CSE221

Lecture 18:
Graph Traversals

Acknowledgment: The content of this file is based on the slides of the textbook as well as the slides provided in former lectures at UNIST.

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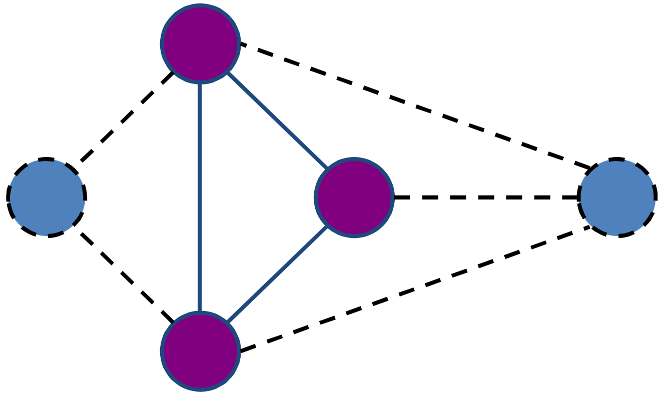
Outline

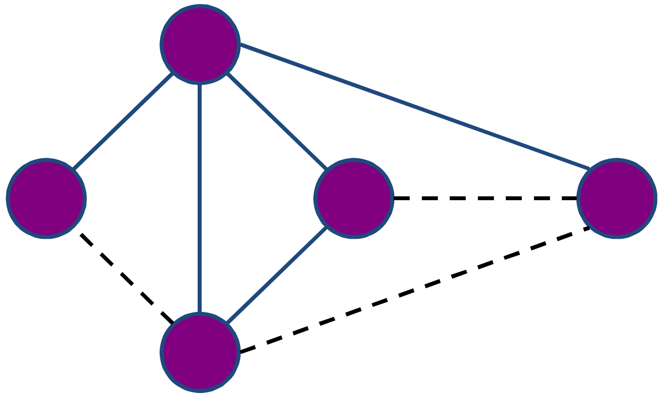
- Depth First Search (DFS)
- Breadth First Search (BFS)

Subgraphs

- A **subgraph** S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A **spanning subgraph** of G is a subgraph that contains all the vertices of G



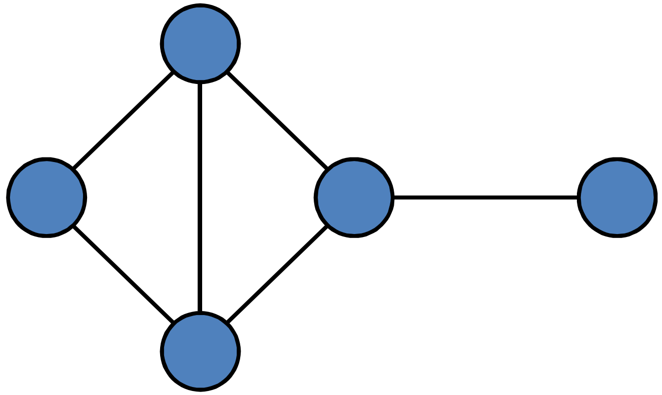
Subgraph



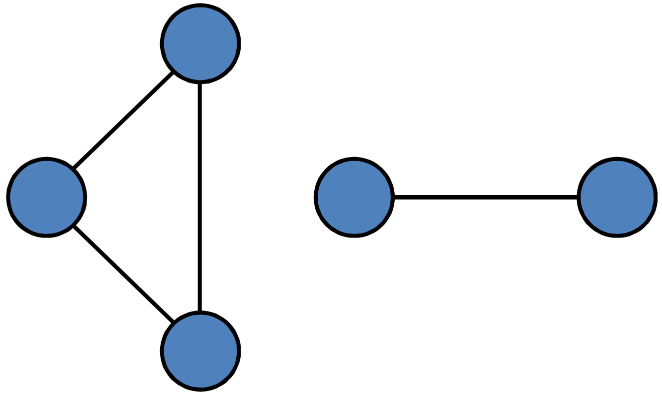
Spanning subgraph

Connectivity

- A graph is **connected** iif there is a path between every pair of vertices
- A **connected component** of a graph G is a maximal connected subgraph of G



