CSE221

Lecture 19:
Directed Graphs and Graph Algorithms

Acknowledgment: The content of this file is based on the slides of the textbook as well as the slides provided in former lectures at UNIST.

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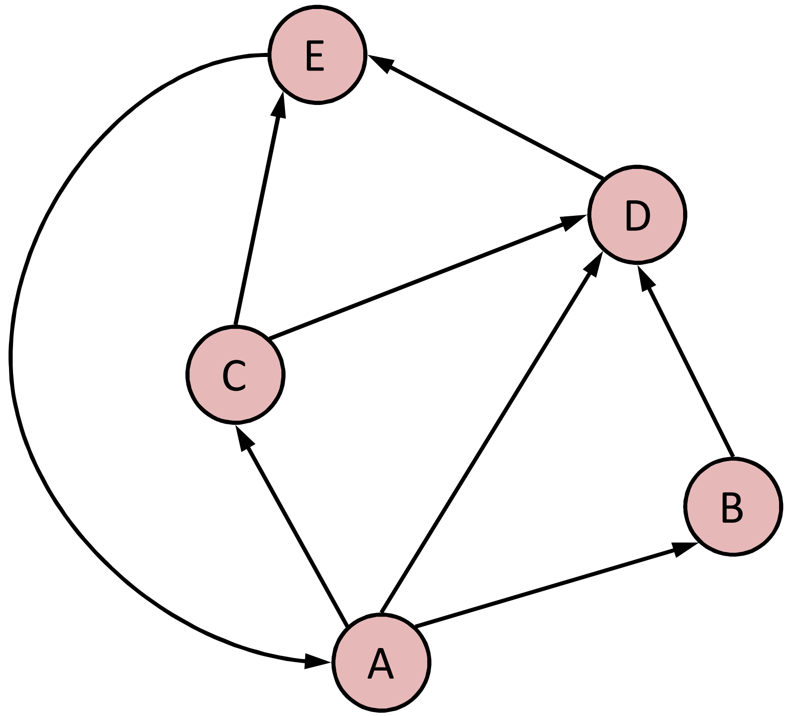
Outline

- Directed graphs
  - Digraph properties
  - Reachability
  - Topological sorting
- Shortest path algorithms

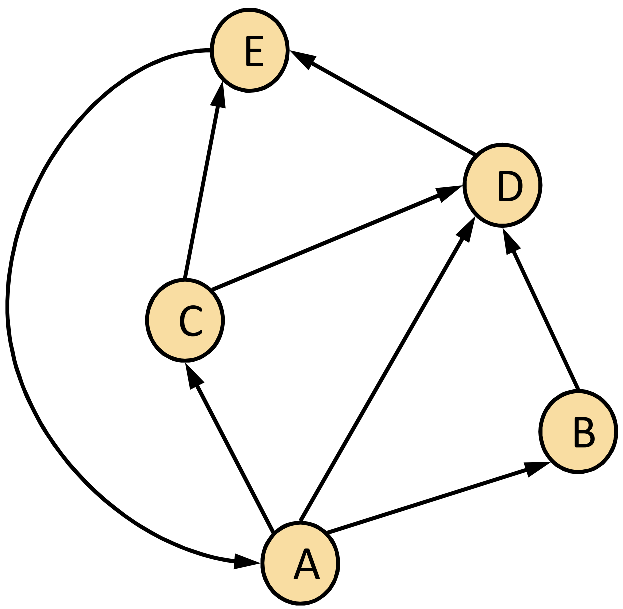
Digraphs

- A digraph is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - flights
  - one-way streets
  - work decompositions



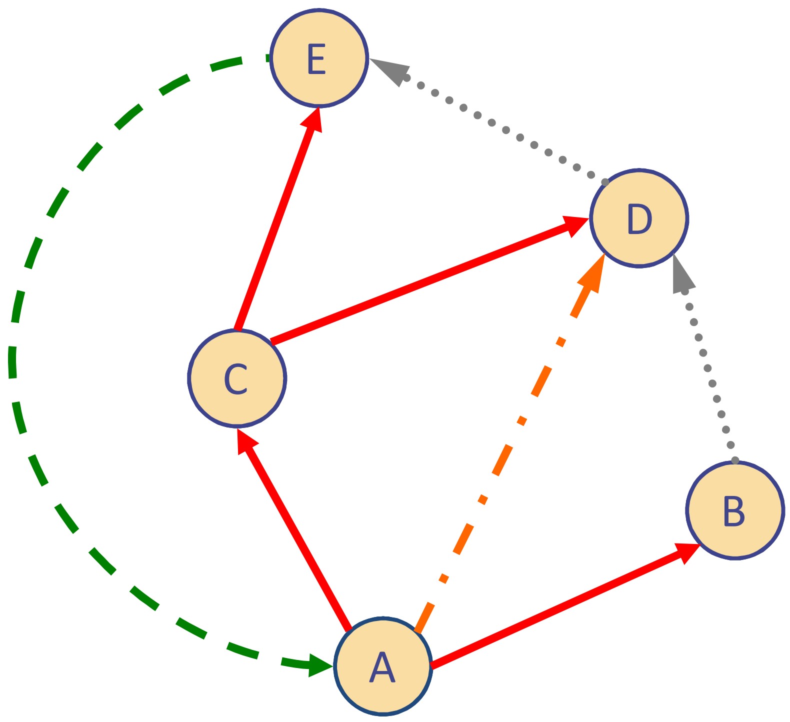
Digraph Properties

- A graph G=(V,E) such that
  - Each edge goes in one direction
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n\,(n - 1)$
- If we keep in-edges and out-edges in separate adjacency lists
  - We can perform listing of incoming edges and outgoing edges in time proportional to their size



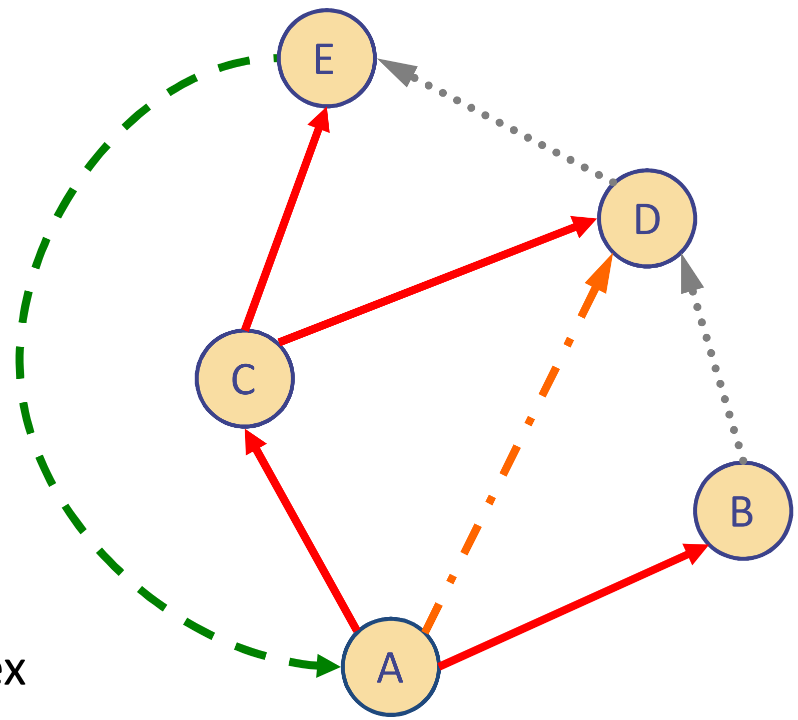
Directed DFS

- DFS traversing edges only along directions in digraphs
- A directed DFS starting at a vertex s determines the vertices reachable from s



