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IE 313 Time Series Analysis

4. Models for Stationary Time Series

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Chapter 4. Models for Stationary Time Series

- 4.1 General Linear Processes
- 4.2 Moving Average (MA) Processes
- 4.3 Autoregressive (AR) Processes
- 4.4 The Mixed Autoregressive Moving Average (ARMA) Model
- 4.5 Invertibility

Chapter 4.1

General Linear Processes

Before we start

- From now on,
  - $\{Y_t\}$ denotes the observed time series
  - $\{e_t\}$ represents an unobserved white noise series
    - A sequence of i.i.d. zero-mean random variables
    - In many cases, the assumption of independence could be replaced by the weaker assumption of 'uncorrelated'
      - Independent r.v.s are uncorrelated,
        but uncorrelated r.v.s are not always independent

General linear process

- A general linear process $\{Y_t\}$ is one that can be represented as a weighted linear combination of present and past white noise terms as

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

  - If the right-hand side is an infinite series, then the following condition is usually assumed for mathematical tractability

$$\sum_{i=1}^{\infty} \psi_i^2 < \infty$$

  - Without loss of generality, we will assume that the coefficient on e_t to be 1 (i.e., $\psi_0 = 1$)

General linear process

- Example
  - Consider the case where the ψ's form an exponentially decaying sequence

$$\psi_j = \phi^j, \qquad \phi \in (-1,1)$$

  - Then,

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

  - For this example,

$$E(Y_t) = E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) = 0$$

General linear process

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

- Example
  - For this example,

$$E(Y_t) = E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) = 0$$

$$\begin{aligned} Var(Y_t) &= Var(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) \\ &= Var(e_t) + \phi^2 Var(e_{t-1}) + \phi^4 Var(e_{t-2}) + \cdots \\ &= \sigma_e^2(1 + \phi^2 + \phi^4 + \cdots) \\ &= \frac{\sigma_e^2}{1-\phi^2} \quad \text{(by summing a geometric series)} \end{aligned}$$

General linear process

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

- Example
  - For this example,

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots, \\ &\qquad e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \cdots) \\ &= Cov(\phi e_{t-1}, e_{t-1}) + Cov(\phi^2 e_{t-2}, \phi e_{t-2}) + \cdots \\ &= \phi\sigma_e^2 + \phi^3\sigma_e^2 + \phi^5\sigma_e^2 + \cdots \\ &= \phi\sigma_e^2(1 + \phi^2 + \phi^4 + \cdots) \\ &= \frac{\phi\sigma_e^2}{1-\phi^2} \end{aligned}$$

$$Corr(Y_t, Y_{t-1}) = \left[\frac{\phi\sigma_e^2}{1-\phi^2}\right] \Big/ \left[\frac{\sigma_e^2}{1-\phi^2}\right] = \phi$$

General linear process

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

- Example
  - Similarly,

$$Cov(Y_t, Y_{t-k}) = \frac{\phi^k \sigma_e^2}{1-\phi^2}$$

$$Corr(Y_t, Y_{t-k}) = \phi^k$$

  - Therefore, this process is stationary
    - Mean is constant
    - Autocovariance depends only on time lag

- For a general linear process $Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$,

$$E(Y_t) = 0, \;\; \gamma_k = Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \quad \text{for } k \geq 0$$

Chapter 4.2

Moving Average Processes

Moving average process

- **Moving Average (MA) process**
  - Only a finite number of the ψ-weights are nonzero
  - For moving average processes, we will change notation as

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

    - Our textbook puts negative signs before θ's, but some others put plus signs. So you should be careful when dealing with MA processes on other books or softwares
  - We call the above process as **a moving average of order $\boldsymbol{q}$**, or $\boldsymbol{MA(q)}$

First-order MA process

- Consider a first-order MA process

$$Y_t = e_t - \theta e_{t-1}$$

  – Then,

$$E(Y_t) = E(e_t - \theta e_{t-1}) = 0$$

$$\begin{aligned} Var(Y_t) &= Var(e_t - \theta e_{t-1}) \\ &= Var(e_t) + \theta^2 Var(e_{t-1}) \\ &= \sigma_e^2(1 + \theta^2) \end{aligned}$$

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) \\ &= Cov(-\theta e_{t-1}, e_{t-1}) \\ &= -\theta \sigma_e^2 \end{aligned}$$

$$Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) = 0$$

First-order MA process

- Consider a first-order MA process

$$Y_t = e_t - \theta e_{t-1}$$

  – Similarly, $Cov(Y_t, Y_{t-k}) = 0$ for $k \geq 2$

    - That is, MA(1) process has no correlation beyond lag 1

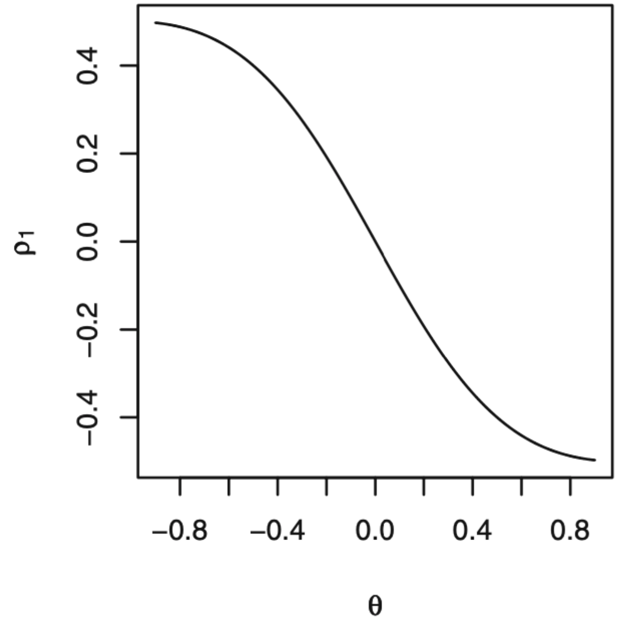
  – In summary, for an MA(1) model,

$$\begin{aligned}
&E(Y_t) = 0 \\
&\gamma_0 = Var(Y_t) = \sigma_e^2(1 + \theta^2) \\
&\gamma_1 = Cov(Y_t, Y_{t-1}) = -\theta\sigma_e^2 \\
&\rho_1 = (-\theta)/(1 + \theta^2) \\
&\gamma_k = \rho_k = 0, \qquad k \geq 2
\end{aligned}$$

← **stationary!**

First-order MA process

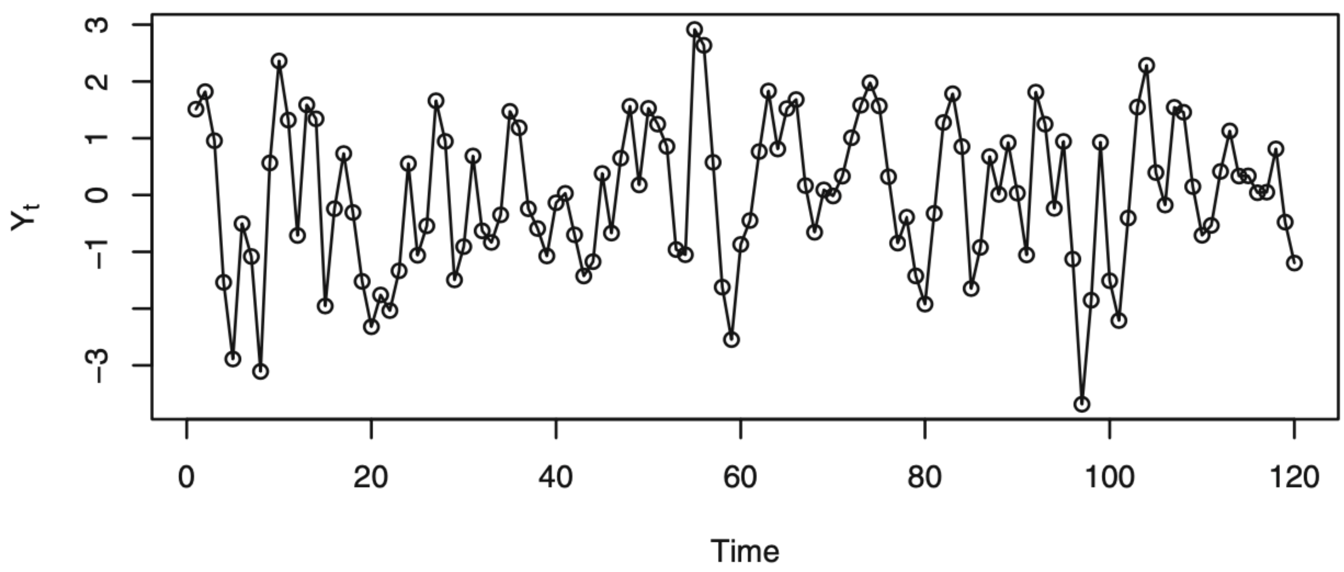
Exhibit 4.1 Lag 1 Autocorrelation of an MA(1) Process for Different θ



