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IE 313 Time Series Analysis

3. Trends

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Chapter 3. Trends

- 3.1 Deterministic vs. Stochastic Trends
- 3.2 Estimation of a Constant Mean
- 3.3 Regression Methods
- 3.4 Reliability and Efficiency of Regression Estimates
- 3.5 Interpreting Regression Output
- 3.6 Residual Analysis

Chapter 3.1

Deterministic vs. Stochastic Trends

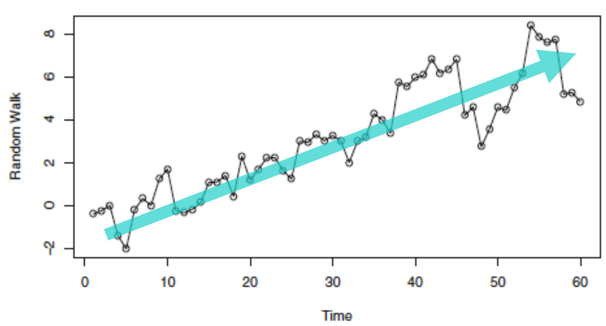
Trends

- In a **general** time series,
  - The mean function is a **totally arbitrary** function of time
- In a **stationary** time series,
  - The mean function must be **constant** in time
- Frequently we need to take the **middle ground**
  - The mean functions that are **relatively simple** (but not constant) functions of time

Stochastic trends

- **Trends can be quite elusive**
  - The same time series may be viewed quite differently by different analysts
  - E.g. the simulated random walk might be considered to display a general upward trend

Exhibit 2.1 Time Series Plot of a Random Walk



    - However, we know that the random walk process has zero mean for all time
    - The perceived trend is just an artifact of the strong positive correlation between the series values at nearby time points
    - Another simulation of exactly the same process would show completely different trends

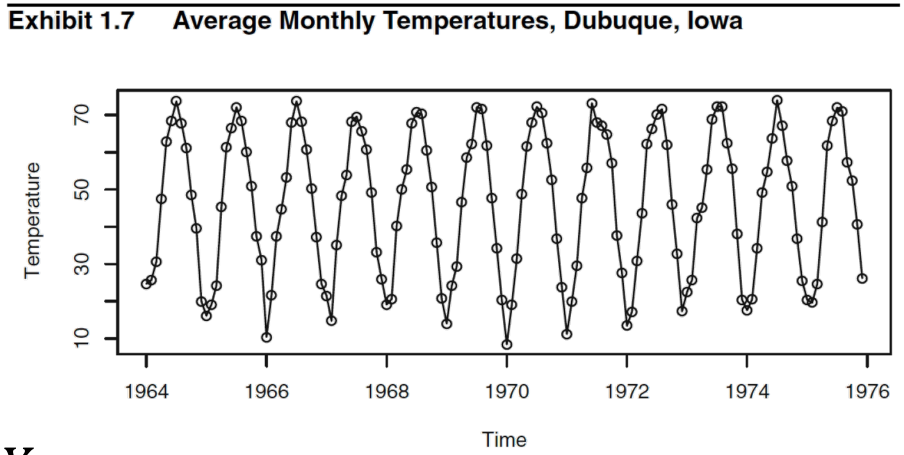
will be discussed in Ch. 5

  - Such trends are often referred to as **stochastic trends**

Deterministic trends

- Consider the average monthly temperature series

  – It shows a cyclical or seasonal trend (yearly)



  – A possible model would be

$$Y_t = \mu_t + X_t$$

  - μ_t is deterministic that is periodic with period 12 (months) (i.e., $\mu_t = \mu_{t-12}$ for all t)
  - X_t might be assumed to have zero mean for all t

  – This kind of μ_t represent **deterministic trends**

Deterministic trends

- **For deterministic trends, the same trend applies for all time**
  - We should have good reasons for assuming such a model

- We may assume **different forms** of μ_t
  - *Linear*

$$\mu_t = \beta_0 + \beta_1 t$$

  - *Quadratic*

$$\mu_t = \beta_0 + \beta_1 t + \beta_2^2 t^2$$

Chapter 3.2

Estimation of a Constant Mean

Estimation of constant mean

- Consider when a **constant mean function** is assumed

$$Y_t = \mu + X_t$$

  - $E(X_t) = 0$ for all t

- We wish to estimate μ with our observed time series $Y_1, Y_2, \dots, Y_n$

$$\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$$

  - The sample mean would be the most common estimate of μ
  - Under the minimal assumptions of $Y_t = \mu + X_t$, $E(\bar{Y}) = \mu$

Estimation of constant mean

- **Precision of $\bar{Y}$**

This can be obtained by solving Exercise 2.17

$$Var(\bar{Y}) = \frac{\gamma_0}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\rho_k\right]$$

– More precise as we have **more samples** (i.e., n is large)

– More precise as we have **small (even negative) $\boldsymbol{\rho_k}$'s**

– That is, if ρ_k's are large, $\bar{Y}$ will be an unstable estimate of μ

- Fortunately, for many stationary processes, the autocorrelation function decays quickly enough with increasing lags that $\sum_{k=0}^{\infty}|\rho_k| < \infty$