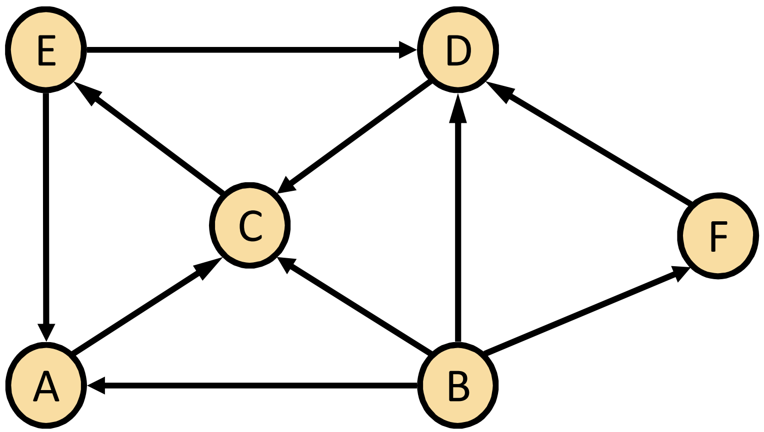
Directed DFS

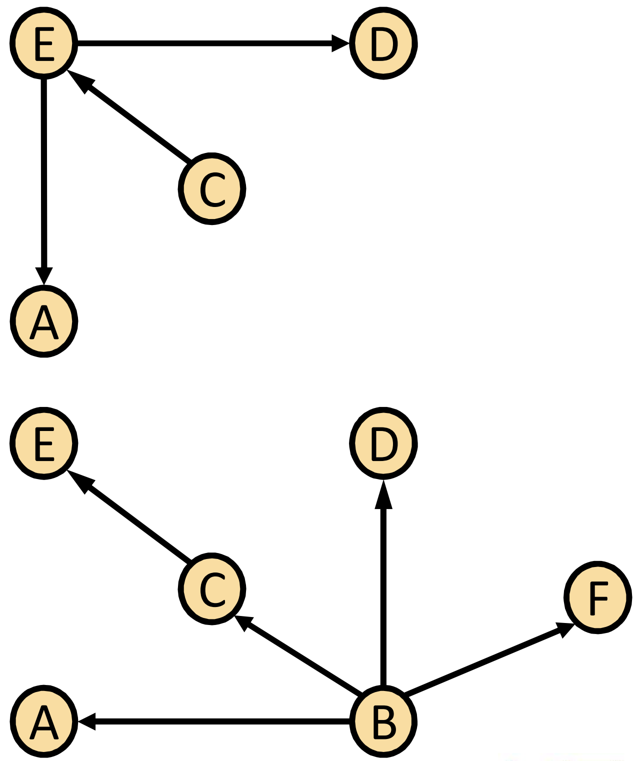
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
    - Tree edges
  - back edges
    - which connect a vertex to an ancestor in the DFS tree
  - forward edges
    - which connect a vertex to a descendent in the DFS tree
  - cross edges
    - which connect a vertex to a vertex that is neither its ancestor nor its descendent



Reachability

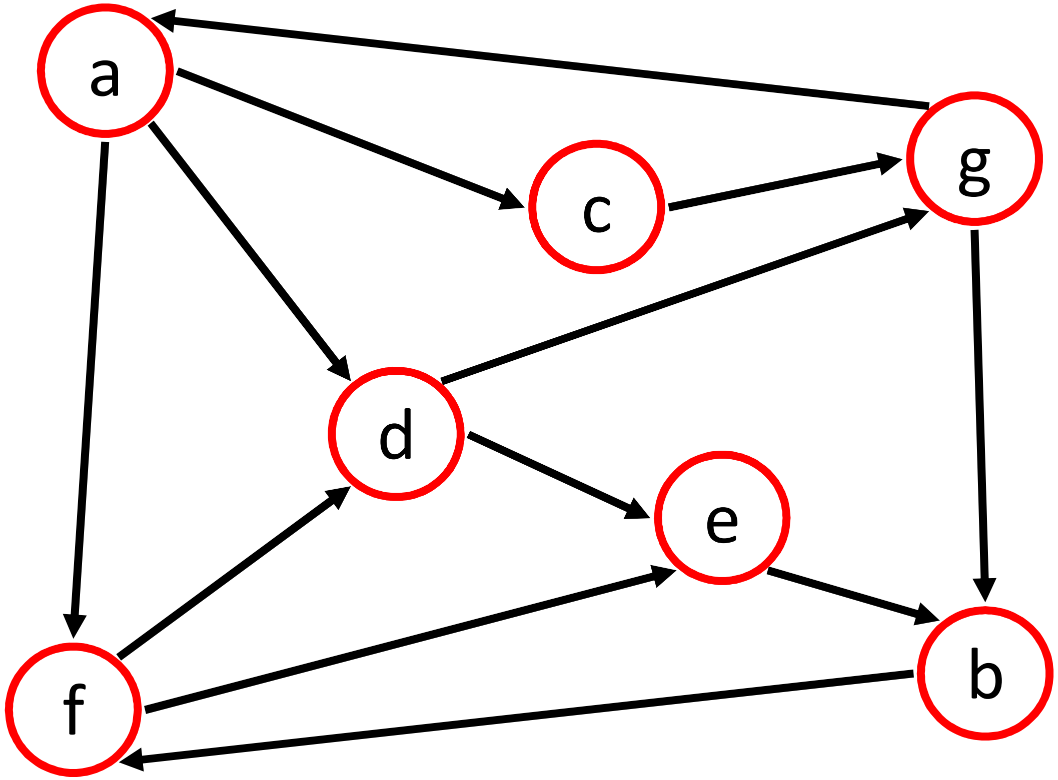
- DFS tree rooted at v: vertices reachable from v via directed paths