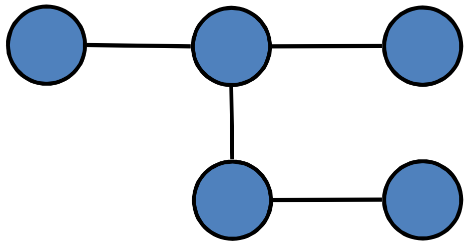
Connected graph



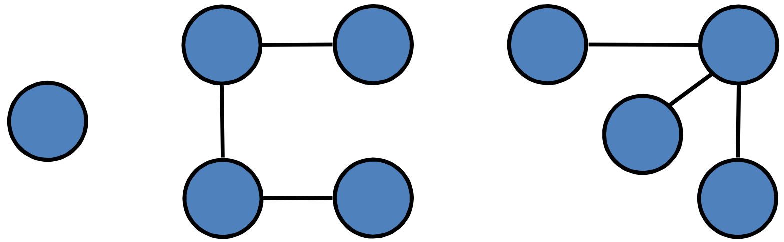
Non connected graph with two connected components

Trees and Forests

- A **tree** is an undirected graph T such that
  - T is connected
  - T has no cycles
  - Note: T is not a rooted tree
- A **forest** is an undirected graph without cycles
  - The connected components of a forest are trees



Tree



Forest

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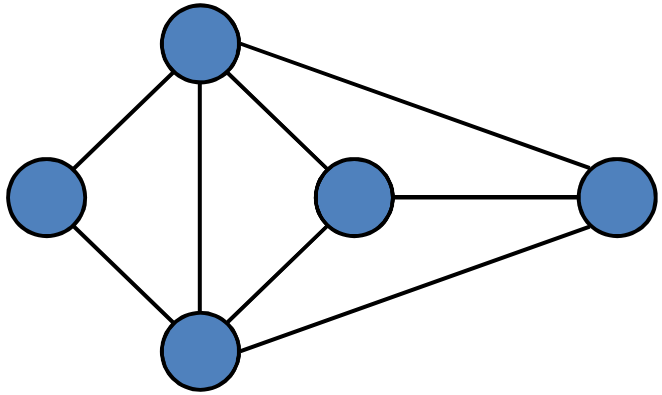
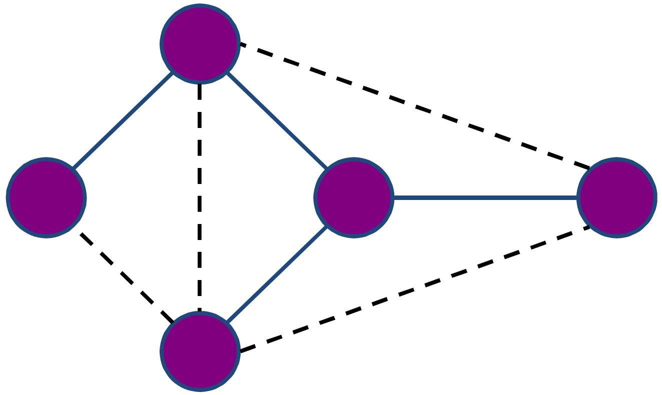
Spanning Trees and Forests

- A **spanning tree** of a connected graph is a spanning subgraph that is a tree
  - –A spanning tree is not "unique" unless the graph is a tree
- A **spanning forest** of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G in a depth-first fashion
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time

- DFS can solve many graph problems
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
  - Find and report a path between two given vertices
  - Find a cycle in the graph

DFS Algorithm

Algorithm ***DFS*(*G*)**

Input graph ***G***

Output labeling of the edges of ***G*** as discovery edges and back edges

```
for all u ∈ G.vertices()
    u.setLabel(UNEXPLORED)
for all e ∈ G.edges()
    e.setLabel(UNEXPLORED)
for all v ∈ G.vertices()
    if v.getLabel() = UNEXPLORED
        DFS(G, v)
```

