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IE 313 Time Series Analysis

9. Forecasting

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Forecasting

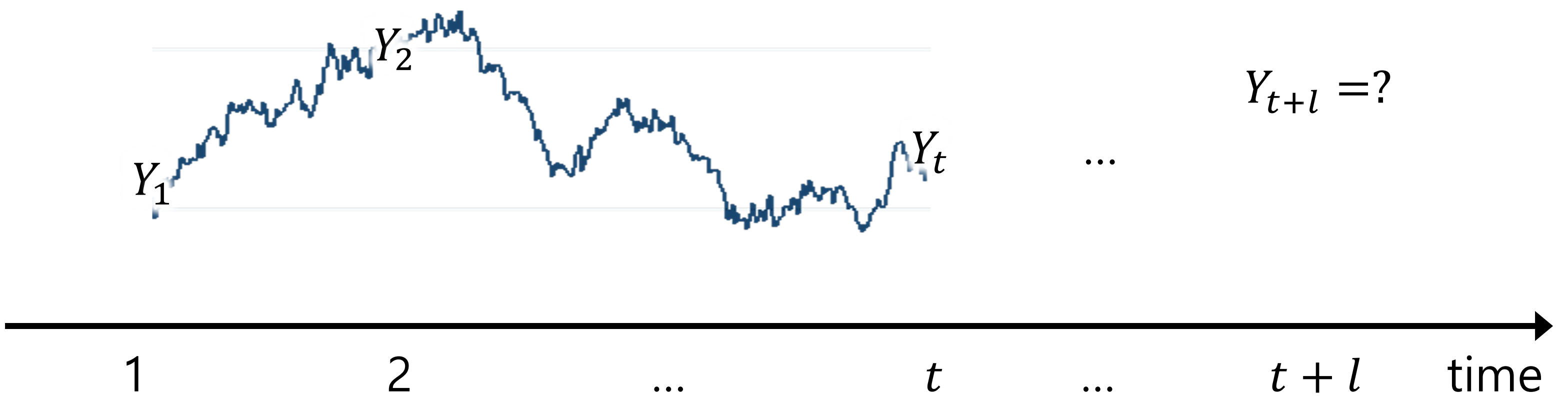
- One of the primary objectives of building a model for a time series is to be able to forecast the values for that series at future times

- Of equal importance is the assessment of the precision of those forecasts

- Let’s briefly see how we can do it

Minimum mean square error forecasting

- Based on the available history of the series up to time t, namely $Y_1, Y_2, \dots, Y_{t-1}, Y_t$, we would like to forecast the value of Y_{t+l} that will occur l time units into the future
  - Time t: **forecast origin**
  - Time l: **lead time** for the forecast



- The minimum mean square error forecast is given by

$$\hat{Y}_t(l) = E(Y_{t+l} \mid Y_1, Y_2, \dots, Y_t)$$

Deterministic trends

- Consider the deterministic trend model of Chapter 3

$$Y_t = \mu_t + X_t$$

  - μ_t: deterministic trend
  - X_t: white noise with variance γ_0
  - Then,

$$\begin{aligned}\hat{Y}_t(l) &= E(Y_{t+l} \mid Y_1, Y_2, \dots, Y_t) \\ &= E(\mu_{t+l} + X_{t+l} \mid Y_1, Y_2, \dots, Y_t) \\ &= E(\mu_{t+l} \mid Y_1, Y_2, \dots, Y_t) + E(X_{t+l} \mid Y_1, Y_2, \dots, Y_t) \\ &= \mu_{t+l}\end{aligned}$$

Deterministic trends

- Consider the deterministic trend model of Chapter 3

$$Y_t = \mu_t + X_t$$

  - μ_t: deterministic trend
  - X_t: white noise with variance γ_0
  - Then, **forecast** $\hat{Y}_t(l) = \mu_{t+l}$
  - The **forecast error** $e_t(l)$ would be

$$\begin{aligned} e_t(l) &= Y_{t+l} - \hat{Y}_t(l) \\ &= \mu_{t+l} + X_{t+l} - \mu_{t+l} \\ &= X_{t+l} \end{aligned}$$