θ	$\rho_1 = -\theta/(1+\theta^2)$	θ	$\rho_1 = -\theta/(1+\theta^2)$
0.1	−0.099	0.6	−0.441
0.2	−0.192	0.7	−0.470
0.3	−0.275	0.8	−0.488
0.4	−0.345	0.9	−0.497
0.5	−0.400	1.0	−0.500

First-order MA process

Exhibit 4.2 Time Plot of an MA(1) Process with θ = −0.9



- MA(1) with $\theta = -0.9$ (i.e., $Y_t = e_t + 0.9e_{t-1}$)
  - For this process, $\rho_1 = 0.4972$ (moderately strong)
  - Consecutive observations tend to be closely related
  - Plot is relatively smooth with occasional large fluctuations

First-order MA process

Exhibit 4.3 Plot of Y_t versus Y_{t-1} for MA(1) Series in Exhibit 4.2

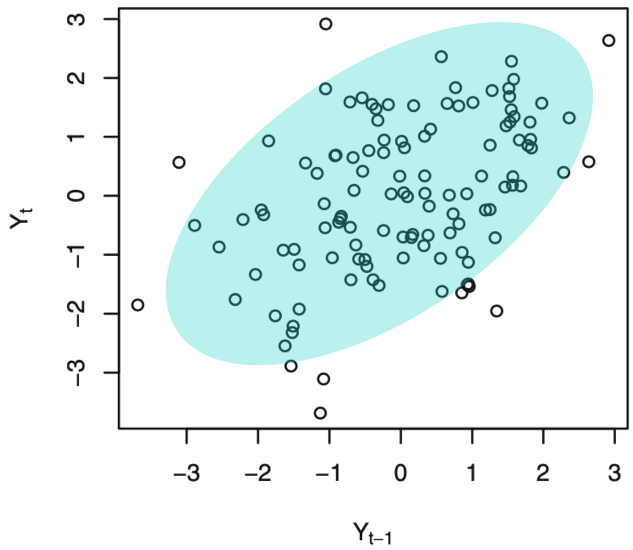
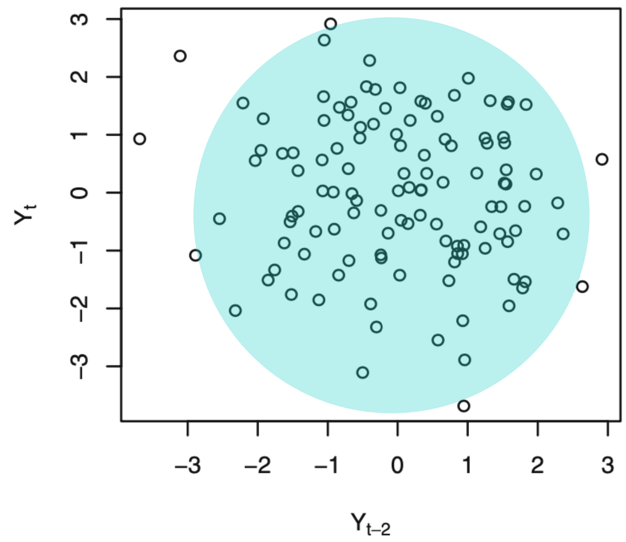


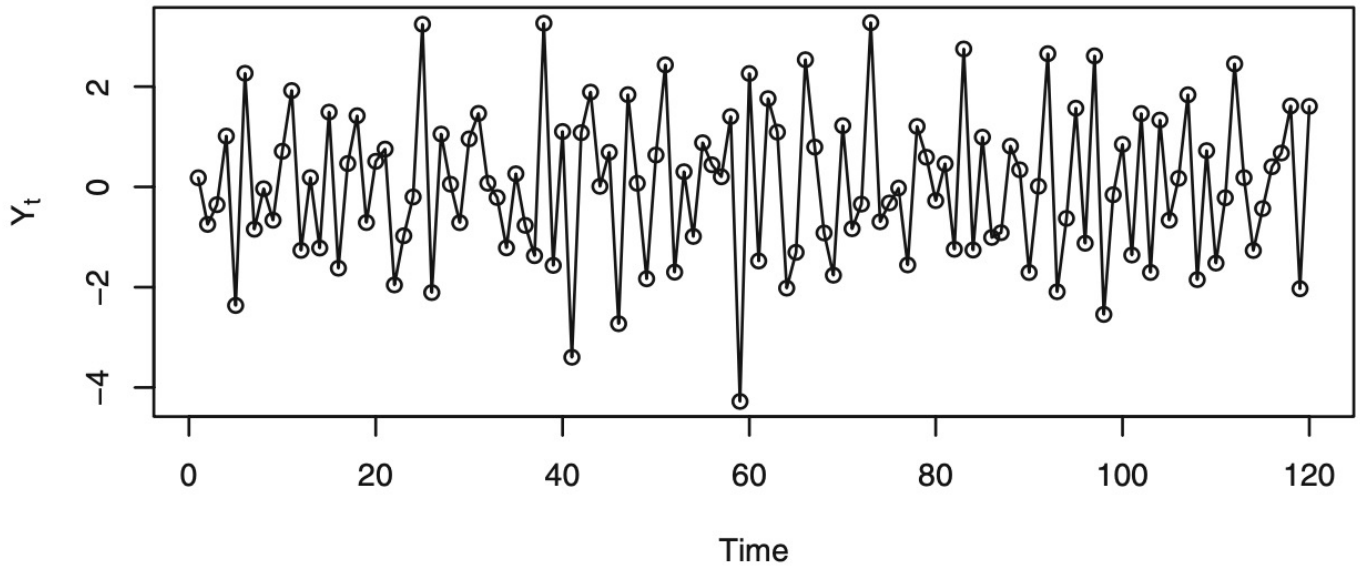
Exhibit 4.4 Plot of Y_t versus Y_{t-2} for MA(1) Series in Exhibit 4.2



- MA(1) with $\theta = -0.9$ (i.e., $Y_t = e_t + 0.9e_{t-1}$)
  - Exhibit 4.3 shows moderate lag 1 autocorrelation
  - Exhibit 4.4 shows zero autocorrelation at lag 2

