CSE221

Recursion

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Acknowledgment: The content of this file is based on the slides of the textbook as well as the slides provided in former lectures at UNIST.

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Outline

- Linear recursion
- Binary recursion
- Multiple recursion

The Recursion Pattern

- **Recursion**: when a method calls itself
  - E.g., factorial function: n! = 1· 2· 3· ··· · (n-1)· n

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- C++ method:

```
// recursive factorial function
int  recursiveFactorial(int n) {
   if  (n  ==  0)  return  1;          // base case
   else return  n  *  recursiveFactorial(n - 1);          // recursive case
}
```

Linear Recursion

- Perform a single recursive call
  - May choose **one** of several recursive cases
  - Should make progress on the base case
- Base cases
  - Should exist at least one
  - Recursion must reach a base case
  - Handling of each base case should not cause recursion

Sum of Array Elements

Algorithm LinearSum(*A, n*):

Input:

An integer array *A* and an integer *n* such that *A* has at least *n* elements

Output:

Sum of the first *n* integers in *A*

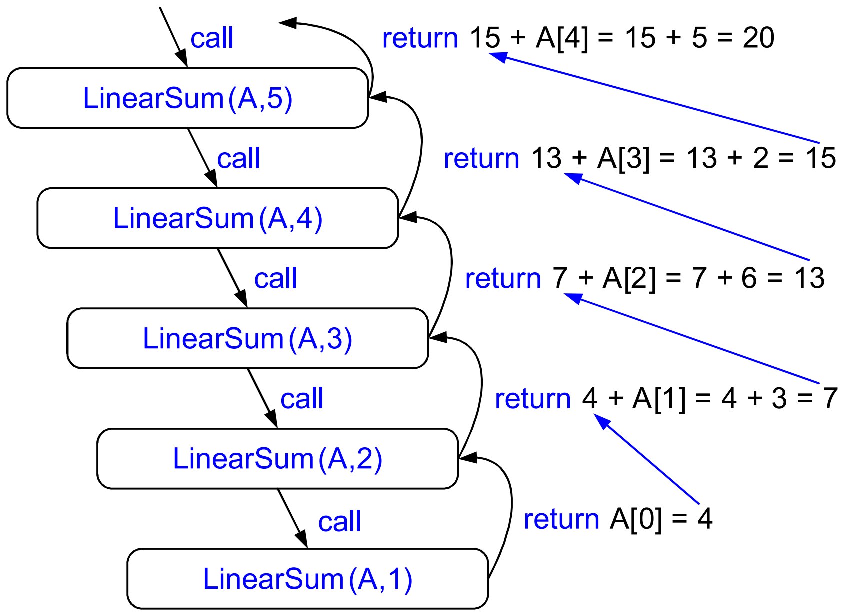
if *n* = 1 **then**

  return *A*[0]

else

  return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



A[ ] = {4,3,6,2,5}

Reversing an Array

Algorithm ReverseArray(*A, i, j*):

Input: An array *A* and non-negative integer indices *i* and *j*

Output: The reversal of the elements in *A* starting at index *i* and ending at *j*

if $i < j$ **then**

  Swap $A[i]$ and $A[j]$

  ReverseArray(A, $i + 1$, $j - 1$)

return

Computing Powers

- The power function, $p(x,n) = x^n$

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- O(n) time (for *n* recursive calls)
- Can we do better?

Recursive Squaring

- More efficient linearly recursive algorithm by using repeated squaring

$$p(x,n)=\begin{cases} 1 & \text{if } x=0 \\ x\cdot p(x,(n-1)/2)^2 & \text{if } x>0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x>0 \text{ is even} \end{cases}$$

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- Examples:

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$

$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$

$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$

$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128$

Recursive Squaring

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

  return 1

if n is odd **then**

  y = Power(x, $(n - 1)/2$)

  return $x \cdot y \cdot y$

else

  y = Power(x, $n/2$)

  return $y \cdot y$

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

Recursive Squaring

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

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if n = 0 **then**

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    return $x \cdot y \cdot y$

else

    y = Power(x, n/ 2)

    return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- **Definition**: A linearly recursive method that makes its recursive call at its <u>last step</u>
- Tail recursion can be easily converted to non-recursive one (saving resources)

Algorithm ReverseArray(*A, i, j*):
  if *i* < *j* **then**
    Swap *A*[*i*] and *A*[*j*]
    ReverseArray(*A, i+1, j-1*)
  return

Tail Recursion

- **Definition**: A linearly recursive method that makes its recursive call at its <u>last step</u>
- Tail recursion can be easily converted to non-recursive one (saving resources)

Algorithm IterativeReverseArray(*A, i, j*):
    while *i* < *j* **do**
        Swap *A*[*i*] and *A*[*j*]
        $i = i + 1$
        $j = j - 1$
    return

