

Gene genealogies in diploid populations evolving according to sweepstakes reproduction

— approximating $\mathbb{E}[R_i(n)]$ for the time-changed Ω - δ_0 -Poisson-Dirichlet($\alpha, 0$)-coalescent

BJARKI ELDON ¹² 

Let $\#A$ denote the number of elements in a finite set A . For a given coalescent $\{\xi^n\}$ write $L_i(n) \equiv \int_0^{\tau(n)} \#\{\xi \in \xi^n(t) : \#\xi = i\} dt$ and $L(n) \equiv \int_0^{\tau(n)} \#\xi^n(t) dt$ where $\tau(n) \equiv \inf\{t \geq 0 : \#\xi^n(t) = 1\}$. Then $L(n) = L_1(n) + \dots + L_{n-1}(n)$. Write $R_i(n) \equiv L_i(n)/L(n)$ for $i = 1, 2, \dots, n-1$. With this C++ simulation code we approximate $\mathbb{E}[R_i(n)]$ when $\{\xi^n(G)\} \equiv \{\xi^n(G(t)) : t \geq 0\}$ is the time-changed continuous-time Ω - δ_0 -Poisson-Dirichlet($\alpha, 0$) coalescent with $0 < \alpha < 1$ and time-change function $G(t)$.

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¹beldon11@gmail.com

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1 Copyright

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2 compilation and output

This CWEB [KL94] document (the `.w` file) can be compiled with `cweave` to generate a `.tex` file, and with `ctangle` to generate a `.c` [KR88] C++ code file.

Use the shell tool `spix` on the script appearing before the preamble (the lines starting with `$$`); simply

```
spix /path/to/the/sourcefile
```

where `sourcefile` is the `.w` file

One may also copy the script into a file and run `parallel` [Tan11] :

```
parallel --gnu -j1 ::: /path/to/scriptfile
```

3 intro

Here we consider the continuous-time Ω - δ_0 -Poisson-Dirichlet($\alpha, 0$)-coalescent corresponding to exponential population growth where the time-change function $G(t) = \int_0^t e^{\rho s} ds$ for some fixed $\rho \geq 0$.

One samples the Ω - δ_0 -Poisson-Dirichlet($\alpha, 0$)-coalescent by sampling group sizes $2 \leq k_1, \dots, k_r \leq n$ with $\sum_i k_i \leq n$, $s = n - \sum_i k_i$, according to

$$\begin{aligned} \lambda_{n;k_1, \dots, k_r; s} &= \mathbb{1}_{\{r=1, k_1=2\}} \frac{C_\kappa}{C_\kappa + c(1-\alpha)} + \frac{c p_{n;k_1, \dots, k_r; s}}{C_\kappa + c(1-\alpha)} \\ C_\kappa &= \frac{2}{m_\infty^2} \left(\mathbb{1}_{\{\kappa=2\}} + \mathbb{1}_{\{\kappa>2\}} \frac{c_\kappa}{2^\kappa (\kappa-2)(\kappa-1)} \right) \\ p_{n;k_1, \dots, k_r; s} &= \frac{\alpha^{r+s-1} (r+s-1)!}{(n-1)!} \prod_{i=1}^r (k_i - 1 - \alpha)_{k_i-1} \end{aligned} \tag{1}$$

where $2 + \kappa < c_\kappa < \kappa^2$ when $\kappa > 2$. The blocks in each group are then split among four subgroups independently and uniformly at random, and the blocks in the same subgroup are merged.

Let $\#A$ denote the number of elements in a finite set A . For a given coalescent $\{\xi^n\}$ write $L_i(n) \equiv \int_0^{\tau(n)} \# \{\xi \in \xi^n(t) : \#\xi = i\} dt$ and $L(n) \equiv \int_0^{\tau(n)} \#\xi^n(t) dt$ where $\tau(n) \equiv \inf \{t \geq 0 : \#\xi^n(t) = 1\}$. Then $L(n) = L_1(n) + \dots + L_{n-1}(n)$. Write $R_i(n) \equiv L_i(n)/L(n)$ for $i = 1, 2, \dots, n-1$. With this C++ code we use simulations to approximate $\mathbb{E}[R_i(n)]$ when the coalescent is the time-changed continuous-time Ω - δ_0 -Poisson-Dirichlet($\alpha, 0$)-coalescent with time-change function $G(t) = \int_0^t e^{\rho s} ds$ for a given fixed $\rho \geq 0$.

