CSE221

Lecture 9: Heaps and Priority Queues

Fall 2021

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Acknowledgment: The content of this file is based on the slides of the textbook as well as the slides provided in former lectures at UNIST.

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Outline

- Priority queues
- Heaps

Priority Queue

- Queue with priority order for *pop*
  - –Entries pushed upon their arrivals
  - –But not FIFO (entries ordered by priority)
- Max priority queue
  - –Pop the entry with a highest priority first
- Min priority queue
  - –Pop the entry with a lowest priority first

Priority Queue ADT

- A priority queue stores a collection of entries
- Typically, an entry is a (key, value) pair, where the key indicates the priority

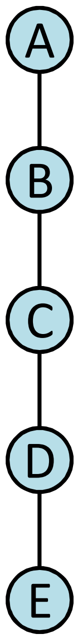
- Main methods
  - insert(e)
    inserts an entry e
  - removeMin() / removeMax()
    removes the entry with smallest / largest key
  - min() / max()
    returns with smallest / largest key, but does not remove the entry
  - size(), empty()

Total Order Relations

- Keys in a priority queue are ALL pairwise comparable and ordered
- When $x \leq y$, we say entry x is related to (or comparable to) entry y
- A binary relation $\leq$ is a total order on all pairs
  - Reflexive property: $\boldsymbol{x} \leq \boldsymbol{x}$
  - Antisymmetric property: $\boldsymbol{x} \leq \boldsymbol{y} \wedge \boldsymbol{y} \leq \boldsymbol{x} \Rightarrow \boldsymbol{x} = \boldsymbol{y}$
  - Transitive property: $\boldsymbol{x} \leq \boldsymbol{y} \wedge \boldsymbol{y} \leq \boldsymbol{z} \Rightarrow \boldsymbol{x} \leq \boldsymbol{z}$
- Connex property: $\boldsymbol{x} \leq \boldsymbol{y}$ or $\boldsymbol{y} \leq \boldsymbol{x}$

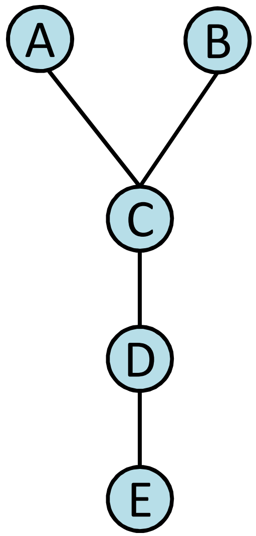
Total Order Example

Total order



Partial order

A and B are not comparable



Priority Queue Sorting

- Use a priority queue to sort a set of comparable entries
  1. Insert the entries one by one with insert operations
  2. Remove the entries in sorted order with removeMin operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm ***PQ-Sort*(*S, C*)**

Input sequence ***S***, comparator ***C*** for the entries of ***S***

Output sequence ***S*** sorted in increasing order according to ***C***

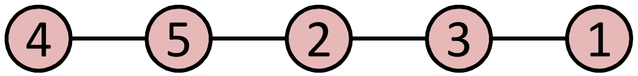
P ← priority queue with comparator ***C***

while !*S.empty* ()
  e ← *S.front*(); *S.eraseFront*()
  P.insert (*e*)

while !*P.empty*()
  e ← *P.min(); P.removeMin*()
  S.insertBack(*e*)

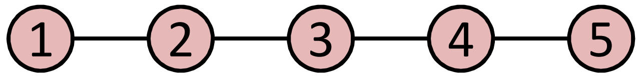
Two Ways of Implementation

- Implementation with an unsorted list



- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list



- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Selection-Sort

- PQ-sorting where the priority queue is implemented with an unsorted sequence
- Running time of Selection-sort:
  1. Inserting the entries with $\boldsymbol{n}$ insert operations takes $\boldsymbol{O}(\boldsymbol{n})$ time
  2. Removing the elements in sorted order with $\boldsymbol{n}$ removeMin operations takes time proportional to

$$\boldsymbol{n} + \boldsymbol{n}\text{-}1 \ldots + 2 + 1$$

- Selection-sort runs in $\boldsymbol{O}(\boldsymbol{n}^2)$ time

Selection-Sort Example

	Sequence S	Priority Queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(7,4)
..	.. ..	
(g)	()	(7,4,8,2,5,3,9)
Phase 2		
(a)	(2)	(7,4,8,5,3,9)
(b)	(2,3)	(7,4,8,5,9)
(c)	(2,3,4)	(7,8,5,9)
(d)	(2,3,4,5)	(7,8,9)
(e)	(2,3,4,5,7)	(8,9)
(f)	(2,3,4,5,7,8)	(9)
(g)	(2,3,4,5,7,8,9)	()

Insertion-Sort

- PQ-sorting where the priority queue is implemented with a sorted sequence
- Running time of Insertion-sort:
  1. Inserting the entries with n insert operations takes time proportional to

     $$1 + 2 \ldots + n\text{-}1 + n$$

  2. Removing the entries in sorted order with n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

Insertion-Sort Example

	Sequence S	Priority queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(4,7)
(c)	(2,5,3,9)	(4,7,8)
(d)	(5,3,9)	(2,4,7,8)
(e)	(3,9)	(2,4,5,7,8)
(f)	(9)	(2,3,4,5,7,8)
(g)	()	(2,3,4,5,7,8,9)
Phase 2		
(a)	(2)	(3,4,5,7,8,9)
(b)	(2,3)	(4,5,7,8,9)
..	..	..
(g)	(2,3,4,5,7,8,9)	()

Outline

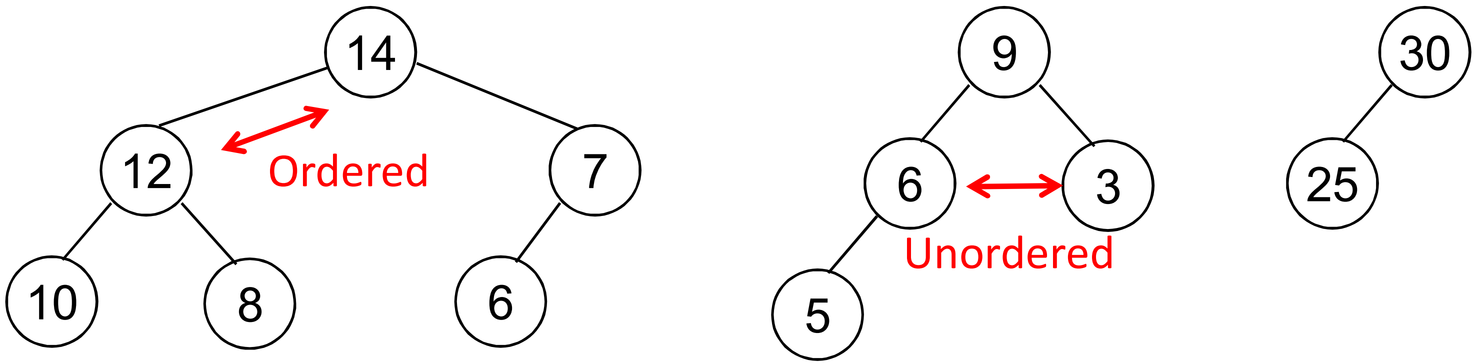
- Priority queues
- Heaps

Heap

- A heap is a tree-based data structure that satisfies the heap property
  - If A is a parent node of B, then key(A) is ordered with respect to key(B)
- Max (min) heap
  - A complete binary tree with the heap property
  - Key in each node is not smaller (not larger) than the keys in its children