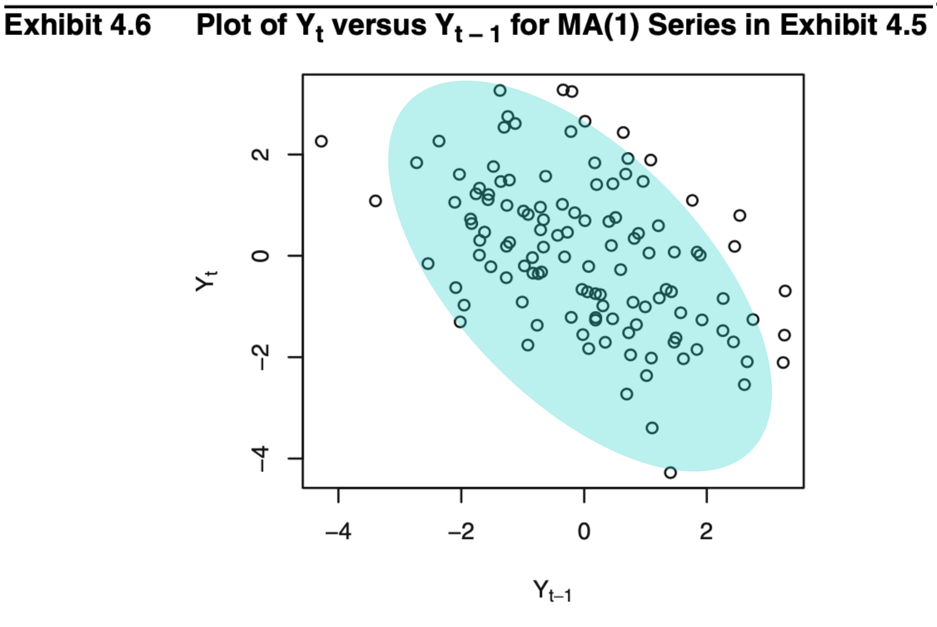
First-order MA process

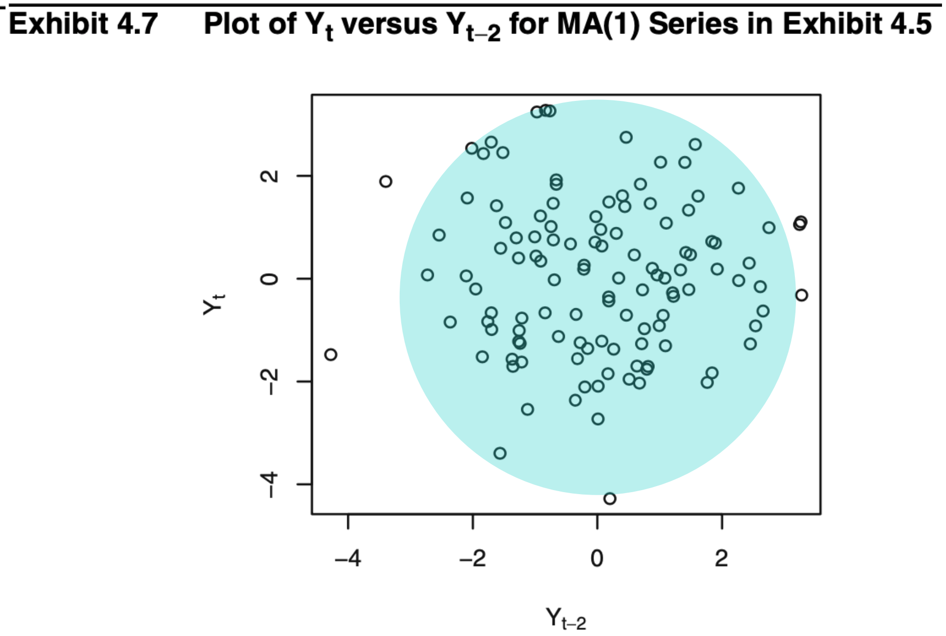
Exhibit 4.5 Time Plot of an MA(1) Process with θ = +0.9



- MA(1) with $\theta = +0.9$ (i.e., $Y_t = e_t - 0.9e_{t-1}$)
  - For this process, $\rho_1 = -0.4972$ (moderately strong)
  - Consecutive observations tend to be negatively related
  - Plot is quite jagged over time

First-order MA process





- MA(1) with $\theta = +0.9$ (i.e., $Y_t = e_t - 0.9e_{t-1}$)
  - Exhibit 4.6 shows strong negative lag 1 autocorrelation
  - Exhibit 4.7 shows zero autocorrelation at lag 2

Second-order MA process

- Consider a second-order MA process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

  – Then,

$$E(Y_t) = E(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) = 0$$

$$\begin{aligned} Var(Y_t) &= Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\ &= Var(e_t) + \theta_1^2 Var(e_{t-1}) + \theta_2^2 Var(e_{t-2}) \\ &= \sigma_e^2(1 + \theta_1^2 + \theta_2^2) \end{aligned}$$

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\ &= Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_2 e_{t-2}, -\theta_1 e_{t-2}) \\ &= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_e^2 = (-\theta_1 + \theta_1\theta_2)\sigma_e^2 \end{aligned}$$

Second-order MA process

- Consider a second-order MA process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} \qquad \leftarrow \textbf{stationary!}$$

  – But,

$$\begin{aligned} Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\ &= Cov(-\theta_2 e_{t-2}, e_{t-2}) = -\theta_2 \sigma_e^2 \qquad \leftarrow \textbf{non-zero!} \end{aligned}$$

  – Thus, for an MA(2) process,

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

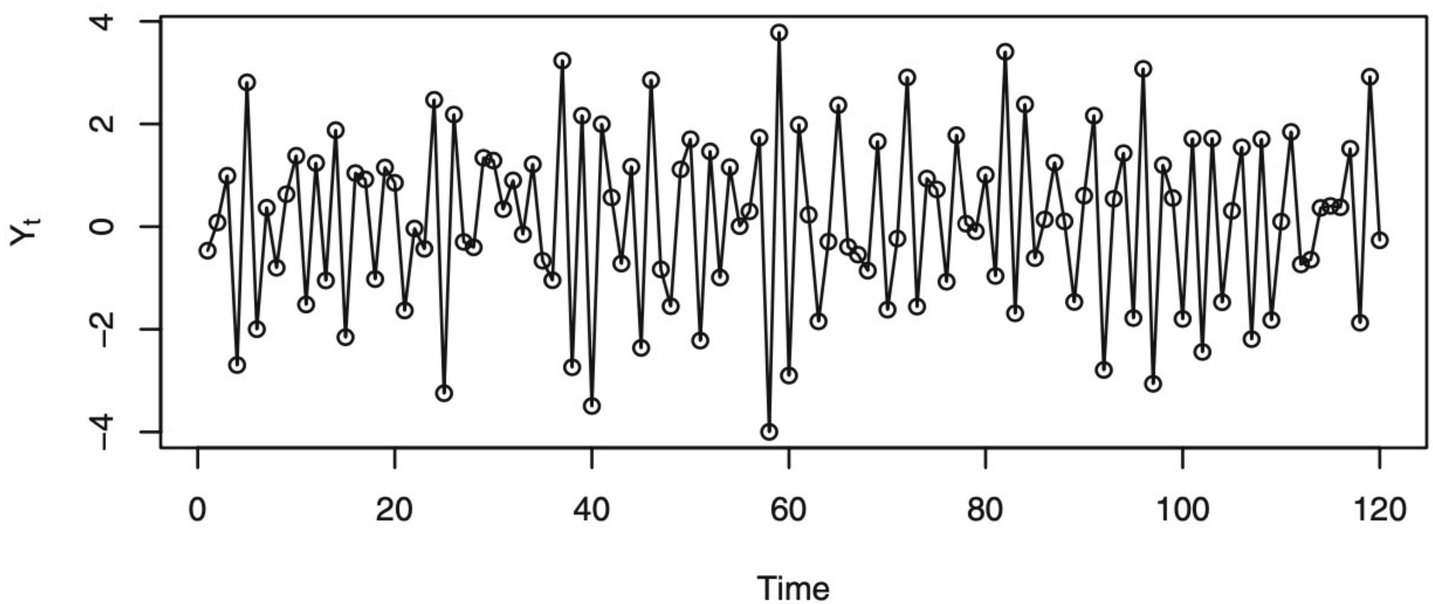
$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_k = 0, \qquad \text{for } k \geq 3$$

    - That is, MA(2) process has no correlation beyond lag 2

Second-order MA process

Exhibit 4.8 Time Plot of an MA(2) Process with $\theta_1 = 1$ and $\theta_2 = -0.6$



- MA(2) with $\theta_1 = 1$ and $\theta_2 = -0.6$ (i.e., $Y_t = e_t - e_{t-1} + 0.6e_{t-2}$)
  - For this process, $\rho_1 = -0.678$ and $\rho_2 = 0.254$
  - Consecutive observations tend to be negatively related

Second-order MA process

Exhibit 4.9 Plot of Y_t versus Y_{t-1} for MA(2) Series in Exhibit 4.8

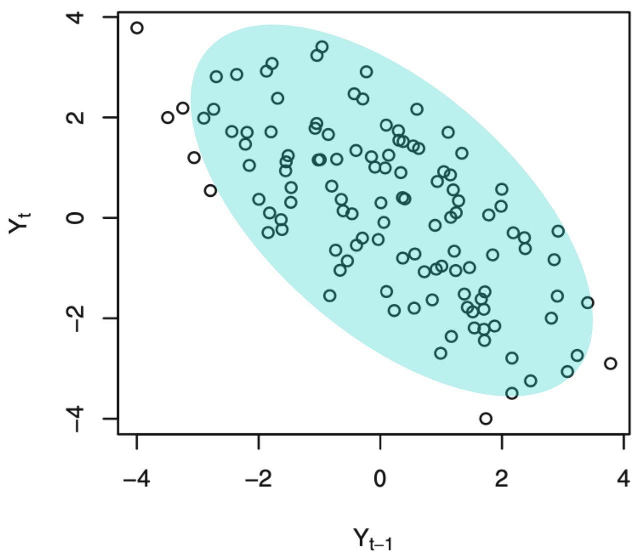
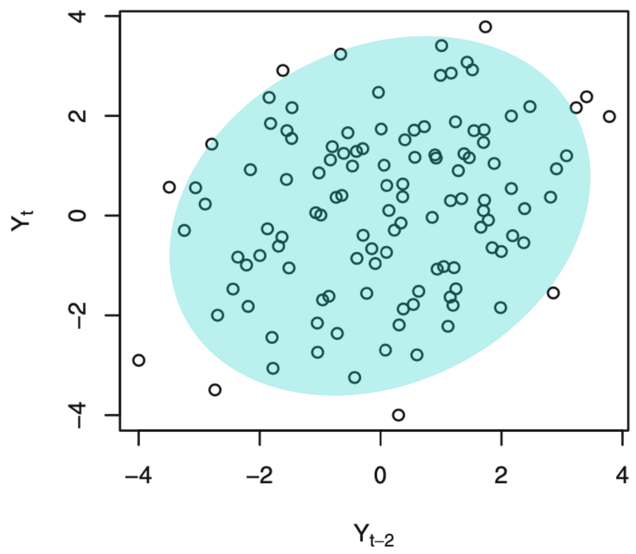