    - $E\big(e_t(l)\big) = E(X_{t+l}) = 0$
    - $Var\big(e_t(l)\big) = Var(X_{t+l}) = \gamma_0$

ARIMA forecasting

- AR(1) with a nonzero mean: $Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$
  - Then, we naturally have $Y_{t+1} - \mu = \phi(Y_t - \mu) + e_{t+1}$
  - Taking conditional expectations given $Y_1, \ldots, Y_t$,

$$\begin{aligned} \hat{Y}_t(1) - \mu &= \phi(E(Y_t|Y_1, \ldots, Y_t) - \mu) + E(e_{t+1}|Y_1, \ldots, Y_t) \\ &= \phi(Y_t - \mu) \end{aligned}$$

  - Similarly, we have $\hat{Y}_t(l) - \mu = \phi\left(\hat{Y}_t(l-1) - \mu\right)$ for $l \geq 1$
  - Then,

$$\begin{aligned} \hat{Y}_t(l) - \mu &= \phi\left(\hat{Y}_t(l-1) - \mu\right) \\ &= \phi^2\left(\hat{Y}_t(l-2) - \mu\right) \\ &\vdots \\ &= \phi^{l-1}\left(\hat{Y}_t(1) - \mu\right) \\ &= \phi^l(Y_t - \mu) \end{aligned}$$

  - Hence, $\hat{Y}_t(l) = \mu + \phi^l(Y_t - \mu)$ ($\hat{Y}_t(l) \approx \mu$ for large l)

ARIMA forecasting

- AR(1) with a nonzero mean: $Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$
  - $\hat{Y}_t(l) = \mu + \phi^l(Y_t - \mu)$
  - Note that stationary Y_t can be represented in MA form
    $$Y_t - \mu = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots$$
  - Then, forecast error is
    $$\begin{aligned} e_t(l) &= Y_{t+l} - \hat{Y}_t(l) \\ &= Y_{t+l} - \mu - \phi^l(Y_t - \mu) \\ &= e_{t+l} + \phi(Y_{t+l-1} - \mu) - \phi^l(Y_t - \mu) \\ &= e_{t+l} + \phi e_{t+l-1} + \cdots + \phi^{l-1} e_{t+1} + \phi^l e_t + \cdots \\ &\quad -\phi^l(e_t + \phi e_{t-1} + \cdots) \\ &= e_{t+l} + \phi e_{t+l-1} + \cdots + \phi^{l-1} e_{t+1} \end{aligned}$$
    - $Var(e_t(l)) = \sigma_e^2 \left[\frac{1-\phi^{2l}}{1-\phi^2}\right] \approx \frac{\sigma_e^2}{1-\phi^2} = Var(Y_t) = \gamma_0$ for large l

ARIMA forecasting

- MA(1) with a nonzero mean: $Y_t - \mu = e_t - \theta e_{t-1}$
  - Then, we naturally have $Y_{t+1} = \mu + e_{t+1} - \theta e_t$

  - Taking conditional expectations given $Y_1, \dots, Y_t$,

$$\begin{aligned}\hat{Y}_t(1) &= \mu + E(e_{t+1} | Y_1, \dots, Y_t) - \theta E(e_t | Y_1, \dots, Y_t) \\ &= \mu - \theta e_t\end{aligned}$$

  - Similarly, for $l > 1$, we have

$$\hat{Y}_t(l) = \mu + E(e_{t+l} | Y_1, \dots, Y_t) - \theta E(e_{t+l-1} | Y_1, \dots, Y_t) = \mu$$

ARIMA forecasting

- Random walk with drift: $Y_t = Y_{t-1} + \theta_0 + e_t$
  - Then, we naturally have $Y_{t+1} = Y_t + \theta_0 + e_{t+1}$
  - Taking conditional expectations given $Y_1, \dots, Y_t$,

$$\begin{aligned}\hat{Y}_t(1) &= E(Y_t|Y_1, \dots, Y_t) + \theta_0 + E(e_{t+1}|Y_1, \dots, Y_t) \\ &= Y_t + \theta_0\end{aligned}$$