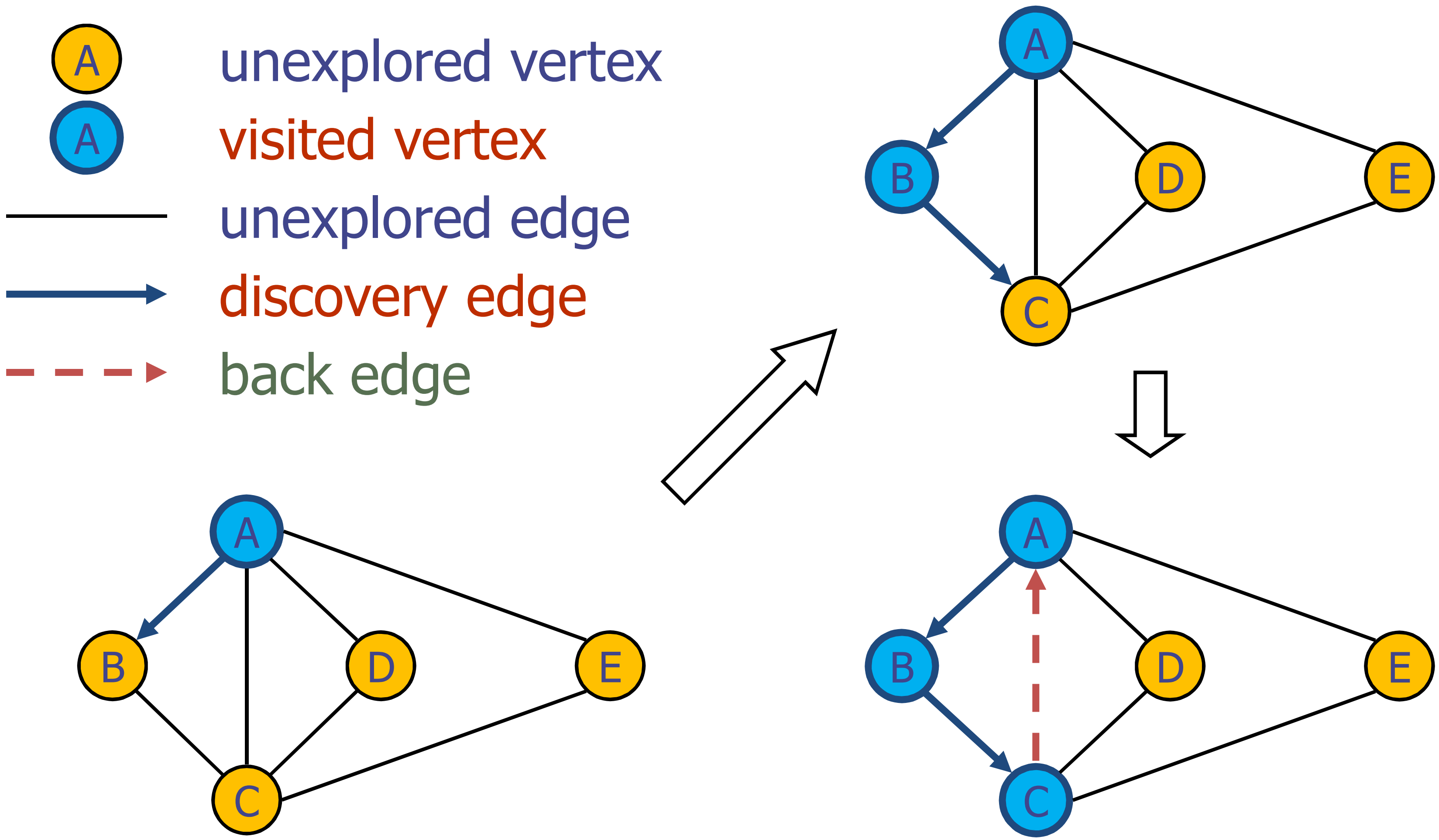
Algorithm ***DFS*(*G, v*)**

Input graph ***G*** and a start vertex ***v*** of ***G***

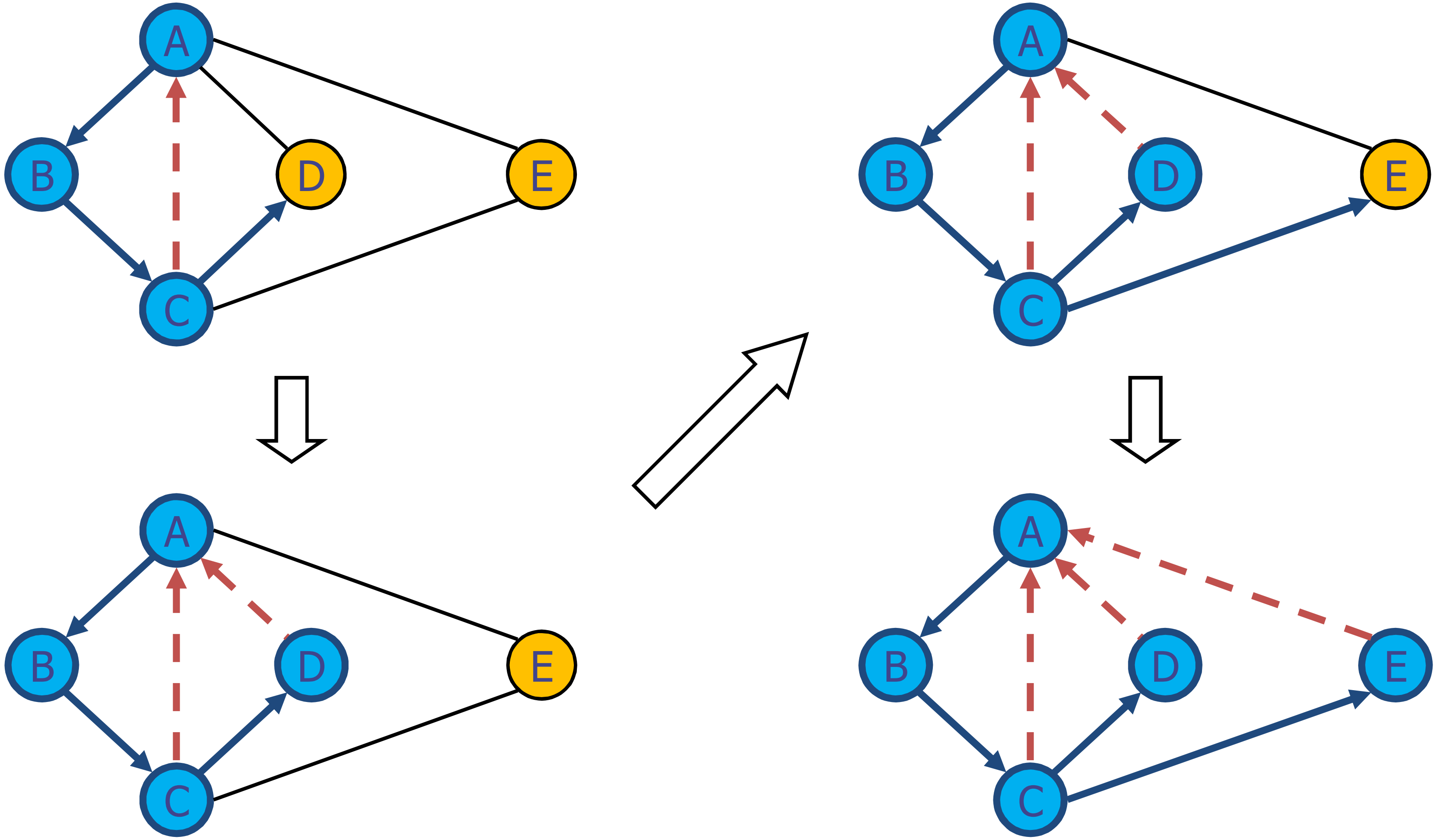
Output labeling of the edges of ***G*** in the connected component of ***v*** as discovery edges and back edges

```
v.setLabel(VISITED)
for all e ∈ G.incidentEdges(v)
    if e.getLabel() = UNEXPLORED
        w ← e.opposite(v)
        if w.getLabel() = UNEXPLORED
            e.setLabel(DISCOVERY)
            DFS(G, w)
        else
            e.setLabel(BACK)
```

