }
        ) ); /* compute  $\lambda_{n;k_1,\dots,k_r;s}$  (1) § 4.9 */
        l = lambdanks(n, v__k);
        v__lambdan[n] += l;
        v__rates.push_back(l); }
        f.close(); }
```

This code is used in chunk 23.

4.11 order the rates in descending order

order the rates in descending order for generating the cmf for sampling merger sizes

```
1 static void order_rates( const std::vector< long double > &
2   v__rates_for_sorting, std::vector< unsigned int > & v__indx )
3 {
4   assert( v__rates_for_sorting.size() > 0 ) ;
5   v__indx.clear();
6   v__indx.resize( v__rates_for_sorting.size() ) ;
7   std::iota( v__indx.begin(), v__indx.end(), 0 ) ;
8   std::stable_sort(v__indx.begin(), v__indx.end(), [&v__rates_for_sorting](const
9   unsigned int x, const unsigned int y){return v__rates_for_sorting[x] >
10  v__rates_for_sorting[y];});
11 }
```

15 ⟨order rates 15⟩ ≡

```
static void order_rates ( const std::vector < long double > &v__rates_for_sorting,
    std::vector < unsigned int > &v__indx ) {
    assert(v__rates_for_sorting.size() > 0);
    v__indx.clear();
    v__indx.resize(v__rates_for_sorting.size());
    std::iota(v__indx.begin(), v__indx.end(), 0); std::stable_sort (v__indx.begin(),
        v__indx.end(), [&v__rates_for_sorting](const unsigned int x, const unsigned
        int y)
    {
        return v__rates_for_sorting[x] > v__rates_for_sorting[y];
    }
} ; }
```

This code is used in chunk 23.

4.12 generate cmf

```

generate cmf for a given set of merger sizes
1 static void generate_cmf_n_blocks( const long double lambdan,    const
2 std::vector<long double>& v__rates, const std::vector<unsigned int> & v__indx,
3 std::vector<long double>& v__cmf )
4 {
5     v__cmf.clear();
6     v__cmf.resize( v__rates.size() );
7     v__cmf[0] = v__rates[v__indx[0]] / lambdan ;
8     for( unsigned int i = 1; i < v__indx.size(); ++i)
9     {
10        v__cmf[i] = v__cmf[ i-1 ] + ( v__rates[ v__indx[i] ] / lambdan );
11    }
12    assert( fabsl(v__cmf.back() - 1.0L) < 1.0e-9L ) ;
13 }

16 <generate cmf 16> ≡
static void generate_cmf_n_blocks (const long double lambdan, const std::vector < long
double > &v__rates, const std::vector < unsigned int > &v__indx, std::vector
< long double > &v__cmf )
{
    /*  

     * lambdan is the total merger rate */  

    v__cmf.clear();  

    v__cmf.resize(v__rates.size());  

    v__cmf[0] = v__rates[v__indx[0]]/lambdan;  

    for (unsigned int i = 1; i < v__indx.size(); ++i) {  

        v__cmf[i] = v__cmf[i - 1] + (v__rates[v__indx[i]]/lambdan);  

    }  

    assert(fabsl(v__cmf.back() - 1.0L) < 1.0 · 10-9 L);
}

```

This code is used in chunk 23.

4.13 compute all merger rates

compute the rate (1) for all possible mergers

17 $\langle \text{all rates } 17 \rangle \equiv$

```
void allrates ( std::vector < long double > &v__lambdan, std::vector < std::vector <
    unsigned int >> &a__indx, std::vector < std::vector < long double >> &a__cmfs ) {
    std::vector < std::vector < long double >> a__rates (SAMPLE_SIZE + SAMPLE_SIZE + 1,
    std::vector < long double > { } ) ; /* SAMPLE_SIZE § 4.2 is number of diploid individuals sampled */
    for (unsigned int m = 2; m ≤ SAMPLE_SIZE + SAMPLE_SIZE; ++m) { /* § 4.10 */
        allrates_when_n_blocks(m, v__lambdan, a__rates[m]);
        assert(v__lambdan[m] > LDBL_EPSILON); /* § 4.11 */
        order_rates(a__rates[m], a__indx[m]); /* § 4.12 */
        generate_cmfn_blocks(v__lambdan[m], a__rates[m], a__indx[m], a__cmfs[m]);
        a__rates[m].resize(0);
        std::vector < long double > ( ).swap(a__rates[m]); } /* clean up */
    for (auto &v:a__rates) { v.resize(0); std::vector < long double > ( ).swap(v); }
    std::vector < std::vector < long double >> ( ).swap(a__rates); }
```

This code is used in chunk 23.

4.14 get time until next merger

get the time

$$t = \mathbb{1}_{\{\rho>0\}} \frac{1}{\rho} \log(1 - \rho \log(1 - U) e^{-\rho s} / \lambda_n) + \mathbb{1}_{\{\rho=0\}} \text{Exp}(\lambda_n)$$

where U is a standard random uniform and s is the sum of the previous times