  - Similarly, for $l \geq 1$, we have

$$\hat{Y}_t(l) = \hat{Y}_t(l-1) + \theta_0 = Y_t + \theta_0 l$$

ARIMA forecasting

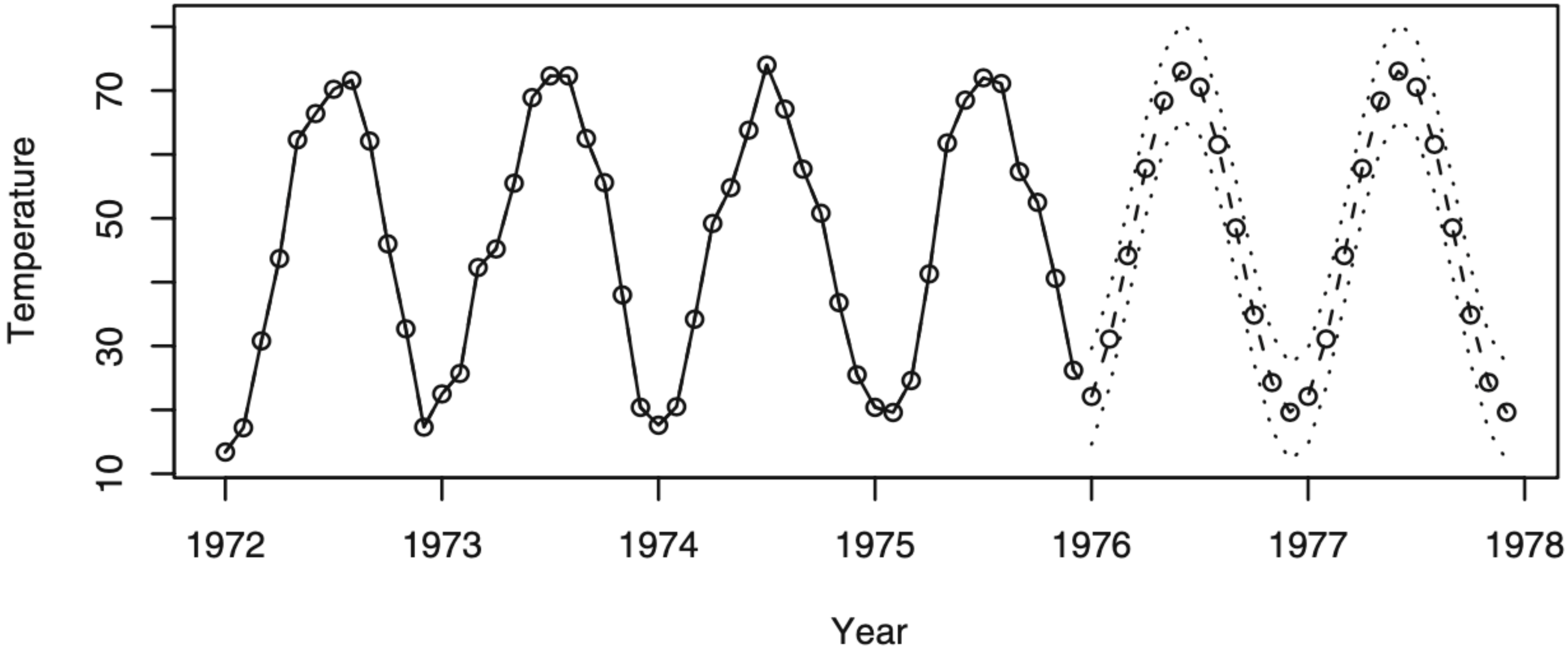
- Random walk with drift: $Y_t = Y_{t-1} + \theta_0 + e_t$
  - $\hat{Y}_t(l) = \hat{Y}_t(l-1) + \theta_0 = Y_t + \theta_0 l$
  - Then, forecast error would be

$$
\begin{aligned}
e_t(l) &= Y_{t+l} - \hat{Y}_t(l) \\
&= (Y_t + \theta_0 l + e_{t+1} + \cdots + e_{t+l}) - (Y_t + \theta_0 l) \\
&= e_{t+1} + \cdots + e_{t+l}
\end{aligned}
$$