Example

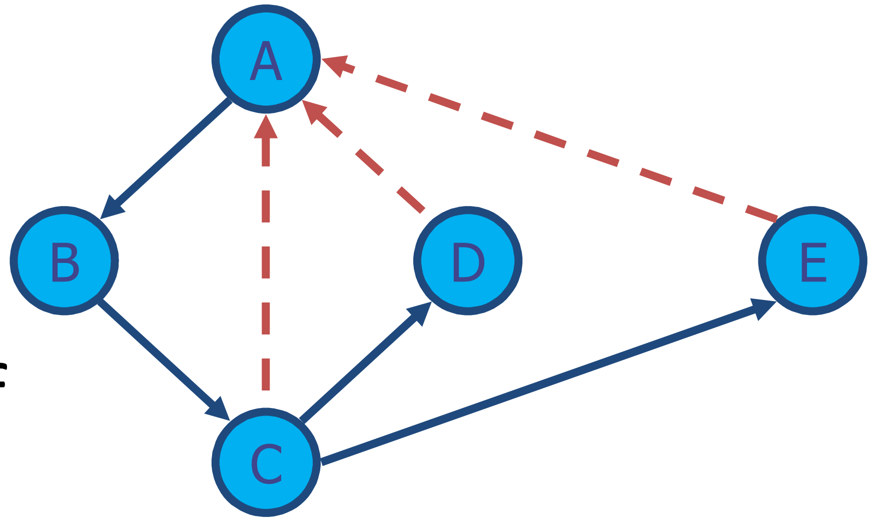

A
unexplored vertex
A
visited vertex
unexplored edge
discovery edge
back edge
A
B
D
E
C
A
B
D
E
C
A
B
D
E
C

Example



Properties of DFS

1) $\boldsymbol{DFS(G, v)}$ visits all the vertices and edges in the connected component of $\boldsymbol{v}$
2) The discovery edges labeled by $\boldsymbol{DFS(G, v)}$ form a spanning tree of the connected component of $\boldsymbol{v}$



Analysis of DFS

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- DFS is called exactly once on each vertex, and that every edge is examined exactly twice, once from each of its end vertices.
- Method incidentEdges which is called once for each vertex v method takes O(degree(v)) for v provided that the graph is represented by the adjacency list structure
    - Recall that $\Sigma_v \deg(v) = 2m$
- DFS runs in $O(n + m)$ time

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Path Finding

- Find a path between two given vertices ***v*** and ***z***
- Use a stack ***S***
  - Keep track of the path between vertex ***v*** and the current vertex
  - As soon as vertex ***z*** is encountered, return the contents of the stack as the path from ***v*** to ***z***

```
Algorithm pathDFS(G, v, z)
  v.setLabel(VISITED)
  S.push(v)
  if v = z
    return S
  for all e ∈ v.incidentEdges()
    if e.getLabel() = UNEXPLORED
      w ← e.opposite(v)
      if w.getLabel() = UNEXPLORED
        e.setLabel(DISCOVERY)
        pathDFS(G, w, z)
      else
        e.setLabel(BACK)
  S.pop()
```

Cycle Finding

- Find a cycle
- Use a stack S
  - Keep track of the path between vertex v and the current vertex
  - As soon as a back edge (v, w) is encountered, return the portion of the stack from the top to vertex w as a cycle

Algorithm *cycleDFS*(G, v)
    v.setLabel(*VISITED*)
    S.push(v)
    for all $e \in$ *v.incidentEdges*()
        if *e.getLabel*() = *UNEXPLORED*
            $w \leftarrow$ *e.opposite*(v)
            if *w.getLabel*() = *UNEXPLORED*
                e.setLabel(*DISCOVERY*)
                cycleDFS(G, w)
            else
                S.push(w)
                return S
    S.pop()