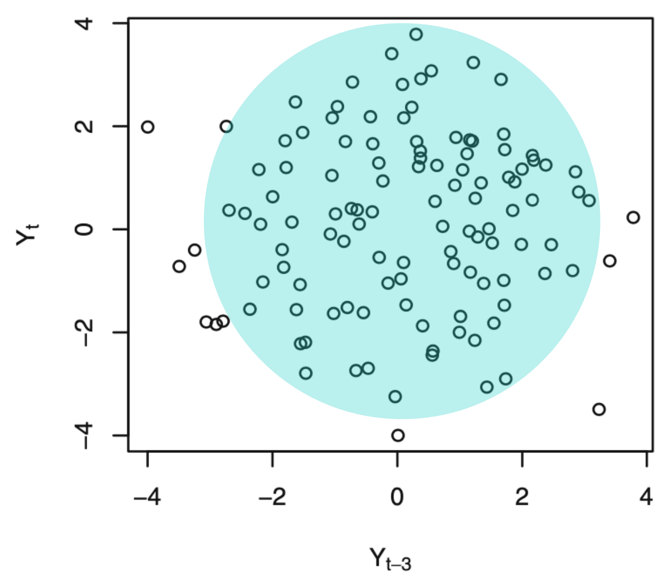
Exhibit 4.10 Plot of Y_t versus Y_{t-2} for MA(2) Series in Exhibit 4.8



- MA(2) with $\theta_1 = 1$ and $\theta_2 = -0.6$ (i.e., $Y_t = e_t - e_{t-1} + 0.6e_{t-2}$)
  - Exhibit 4.9 shows strong negative lag 1 autocorrelation
  - Exhibit 4.10 shows weak positive autocorrelation at lag 2

Second-order MA process

Exhibit 4.11 Plot of Y_t versus Y_{t-3} for MA(2) Series in Exhibit 4.8



- MA(2) with $\theta_1 = 1$ and $\theta_2 = -0.6$ (i.e., $Y_t = e_t - e_{t-1} + 0.6e_{t-2}$)
  - Exhibit 4.9 shows strong negative lag 1 autocorrelation
  - Exhibit 4.10 shows weak positive autocorrelation at lag 2
  - Exhibit 4.11 shows zero autocorrelation at lag 3

General MA(q) process

- Consider a general MA(q) process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$ ← **stationary!**

  – Similar calculations show that

    - $\gamma_0 = Var(Y_t) = \left(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2\right)\sigma_e^2$
    - $\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } 1 \le k \le q \\ 0 & \text{for } k > q \end{cases}$

  – The autocorrelation function *"cuts off"* after lag q (become zero)

  – Its shape can be almost anything for the earlier lags

Chapter 4.3

Autoregressive Processes

Autoregressive process

- **Autoregressive (AR) process**
  - As its name suggests, regression on itself
  - **A $\boldsymbol{p}$th-order autoregressive process**, or $\boldsymbol{AR(p)}$ can be written as

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

    - The current value of Y_t is a linear combination of the p most recent past values of itself plus an "innovation" term e_t
      - e_t incorporates everything new in the series at time t that is not explained by the past values
      - Thus, for every t, we assume that e_t is independent of $Y_{t-1}, Y_{t-2}, \ldots$

First-order AR process

- Consider a first-order AR process

$$Y_t = \phi Y_{t-1} + e_t$$

  – Assume that the process is stationary and its mean is zero
    - If the process has nonzero mean, we can subtract out its mean
    - Conditions for stationarity will be considered later

  – Then,

$$\gamma_0 = Var(Y_t) = Var(\phi Y_{t-1} + e_t) = \phi^2 \gamma_0 + \sigma_e^2$$

  – Solving for γ_0 yields

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi^2}$$

→ we can see that $\phi^2 < 1$ or $|\phi| < 1$

(Note that when $\phi = 1$, it becomes a random walk, which is non-stationary)

First-order AR process

- Consider a first-order AR process

$$Y_t = \phi Y_{t-1} + e_t$$

  – Now multiply Y_{t-k} to the both sides of the above equation and take expected values

$$E(Y_{t-k}Y_t) = \phi E(Y_{t-k}Y_{t-1}) + E(e_t Y_{t-k})$$

  or

$$\gamma_k = \phi \gamma_{k-1} + E(e_t Y_{t-k})$$

$$\begin{aligned}\gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= E(Y_t Y_{t-k}) - E(Y_t)E(Y_{t-k}) \\ &= E(Y_t Y_{t-k}) \ (\because E(Y_t) = E(Y_{t-k}) = 0)\end{aligned}$$

  – Since e_t is independent of Y_{t-k} and Y_t is stationary with zero mean,

$$E(e_t Y_{t-k}) = E(e_t)E(Y_{t-k}) = 0 \quad \Rightarrow \quad \gamma_k = \phi \gamma_{k-1}, \text{ for } k = 1,2,3, \ldots$$