Chapter 3.3

Regression Methods

Regression methods

- Classical statistical method of **regression analysis** may be readily used to **estimate the parameters of common nonconstant mean trend models**

- We shall consider the most useful ones
  - Linear trends
  - Quadratic trends
  - Cyclical or Seasonal trends
  - Cosine trends

Linear and quadratic trends

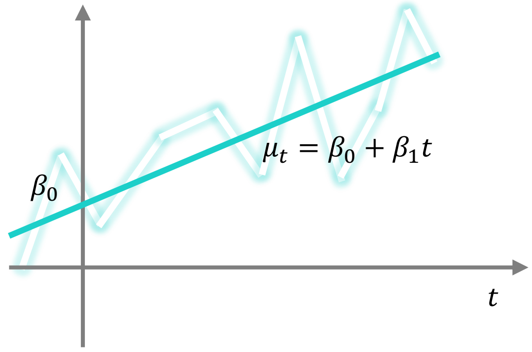
$$Y_t = \mu_t + X_t$$
$(E(X_t) = 0$ for all $t)$

- Consider the **deterministic linear trend** expressed as

$$\mu_t = \beta_0 + \beta_1 t$$

  - β_0: intercept
  - β_1: slope

Unknown parameters



- The classical **least squares** (or **regression**) method can be used to find estimates of β_0 and β_1 that minimize

$$Q(\beta_0, \beta_1) = \sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t)]^2$$

Linear and quadratic trends

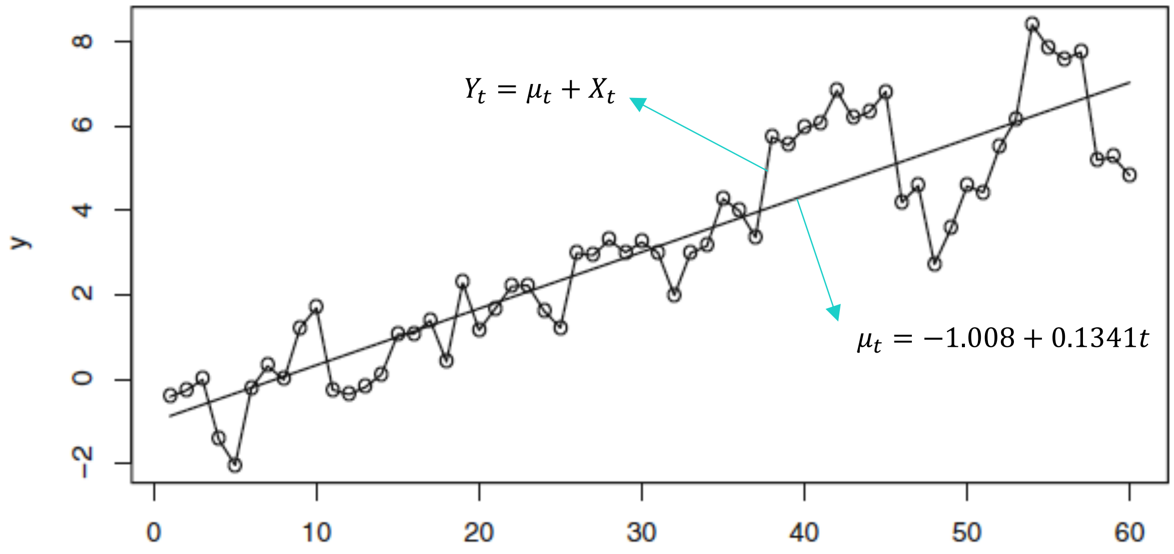
$$Q(\beta_0, \beta_1) = \sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t)]^2$$

- **Example**

Exhibit 3.1 Least Squares Regression Estimates for Linear Time Trend

		Estimate	Std. Error	*t* value	*Pr*(>|*t*|)
β_0	Intercept	−1.008	0.2972	−3.39	0.00126
β_1	Time	0.1341	0.00848	15.82	< 0.0001

Exhibit 3.2 Random Walk with Linear Time Trend



Linear and quadratic trends

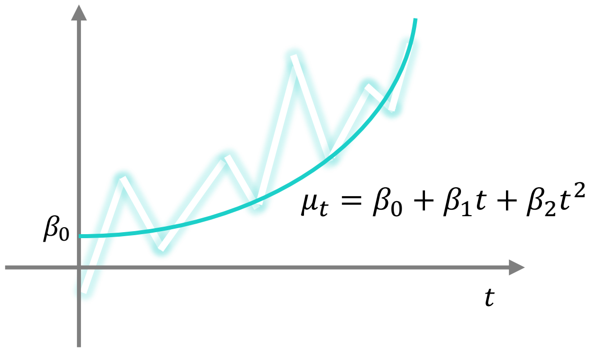
$Y_t = \mu_t + X_t$
$(E(X_t) = 0$ for all $t)$