The code follows in § 4.1–§ 4.17, we conclude in § 5. Comments within the code in **this font and color**

4 code

4.1 includes

the included libraries and header file with the global constants

```
5 <includes 5> ≡  
#include <iostream>  
#include <cstdlib>  
#include <iterator>  
#include <random>  
#include <fstream>  
#include <iomanip>  
#include <vector>  
#include <numeric>  
#include <functional>  
#include <algorithm>  
#include <cmath>  
#include <unordered_map>  
#include <assert.h>  
#include <float.h>  
#include <fenv.h>  
#include <gsl/gsl_rng.h>  
#include <gsl/gsl_randist.h>  
#include <gsl/gsl_math.h>  
#include <boost/math/special_functions/factorials.hpp>  
#include "headerfile.hpp"
```

This code is used in chunk 21.

4.2 random number generators

```
6  ⟨rngs 6⟩ ≡      /*
    define a random seed object */
    std::random_device randomseed;      /*
    Standard Mersenne twister random number engine */
    std::mt19937_64 rng(randomseed());
    gsl_rng * rngtype;
    static void setup_rng(unsigned long int s)
    {
        const gsl_rng_type *T;
        gsl_rng_env_setup();
        T = gsl_rng_default;
        rngtype = gsl_rng_alloc(T);
        gsl_rng_set(rngtype, s);
    }
```

This code is used in chunk 21.

4.3 e^x with checks

compute e^x checking for over- and underflow

7 $\langle e^x \rangle \equiv$

```
static double veldi(const double x, const double y)
{
    feclearexcept(FE_ALL_EXCEPT);
    const double d = pow(x, y);
    return (fetestexcept(FE_UNDERFLOW) ? 0. : (fetestexcept(FE_OVERFLOW) ? FLT_MAX : d));
}
```

This code is used in chunk 21.

4.4 $\lambda_{n;k_1,\dots,k_r;s}$

compute $\lambda_{n;k_1,\dots,k_r;s}$ (1)

8 $\langle \lambda_{n;k_1,\dots,k_r;s} (1) \ 8 \rangle \equiv$

```

static double lambdanks (const double n, const std::vector < unsigned int > &v_k ) {
    assert(v_k.size() > 0);
    assert ( std::all_of (v_k.cbegin(), v_k.cend(), [](const auto k)
    {
        return k > 1;
    }
    ) ) ;
    double d
    {}
    ;
    double k
    {}
    ;
    double f
    {1};
    const double r = static_cast<double>(v_k.size());
    std::unordered_map < unsigned int , unsigned int > counts
    {}
    ;
    /*
    (x)m ≡ x(x-1)⋯(x-m+1) */
    auto ff = [](const double x, const unsigned int m)
    {
        return static_cast<double>(boost::math::falling_factorial(x, m));
    }
    ;
    for (std::size_t i = 0; i < v_k.size(); ++i) {
        f *= ff(static_cast<double>(v_k[i]) - 1. - ALPHA, v_k[i] - 1);    /*
        count occurrence of each merger size */
        ++counts[v_k[i]];
        k += static_cast<double>(v_k[i]);
        d += lgamma(static_cast<double>(v_k[i] + 1));
    }
    assert(k < n + 1);
    const double s = n - k; const double p = static_cast<double> ( std::accumulate
        (counts.cbegin(), counts.cend(), 0, [](double a, const auto &x)
        {
            return a + lgamma((double) x.second + 1);
        }
        ) ) ;
    const double l = ((v_k.size() < 2 ? (v_k[0] < 3 ? 1. : 0) : 0) * CKAPPA) + (CEPS * veldi(ALPHA,
        r + s - 1) * tgamma(r + s) * f / tgamma(n));    /*
    § 4.3 */
    return (veldi(exp(1),
        (lgamma(n + 1.) - d) - lgamma(n - k + 1) - p) * l / (CKAPPA + (CEPS * (1 - ALPHA)))); }

```

This code is used in chunk 21.