First-order AR process

$$\gamma_k = \phi\gamma_{k-1}, \quad \text{for } k = 1,2,3,\ldots$$

- Consider a first-order AR process

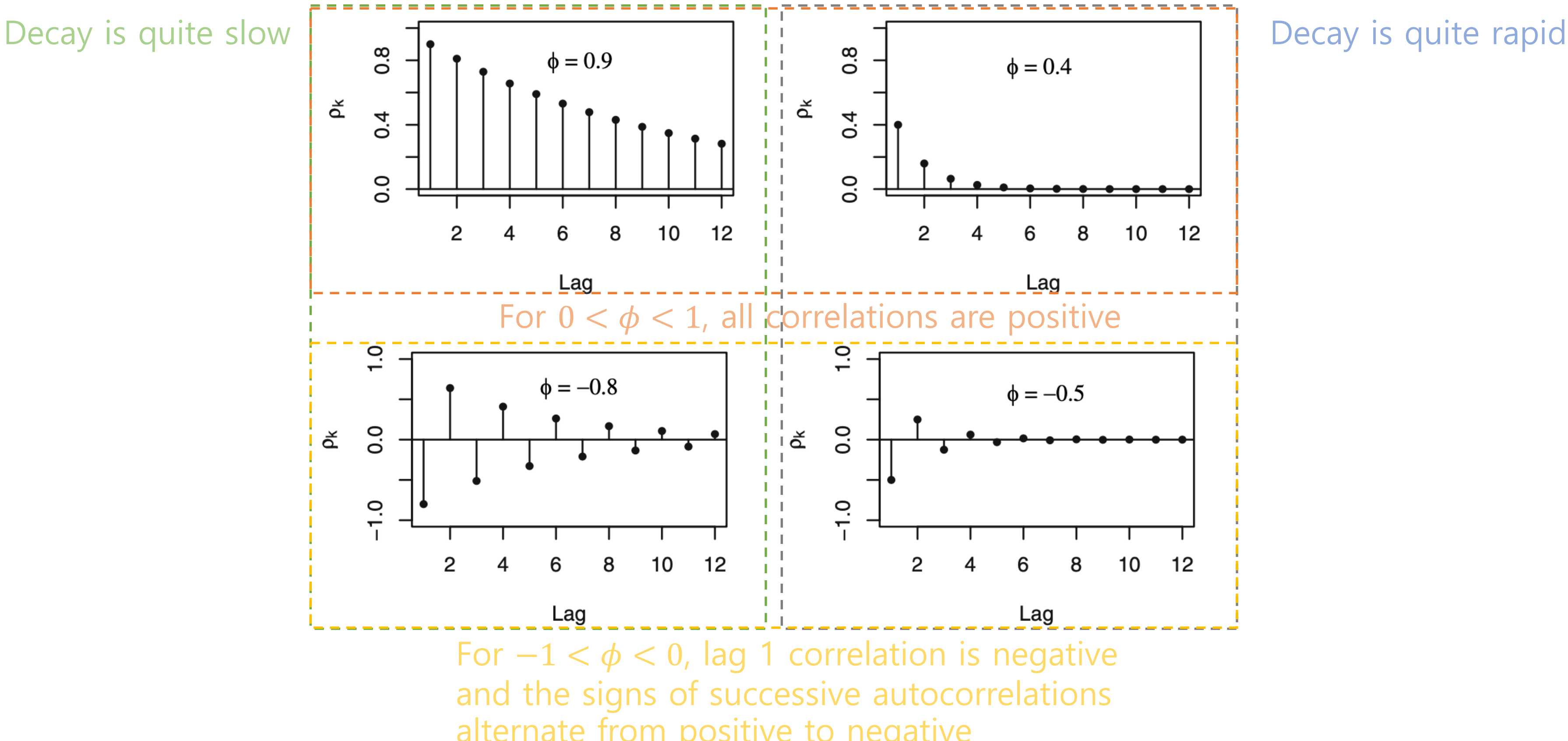
$$Y_t = \phi Y_{t-1} + e_t$$

  - Setting $k = 1$,
    - $\gamma_1 = \phi\gamma_0 = \frac{\phi\sigma_e^2}{1-\phi^2}$
  - With $k = 2$,
    - $\gamma_2 = \phi\gamma_1 = \phi^2\gamma_0 = \frac{\phi^2\sigma_e^2}{1-\phi^2}$
  - In general,
    - $\gamma_k = \phi^k\gamma_0 = \frac{\phi^k\sigma_e^2}{1-\phi^2}$
  - And thus,
    - $\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k, \text{ for } k = 1,2,3,\ldots$

First-order AR process

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k, \text{ for } k = 1,2,3,\ldots$$

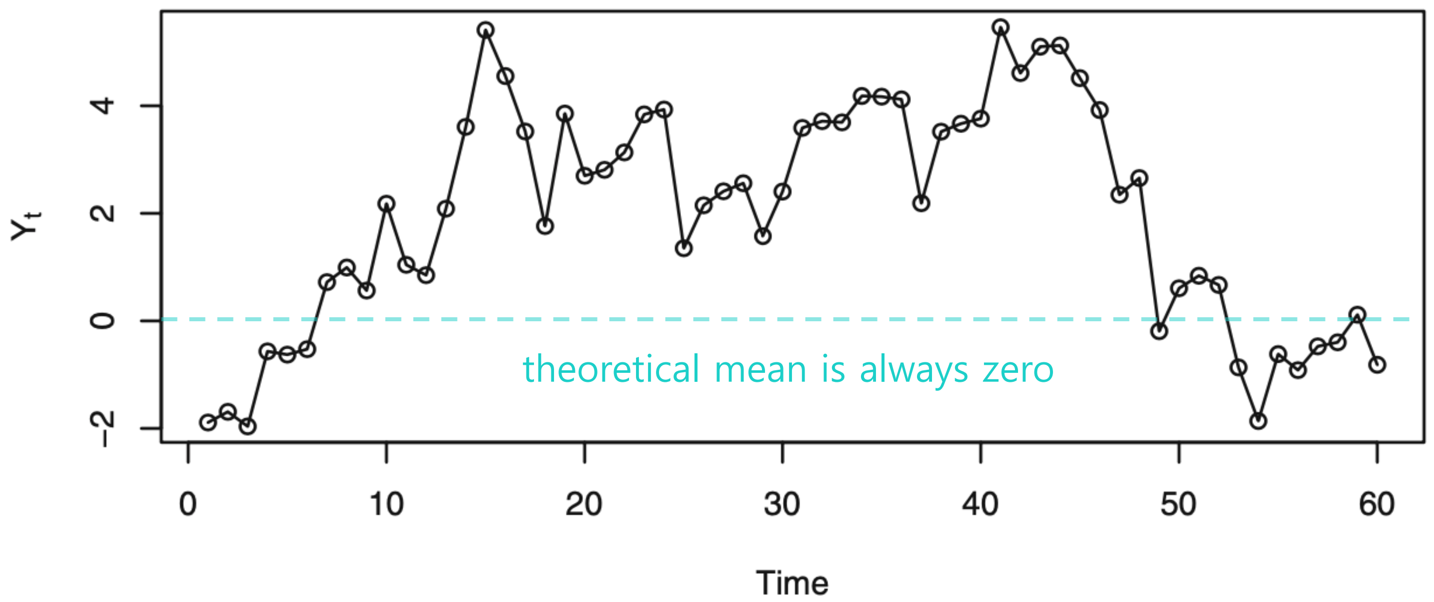
Exhibit 4.12 Autocorrelation Functions for Several AR(1) Models



- Since $|\phi| < 1$, the magnitude of AFC decreases exponentially as the number of lags, k, increases

First-order AR process

Exhibit 4.13 Time Plot of an AR(1) Series with $\phi = 0.9$



- AR(1) with $\phi = 0.9$ (i.e., $Y_t = 0.9Y_t + e_t$)
  - Hangs together
  - Remains on the same side of the mean for extended periods

First-order AR process

Exhibit 4.14 Plot of Y_t vs Y_{t-1} for AR(1)

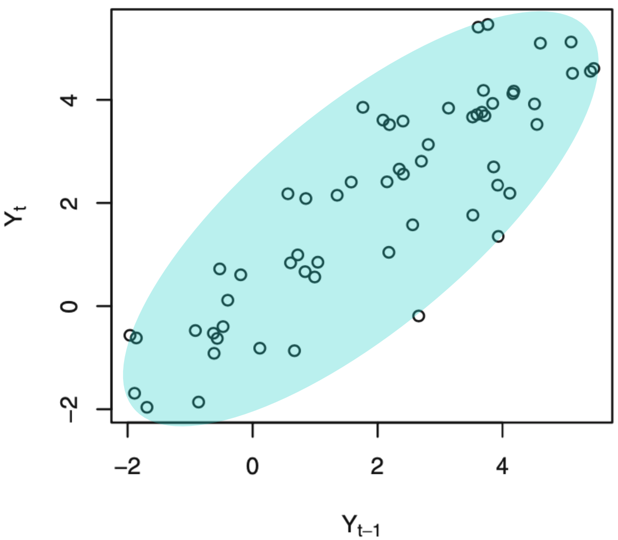


Exhibit 4.15 Plot of Y_t vs Y_{t-2} for AR(1)

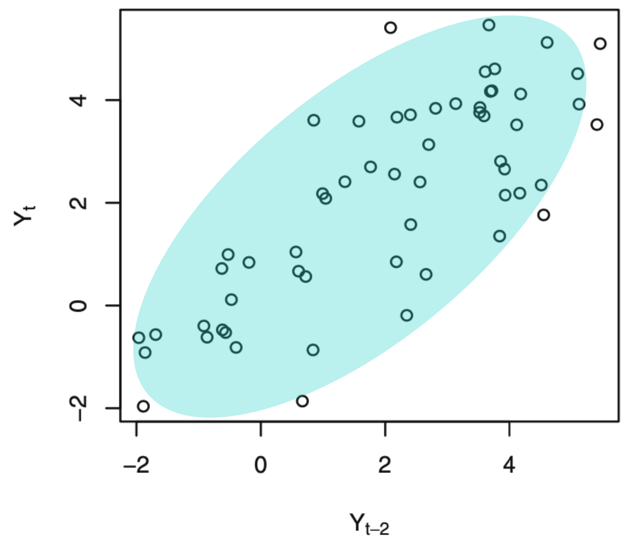
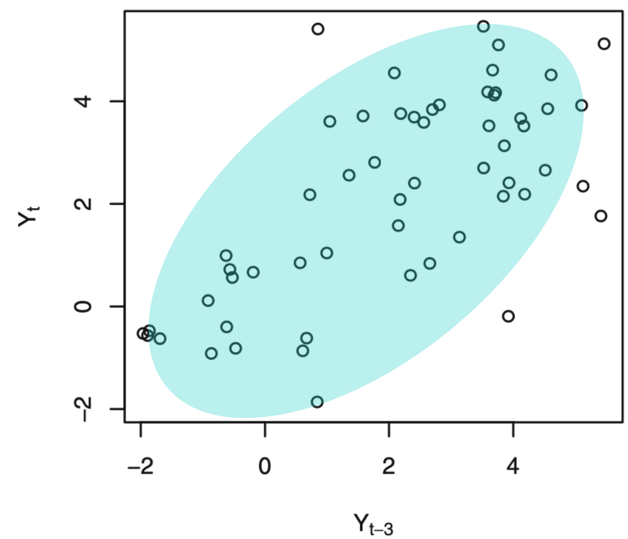