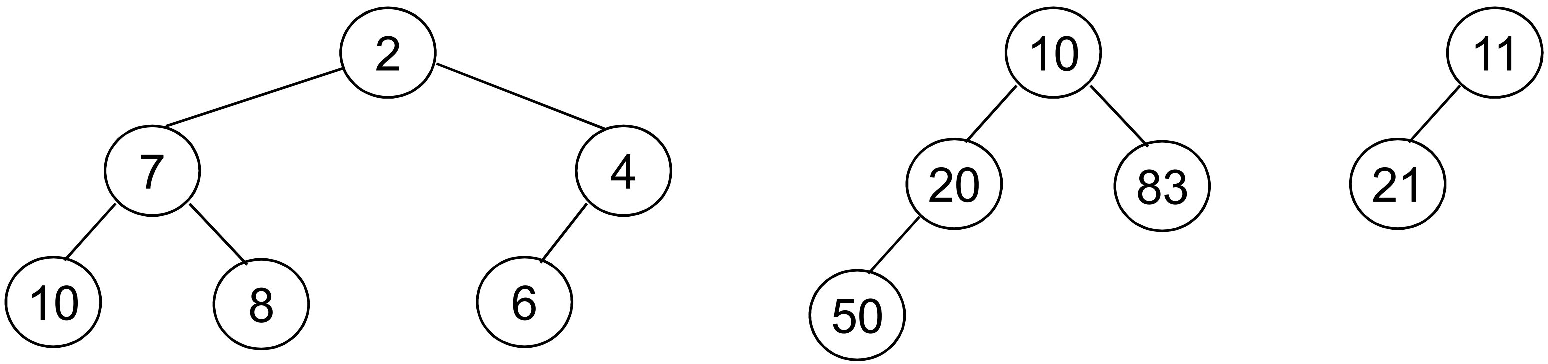
Heap: Complete Binary Tree

Max heap



Complete binary trees!

Min heap

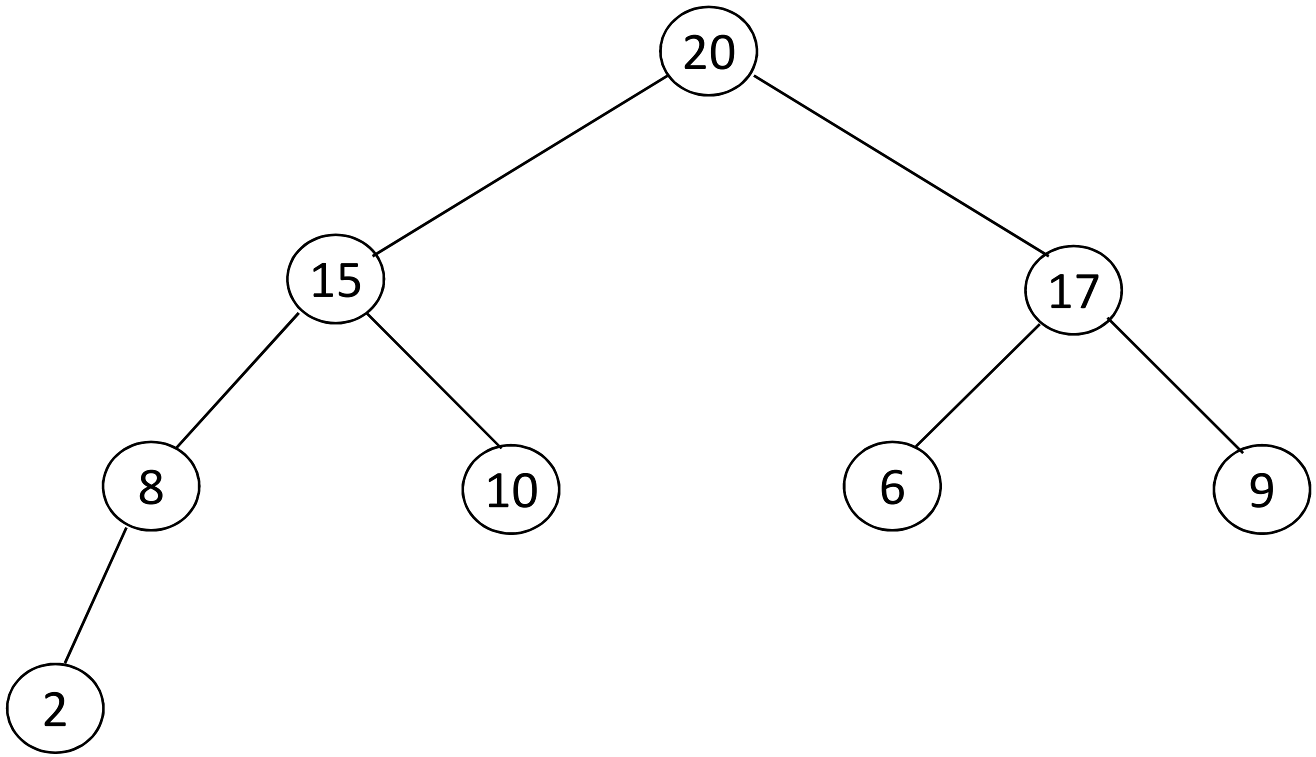


Max Heap

- Push operation
  - Add new element at the end of the tree
  - Bubble up
  - We keep maintaining the max heap property
    - key in each node is not smaller than the keys in its children