4.5 generate group sizes

generate group sizes k_1, \dots, k_r summing to $myInt$

9 $\langle \text{sizes of groups } 9 \rangle \equiv$

```

static double GenPartitions (const unsigned int m, const unsigned int myInt, const
    unsigned int PartitionSize, unsigned int MinVal, unsigned int MaxVal,
    std::vector < std::pair < double , std::vector < unsigned int >>> &v_l_k, std::vector
    < double > &lrates_sorting ) {    /*
    m is the given number of blocks; the partitions sum to myInt */
double lrate
{}
;
double sumrates
{}
; std::vector < unsigned int > partition(PartitionSize);
unsigned int idx_Last = PartitionSize - 1;
unsigned int idx_Dec = idx_Last;
unsigned int idx_Spill = 0;
unsigned int idx_SpillPrev;
unsigned int LeftRemain = myInt - MaxVal - (idx_Dec - 1) * MinVal;
partition[idx_Dec] = MaxVal + 1;
do {
    unsigned int val_Dec = partition[idx_Dec] - 1;
    partition[idx_Dec] = val_Dec;
    idx_SpillPrev = idx_Spill;
    idx_Spill = idx_Dec - 1;
    while (LeftRemain > val_Dec) {
        partition[idx_Spill--] = val_Dec;
        LeftRemain -= val_Dec - MinVal;
    }
    partition[idx_Spill] = LeftRemain;
    const char a = (idx_Spill) ? ~((-3 >> (LeftRemain - MinVal)) << 2) : 11;
    const char b = (-3 >> (val_Dec - LeftRemain));
    switch (a & b) {
    case 1: case 2: case 3: idx_Dec = idx_Spill;
        LeftRemain = 1 + (idx_Spill - idx_Dec + 1) * MinVal;
        break;
    case 5:
        for (++idx_Dec, LeftRemain = (idx_Dec - idx_Spill) * val_Dec;
            (idx_Dec ≤ idx_Last) ∧ (partition[idx_Dec] ≤ MinVal); idx_Dec++)
            LeftRemain += partition[idx_Dec];
        LeftRemain += 1 + (idx_Spill - idx_Dec + 1) * MinVal;
        break;
    case 6: case 7: case 11: idx_Dec = idx_Spill + 1;
        LeftRemain += 1 + (idx_Spill - idx_Dec + 1) * MinVal;
        break;
    case 9:
        for (++idx_Dec, LeftRemain = idx_Dec * val_Dec;
            (idx_Dec ≤ idx_Last) ∧ (partition[idx_Dec] ≤ (val_Dec + 1));
            idx_Dec++) LeftRemain += partition[idx_Dec];
        LeftRemain += 1 - (idx_Dec - 1) * MinVal;
        break;
    }
}

```

```

case 10:
  for (LeftRemain += idx_Spill * MinVal + (idx_Dec - idx_Spill) * val_Dec + 1, ++idx_Dec;
      (idx_Dec ≤ idx_Last) ∧ (partition[idx_Dec] ≤ (val_Dec - 1)); idx_Dec++)
    LeftRemain += partition[idx_Dec];
  LeftRemain -= (idx_Dec - 1) * MinVal;
  break;
}
while (idx_Spill > idx_SpillPrev) partition[--idx_Spill] = MinVal;
assert(static_cast<unsigned int>(std::accumulate(partition.begin(), partition.end(),
    0)) ≡ myInt);
lrate = lambdanks(static_cast<double>(m), partition);
assert(lrate ≥ 0);
v_l_k.push_back(std::make_pair(lrate, partition));
rates_sorting.push_back(lrate);
sumrates += lrate;
} while (idx_Dec ≤ idx_Last);
assert(sumrates ≥ 0);
return sumrates; }

```

This code is used in chunk 21.

4.6 group sizes up to a given number

get all partitions summing to a given number

10 $\langle \text{sizes up to number } 10 \rangle \equiv$

```
static double allmergers_sum_m (const unsigned int n, const unsigned
    int m, std::vector < std::pair < double , std::vector < unsigned
    int >>> &v__l_k, std::vector < double > &v_lrates_sort ) {    /*
    n is number of blocks ; all partitions summing to  $m \leq n$  */
const std::vector < unsigned int > v__m
{m};    /*
    § 4.4 */
double sumr = lambdanks(static_cast<double>(n), v__m);
v__l_k.push_back(std::make_pair(sumr, v__m));
v_lrates_sort.push_back(sumr);
if (m > 3) {
    for (unsigned int s = 2; s ≤ m/2; ++s) {
        assert(m > 2 * (s - 1));    /*
            § 4.5 */
        sumr += GenPartitions(n, m, s, 2, m - (2 * (s - 1)), v__l_k, v_lrates_sort);
    }
}
assert(sumr ≥ 0);
return sumr; }
```

This code is used in chunk 21.