Exhibit 4.16 Plot of Y_t vs Y_{t-3} for AR(1)



- AR(1) with $\phi = 0.9$ (i.e., $Y_t = 0.9Y_t + e_t$)
  - Exhibit 4.14 shows strong lag 1 autocorrelation ($\rho_1 = \phi = 0.9$)
  - Exhibit 4.15 shows still strong autocorrelation at lag 2 ($\rho_2 = \phi^2 = 0.81$)
  - Exhibit 4.16 shows still strong autocorrelation at lag 3 ($\rho_3 = \phi^3 = 0.729$)

General linear process version of AR(1) model

- The recursive definition of the AR(1) process is extremely useful for interpreting the model
- For other purposes (e.g., calculating ACF), it is convenient to express the AR(1) model as a **general linear process**
  - Notice that $Y_{t-1} = \phi Y_{t-2} + e_{t-1}$
  - Then,

$$\begin{aligned} Y_t &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 Y_{t-2} \end{aligned}$$

  - Repeat this for $k-1$ times, we get

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k}$$

General linear process version of AR(1) model

- The recursive definition of the AR(1) process is extremely useful for interpreting the model
- For other purposes (e.g., calculating ACF), it is convenient to express the AR(1) model as a **general linear process**
  - Assuming $|\phi| < 1$ and letting k increase without bound, it seems reasonable that we should obtain the infinite series representation

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots$$

  - Notice that this is in the form of the general linear process with $\psi_j = \phi^j$, which we already investigated in Section 4.1

Second-order AR process

- Now consider the AR(2) series satisfying

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

  – e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$

  – To discuss stationarity, we introduce **AR characteristic polynomial**

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2$$

  – And the corresponding **AR characteristic equation**

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

Second-order AR process

For a complex number $z = x + iy$, its modulus is $|z| = \sqrt{x^2 + y^2}$

- **Stationarity of the AR(2) process**
  - It can be shown that AR(2) process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

    is stationary if and only if **the roots of the AR characteristic equation**

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

    exceed 1 in absolute value (modulus for complex roots)

  - This statement will be generalized into the pth-order case without change
    - Also applies in AR(1) case. Its AR characteristic equation is $1 - \phi x = 0$ with root $1/\phi$, which exceeds 1 in absolute value if and only if $|\phi| < 1$

Second-order AR process

- **Stationarity of the AR(2) process**
  - Note that the roots of the AR characteristic equation

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

Quadratic formula (근의 공식)

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

can be easily found to be

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

Second-order AR process

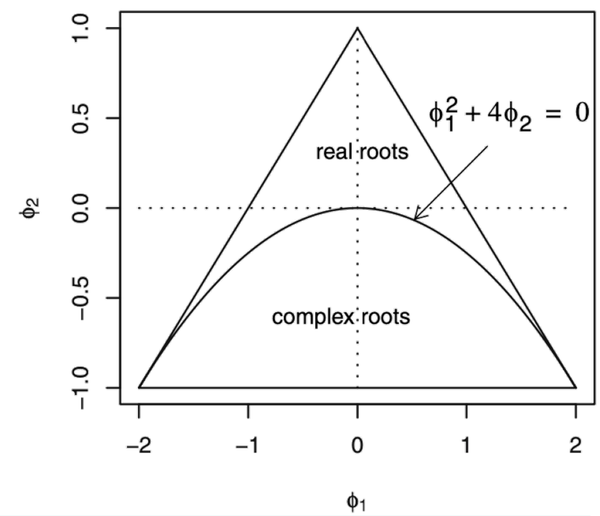
$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

- **Stationarity of the AR(2) process**
  – For stationarity, the roots should exceed 1 in absolute value. It can be shown that this is true if and only if **the following three conditions are satisfied**:

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad |\phi_2| < 1$$

  – These are called the **stationarity conditions** for the AR(2) model

Exhibit 4.17 Stationarity Parameter Region for AR(2) Process



Second-order AR process

- **ACF for the AR(2) process**
  - As we have done for AR(1) (in pages 27 - 28), multiply Y_{t-k} to the both sides of

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

and take expectations. Assuming stationarity, zero means, and that e_t is independent of Y_{t-k}, we get

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}, \qquad \text{for } k = 1,2,3, \dots$$

or, dividing through by γ_0,

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}, \qquad \text{for } k = 1,2,3, \dots$$

These are called **the Yule-Walker equations**

Second-order AR process

Yule-Walker equations

$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}, \text{ for } k = 1,2,3,\ldots$

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}, \text{ for } k = 1,2,3,\ldots$

- **ACF for the AR(2) process**
  - If we set $k = 1$ for the Yule-Walker equation,

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1}$$

  - Since $\rho_0 = 1$ and $\rho_{-1} = \rho_1$,

$$\rho_1 = \phi_1 + \phi_2 \rho_1 \;\Rightarrow\; \rho_1 = \frac{\phi_1}{1 - \phi_2}$$

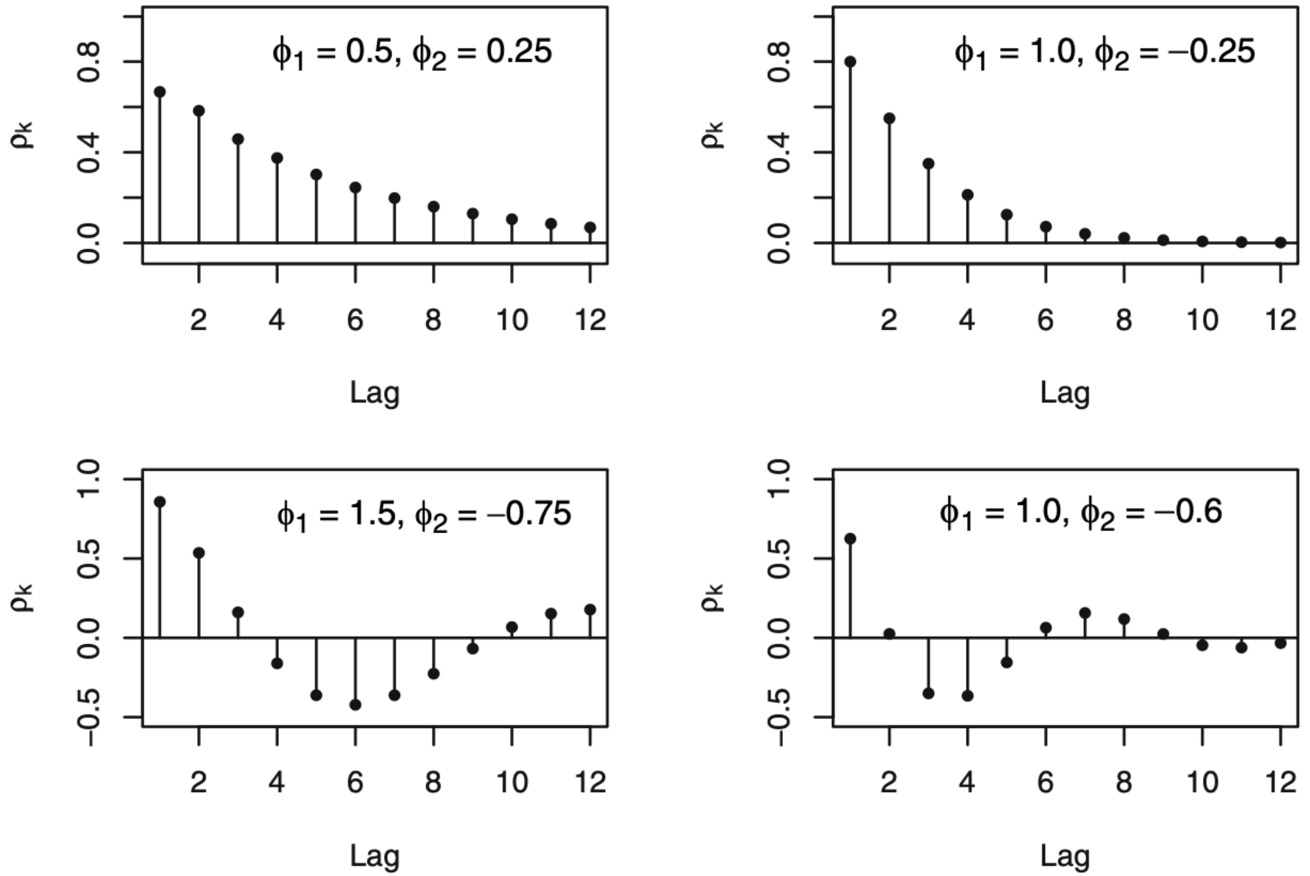
  - Then, we can plug the values of ρ_0 and ρ_1 into the Yule-Walker equation for $k = 2$ to find ρ_2

$$\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0 = \frac{\phi_2 (1 - \phi_2) + \phi_1^2}{1 - \phi_2}$$