```

1 long double newtime(const long double lambdab, const long double oldtime)
2 {
3     return (RHO > LDBL_EPSILON ? log1pl(-(RHO * expl(-RHO * oldtime) / lambdab) \
+   ↪ *
4     logl(static_cast<long double>(gsl_rng_uniform_pos(rngtype))))/RHO :
5     static_cast<long double>(gsl_ran_exponential(rngtype, 1./lambdab))) ;
6 }
```

18 ⟨new time 18⟩ ≡

```

long double newtime(const long double lambdab, const long double oldtime)
{
    /*
        log1p(x) is log(1 + x) */
    return (RHO > LDBL_EPSILON ? log1pl(-(RHO * expl(-RHO * oldtime)/lambdab) *
        logl(static_cast<long double>(gsl_rng_uniform_pos(rngtype)))/RHO :
        static_cast<long double>(gsl_ran_exponential(rngtype, 1./lambdab)));
}
```

This code is used in chunk 23.

4.15 update lengths

update functionals $L_i(n)$ given current block sizes

19 $\langle \text{update lengths } 19 \rangle \equiv$

```
void update_lengths ( const std::vector<unsigned int> &v__tree, std::vector<long
                     double> &v__lengths, const long double t )
{
    for (unsigned int i = 0; i < v__lengths.size(); ++i) {
        v__lengths[0] += t;
        assert(v__tree[i] > 0);
        assert(v__tree[i] < SAMPLE_SIZE + SAMPLE_SIZE);
        v__lengths[v__tree[i]] += t;
    }
}
```

This code is used in chunk 23.

4.16 update block sizes

merge blocks and record the continuing ones

```

1 void update_tree( std::vector<unsigned int> & v__tree, const
2 std::vector<unsigned int>& merger_sizes )
3 {
4     assert( merger_sizes.size() > 0) ;
5     assert( static_cast<unsigned int>(std::accumulate( merger_sizes.cbegin(),
6 merger_sizes.cend(),0)) <= v__tree.size() );
7     std::shuffle( v__tree.begin(), v__tree.end(), rng );
8     std::vector<unsigned int > newblocks {};
9     newblocks.clear();
10    newblocks.reserve( merger_sizes.size() ) ;
11    assert( newblocks.size() < 1) ;
12    for( const auto &s : merger_sizes){
13        newblocks.push_back( std::accumulate( v__tree.cbegin(), v__tree.cbegin() + \
+ s,
14 0));
15    v__tree.resize( v__tree.size() - s) ; }

16    v__tree.insert( v__tree.end(), newblocks.cbegin(), newblocks.cend() );
17 }
20 {update block sizes 20} ≡
void update_tree ( std::vector < unsigned int > &v__tree, const std::vector < unsigned
int > &merger_sizes ) {
    assert(merger_sizes.size() > 0);
    assert(static_cast<unsigned int>(std::accumulate(merger_sizes.cbegin(),
merger_sizes.cend(),0)) ≤ v__tree.size());
    std::shuffle(v__tree.begin(),v__tree.end(),rng); std::vector < unsigned int >
newblocks
    {}
    ;
    newblocks.clear();
    newblocks.reserve(merger_sizes.size());
    assert(newblocks.size() < 1);
    for (const auto &s:merger_sizes)
    {
        newblocks.push_back(std::accumulate(v__tree.cbegin(),v__tree.cbegin() + s,0));
        v__tree.resize(v__tree.size() - s);
    }
    v__tree.insert(v__tree.end(),newblocks.cbegin(),newblocks.cend()); }
```

This code is used in chunk 23.

4.17 one experiment