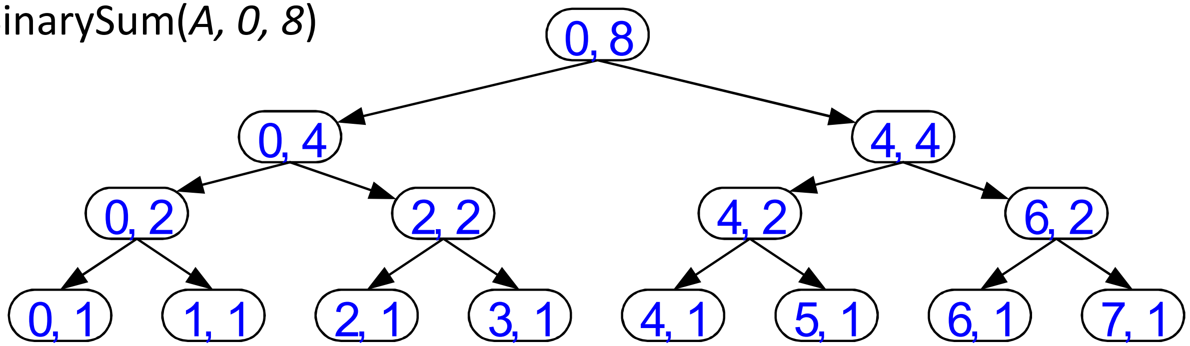
Binary Recursion

- **Definition**: There are **two** recursive calls for each non-base case
- Add all the numbers in an integer array A

Algorithm BinarySum(*A, i, n*):
  Input: An array *A* and integers *i* and *n*
  Output: The sum of the *n* integers in *A* starting at index *i*

  if *n* = 1 **then**
    return *A*[*i*]
  return BinarySum(*A, i, n/* 2) + BinarySum(*A, i* + *n/* 2, *n/* 2)

e.g., BinarySum(*A, 0, 8*)



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Computing Fibonacci Numbers

- **Definition**:

  $n_0 = 0$
  $n_1 = 1$
  $n_i = n_{i-1} + n_{i-2}$ for $i > 1$

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):
Input: Nonnegative integer k
Output: The kth Fibonacci number F_k
if k <= 1 **then**
  return k
else
  return BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$

Distance of 2

- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

More than 2 times

A Better Fibonacci Algorithm

- Use linear recursion instead
  - k-1 recursive calls

$F_0 = 0$
$F_1 = 1$
$F_i = F_{i-1} + F_{i-2}$ for $i > 1$

Algorithm LinearFibonacci(k):

Input: A non-negative integer k
Output: Pair of Fibonacci numbers (F_k , F_{k-1})

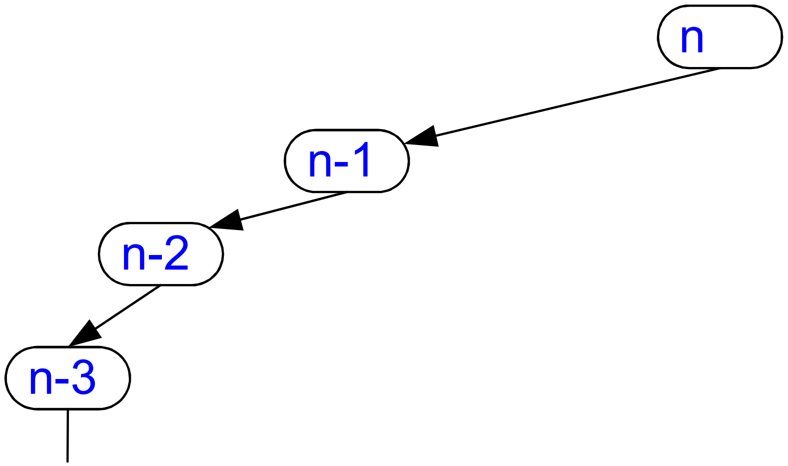
if *k* <= 1 **then**
    return (k, 0)
else
    (i, j) = LinearFibonacci(*k* - 1)
    return (i+j, i)

Multiple Recursion

- Multiple recursion:
  - Makes potentially many recursive calls
  - Not just one or two

Recursion Cost Analysis

- Number of function calls: O(n)
- Height: O(n)

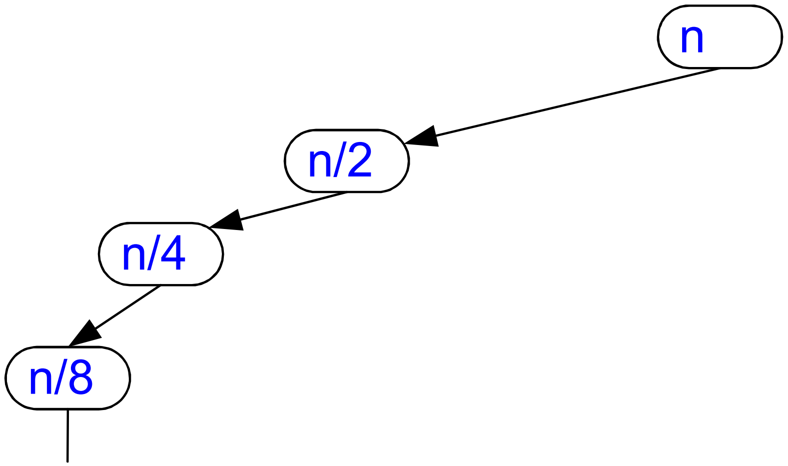


Alright!