  - Successive values of ρ_k can be calculated recursively

Second-order AR process

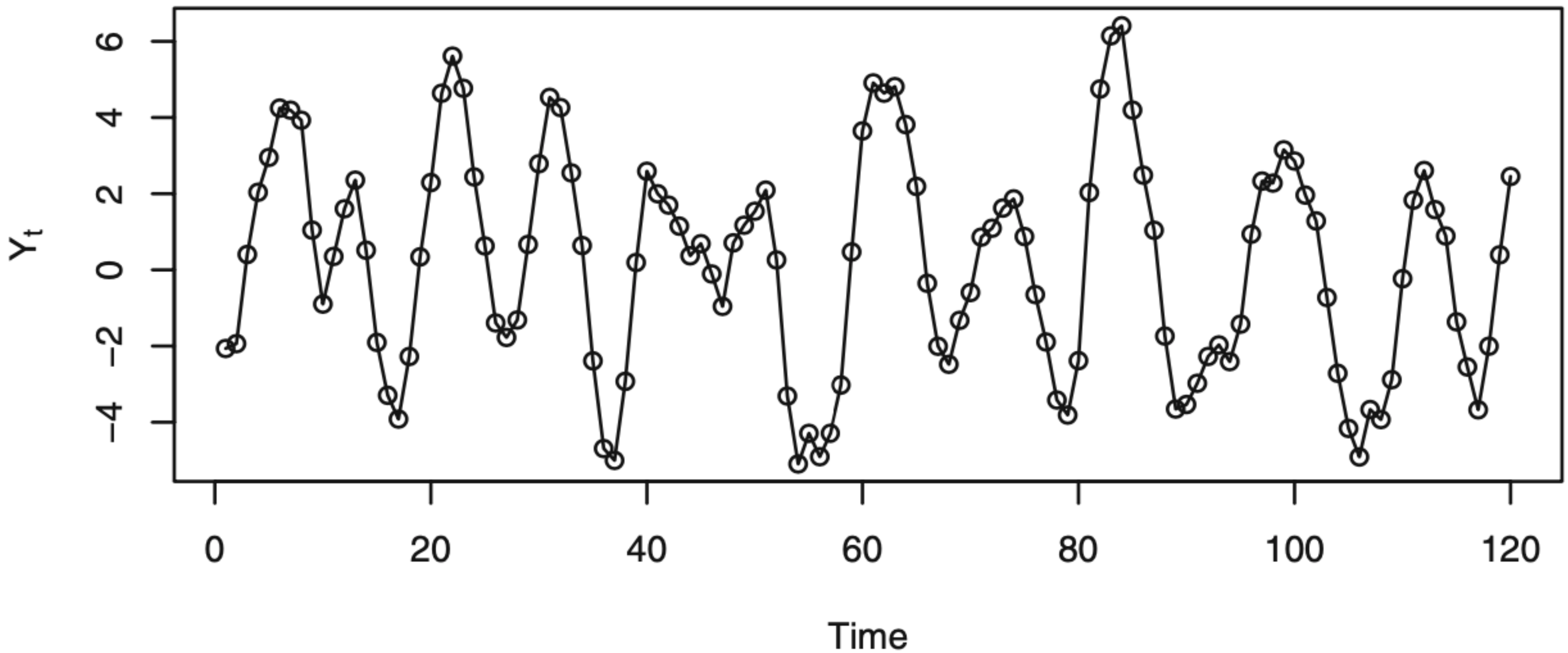
Exhibit 4.18 Autocorrelation Functions for Several AR(2) Models



– ACF of AR(2) can have many different shapes

Second-order AR process

Exhibit 4.19 Time Plot of an AR(2) Series with $\phi_1 = 1.5$ and $\phi_2 = -0.75$



– Periodic behavior of ρ_k shown in Exhibit 4.18 is clearly reflected in the **nearly periodic behavior** of the series

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Yule-Walker equations

$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$, for $k = 1,2,3,\ldots$

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$, for $k = 1,2,3,\ldots$

Second-order AR process

- Variance for the AR(2) model
  - If we take the variance of the AR(2) equation,

$$\begin{aligned}\gamma_0 &= Var(Y_t) \\ &= \phi_1^2 Var(Y_{t-1}) + \phi_2^2 Var(Y_{t-2}) + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) + \sigma_e^2\end{aligned}$$

$$\boxed{\gamma_0 = (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2}$$

  - Setting $k = 1$ for the Yule-Walker equation gives

$$\boxed{\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_{-1} = \phi_1\gamma_0 + \phi_2\gamma_1}$$

  - Then, we have two equations with two unknown variables γ_0, γ_1

$$\gamma_0 = \frac{(1-\phi_2)\sigma_e^2}{(1-\phi_2)(1-\phi_1^2-\phi_2^2)-2\phi_2\phi_1^2} = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2}$$

$$\gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0 = \left(\frac{\phi_1}{1+\phi_2}\right)\frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2}$$

General linear process version of AR(2) model

- General liner process representation for an AR(2) series
  - We can substitute the general linear process representation using the following equation

$$Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

for Y_t, for Y_{t-1}, and for Y_{t-2} into $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$

  - That is, we have
    - $Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$
    - $Y_{t-1} = \psi_0 e_{t-1} + \psi_1 e_{t-2} + \psi_2 e_{t-3} + \cdots$
    - $Y_{t-2} = \psi_0 e_{t-2} + \psi_1 e_{t-3} + \psi_2 e_{t-4} + \cdots$

General linear process version of AR(2) model

- General liner process representation for an AR(1) series
  - We can substitute the general linear process representation using the following equation

$$Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

  for Y_t, for Y_{t-1}, and for Y_{t-2} into $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$

  - That is, we have
    - $Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$
    - $\phi_1 Y_{t-1} = \phi_1 \psi_0 e_{t-1} + \phi_1 \psi_1 e_{t-2} + \phi_1 \psi_2 e_{t-3} + \cdots$
    - $\phi_2 Y_{t-2} = \phi_2 \psi_0 e_{t-2} + \phi_2 \psi_1 e_{t-3} + \phi_2 \psi_2 e_{t-4} + \cdots$

General linear process version of AR(2) model

- General liner process representation for an AR(1) series
  - We can substitute the general linear process representation using the following equation

$$Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

for Y_t, for Y_{t-1}, and for Y_{t-2} into $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$

  - That is, we have

=

- $\boxed{Y_t} = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$
- $\boxed{e_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2}} = e_t + \phi_1 \psi_0 e_{t-1} + (\phi_1 \psi_1 + \phi_2 \psi_0) e_{t-2} + (\phi_1 \psi_2 + \phi_2 \psi_1) e_{t-3} + \cdots$

  - Therefore,
    - $\psi_0 = 1$
    - $\psi_1 = \phi_1 \psi_0$ or $\psi_1 - \phi_1 \psi_0 = 0$
    - $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}$ or $\psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2} = 0$ for $j = 2,3, \dots$

General AR(p) process

- Consider the pth-order autoregressive model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

  – with AR characteristic polynomial

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

  – and corresponding AR characteristic equation

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

  – e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$

General AR(p) process

AR(p)
$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$

AR characteristic equation
$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$

- **Stationarity of AR(p) process**
  - AR(p) is stationary if and only if **the p roots of the AR characteristic equation each exceed 1 in absolute value** (modulus for complex roots)

For a complex number $z = x + iy$, its modulus is $|z| = \sqrt{x^2 + y^2}$

  - Necessary conditions for stationarity (not sufficient)
    - $\phi_1 + \phi_2 + \cdots + \phi_p < 1$
    - $|\phi_p| < 1$

General AR(p) process

AR(p)
$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$

AR characteristic equation
$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$

- **Yule-Walker equations for AR(p) process**
  - Multiply Y_{t-k} to the AR(p) equation and take expectations and divide by γ_0. Then we get

    $$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p} \quad \text{for } k \geq 1$$