```

1 void one_experiment( const std::vector<long double>& v__lambdan,
2 std::vector<long double>& v__lengths, const std::vector< std::vector< long
3 double >> & a__cmfs, const std::vector< std::vector< unsigned int > > &
4 a__indx, std::vector<long double>& v__ri )
5 {
6 std::vector< unsigned int > v__tree( 2*SAMPLE_SIZE, 1 );
7 std::fill( v__lengths.begin(), v__lengths.end(), 0.0L ) ;
8 unsigned int merger_lina {} ;
9 std::vector< unsigned int > merger_sizes {} ;
10 long double otimi {} ;
11 long double it {} ;

12 auto mlina = [] (const std::vector<long double>& f)
13 {
14 unsigned int j = 0;
15 const long double u = static_cast<long double>(gsl_rng_uniform(rngtype));
16 assert( u <= 1.0L );

17 while( u > f[j]) {++j; }

18 return j ;
};

19 merger_sizes.reserve(4);
20 while( v__tree.size() > 1)
21 {
22 it = newtime( v__lambdan[v__tree.size()], otimi );
23 otimi += it ;
24 update_lengths(v__tree, v__lengths, it);
25 merger_lina = mlina( a__cmfs[v__tree.size()] ) ;
26 readmergersizes( v__tree.size(), a__indx[v__tree.size()][merger_lina]+1,
27 merger_sizes);
28 update_tree( v__tree, merger_sizes ) ;
29 }
30 assert( v__tree.back() == (2*SAMPLE_SIZE));

31 assert( v__lengths[0] > LDBL_EPSILON) ;
32 const long double d = v__lengths[0];
33 std::transform( v__lengths.cbegin(), v__lengths.cend(), v__ri.begin(),
34 v__ri.begin(), [&d](const auto x, const auto y){return y + (x/d);});
35 }

21 {one experiment 21} ≡
void one_experiment ( const std::vector < long double > &v__lambdan, std::vector
< long double >&v__lengths, const std::vector < std::vector < long
double >> &a__cmfs, const std::vector < std::vector < unsigned
int >> &a__indx, std::vector < long double >&v__ri ) {
    std::vector < unsigned int > v__tree(2 * SAMPLE_SIZE, 1);
    std::fill(v__lengths.begin(), v__lengths.end(), 0.0L);
    unsigned int merger_lina
    { }
```

```

; std::vector < unsigned int > merger_sizes
{ }
;
long double otimi
{ }
;
long double it
{ }
; auto mlina = [] ( const std::vector < long double > &f )
{
    unsigned int j = 0;
    const long double u = static_cast<long double>(gsl_rng_uniform(rngtype));
    assert(u ≤ 1.0_L);
    while (u > f[j]) {
        ++j;
    }
    return j;
}
;
merger_sizes.reserve(4);
while (v__tree.size() > 1) { /* § 4.14 */
    it = newtime(v__lambdan[v__tree.size()], otimi);
    otimi += it; /* § 4.15 */
    update_lengths(v__tree, v__lengths, it);
    merger_lina = mlina(a__cmfs[v__tree.size()]); /* § 4.8 */
    readmergersizes(v__tree.size(), a__indx[v__tree.size()][merger_lina] + 1,
                    merger_sizes); /* § 4.16 */
    update_tree(v__tree, merger_sizes);
}
assert(v__tree.back() ≡ (2 * SAMPLE_SIZE));
assert(v__lengths[0] > LDBL_EPSILON);
const long double d = v__lengths[0];
std::transform (v__lengths.cbegin(), v__lengths.cend(), v__ri.begin(),
               v__ri.begin(), [&d](const auto x, const auto y)
{
    return y + (x/d);
} );
)
;

```

This code is used in chunk 23.

4.18 approximate