Strong Connectivity

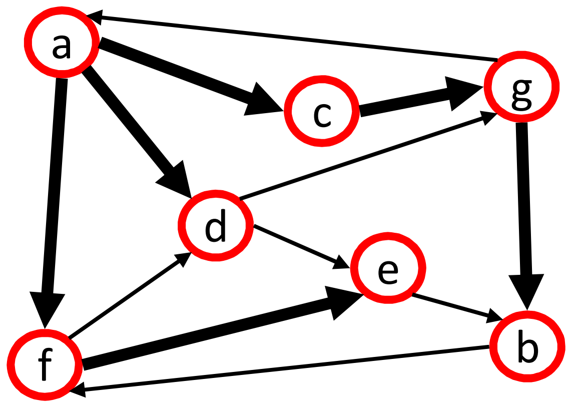
- Each vertex can reach all other vertices
  - Run directed DFS for every vertex : O(n(n+m))



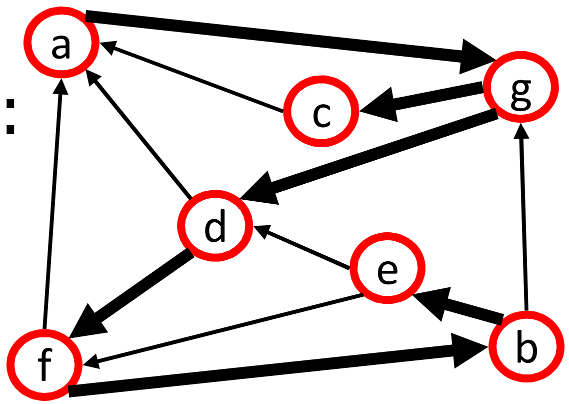
Strong Connectivity Algorithm

- Pick any vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G′ be G with edges reversed
- Perform a DFS from v in G′
  - If there's a w not visited, print "no"
  - Else, print "yes"
    - Each of the vertices visited can reach v
- Running time: O(n+m)
  - Requires only two directed DFS
    - If every node can be reached from a vertex v, and every node can reach v, then the graph is strongly connected.

G:



G' :



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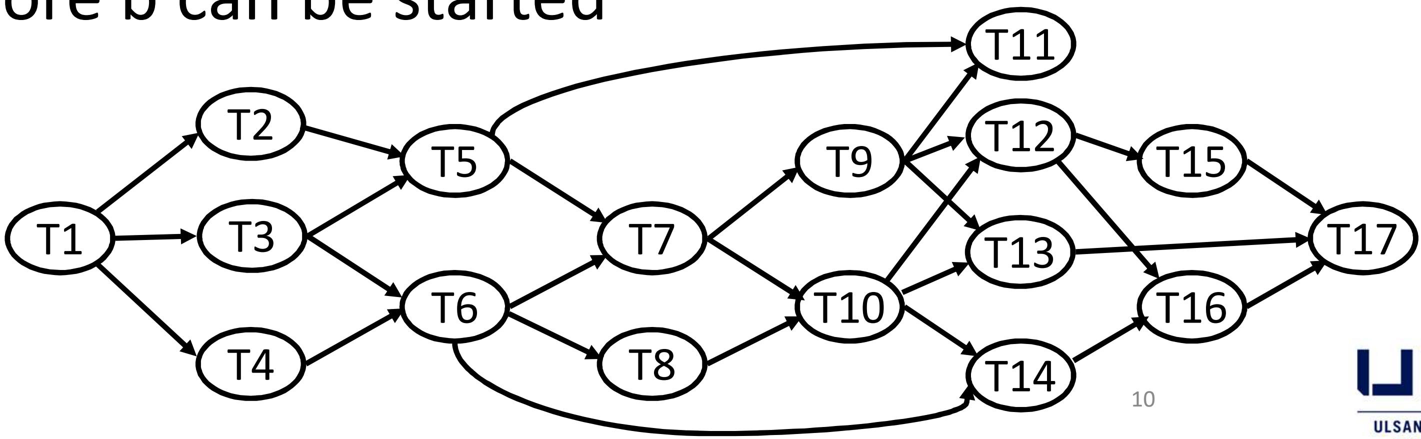
DAGs and Topological Ordering

- Decompose work into tasks that can be executed
- Conceptualize tasks and ordering as task dependency DAG (directed acyclic graph)
  - vertex = task
  - edge = control dependency
- Scheduling: edge (a,b) means task a must be completed before b can be started



DAGs and Topological Ordering

- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

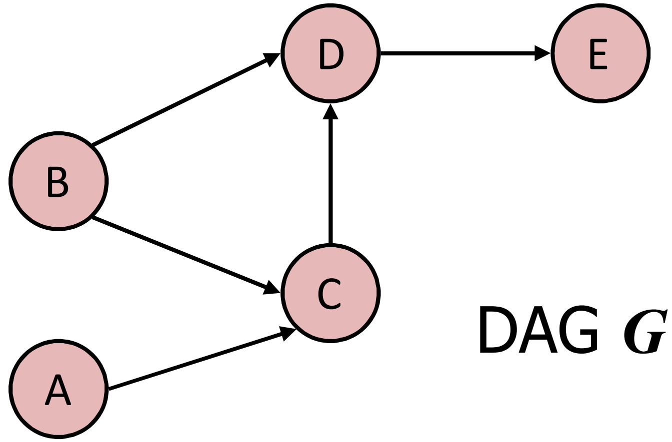
$$v_1, \dots, v_n$$

of the vertices such that for every edge (v_i, v_j), we have $i < j$

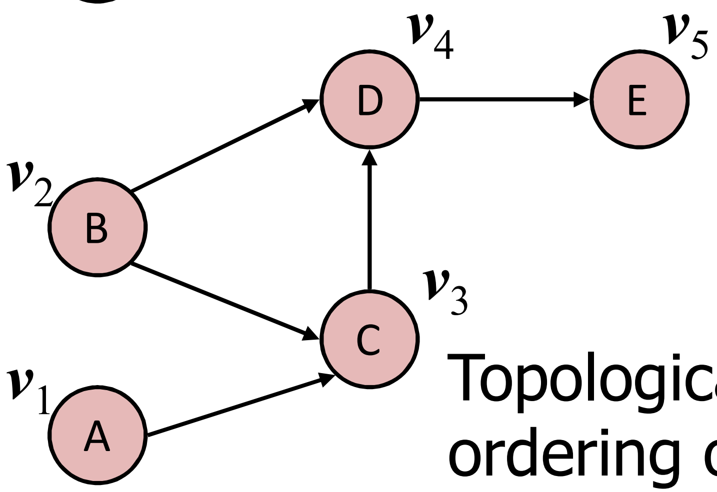
- In a task scheduling digraph, a topological ordering is a task sequence that satisfies the control dependency across all tasks