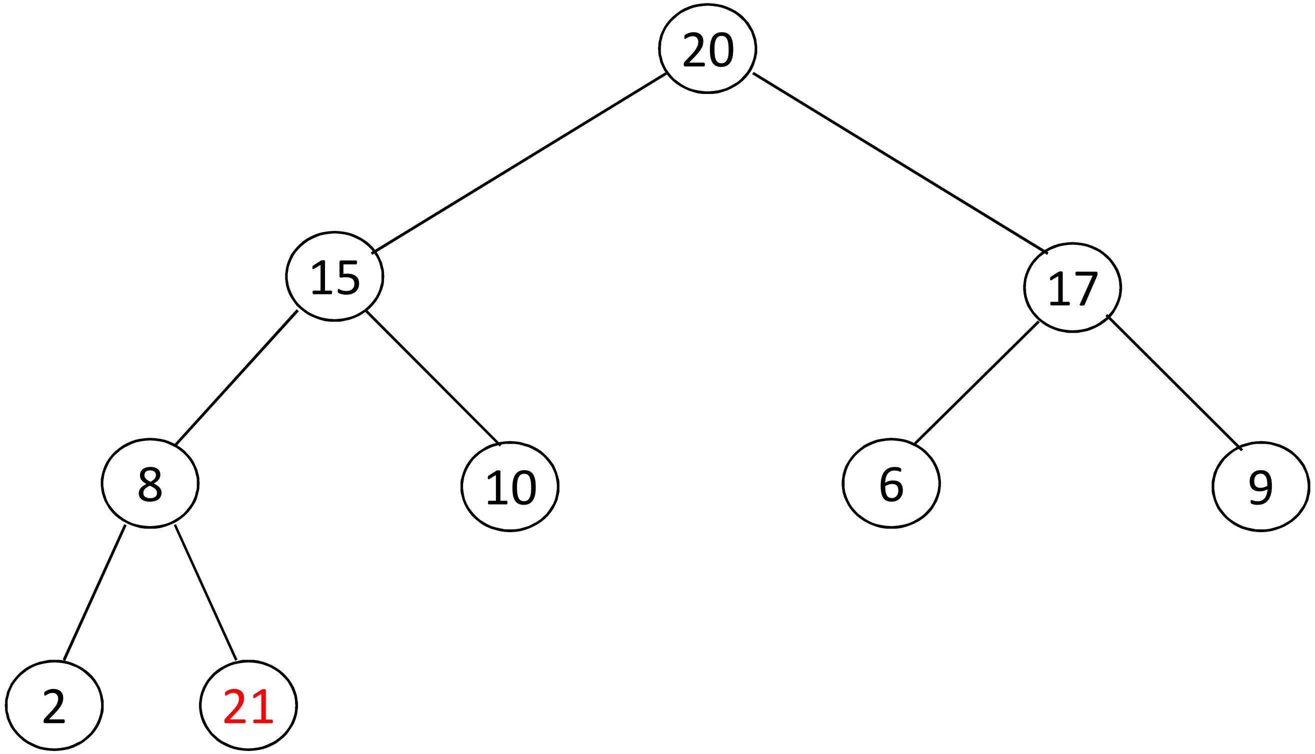
Push Example

- Push 21



Push Example

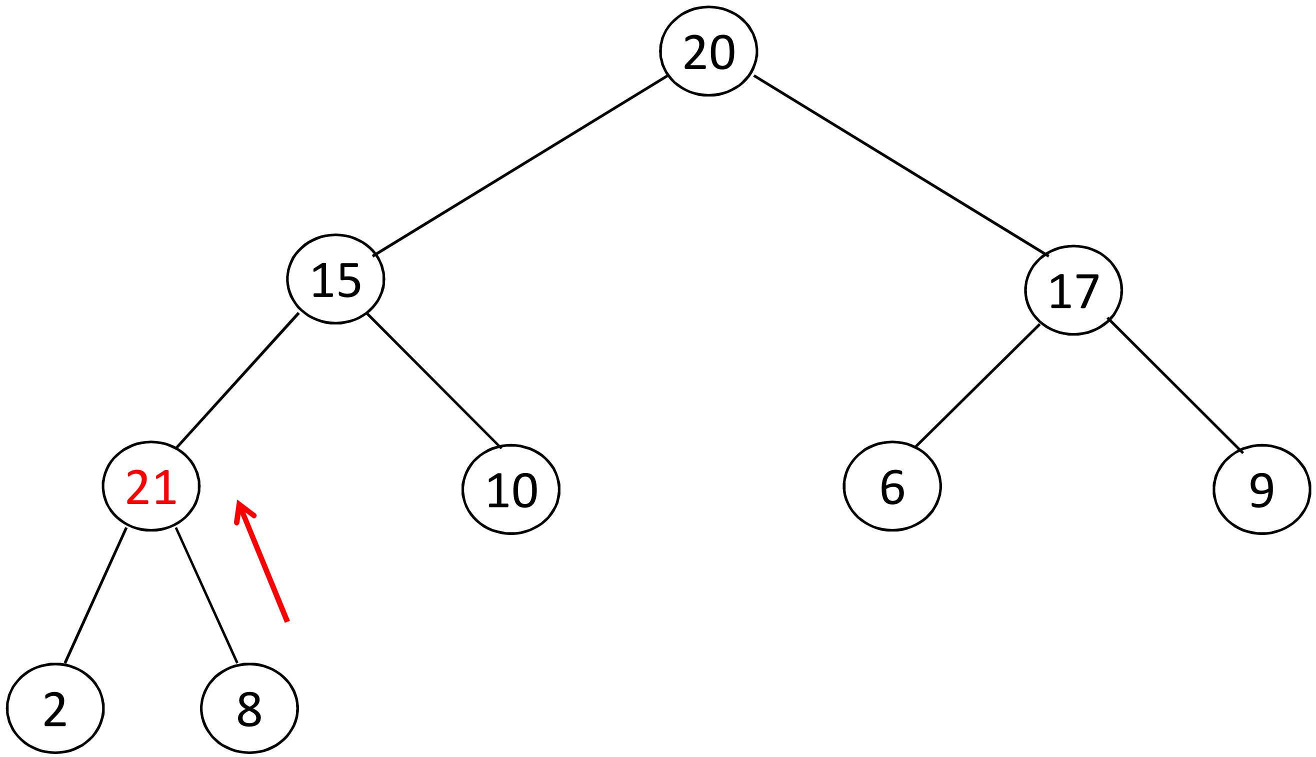
- Push 21



Create node at the end of the heap (complete binary tree)

Push Example

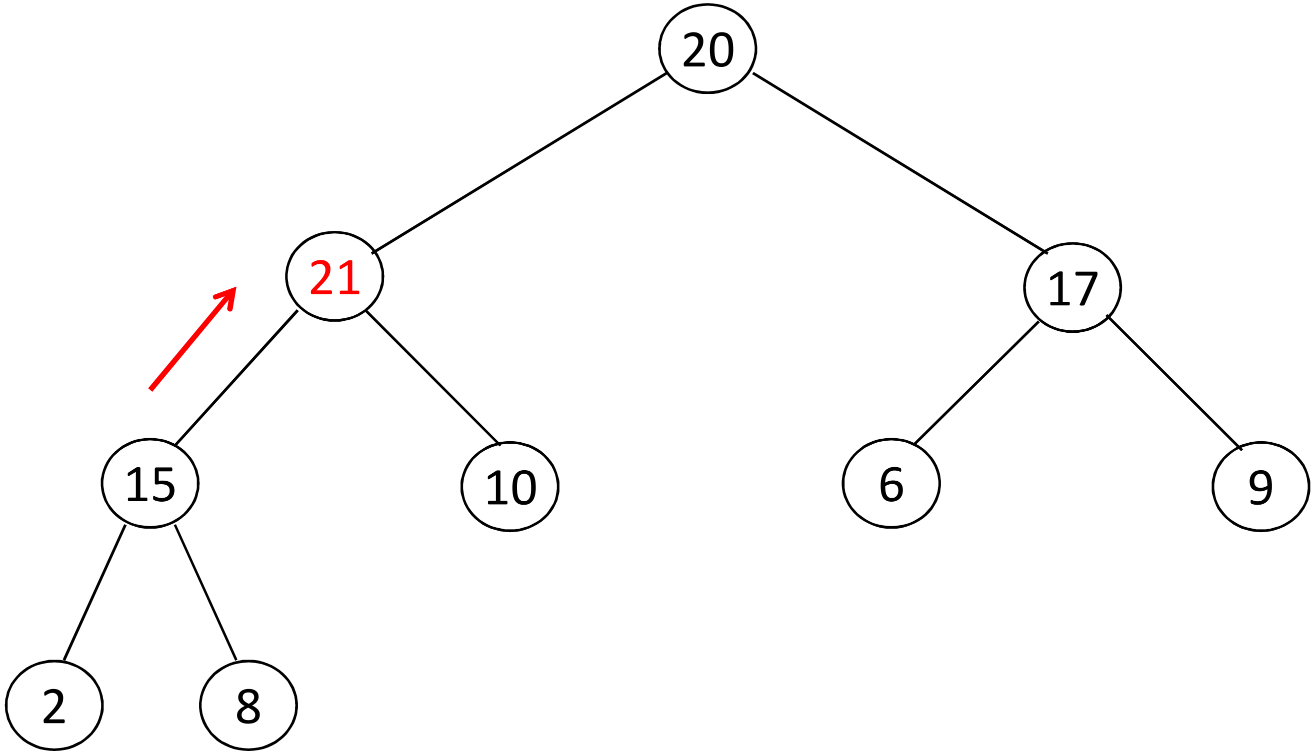
- Push 21



Bubbling up : 8 <-> 21, heap property!