4.7 write partitions to files

write partitions to files `gg_n_.txt`

11 \langle partitions to files 11 $\rangle \equiv$

```
static void ratesmergersfile (const unsigned int n, const std::vector < unsigned
    int > &v___indx, const std::vector < std::pair < double , std::vector <
    unsigned int >>> &vlk, const double s, std::vector < std::vector <
    double >> &a___cmf ) {
    assert(s > 0);
    double cmf
    {}
    ;
    std::ofstream f;
    f.open("gg_" + std::to_string(n) + "__.txt", std::ios::app);
    a___cmf[n].clear();
    for (const auto &i:v___indx) { cmf += (vlk[i].first)/s;
    assert(cmf ≥ 0);
    a___cmf[n].push_back(cmf);
    assert((vlk[i].second).size() > 0);
    for (const auto &x:vlk[i].second)
    {
        f << x << ' ';
    }
    f << '\n'; }
    f.close();
    assert(abs(cmf - 1.) < 0.999999); }
```

This code is used in chunk 21.

4.8 partitions when given number of blocks

12 \langle all partitions given number of blocks 12 $\rangle \equiv$

```

static void allmergers_when_n_blocks (const unsigned int n, std::vector <
    double > &v__lambdan, std::vector < std::vector < double >> &a__cmf ) {
    std::vector < std::pair < double , std::vector < unsigned int >>> vlk
    {}
    ; std::vector < double > ratetosort
    {}
    ;
    ratetosort.clear();
    double lambdan
    {}
    ;
    vlk.clear();
    assert(n > 1);
    for (unsigned int k = 2; k ≤ n; ++k) { /*
        the partition sums to k; the number of blocks is n; § 4.6 */
        lambdan += allmergers_sum_m(n, k, vlk, ratetosort);
    } /*
        record the total rate when n blocks; use for sampling time */
    assert(lambdan > 0);
    v__lambdan[n] = lambdan;
    std::vector < unsigned int > indx(ratetosort.size());
    std::iota(indx.begin(), indx.end(), 0);
    std::stable_sort (indx.begin(), indx.end(), [&ratetosort](const unsigned int x, const
        unsigned int y)
    {
        return ratetosort[x] > ratetosort[y];
    }
    ); /*
        § 4.7 */
    ratesmergersfile(n, indx, vlk, v__lambdan[n], a__cmf); }

```

This code is used in chunk 21.

4.9 all partitions

generate all partitions and rates

13 \langle partitions generate all 13 $\rangle \equiv$

```
static void all_partitions_rates ( std::vector < double > &vlmn, std::vector < std::vector  
    < double >> &acmf )  
{  
    for (unsigned int tmpn = 2; tmpn ≤ SAMPLE_SIZE; ++tmpn) {      /*  
        § 4.8 */  
        allmergers_when_n_blocks(tmpn, vlmn, acmf);  
    }  
}
```

This code is used in chunk 21.

4.10 read in partitions

read in partitions k_1, \dots, k_r from files `g_<m>_.txt`

14 `<partitions read in 14> ≡`

```
static void readmersizes (const unsigned int n, const unsigned int j, std::vector <
    unsigned int > &v__mers) {
    std::ifstream f("gg_" + std::to_string(n) + "_.txt");
    std::string line {}
    ;
    v__mers.clear();
    for (unsigned int i = 0; std::getline (f, line) ∧ i < j; ++i) { if (i ≥ j - 1) {
    std::stringstream ss (line) ;
    v__mers = std::vector < unsigned int > ( std::istream_iterator < unsigned int > (ss),
    {} ) ; } }
    assert(v__mers.size() > 0);
    assert ( std::all_of (v__mers.cbegin(), v__mers.cend(), [](const auto &x)
    {
        return x > 1;
    }
    ) ) ;
    f.close(); }
```

This code is used in chunk 21.