Theorem

A digraph admits a topological ordering if and only if it is a DAG



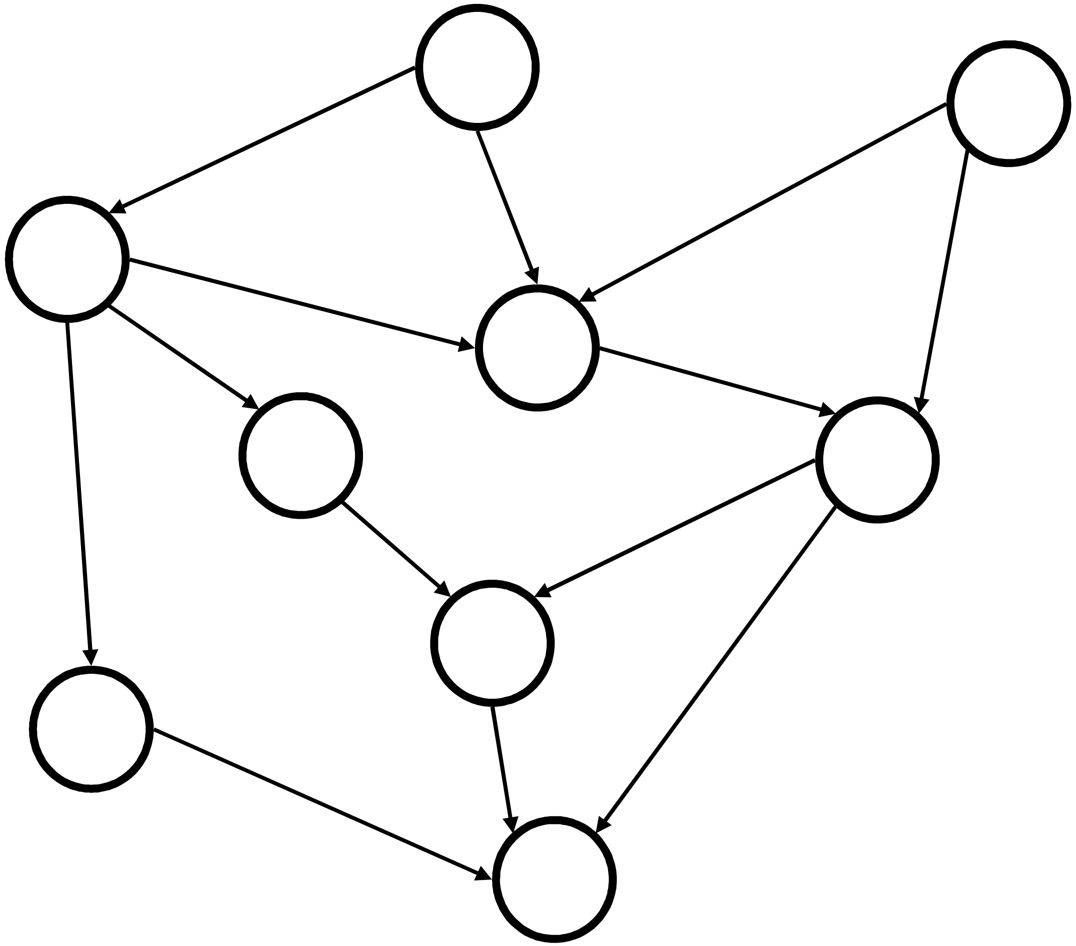
DAG G



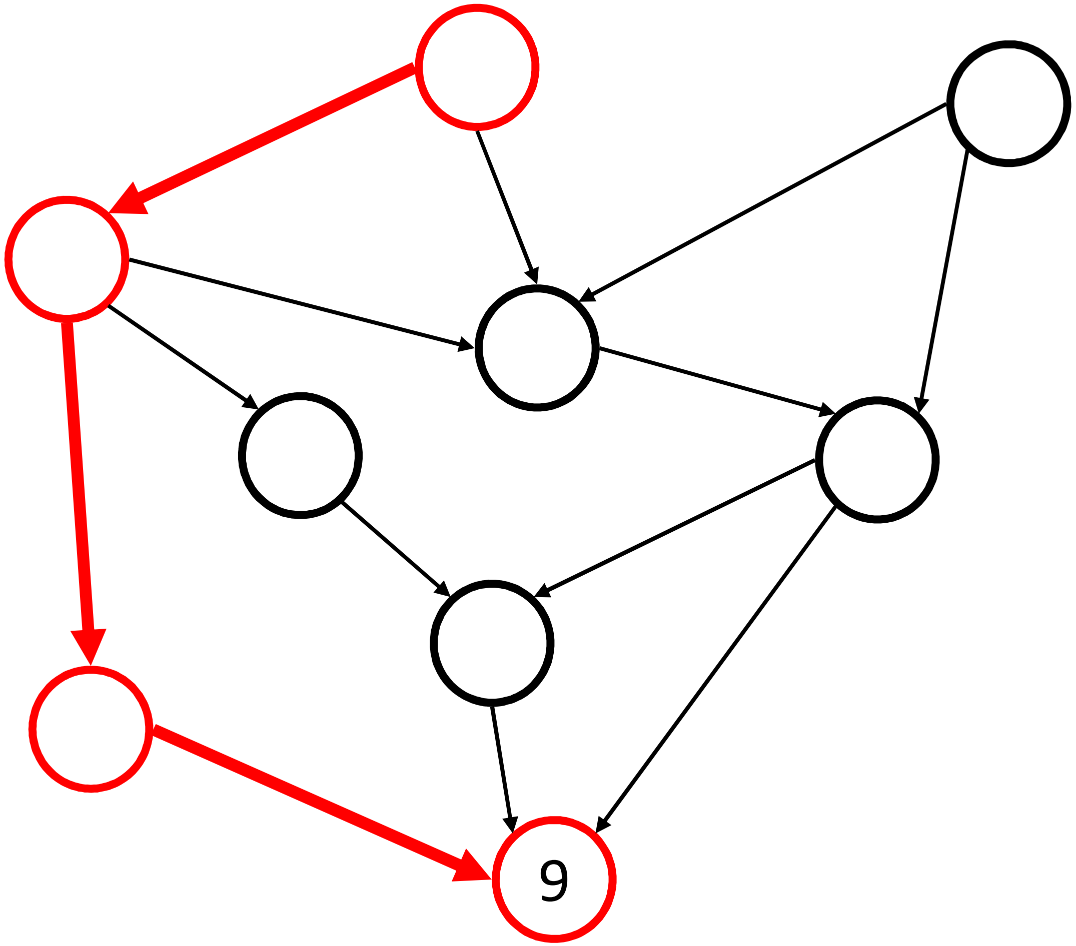
Topological ordering of G

Q: is topological sorting unique?