- Consider the **deterministic quadratic trend** expressed as

$$\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

  - β_0: intercept
  - β_1, β_2: coefficients

Unknown parameters



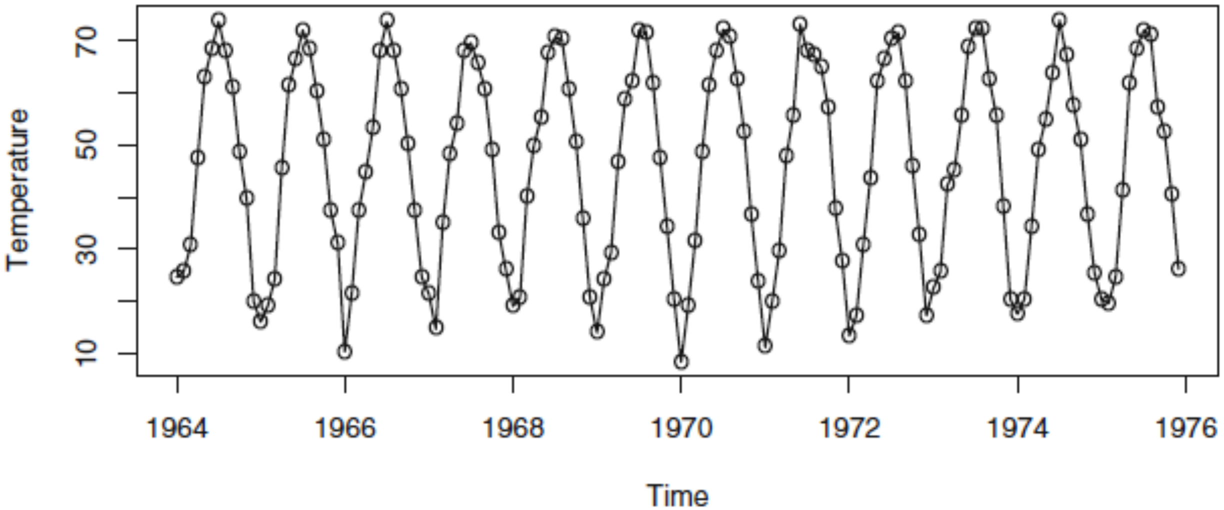
- The classical **least squares** (or **regression**) method can be used to find estimates of β_0, β_1, and β_2 that minimize

$$Q(\beta_0, \beta_1, \beta_2) = \sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t + \beta_2 t^2)]^2$$

Cyclical or seasonal trends

$Y_t = \mu_t + X_t$
$(E(X_t) = 0$ for all $t)$

Exhibit 1.7 Average Monthly Temperatures, Dubuque, Iowa



Cyclical or seasonal trends

$Y_t = \mu_t + X_t$
$(E(X_t) = 0$ for all $t)$

- The most general assumption for μ_t with **monthly seasonal data** is that there are 12 constants (parameters),

$$\beta_1, \beta_2, \dots, \beta_{12}$$

giving the expected average temperature for each of the 12 months

  – That is,

$$\mu_t = \begin{cases} \beta_1 & \text{for } t = 1, 13, 25, \dots \\ \beta_2 & \text{for } t = 2, 14, 26, \dots \\ \vdots & \vdots \\ \beta_{12} & \text{for } t = 12, 24, 36, \dots \end{cases}$$