4.11 split partitions

```

1 void split_blocks(const unsigned int k, std::vector<unsigned int>&
2 split_partition )
3 {
4     auto sample_box = [] (void)
5     { const double u = gsl_rng_uniform(rngtype);
6     return static_cast<unsigned int>(u < 0.25 ? 0 : ( u < .5 ? 1 : ( u < .75 ? 2 :
7     3))) ;
8     } ;

9     std::vector<unsigned int> boxes (4);
10    for( unsigned int j = 0; j < k; ++j)
11    {
12        ++boxes[ sample_box() ];
13    }
14    assert( static_cast<unsigned int>( std::accumulate(boxes.cbegin(),
15    boxes.cend(), 0)) == k);

16    const auto t = static_cast<std::size_t>( std::count_if( boxes.cbegin(),
17    boxes.cend(), [](const auto b){ return b > 1;}));

18    if(t > 0){
19        std::copy_if(boxes.cbegin(), boxes.cend(), \
+ ↪ std::back_inserter(split_partition),
20        [](const auto b){return b > 1;});
21    }
22    }

15 <split partition into boxes 15> ≡
    void split_blocks (const unsigned int k, std::vector < unsigned int > &split_partition ) {
        auto sample_box = [] (void)
        {
            const double u = gsl_rng_uniform(rngtype);
            return static_cast<unsigned int>(u < 0.25 ? 0 : (u < .5 ? 1 : (u < .75 ? 2 : 3)));
        }
        ;
        std::vector < unsigned int > boxes(4);
        for (unsigned int j = 0; j < k; ++j) {
            ++boxes[sample_box()];
        }
        assert(static_cast<unsigned int>(std::accumulate(boxes.cbegin(), boxes.cend(), 0)) ≡ k);
        /*
            count how many boxes have at least 2 blocks */
        const auto t = static_cast<std::size_t> ( std::count_if (boxes.cbegin(), boxes.cend(),
            [](const auto b)
            {
                return b > 1;
            }
            ) ) ;
        if (t > 0) { std::copy_if (boxes.cbegin(), boxes.cend(), std::back_inserter(split_partition),
            [](const auto b)
            {

```

```
    return  $b > 1$ ;  
  }  
); } }
```

This code is used in chunk 21.

4.12 split groups in partition

given a partition, return with each group split into boxes

```
1 void split_groups( const std::vector< unsigned int>& partition,
2   std::vector<unsigned int>& split_partition )
3 {
4   split_partition.clear();
5   for( const auto &k:partition){
6     split_blocks( k,  split_partition ) ;
7   }
8
9   assert( std::accumulate(split_partition.cbegin(), split_partition.cend(),0) <=
10  std::accumulate(partition.cbegin(), partition.cend(),0));
11 }
12
13
14
15
16  $\langle$ separate entire partition 16 $\rangle \equiv$ 
    void split_groups ( const std::vector < unsigned int > &partition, std::vector < unsigned
      int > &split_partition ) {
      split_partition.clear();
      for (const auto &k:partition)
      {
        /*
          § 4.11 */
        split_blocks(k, split_partition);
      }
      assert(std::accumulate(split_partition.cbegin(), split_partition.cend(),
        0) ≤ std::accumulate(partition.cbegin(), partition.cend(),0)); }
This code is used in chunk 21.
```

4.13 merge blocks

merge blocks and record continuing ones

```
1 static void update_tree( std::vector<unsigned int>& tree, const
2 std::vector<unsigned int>& mergers )
3 {
4     assert( mergers.size() > 0 );
5
6     assert( static_cast<std::size_t>(std::accumulate(mergers.cbegin(),
7     mergers.cend(), 0)) <= tree.size() ) ;
8     std::shuffle( tree.begin(), tree.end(), rng ) ;
9     std::vector<unsigned int> newblocks (mergers.size());
10    std::size_t j {} ;
11
12    for( const auto &m: mergers ){
13        newblocks[j] = std::accumulate( std::crbegin(tree), std::crbegin(tree) + m, \
14        + ↪ 0);
15        tree.resize( tree.size() - m ) ;
16        ++j ;
17    }
18    tree.reserve( tree.size() + newblocks.size() );
19    tree.insert( tree.end(), newblocks.cbegin(), newblocks.cend() );
20 }
```