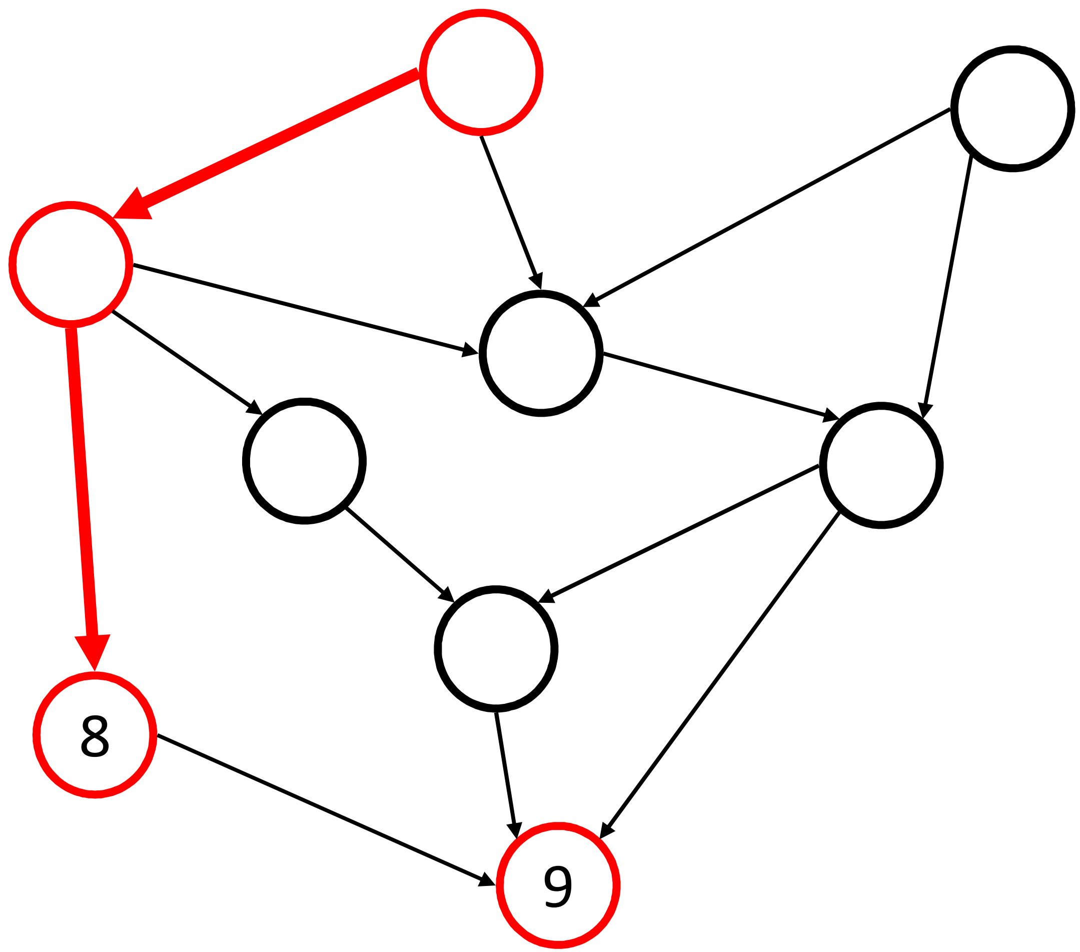
Topological Sorting Example



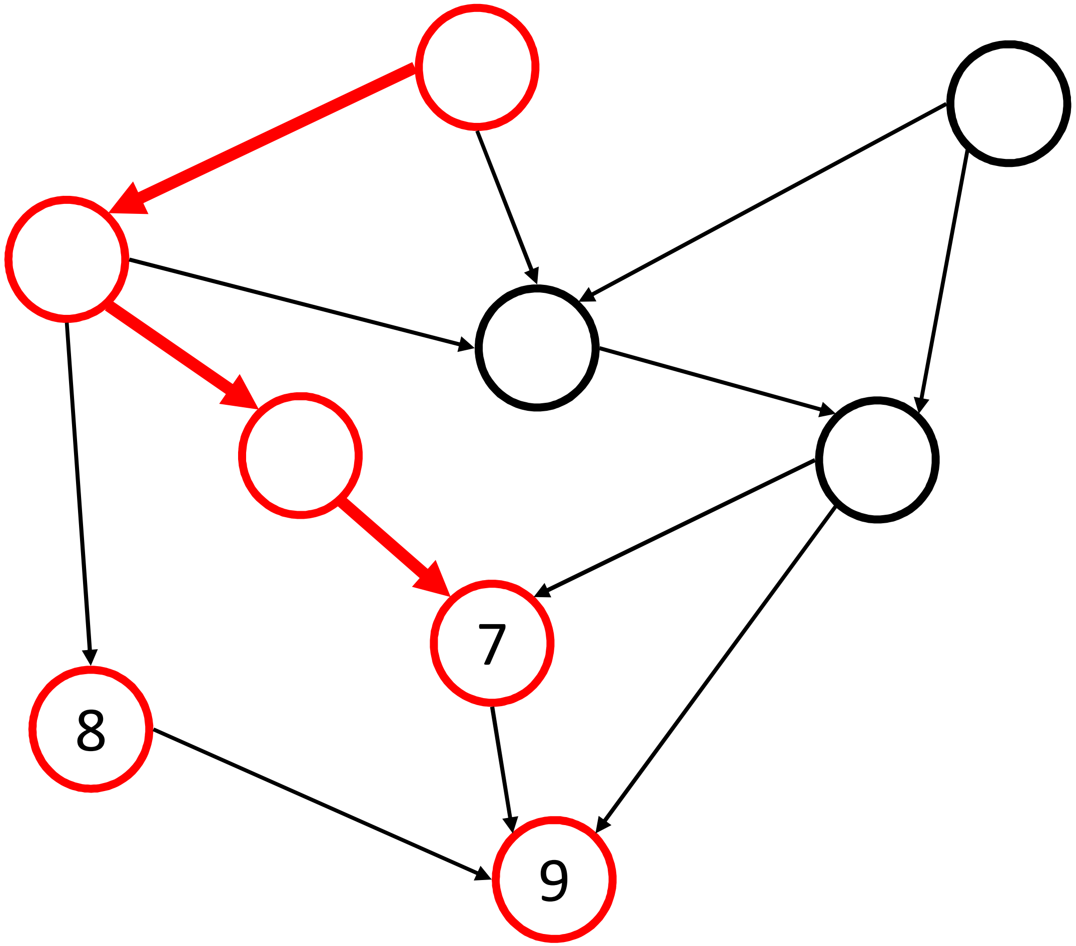
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