January
February
⋮
December

  – This is sometimes called a **seasonal means** model

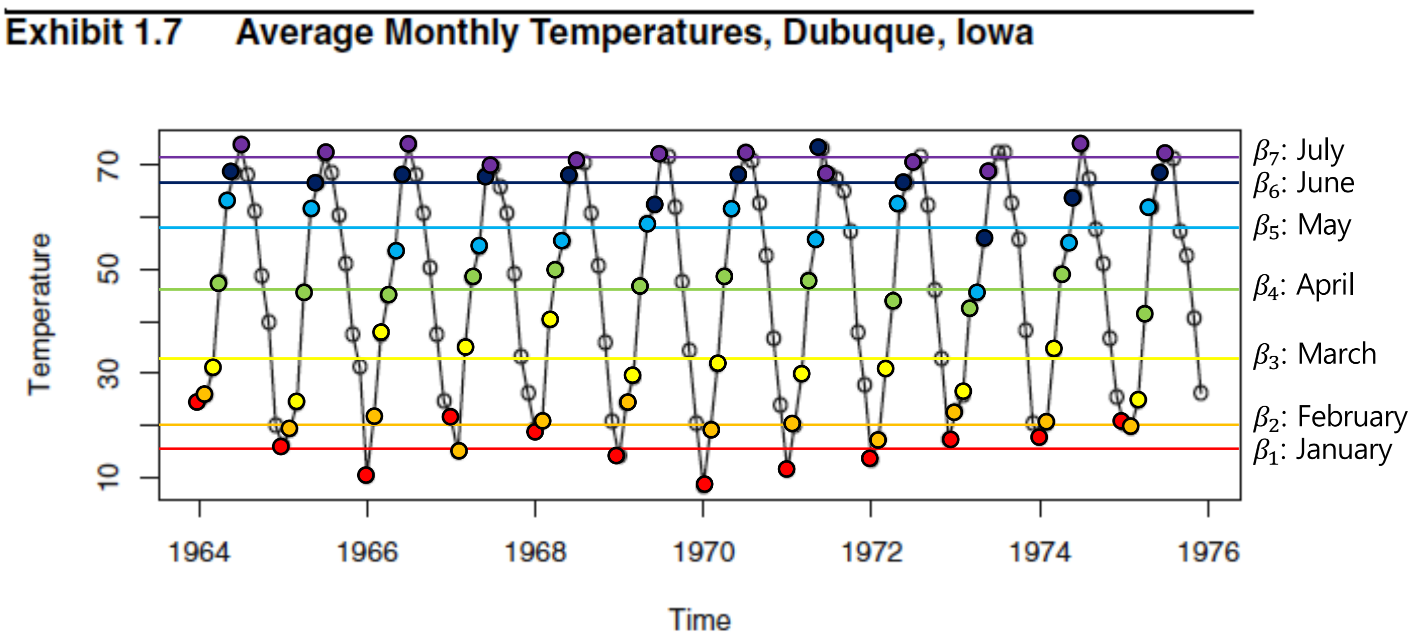
Cyclical or seasonal trends

Exhibit 3.3 Regression Results for the Seasonal Means Model

	Estimate	Std. Error	*t*-value	*Pr*(>|*t*|)
January	16.608	0.987	16.8	< 0.0001
February	20.650	0.987	20.9	< 0.0001
March	32.475	0.987	32.9	< 0.0001
April	46.525	0.987	47.1	< 0.0001
May	58.092	0.987	58.9	< 0.0001
June	67.500	0.987	68.4	< 0.0001
July	71.717	0.987	72.7	< 0.0001
August	69.333	0.987	70.2	< 0.0001
September	61.025	0.987	61.8	< 0.0001
October	50.975	0.987	51.6	< 0.0001
November	36.650	0.987	37.1	< 0.0001
December	23.642	0.987	24.0	< 0.0001

To fit this model, we need to set up indicator variables (or sometimes called dummy variables) that indicate the month which each of the data point pertains.
But you may not have to do this manually depending on the software you use.

Cyclical or seasonal trends



Cosine trends

- In the previous example, we just considered 12 independent parameters to model seasonal trend
- Hence, they do not care about the shape of the seasonal trend at all
- In some cases, seasonal trends can be modeled economically with cosine curves that incorporate the smooth change expected from one period to the next while still preserving the seasonality

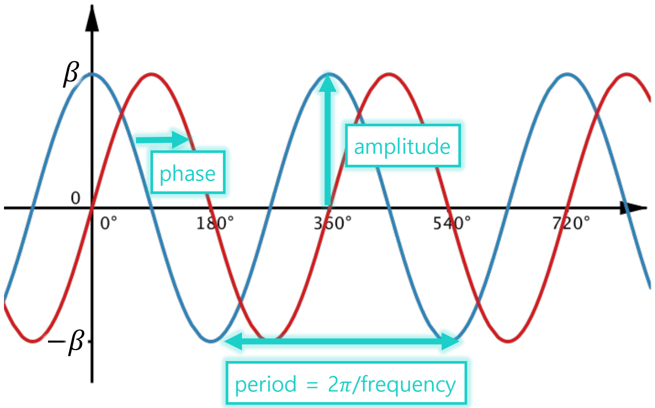
Cosine trends

$Y_t = \mu_t + X_t$
$(E(X_t) = 0$ for all $t)$

- Consider the cosine curve with equation

$$\mu_t = \beta \cos(2\pi f t + \Phi)$$

  – $\beta (> 0)$: amplitude
  – f: frequency
  – Φ: phase



  – Example
    - For monthly data with time indexed as 1, 2, …,
      the most important frequency would be $f = 1/12$
      because such a cosine wave will repeat itself every 12 months
      (Hence, the period is 12)

Cosine trends

$Y_t = \mu_t + X_t$
$(E(X_t) = 0$ for all $t)$

- Consider the cosine curve with equation

$$\mu_t = \beta \cos(2\pi f t + \Phi)$$

- The above can be reparametrized using a trigonometric identity as follows

$$\beta \cos(2\pi f t + \Phi) = \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t)$$

  – $\beta = \sqrt{\beta_1^2 + \beta_2^2}$, $\Phi = \arctan(-\beta_2/\beta_1)$
  – (or $\beta_1 = \beta \cos(\Phi)$, $\beta_2 = \beta \sin(\Phi)$)

Cosine trends

$$Y_t = \mu_t + X_t$$
$(E(X_t) = 0$ for all $t)$

- The simplest model for the cosine trend would be

$$\mu_t = \beta_0 + \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t)$$

  - Here, we must be careful how we represent time
    - For monthly data,
      - If we choose 1,2,3, … as our time scale,
        then 1/12 would be the most interesting frequency
        with a corresponding period of 12 months
      - If we represent time by year and fractional year,
        (e.g. 1980 for January, 1980.08333 for February, …)
        then a frequency of 1 corresponds to an annual or
        12 month periodicity