    - $Var\big(e_t(l)\big) = l\sigma_e^2$
      - Unlike the stationary cases, forecast error increases as l increases

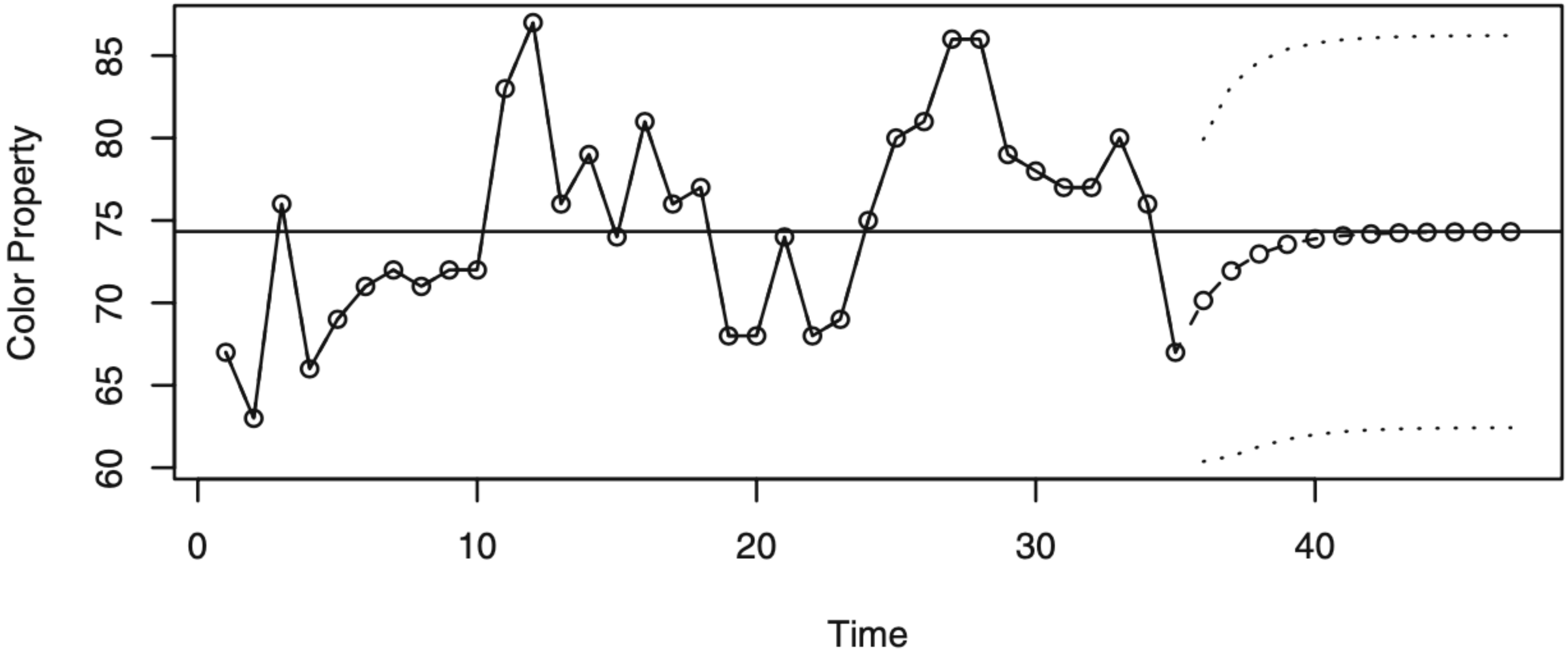
Examples

Exhibit 9.2 Forecasts and Limits for the Temperature Cosine Trend



Examples

Exhibit 9.3 Forecasts and Forecast Limits for the AR(1) Model for Color



Examples

Exhibit 9.4 Forecasts from an AR(3) Model for Sqrt(Hare)