Push Example

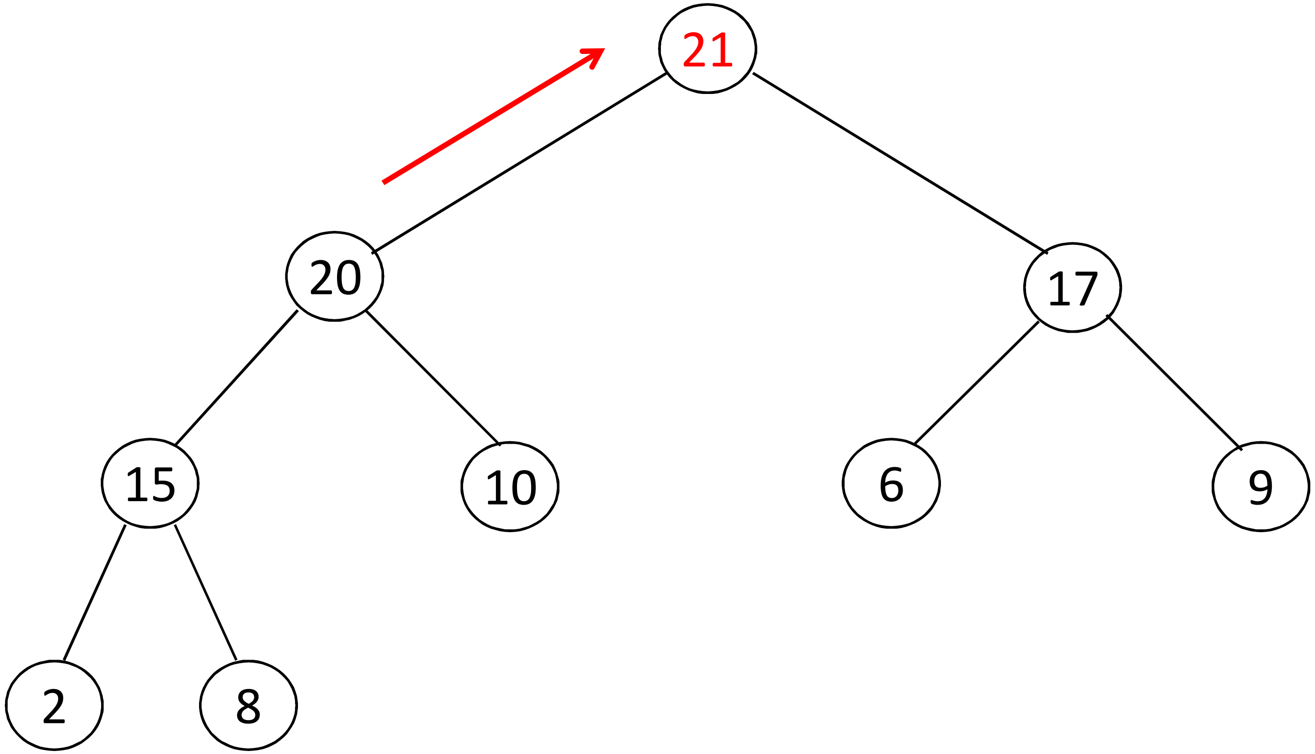
- Push 21



Bubbling up : 15 <-> 21

Push Example

- Push 21



Bubbling up : 20 <-> 21, done!

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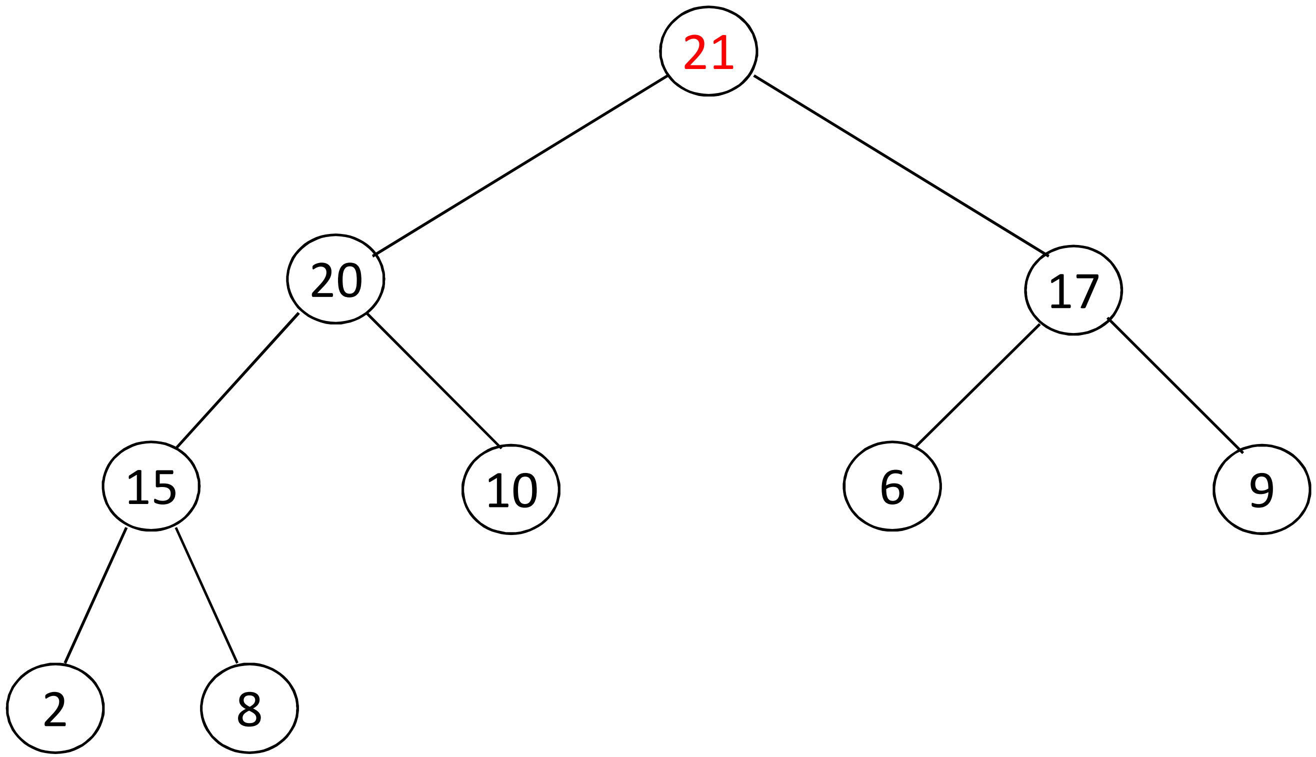
Max Heap

- Pop operation
  - Remove the largest value at root
  - Overwrite root with the last element
  - Remove last element
  - Trickle down while maintaining the max heap property

Pop Example

- Pop



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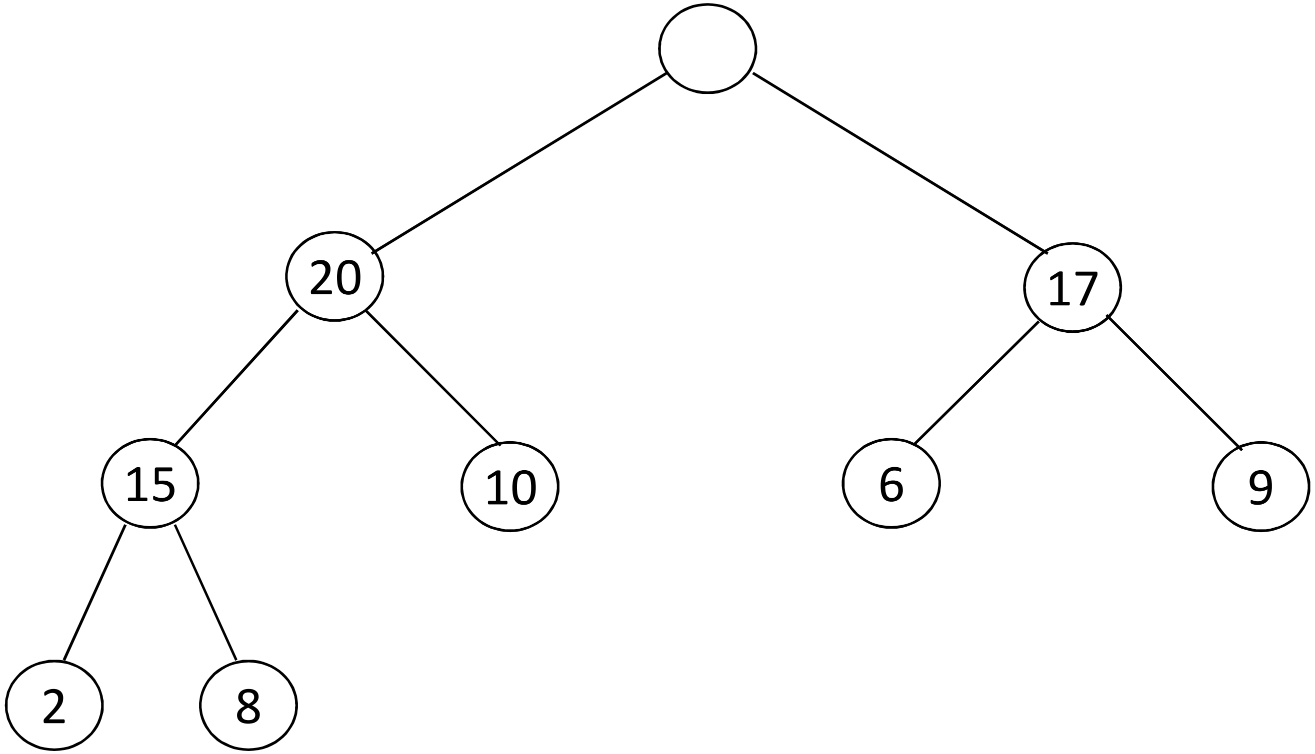
Pop Example