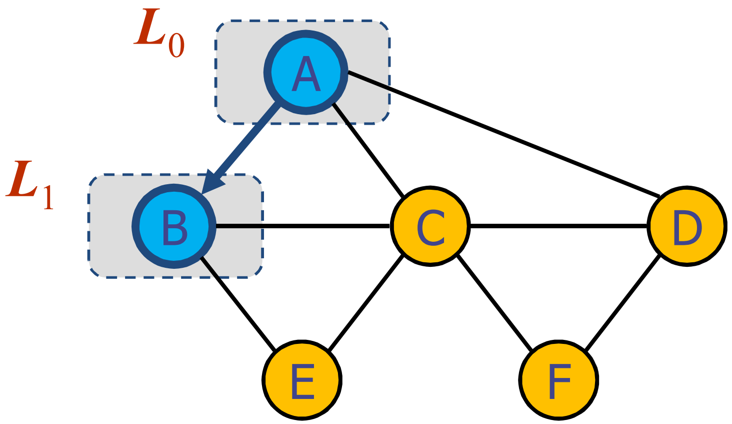
Breadth-First Search

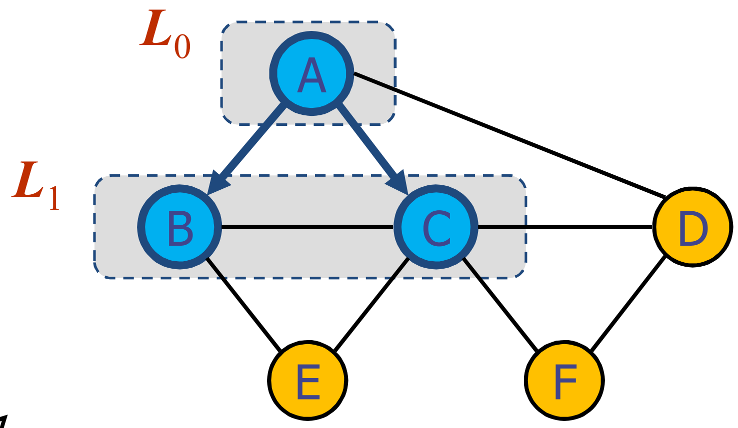
- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G in a breadth-first fashion
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time

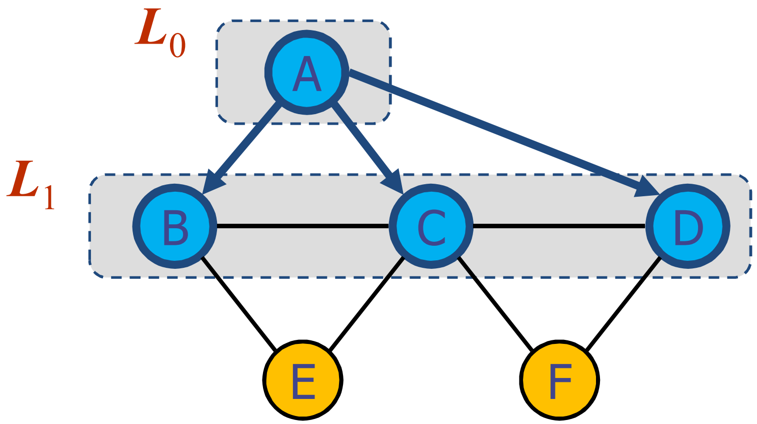
- BFS can solve many graph problems
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
  - Find and report a path with the **minimum number of edges** between two given vertices
  - Find a cycle in the graph