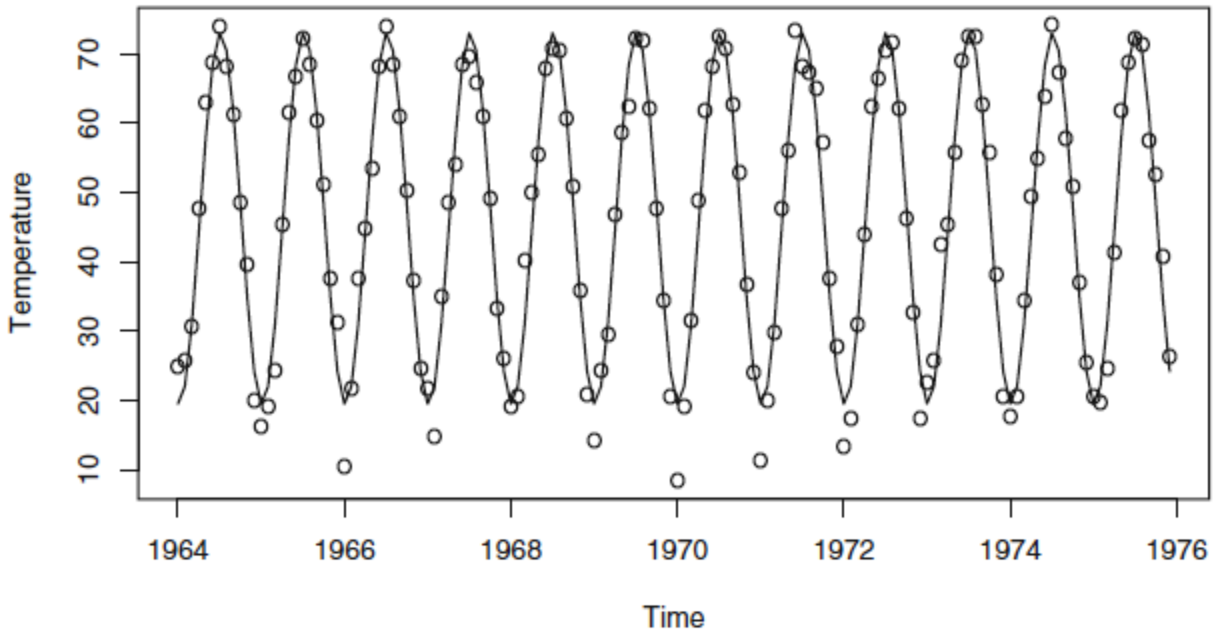
Cosine trends

$$Y_t = \mu_t + X_t$$
$(E(X_t) = 0$ for all $t)$

Exhibit 3.5 Cosine Trend Model for Temperature Series

Coefficient	Estimate	Std. Error	*t*-value	*Pr*(>|*t*|)
Intercept	46.2660	0.3088	149.82	< 0.0001
cos(2π*t*)	−26.7079	0.4367	−61.15	< 0.0001
sin(2π*t*)	−2.1697	0.4367	−4.97	<0.0001