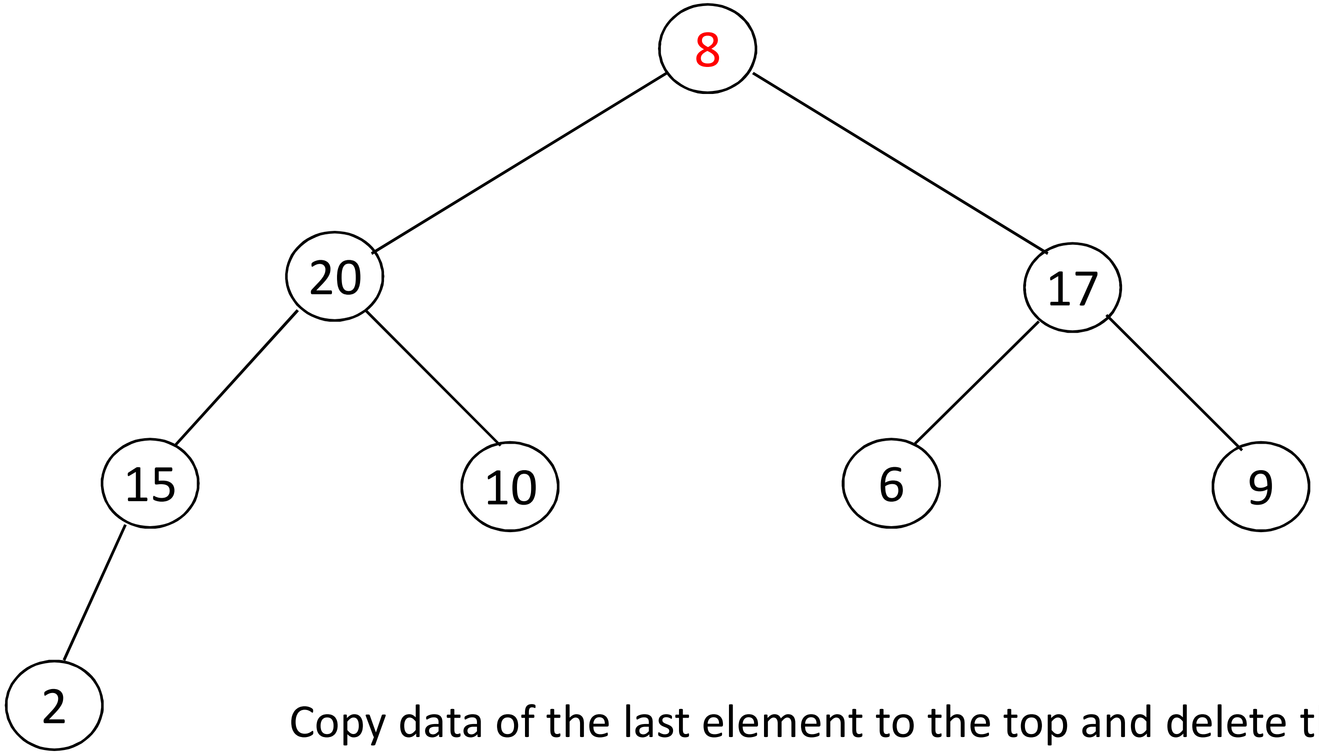
- Pop



Delete 21

Pop Example

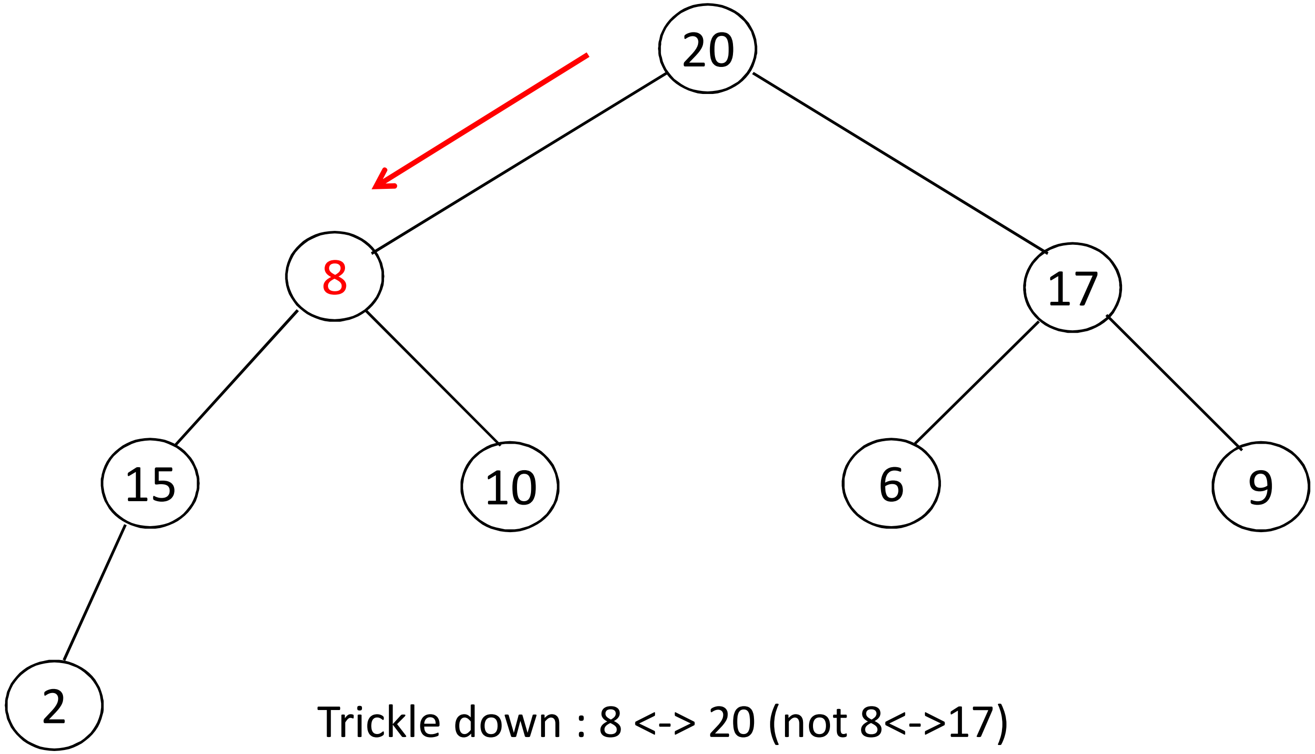
- Pop



Copy data of the last element to the top and delete the last node

Pop Example

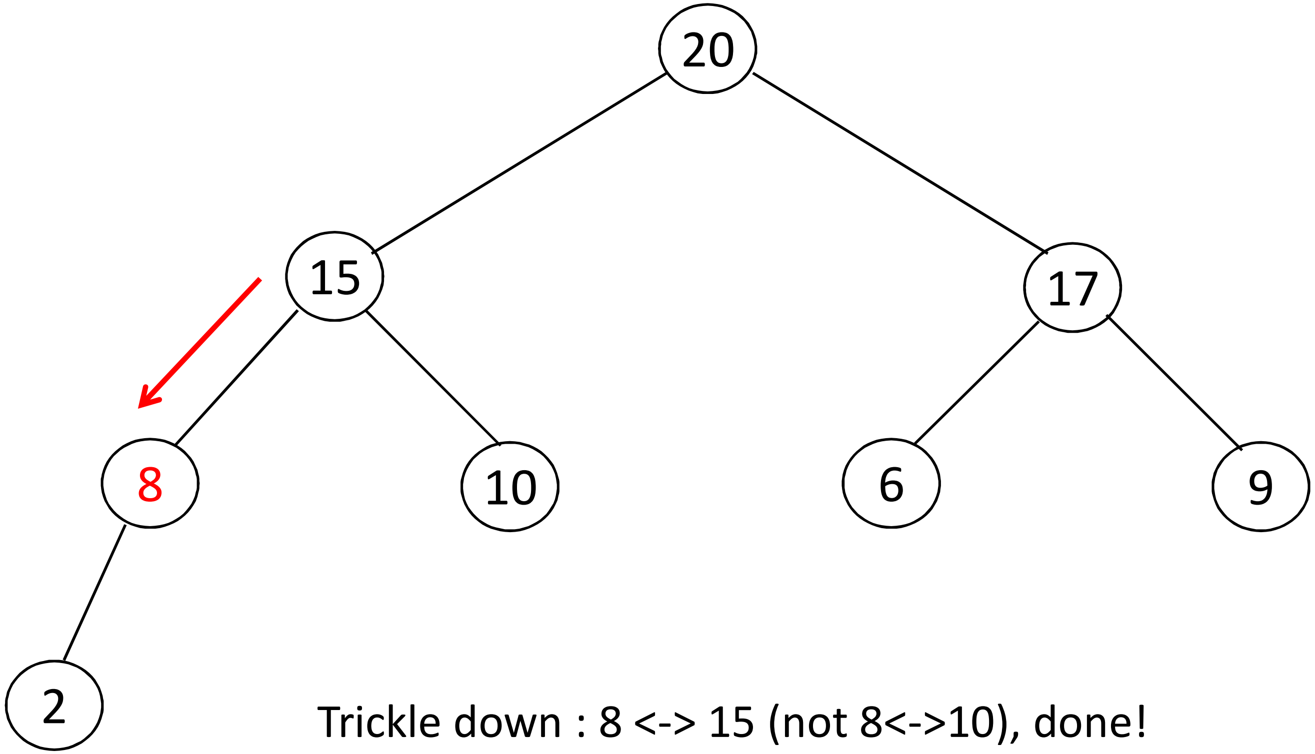
- Pop



Trickle down : 8 <-> 20 (not 8<->17)

Pop Example

- Pop



Trickle down : 8 <-> 15 (not 8<->10), done!

Heap-Sort

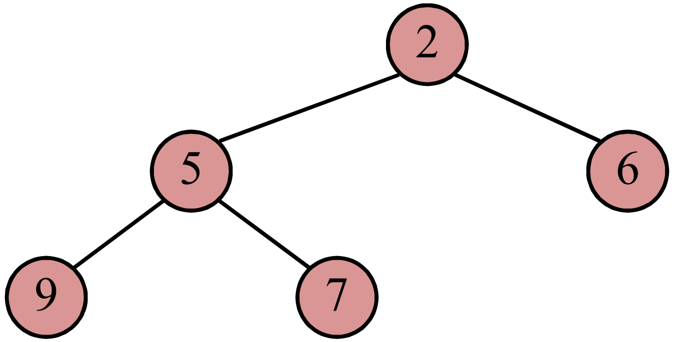
- Consider a priority queue with n items implemented by means of a min heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, empty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

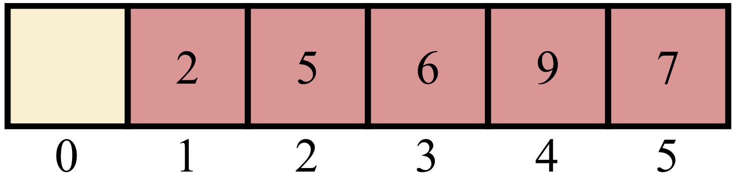
Implementation

- Can be implemented using array
  - Why? Complete binary tree
- Parent / children can be efficiently accessed
  - Parent : floor(i/2)
  - Left child : 2*i
  - Right child : 2*i + 1

Array-based Heap Implementation

- ***n*** keys on an array of length ***n* + 1**
- The cell of at index ***0*** is not used
- For the node at index ***i***
  - the left child is at index **2*i***
  - the right child is at index **2*i* + 1**
- Links between nodes are not explicitly stored
- insert corresponds to inserting at index ***n* + 1**
- removeMin corresponds to removing at index ***1***