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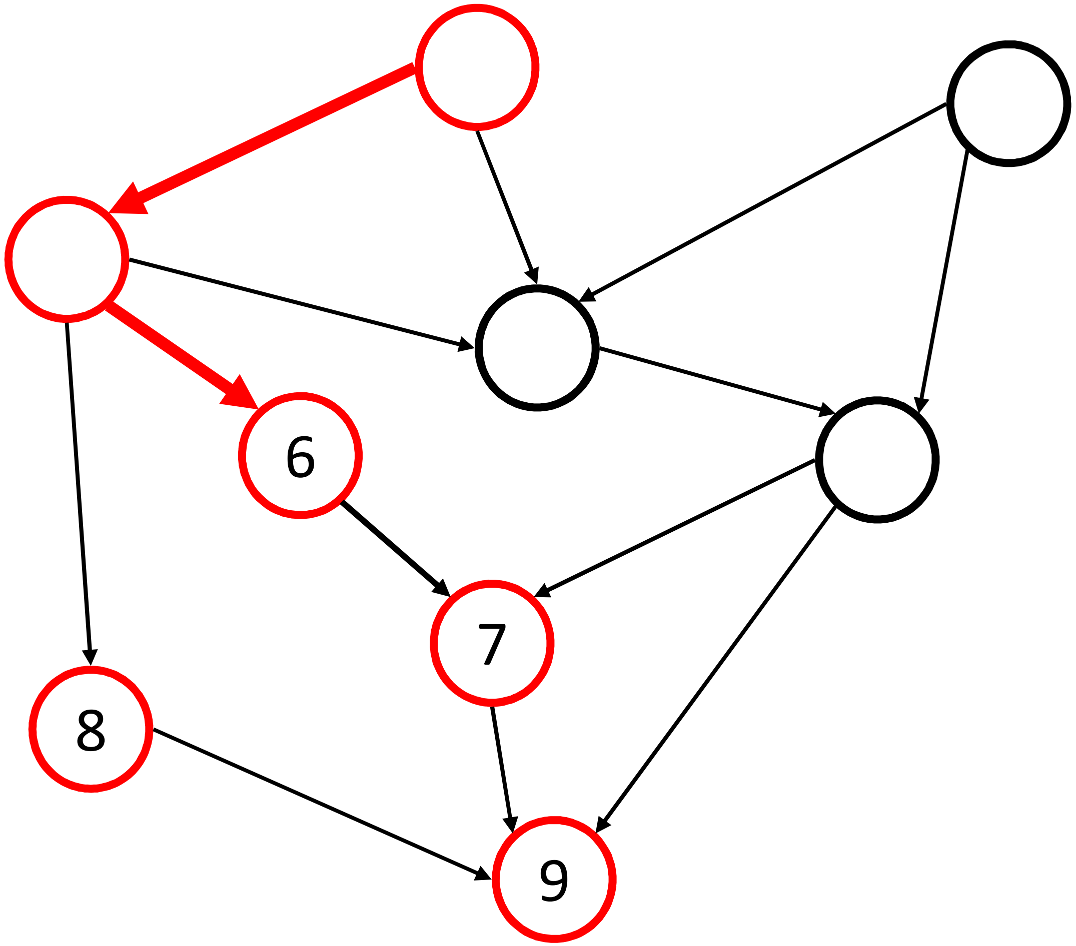
Topological Sorting Example



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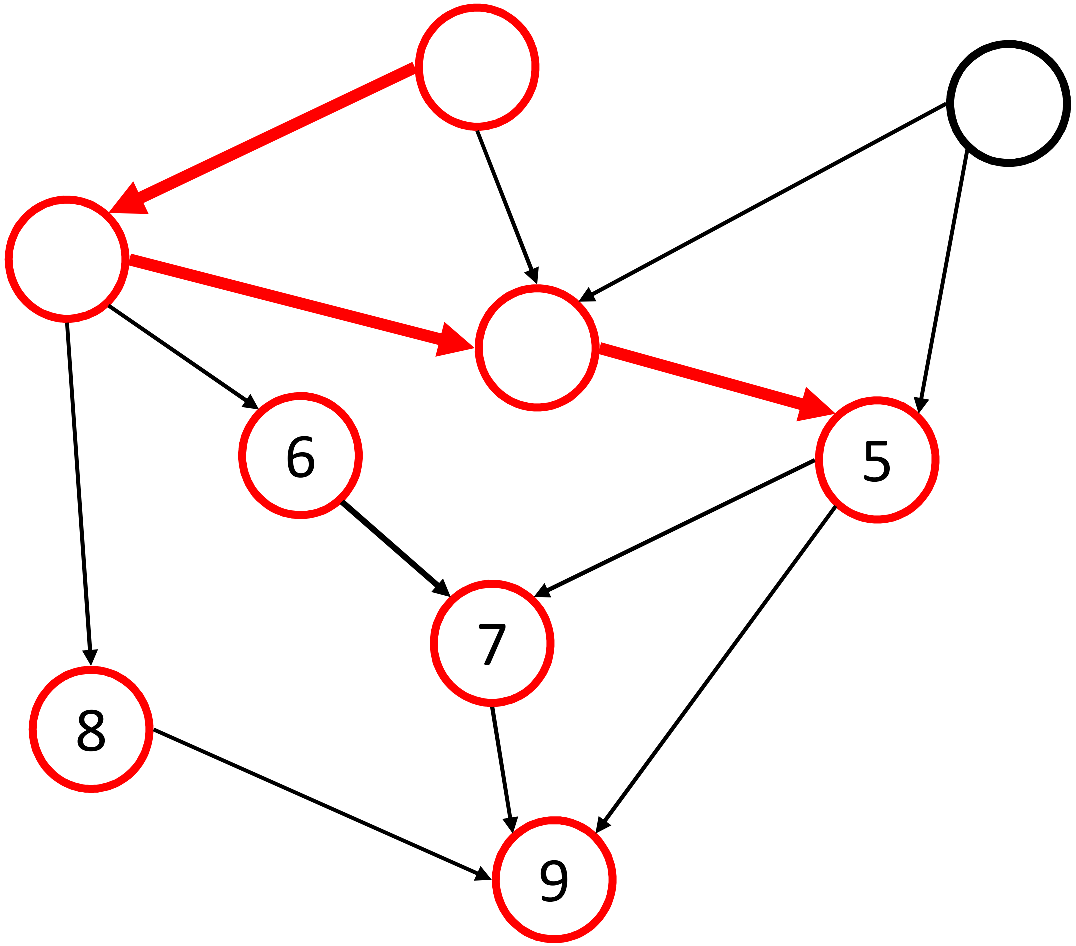