Exhibit 3.6 Cosine Trend for the Temperature Series

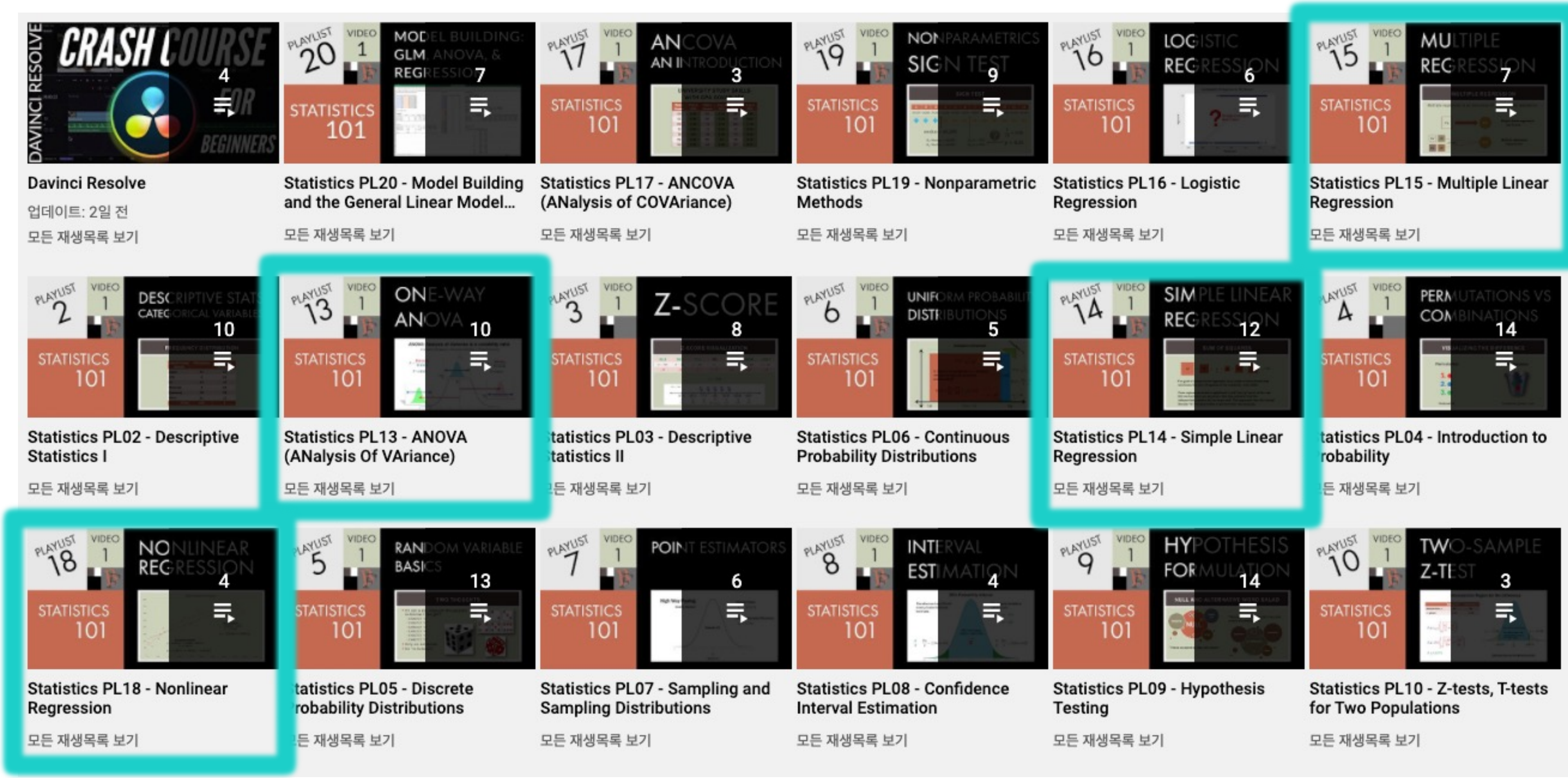


Before we further continue..

- Please review what you have learned about regression
  - Basics of regression
    - ANOVA
    - R^2
    - Standard error
  - Nonlinear regression
  - Multiple regression
  - Multiple regression with dummy variables

Before we further continue..

- In case you haven't learned about these things before..



https://www.youtube.com/c/BrandonFoltz/playlists

Chapter 3.6

Residual Analysis

Residual

- The unobserved stochastic component $\{X_t\}$ can be estimated or predicted by residual

$$\hat{X}_t = Y_t - \hat{\mu}_t$$

  – $\hat{X}_t$: residual corresponding to the t th observation

  – If the trend model is reasonably correct, then the residuals should behave roughly like the true stochastic component
    - Various assumptions about the stochastic component can be assessed by looking at the residuals
    - Ex) if $\{X_t\}$ is white noise, then residuals $\{\hat{X}_t\}$ should behave roughly like independent (normal) random variables with zero mean and standard deviation s

Residual

- The unobserved stochastic component $\{X_t\}$ can be estimated or predicted by residual

$$\hat{X}_t = Y_t - \hat{\mu}_t$$

  – $\hat{X}_t$: residual corresponding to the t th observation

  – If the trend model is reasonably correct, then the residuals should behave roughly like the true stochastic component
    - If μ_t contains a constant term, least squares fit will automatically produce residuals with a zero mean
    - Hence, we might consider **standardized residuals** $\hat{X}_t/s$