```

1 void approximate()
2 {
3     std::vector<long double> v__lambdan ((2*SAMPLE_SIZE) + 1);
4     std::vector< std::vector< long double >> a__cmfs ((2*SAMPLE_SIZE) + 1,
5     std::vector< long double > {} );
6     std::vector< std::vector< unsigned int >> a__indx ((2*SAMPLE_SIZE) + 1,
7     std::vector< unsigned int > {} );
8     std::vector<long double> v__lengths (2*SAMPLE_SIZE) ;
9     std::vector< long double> v__ri (2*SAMPLE_SIZE) ;
10    allrates(v__lambdan, a__indx, a__cmfs);
11    int r = EXPERIMENTS + 1;
12    while( --r > 0)
13    {one_experiment(v__lambdan, v__lengths, a__cmfs, a__indx, v__ri) ; }
14    for( const auto &z:v__ri)
15    {std::cout << z << '\n';}
16 }
22 ⟨go ahead – approximate  $\mathbb{E}[R_i(n)]$  22⟩ ≡
void approximate() {
    std::vector < long double > v__lambdan((2 * SAMPLE_SIZE) + 1);
    std::vector < std::vector < long double >> a__cmfs ((2 * SAMPLE_SIZE) + 1,
    std::vector < long double > {} ); std::vector < std::vector < unsigned
    int >> a__indx ((2 * SAMPLE_SIZE) + 1, std::vector < unsigned int > {} );
    std::vector < long double > v__lengths(2 * SAMPLE_SIZE); std::vector < long
    double > v__ri(2 * SAMPLE_SIZE); /*  

§ 4.13 */
    allrates(v__lambdan, a__indx, a__cmfs);
    int r = EXPERIMENTS + 1;
    while ( --r > 0) { /*  

§ 4.17 */
        one_experiment(v__lambdan, v__lengths, a__cmfs, a__indx, v__ri);
    }
    for (const auto &z:v__ri)
    {
        std::cout << z << '\n';
    }
}

```

This code is used in chunk 23.

4.19 main

the *main* module

```

23      /*
§ 4.1 */
⟨ includes 5 ⟩    /*
§ 4.2 */
⟨ constants 6 ⟩    /*
§ 4.3 */
⟨ rngs 7 ⟩    /*
§ 4.4 */
⟨  $e^x$  with checks 8 ⟩    /*
§ 4.5 */
⟨ numbercollisions 9 ⟩    /*
§ 4.6 */
⟨ multinomial constant 10 ⟩    /*
§ 4.7 */
⟨ the beta function 11 ⟩    /*
§ 4.8 */
⟨ readmergersizes 12 ⟩    /*
§ 4.9 */
⟨  $\lambda_{n;k_1,\dots,k_r;s}$  13 ⟩    /*
§ 4.10 */
⟨ rates when given number of blocks 14 ⟩    /*
§ 4.11 */
⟨ order rates 15 ⟩    /*
§ 4.12 */
⟨ generate cmf 16 ⟩    /*
§ 4.13 */
⟨ all rates 17 ⟩    /*
§ 4.14 */
⟨ new time 18 ⟩    /*
§ 4.15 */
⟨ update lengths 19 ⟩    /*
§ 4.16 */
⟨ update block sizes 20 ⟩    /*
§ 4.17 */
⟨ one experiment 21 ⟩    /*
§ 4.18 */
⟨ go ahead – approximate  $\mathbb{E}[R_i(n)]$  22 ⟩

int main(int argc, const char *argv[])
{
    /*
§ 4.3 */
    setup_rng(static_cast<std::size_t>(atoi(argv[1])));    /*
§ 4.18 */
    approximate();    /*
    rngtype § 4.3 */
    gsl_rng_free(rngtype);
    return GSL_SUCCESS;
}

```

5 conclusion and references

Figure 1 records an example approximation of $\mathbb{E}[R_i(n)]$ given the parameter values in § 4.2

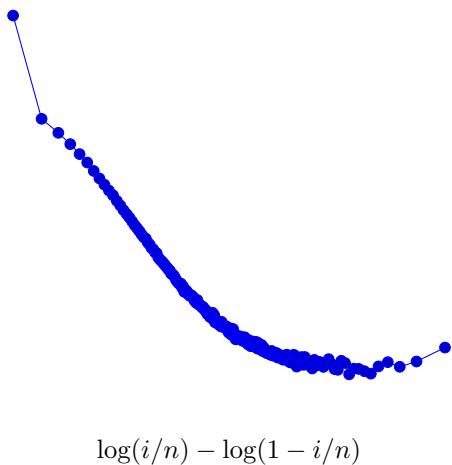


Figure 1: An example approximation of $\mathbb{E}[R_i(n)]$ for the given parameter values and graphed as logits against $\log(i/n) - \log(1 - i/n)$ for $i = 1, 2, \dots, n - 1$ where n is sample size

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