- Yields in-place heap-sort





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Push Algorithm

```
template <class Type>
void MaxHeap<Type>::Push(const Element<Type> &x)
{
   if (n==MaxSize) { HeapFull(); return; }
   n++;
   int i; // i : current node
   for(i=n; 1; ) {
       if (i==1) break; // Root reached
       if (x.key <= heap[i/2].key) break;
       // move parent down
       heap[i] = heap[i/2];
       i /= 2;
   }
   heap[i] = x;
}
```

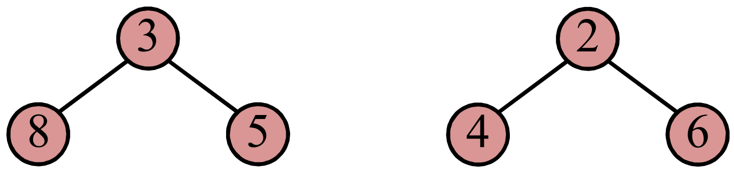
Pop Algorithm

```
template <class Type>
Element<Type> *MaxHeap<Type>::Pop(Element<Type> &x)
{
   if (!n) { HeapEmpty(); return 0;}
   x = heap[1];
   Element<Type> k = heap[n];
   n--;
   int i, j; // i : current node, j : child
   for (i=1, j=2; j<=n; )
   {
      if (j<n)
         if (heap[j].key < heap[j+1].key) j++;
      // j is the larger child
      if (k.key >= heap[j].key) break;
      heap[i] = heap[j]; // move the child up
      i = j; j *= 2; // move down a level
   }
   heap[i] = k;
   return &x;
}
```

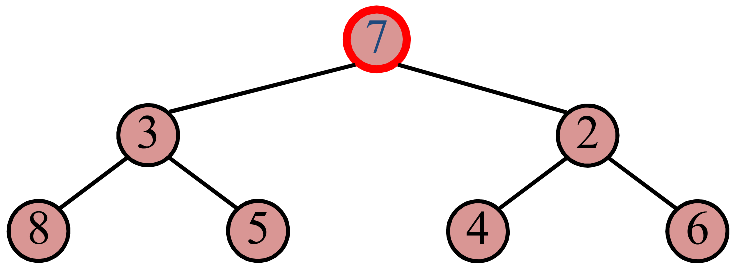
Push/Pop time complexity : O(log n)

Merging Two Heaps

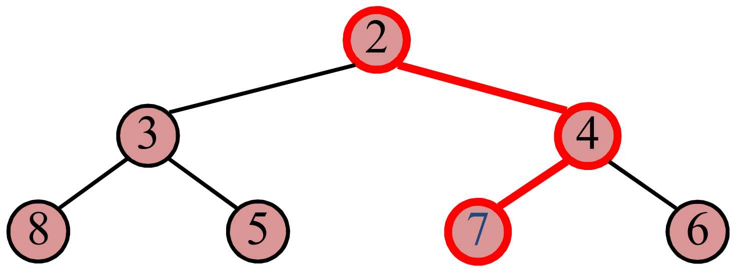
- We are given two heaps and a key ***k***



- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees

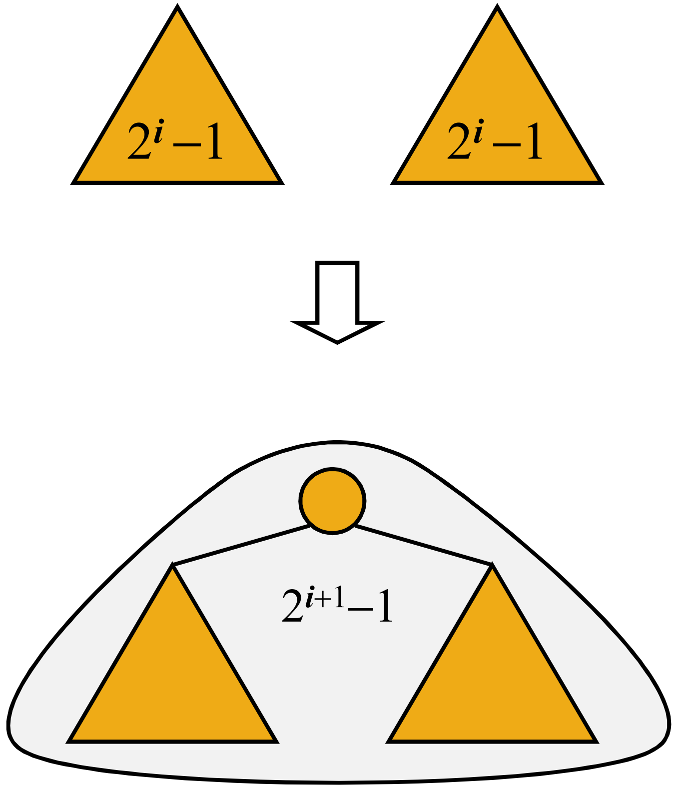


- We trickle down to restore the heap-order property

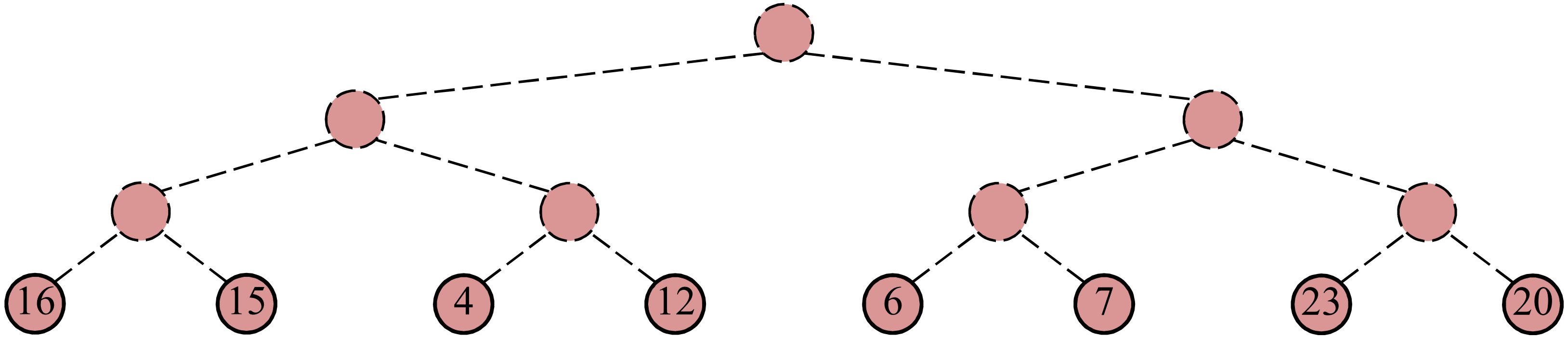


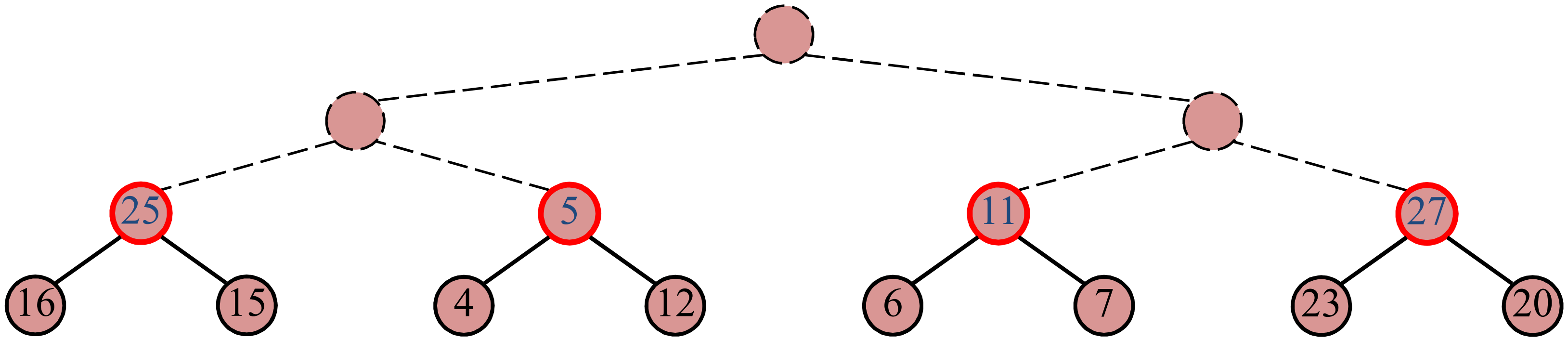
Bottom-up Heap Construction

- We can construct a heap storing ***n*** given keys in using a bottom-up construction with log ***n*** phases
- In phase ***i***, pairs of heaps with 2^i -1 keys are merged into heaps with 2^{i+1}-1 keys
- Higher opportunity for parallel execution



Example (Min Heap)