17 <update tree 17> ≡

```
static void update_tree ( std::vector < unsigned int > &tree, const std::vector <
    unsigned int > &mergers ) {
    assert(mergers.size() > 0);
    assert(static_cast<std::size_t>(std::accumulate(mergers.cbegin(), mergers.cend(),
        0)) ≤ tree.size());
    std::shuffle(tree.begin(), tree.end(), rng);
    std::vector < unsigned int > newblocks(mergers.size());
    std::size_t j
    {}
    ;
    for (const auto &m:mergers)
    {
        newblocks[j] = std::accumulate(std::crbegin(tree), std::crbegin(tree) + m, 0);
        tree.resize(tree.size() - m);
        ++j;
    }
    tree.reserve(tree.size() + newblocks.size());
    tree.insert(tree.end(), newblocks.cbegin(), newblocks.cend()); }
```

This code is used in chunk 21.

4.14 sample time and partitions until a merger occurs

```

1  static double until_merger(const std::size_t current_number_blocks, const
2  std::vector<double>& v_lambdan, const std::vector<double>& v__cmf,
3  std::vector<unsigned int>& v_merger_sizes, double &otimi)
4  {
5  std::vector<unsigned int> v_partition (SAMPLE_SIZE/2);
6  v_partition.reserve(SAMPLE_SIZE/2) ;

7  unsigned int lina {} ;
8  double t {};

9  auto newtime = [](const double l, const double otime){
10 return (RHO > DBL_EPSILON ? log1p( -(RHO * exp(- RHO * otime) / l) *
11 log(gsl_rng_uniform_pos(rngtype)))/RHO : gsl_ran_exponential(rngtype, 1./l)) ;
12 };

13 auto samplemerger = [&v__cmf](void){

14 unsigned int j {} ;
15 const double u = gsl_rng_uniform( rngtype);

16 while( u > v__cmf[j]){ ++j; }

17 return j ;
18 } ;

19 v_merger_sizes.clear() ;
20 while( std::all_of(v_merger_sizes.cbegin(), v_merger_sizes.cend(), [](const
21 auto m){ return m < 2;}))
22 {
23 t += newtime( v_lambdan[current_number_blocks], otimi);
24 otimi += t ;
25 lina = samplemerger();
26 readmergersizes( current_number_blocks, 1 + lina, v_partition) ;
27 split_groups( v_partition, v_merger_sizes) ;
28 }

29 assert( v_merger_sizes.size() > 0) ;
30 assert( std::all_of(v_merger_sizes.cbegin(), v_merger_sizes.cend(), [](const
31 auto m){ return m > 1;}));
32 return t ;
33 }

18 <until merger 18> ≡
    static double until_merger ( const std ::size_t current_number_blocks, const
                                std::vector < double > &v_lambdan, const std::vector <
                                double > &v__cmf, std::vector < unsigned int > &v_merger_sizes,
                                double &otimi ) {
                                std::vector < unsigned int > v_partition(SAMPLE_SIZE/2);
                                v_partition.reserve(SAMPLE_SIZE/2);
                                unsigned int lina
                                {}
                                ;

```

```

double t
{}
; /*
    new time  $t = \log(1 - \log(1 - U)/\phi) / \rho$ ,  $\phi = \lambda_n \exp(\rho s) / \rho$ ,  $s$  is cumulative
    time */
auto newtime = [] (const double l, const double otime)
{
    return (RHO > DBL_EPSILON ? log1p(-(RHO * exp(-RHO *
        otime)/l) * log(gsl_rng_uniform_pos(rngtype)))/RHO :
        gsl_ran_exponential(rngtype, 1./l));
}
;
auto samplemerger = [&v___cmf](void)
{
    unsigned int j
    {}
    ;
    const double u = gsl_rng_uniform(rngtype);
    while (u > v___cmf[j]) {
        ++j;
    }
    return j;
}
;
v_merger_sizes.clear(); while ( std::all_of (v_merger_sizes.cbegin(),
    v_merger_sizes.cend(), [] (const auto m)
    {
        return m < 2;
    }
    ) )
{
    t += newtime(v_lambdan[current_number_blocks], otime);
    otime += t;
    lina = samplemerger(); /*
        § 4.10 */
    readmergersizes(current_number_blocks, 1 + lina, v_partition); /*
        § 4.12 */
    split_groups(v_partition, v_merger_sizes);
}
assert(v_merger_sizes.size() > 0);
assert ( std::all_of (v_merger_sizes.cbegin(), v_merger_sizes.cend(),
    [] (const auto m)
    {
        return m > 1;
    }
    ) ) ;
return t; }

```

This code is used in chunk 21.