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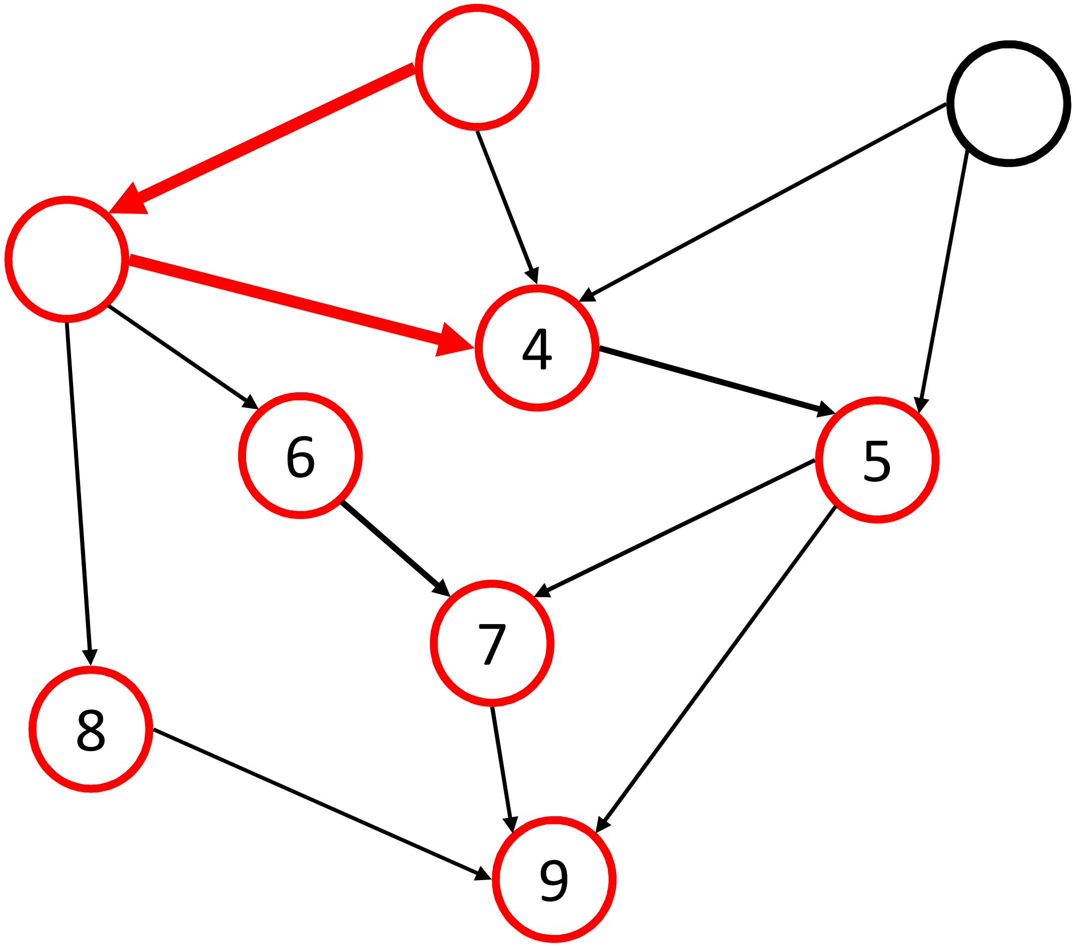

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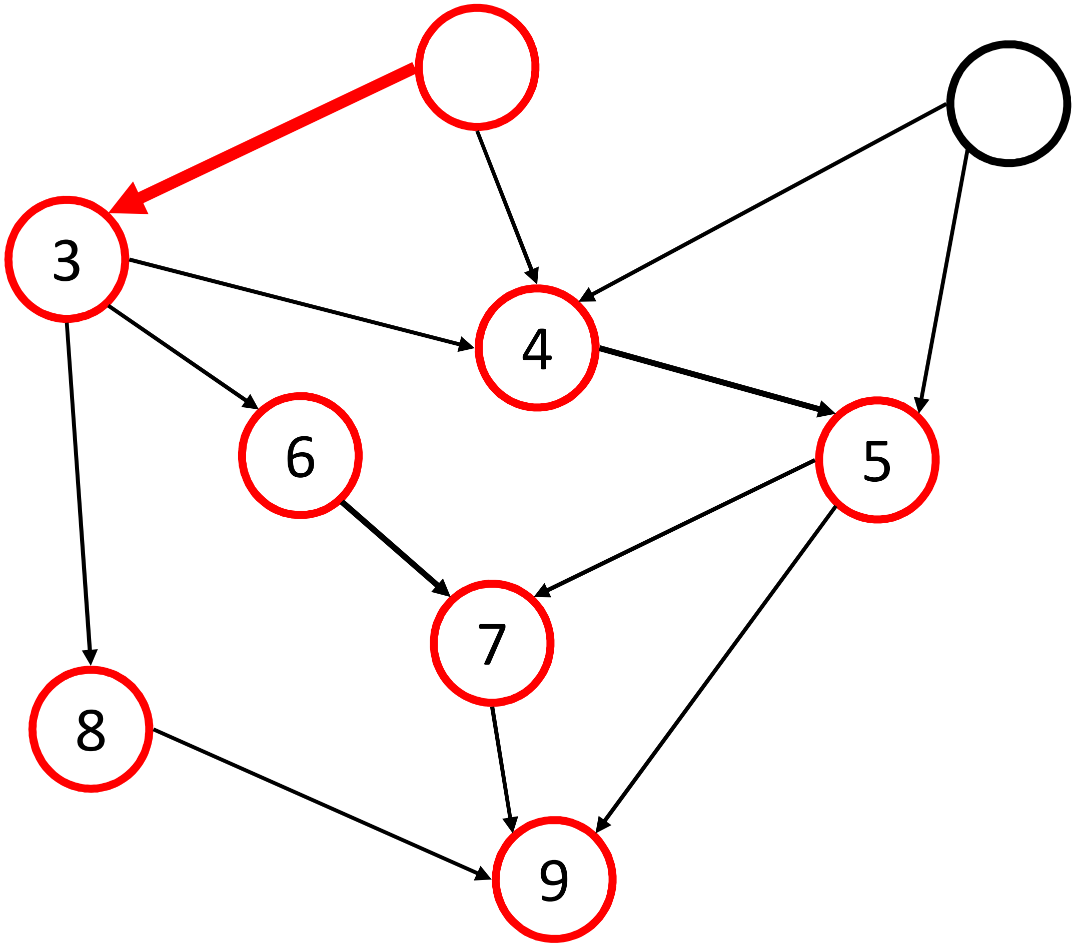