Example

A unexplored vertex
A visited vertex
unexplored edge
discovery edge
cross edge







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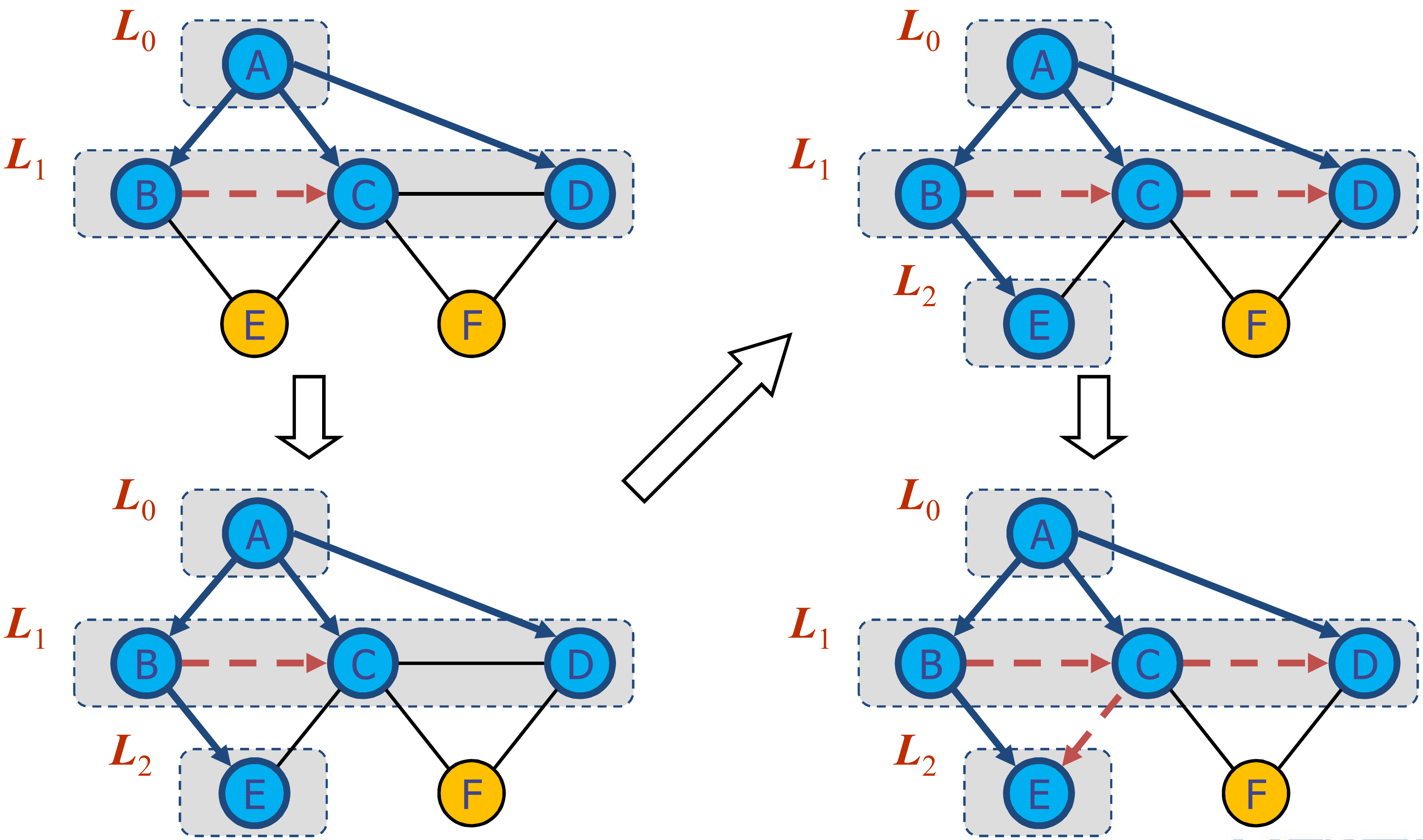
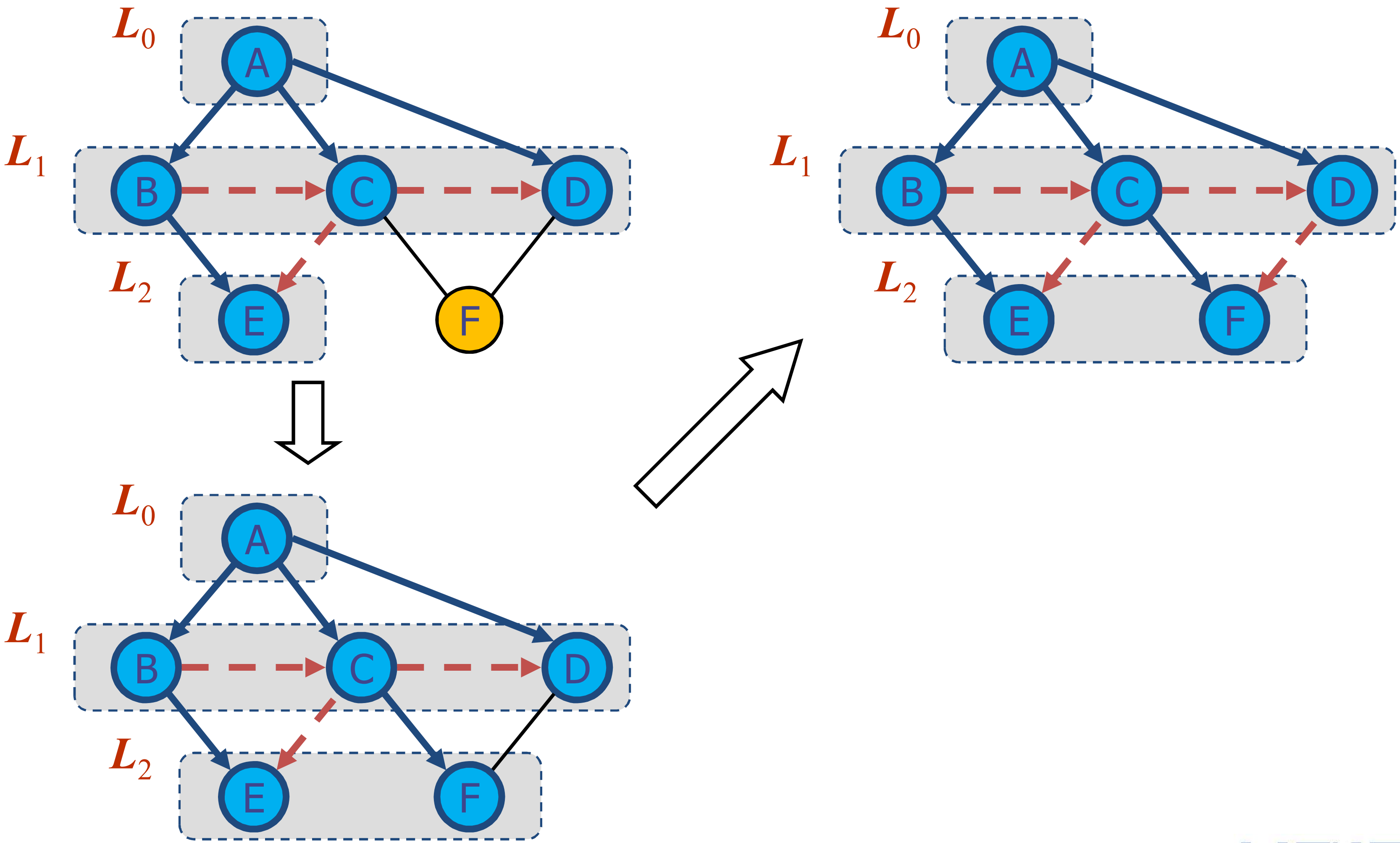
Example