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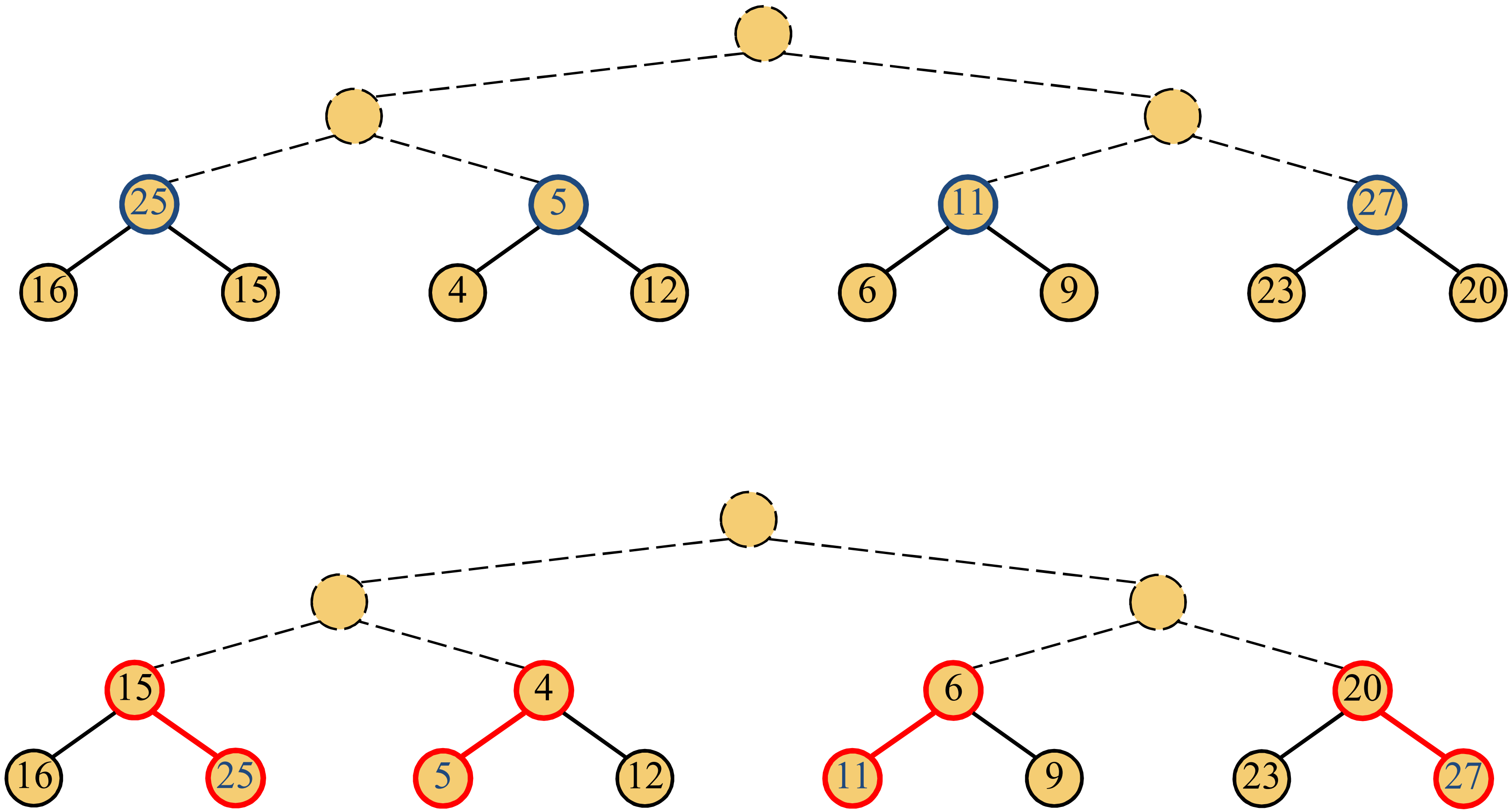
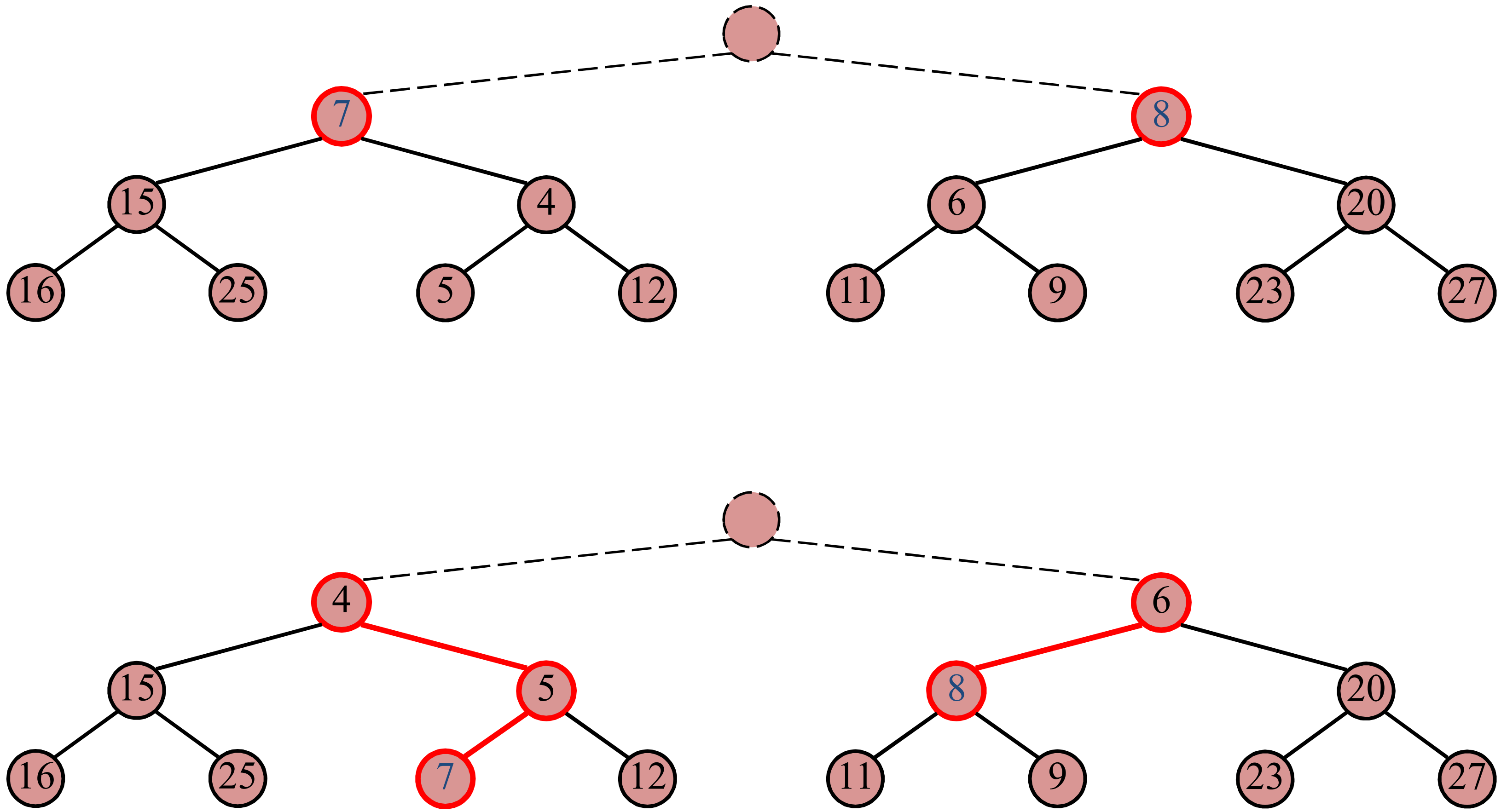
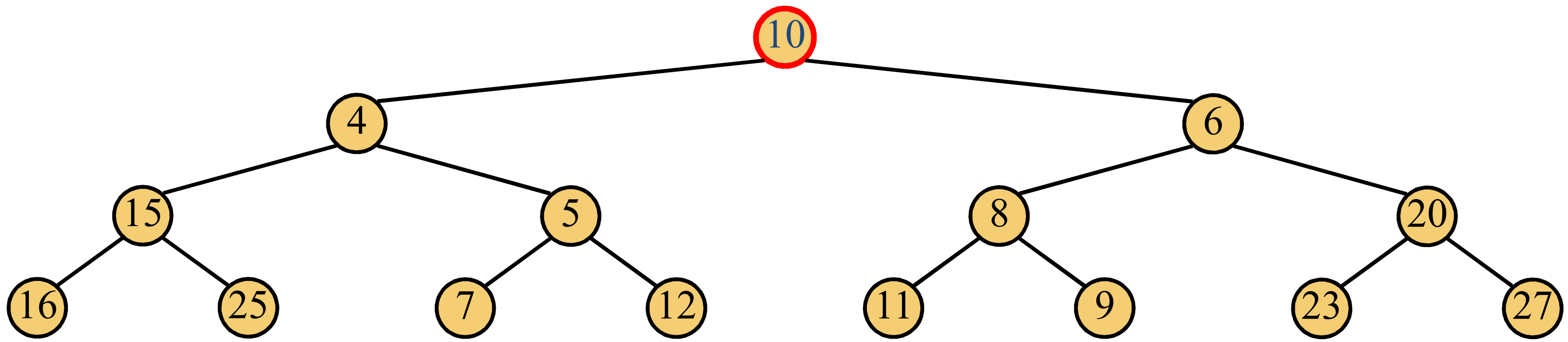
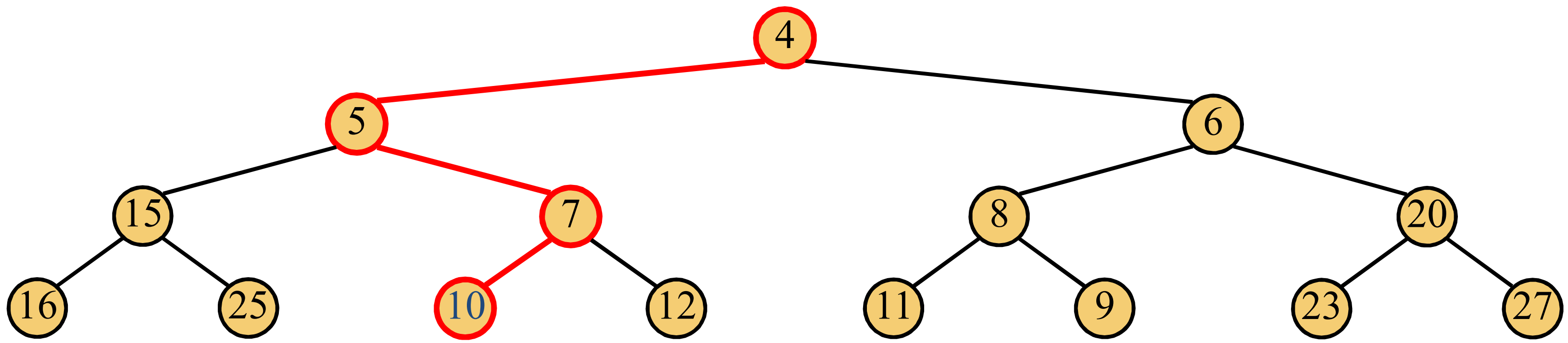


Example (Min Heap)



Example (Min Heap)





Questions?