4.15 one experiment

```

1  static void one_experiment( std::vector<double>& v_l, std::vector<double>& \
+  ↪ v_r,
2  const std::vector<double>& v__lambdan, const std::vector< std::vector< double \
+  ↪ >
3  >& a__cmf)
4  {

5  std::vector<unsigned int> tree (SAMPLE_SIZE, 1) ;
6  tree.reserve(SAMPLE_SIZE) ;
7  std::fill(v_l.begin(), v_l.end(),0);

8  double t {};
9  double ot {} ;

10 std::vector<unsigned int> v__merger_sizes {};
11 v__merger_sizes.clear() ;
12 v__merger_sizes.reserve(2*SAMPLE_SIZE) ;
13 while( tree.size() > 1)
14 {

15 t = until_merger(tree.size(), v__lambdan, a__cmf[tree.size()], \
+  ↪ v__merger_sizes,
16 ot);
17 update_lengths(tree, v_l, t);
18 update_tree(tree, v__merger_sizes);
19 }
20 assert( tree.back() == SAMPLE_SIZE);
21 update_ri( v_r, v_l);
22 }

19 ⟨ a single experiment 19 ⟩ ≡
    static void one_experiment ( std::vector < double > &v_l, std::vector < double > &v_r,
        const std::vector < double > &v__lambdan, const std::vector < std::vector
        < double >> &a__cmf ) {

        std::vector < unsigned int > tree(SAMPLE_SIZE,1);
        tree.reserve(SAMPLE_SIZE);
        std::fill(v_l.begin(), v_l.end(),0);
        double t
        {}
        ;
        double ot
        {}
        ;
        std::vector < unsigned int > v__merger_sizes
        {}
        ;
        v__merger_sizes.clear();
        v__merger_sizes.reserve(2 * SAMPLE_SIZE);
        auto update_lengths = [&tree,&v_l](const double t) { for (const auto &b:tree)
            {
                v_l[0] += t;

```

```

    v_l[b] += t;
}
};
while (tree.size() > 1) { /*
    § 4.14 */
    t = until_merger(tree.size(), v__lambdan, a__cmf[tree.size()],
        v__merger_sizes, ot);
    update_lengths(t);
    update_tree(tree, v__merger_sizes);
}
assert(tree.back() == SAMPLE_SIZE);
const double d = v_l[0];
std::transform (v_l.cbegin(), v_l.cend(), v_r.begin(), v_r.begin(), [&d](const
    auto &x, const auto &y)
{
    return y + (x/d);
}) ; }

```

This code is used in chunk 21.

4.16 approximate $\mathbb{E}[R_i(n)]$

approximate $\mathbb{E}[R_i(n)]$

```

1  static void approximate_eri()
2  {
3      std::vector<double> vri (SAMPLE_SIZE) ;
4      vri.reserve(SAMPLE_SIZE) ;

5      std::vector<double> v__l (SAMPLE_SIZE);
6      v__l.reserve(SAMPLE_SIZE) ;

7      std::vector<double> v__lambdan (SAMPLE_SIZE + 1) ;
8      v__lambdan.reserve(SAMPLE_SIZE + 1) ;

9      std::vector< std::vector< double > > a__cmfs (SAMPLE_SIZE + 1,
10     std::vector<double> {} ) ;

11     all_partitions_rates(v__lambdan, a__cmfs );

12     int r = EXPERIMENTS + 1 ;

13     while( --r > 0)
14     {
15         one_experiment(v__l,  vri, v__lambdan,  a__cmfs);
16     }

17     for( std::size_t i = 1; i < SAMPLE_SIZE; ++i){
18         std::cout << (log( vri[i]/static_cast<double>(EXPERIMENTS)) - log1p(-
19         vri[i]/static_cast<double>(EXPERIMENTS))) << '\n'; }
20     }