  - Then put $k = 1,2, \dots,$ and p into the above equation, and use $\rho_0 = 1$ and $\rho_{-k} = \rho_k$ to get the general **Yule-Walker equations**
    - $\rho_1 = \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \cdots + \phi_p \rho_{p-1}$
    - $\rho_2 = \phi_1 \rho_1 + \phi_2 + \phi_3 \rho_1 + \cdots + \phi_p \rho_{p-2}$
    - $\vdots$
    - $\rho_p = \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \cdots + \phi_p$
  - Note that these are a system of p linear equations with p unknowns $(\rho_1, \rho_2, \ldots, \rho_p)$
    - Can solve for $\rho_1, \rho_2, \ldots, \rho_p$ with the values of $\phi_1, \dots, \phi_p$

General AR(p) process

AR(p)
$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$

AR characteristic equation
$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$

- **Yule-Walker equations for AR(p) process**
  - Noting that

$$E(e_t Y_t) = E\left[e_t\left(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t\right)\right]$$
$$= E(e_t^2) = \sigma_e^2$$

  - We may multiply the AR(p) equation by Y_t and take expectations to find

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p + \sigma_e^2$$

  - Using $\rho_k = \gamma_k / \gamma_0$, the above can be rewritten as

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2 - \cdots - \phi_p \rho_p}$$

    - i.e., Variance can be found using $\sigma_e^2, \phi_1, \ldots, \phi_p$ and $\rho_1, \ldots, \rho_p$

General AR(p) process

AR(p)
$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$

AR characteristic equation
$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$

- **General linear process representation for AR(p) process**
  - Assuming stationarity, AR(p) can also be expressed in the general linear process form of $Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$
  - But, ψ-coefficients are complicated functions of the parameters $\phi_1, \dots, \phi_p$
    - They can be found numerically (see Appendix C on page 85 of the textbook)

Chapter 4.4

The Mixed Autoregressive Moving Average Model

Autoregressive moving average model

- A series is partly autoregressive and partly moving average would become a quite general time series model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p}$$
$$+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

  – $\{Y_t\}$ is called autoregressive moving average process of orders p and q (or ARMA(p,q))

ARMA(1,1) model

- Consider the following defining equation

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

- To derive Yule-Walker type equations, first note that

$$\begin{aligned} E(e_t Y_t) &= E[e_t(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\ &= \sigma_e^2 \end{aligned}$$

and

$$\begin{aligned} E(e_{t-1} Y_t) &= E[e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\ &= \phi\sigma_e^2 - \theta\sigma_e^2 = (\phi - \theta)\sigma_e^2 \end{aligned}$$

ARMA(1,1) model

$E(e_t Y_t) = \sigma_e^2$
$E(e_{t-1} Y_t) = (\phi - \theta)\sigma_e^2$

- Consider the following defining equation

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

- To derive Yule-Walker type equations,
  - Now multiply Y_{t-k} to ARMA(1,1) equation and take expectation

$$E(Y_{t-k} Y_t) = E(\phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t - \theta Y_{t-k} e_{t-1})$$

- For $k = 0$,

$$\gamma_0 = \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2$$

- For $k = 1$,

$$\gamma_1 = \phi\gamma_0 - \theta\sigma_e^2$$

- For $k \geq 2$,

$$\gamma_k = \phi\gamma_{k-1}$$

ARMA(1,1) model

$$\gamma_0 = \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2$$
$$\gamma_1 = \phi\gamma_0 - \theta\sigma_e^2$$
$$\gamma_k = \phi\gamma_{k-1}$$

- Consider the following defining equation

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

- To derive Yule-Walker type equations,
  - Solving the first two equations yields

$$\gamma_0 = \frac{1 - 2\phi\theta + \theta^2}{1 - \phi^2}\sigma_e^2$$

  - Solving the simple recursion gives

$$\rho_k = \frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\theta\phi + \theta^2}\phi^{k-1} \quad \text{for } k \geq 1$$

It decays exponentially as the lag k increases.

ARMA(1,1) model

$$\gamma_0 = \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2$$
$$\gamma_1 = \phi\gamma_0 - \theta\sigma_e^2$$
$$\gamma_k = \phi\gamma_{k-1}$$

- Consider the following defining equation

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

- The general linear process form of the model be obtained (In a recursive manner that we did for AR(1))

$$Y_t = e_t + (\phi - \theta)\sum_{j=1}^{\infty} \phi^{j-1} e_{t-j}$$

  – That is, $\psi_j = (\phi - \theta)\phi^{j-1}$ for $j \geq 1$

- Note that the AR characteristic equation is $1 - \phi x = 0$, hence, the stationarity condition is $|\phi| < 1$

ARMA(p,q) model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p}$$
$$+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

- For the general ARMA(p,q) model, the following facts are stated without proof
  - ARMA(p,q) equation is stationary if and only if all the roots of the AR characteristic equation $\phi(x) = 0$ exceeds 1 in absolute value (modulus for complex roots)
  - If the stationarity conditions are satisfied, then the model can also be written as the general linear process with ψ-coefficients
    - $\psi_0 = 1$
    - $\psi_1 = -\theta_1 + \phi_1$
    - $\psi_2 = -\theta_2 + \phi_2 + \phi_1 \psi_1$
    - $\vdots$
    - $\psi_j = -\theta_j + \phi_p \psi_{j-p} + \phi_{p-1} \psi_{j-p+1} + \cdots + \phi_1 \psi_{j-1}$

$\psi_j = 0$ for $j < 0$
$\theta_j = 0$ for $j > q$

ARMA(p,q) model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p}$$
$$+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

- For the general ARMA(p,q) model, the following facts are stated without proof
  - If the stationarity conditions are satisfied, ACF can easily be shown to satisfy

    $$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p} \quad \text{for } k > q$$

  - Similar equations can be derived for $k = 1,2,3, \ldots, q$ that involve $\theta_1, \theta_2, \ldots, \theta_q$ (but very complex). An algorithm suitable for numerical computation of the complete ACF is given in Appendix C on page 85 of the textbook

Chapter 4.5

Invertibility

Invertibility

- We know that an AR process can always be reexpressed as a general linear process through the ψ-coefficients
  - AR process may also be thought of as an infinite-order MA process

- For some purposes, the autoregressive representations are also convenient

- Can an MA process be reexpressed as an AR?

Invertibility

- Consider an MA(1) model

$$Y_t = e_t - \theta e_{t-1}$$

  – First rewriting this as $e_t = Y_t + \theta e_{t-1}$

  – Then, since $e_{t-1} = Y_{t-1} + \theta e_{t-2}$,

$$\begin{aligned} e_t &= Y_t + \theta(Y_{t-1} + \theta e_{t-2}) \\ &= Y_t + \theta Y_{t-1} + \theta^2 e_{t-2} \end{aligned}$$

  – If $|\theta| < 1$, we may continue this substitution “infinitely” into the past and obtain the expression

$$e_t = Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \cdots$$

or

$$Y_t = (-\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots) + e_t$$

Invertibility

- If $|\theta| < 1$, we see that MA(1) model can be inverted into an infinite-order AR model
  - MA(1) model is **invertible** if and only of $|\theta| < 1$
- For a general MA(q) or ARMA(p,q) model, we define the **MA characteristic polynomial** as

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \cdots - \theta_q x^q$$

  - And the corresponding **MA characteristic equation**

$$1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \cdots - \theta_q x^q = 0$$

Invertibility

- MA(q) model is **invertible**; that is, there are coefficients π_j such that

$$Y_t = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \cdots + e_t$$

if and only if the roots of the MA characteristic equation exceed 1 in absolute value (in modulus for complex roots)

Invertibility

- Note that the following two MA(1) models (for $|\theta| < 1$)
  - $Y_t = e_t - \theta e_{t-1}$
  - $Y_t = e_t - \frac{1}{\theta} e_{t-1}$

  have the same ACF (please check it by yourself)

- But, only the first one is invertible with root $\frac{1}{\theta}$

- From here on, we will restrict our attention to the physically sensible class of invertible models
  - For a general ARMA(p,q) model, we require both stationarity and invertibility

Summary

- **General linear process**

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

- **Moving average process: MA(q)**

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

  – Stationary
  – Autocorrelation ***cuts off*** after lag q
  – Can be represented in **AR form**

- **Autoregressive process: AR(p)**

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

  – Not always stationary (Can check stationarity using **AR characteristic equation**)
  – **Yule-Walker equations** can be used to find its autocorrelations and autocovariances
  – Can be represented in **general linear process form** (just like MA)

Summary

- **Autoregressive Moving average process: ARMA(p,q)**

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

  – Not always stationary (Can check stationarity using **AR characteristic equation**)
  – Can be represented in **general linear process form** (just like MA)
  – Can be represented in **AR form**