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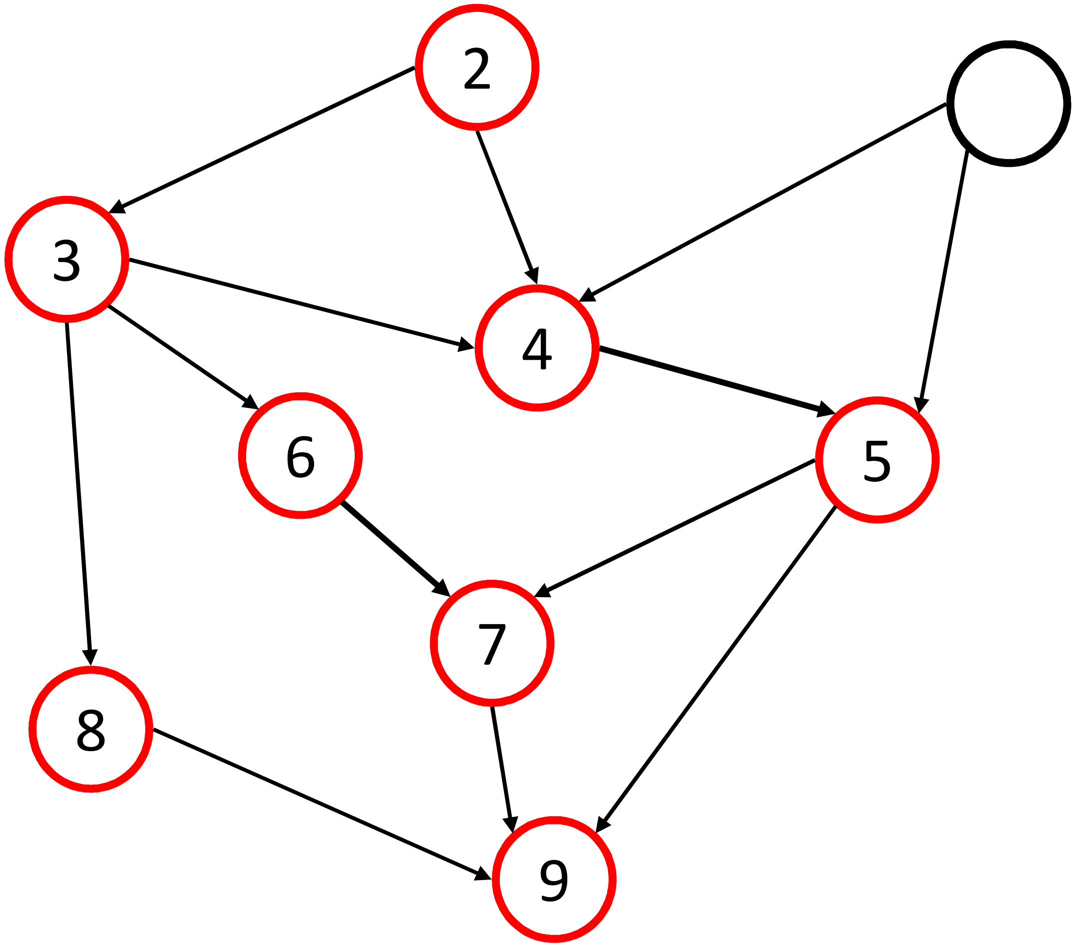
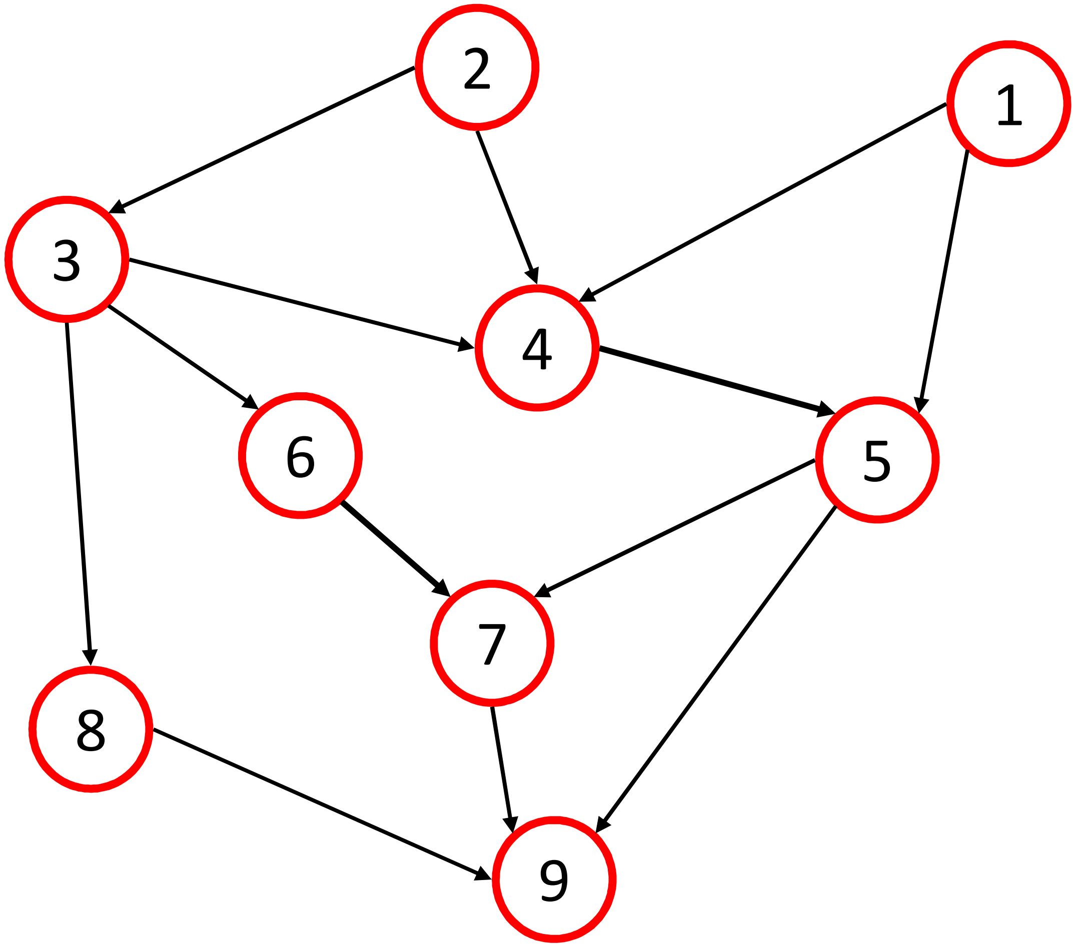


Topological Sorting Example



Algorithm for Topological Sorting

Algorithm TopologicalSort(G)
    $H \leftarrow G$ // Temporary copy of G
    $n \leftarrow G.numVertices()$
    while H is not empty **do**
        Let v be a vertex with no outgoing edges
        Label $v \leftarrow n$
        $n \leftarrow n - 1$
        Remove v from H

Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time

```
Algorithm topologicalDFS(G)
   Input dag G
   Output topological ordering of G

   n ← G.numVertices()
   for all u in G.vertices()
      u.setLabel(UNEXPLORED)
   for all v in G.vertices()
      if v.getLabel() = UNEXPLORED
         topologicalDFS(G, v)
```

```
Algorithm topologicalDFS(G, v)
   Input graph G and a start vertex v of G
   Output labeling of the vertices of G
      in the connected component of v

   v.setLabel(VISITED)
   for all e in v.outEdges()
      { outgoing edges }
      w ← e.opposite(v)
      if w.getLabel() = UNEXPLORED
         { e is a discovery edge }
         topologicalDFS(G, w)
      else
         { e is a forward or cross edge }
   Label v with topological number n
   n ← n - 1
```