Recursion Cost Analysis

- Number of function calls: O(logn)
- Height: O(logn)

Very good!



Recursion Cost Analysis

- Number of function calls: O(n)
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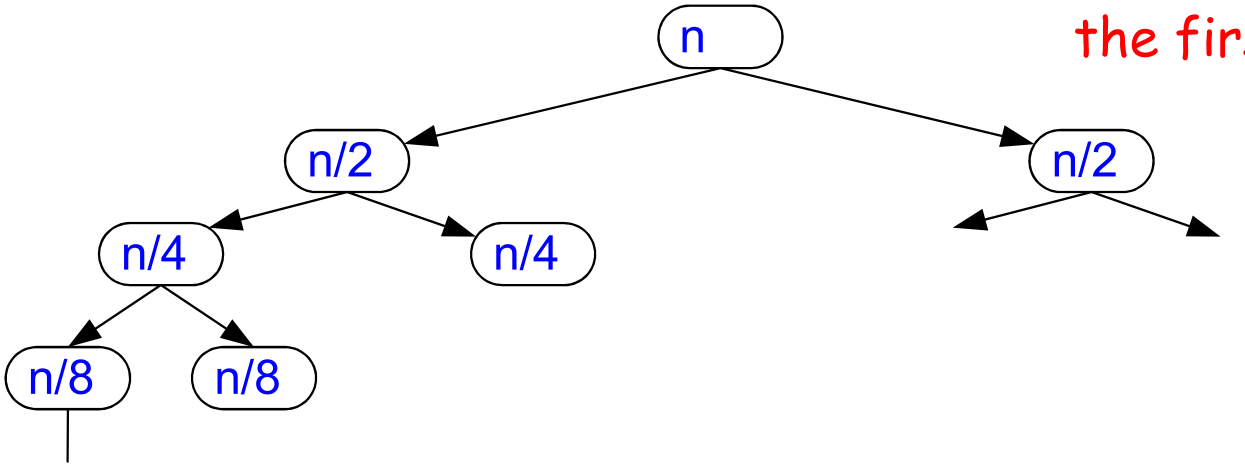

Better than
the first case!

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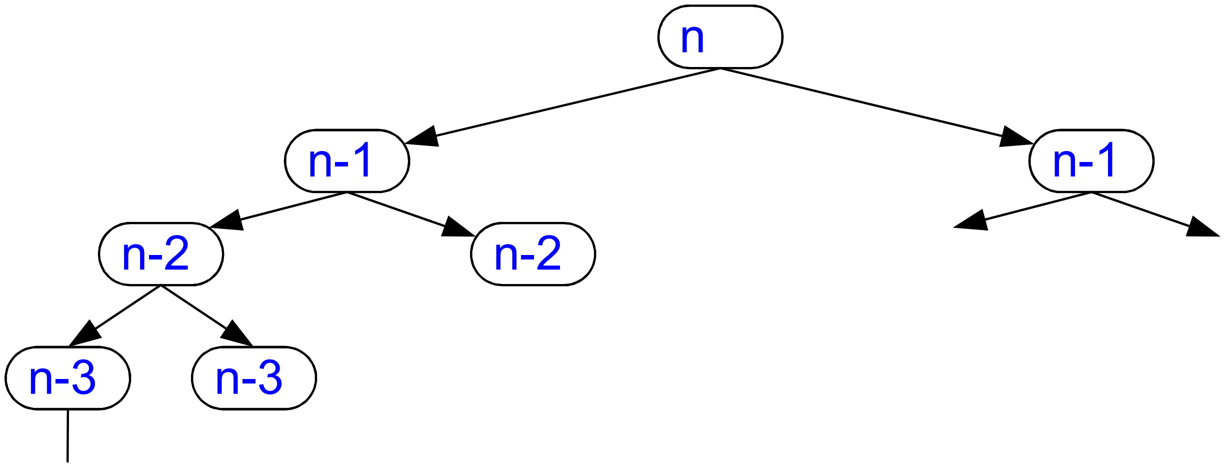
Recursion Cost Analysis

- Number of function calls: $O(2^n)$
- Height: $O(n)$

Go to the hell..



Questions?