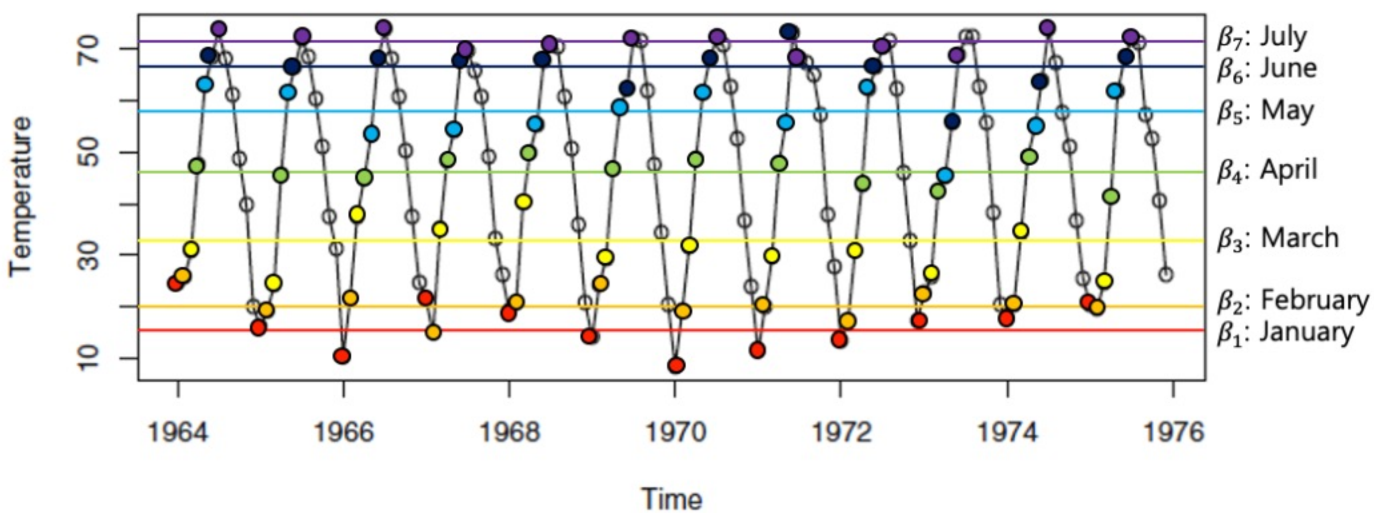
Residual analysis

- Residuals in seasonal means model

$$\mu_t = \begin{cases} \beta_1 & \text{for } t = 1, 13, 25, \dots \\ \beta_2 & \text{for } t = 2, 14, 26, \dots \\ \vdots & \vdots \\ \beta_{12} & \text{for } t = 12, 24, 36, \dots \end{cases}$$

January
February
⋮
December

Exhibit 1.7 Average Monthly Temperatures, Dubuque, Iowa

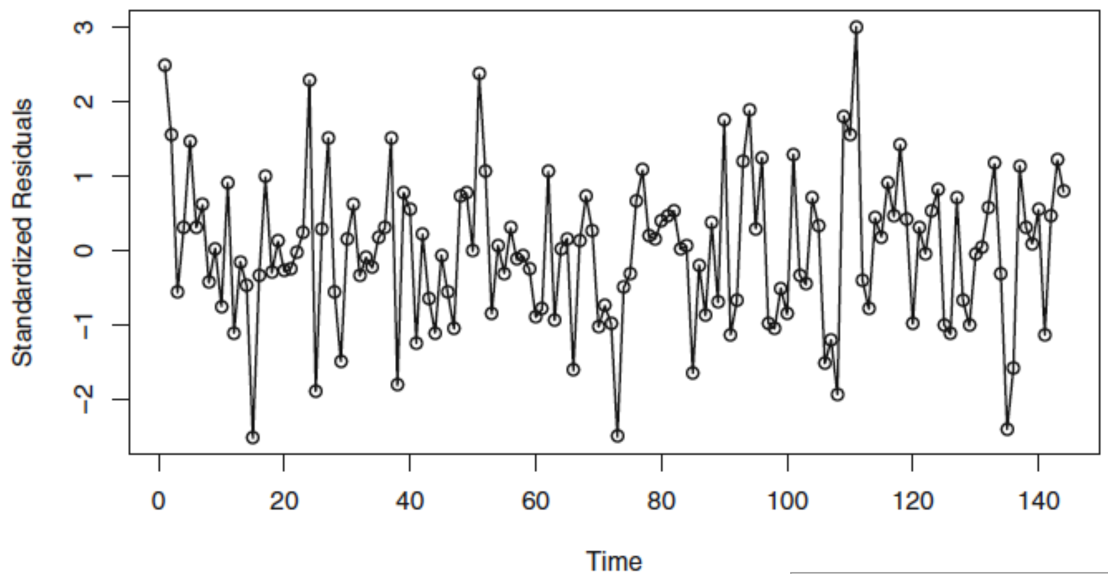


Residual analysis

- Residuals in seasonal means model

$$\hat{X}_t = Y_t - \hat{\mu}_t$$

Exhibit 3.8 Residuals versus Time for Temperature Seasonal Means



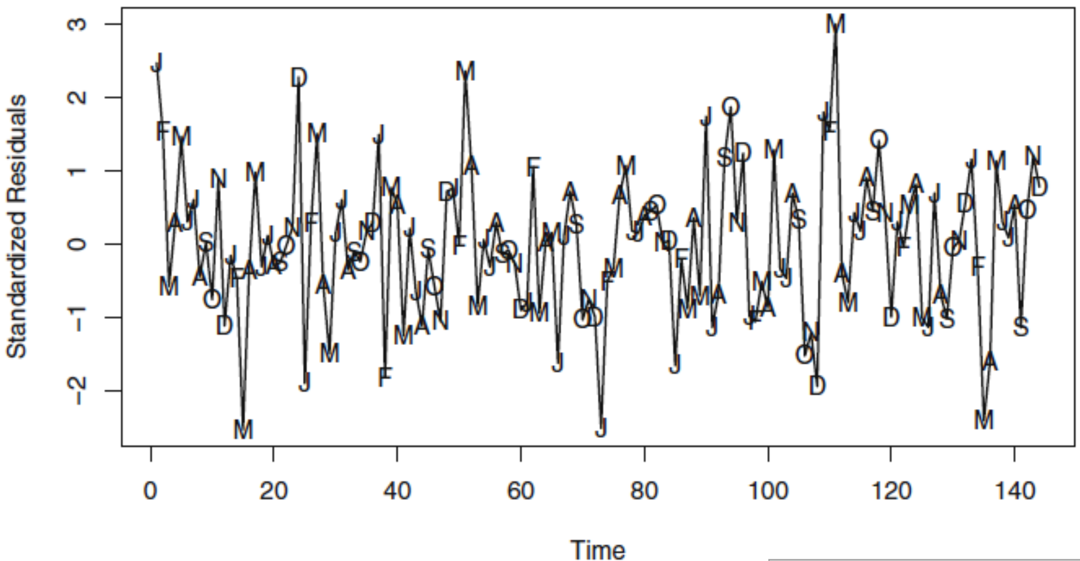
Hard to find any pattern..