```

20 \langle go ahead – approximate $\mathbb{E}[R_i(n)]$ 20 $\rangle \equiv$

```

    static void approximate_eri() {
        std::vector< double > vri(SAMPLE_SIZE);
        vri.reserve(SAMPLE_SIZE); std::vector< double > v__l(SAMPLE_SIZE);
        v__l.reserve(SAMPLE_SIZE); std::vector< double > v__lambdan(SAMPLE_SIZE + 1);
        v__lambdan.reserve(SAMPLE_SIZE + 1); std::vector< std::vector< double >>
            a__cmfs (SAMPLE_SIZE + 1, std::vector< double > {} ) ;    /*
            § 4.9 */
        all_partitions_rates(v__lambdan, a__cmfs);
        int r = EXPERIMENTS + 1;
        while ( --r > 0) {    /*
            § 4.15 */
            one_experiment(v__l, vri, v__lambdan, a__cmfs);
        }
        for (std::size_t i = 1; i < SAMPLE_SIZE; ++i) {
            std::cout << (log(vri[i]/static_cast<double>(EXPERIMENTS)) -
                log1p(-vri[i]/static_cast<double>(EXPERIMENTS))) << '\n';
        }
    }

```

This code is used in chunk 21.

4.17 main

the *main* module

```
21      /*
      § 4.1 */
      <includes 5> /*
      § 4.2 */
      <rngs 6> /*
      § 4.3 */
      < $e^x$  7> /*
      § 4.4 */
      < $\lambda_{n;k_1,\dots,k_r;s}$  (1) 8> /*
      § 4.5 */
      <sizes of groups 9> /*
      § 4.6 */
      <sizes up to number 10> /*
      § 4.7 */
      <partitions to files 11> /*
      § 4.8 */
      <all partitions given number of blocks 12> /*
      § 4.9 */
      <partitions generate all 13> /*
      § 4.10 */
      <partitions read in 14> /*
      § 4.11 */
      <split partition into boxes 15> /*
      § 4.12 */
      <separate entire partition 16> /*
      § 4.13 */
      <update tree 17> /*
      § 4.14 */
      <until merger 18> /*
      § 4.15 */
      <a single experiment 19> /*
      § 4.16 */
      <go ahead – approximate  $\mathbb{E}[R_i(n)]$  20>
      int main(int argc, const char *argv[])
      {
      /*
      § 4.2 */
      setup_rng(static_cast<std::size_t>(atoi(argv[1]))); /*
      § 4.16 */
      approximate_eri();
      gsl_rng_free(rngtype);
      return 0;
      }
```

5 conclusions and bibliography

We approximate $\mathbb{E}[R_i(n)]$ when the coalescent is the time-changed Ω - δ_0 -Poisson-Dirichlet($\alpha, 0$)-coalescent with time-change function $G(t) = \int_0^t e^{\rho s} ds$ for a given fixed $\rho \geq 0$. Figure 1 example approximation of $\mathbb{E}[R_i(n)]$

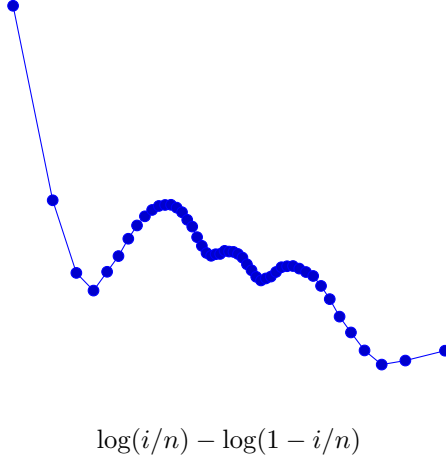


Figure 1: *An example approximation of $\mathbb{E}[R_i(n)]$ for the given parameter values and graphed as logits against $\log(i/n) - \log(1 - i/n)$ for $i = 1, 2, \dots, n - 1$ where n is sample size*

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- $\langle \lambda_{n;k_1,\dots,k_r;s} (1) \ 8 \rangle$ Used in chunk 21.
- $\langle e^x \ 7 \rangle$ Used in chunk 21.
- $\langle \text{a single experiment } 19 \rangle$ Used in chunk 21.
- $\langle \text{all partitions given number of blocks } 12 \rangle$ Used in chunk 21.
- $\langle \text{go ahead – approximate } \mathbb{E}[R_i(n)] \ 20 \rangle$ Used in chunk 21.
- $\langle \text{includes } 5 \rangle$ Used in chunk 21.
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- $\langle \text{sizes up to number } 10 \rangle$ Used in chunk 21.
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- $\langle \text{update tree } 17 \rangle$ Used in chunk 21.