Example



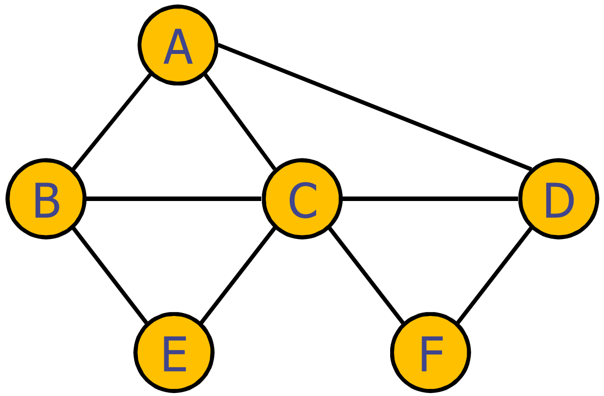
BFS Algorithm

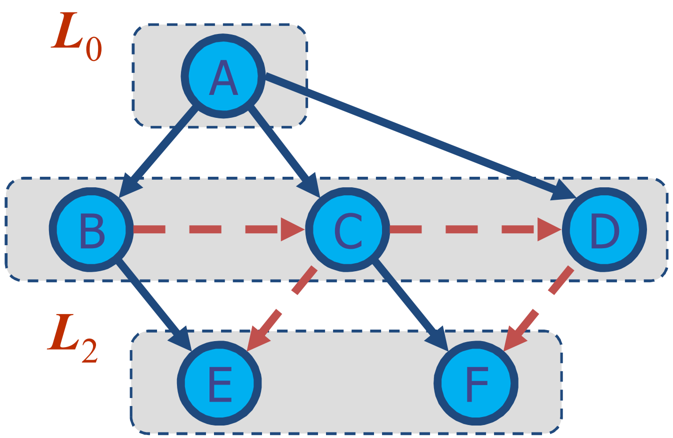
```
Algorithm BFS(G)
  Input graph G
  Output labeling of the edges
      and partition of the
      vertices of G

  for all u ∈ G.vertices()
    u.setLabel(UNEXPLORED)
  for all e ∈ G.edges()
    e.setLabel(UNEXPLORED)
  for all v ∈ G.vertices()
    if v.getLabel() = UNEXPLORED
      BFS(G, v)
```

```
Algorithm BFS(G, s)
  L_0 ← new empty sequence
  L_0.insertBack(s)
  s.setLabel(VISITED)
  i ← 0
  while ¬L_i.empty()
    L_{i+1} ← new empty sequence
    for all v ∈ L_i.elements()
      for all e ∈ v.incidentEdges()
        if e.getLabel() = UNEXPLORED
          w ← e.opposite(v)
          if w.getLabel() = UNEXPLORED
            e.setLabel(DISCOVERY)
            w.setLabel(VISITED)
            L_{i+1}.insertBack(w)
          else
            e.setLabel(CROSS)
    i ← i + 1
```

Properties of BFS

1) $BFS(G, s)$ visits all the vertices and edges in the connected components of s
2) The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of the connected component of s
3) For each vertex v in L_i
   - The path of T_s from s to v has i edges (i.e., shortest path)
   - Every path from s to v in G_s has at least i edges





Questions?

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