Residual analysis

- Residuals in seasonal means model

$$\hat{X}_t = Y_t - \hat{\mu}_t$$

Exhibit 3.9 Residuals versus Time with Seasonal Plotting Symbols



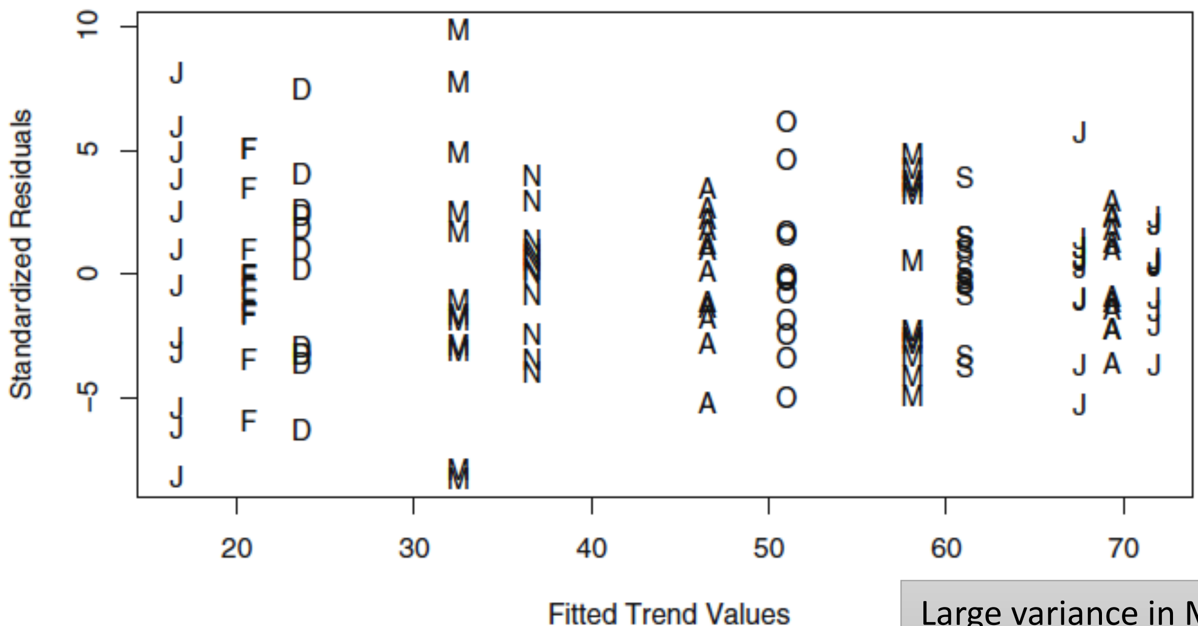
Still hard to find any pattern..

Residual analysis

- Standardized residuals in seasonal means model

$$\hat{X}_t/s = (Y_t - \hat{\mu}_t)/s$$

Exhibit 3.10 Standardized Residuals versus Fitted Values for the Temperature Seasonal Means Model



Large variance in March
Small variance in November
Other than that, still..

Residual analysis

- Standardized residuals in seasonal means model

$$\hat{X}_t/s = (Y_t - \hat{\mu}_t)/s$$

Q-Q plot:
- Displays quantiles of data versus the theoretical quantiles of a normal distribution.
- With normally distributed data, the Q-Q plot looks like a straight line

Exhibit 3.11 Histogram of Standardized Residuals from Seasonal Means Model

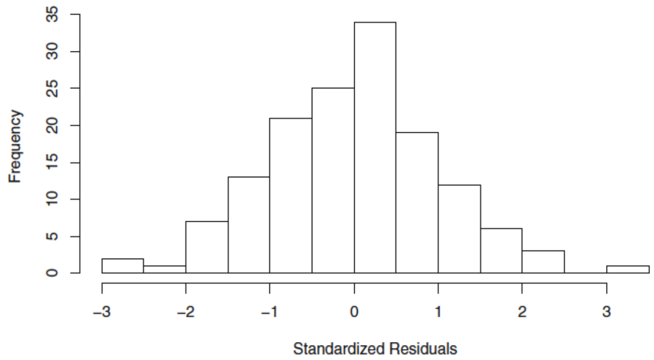
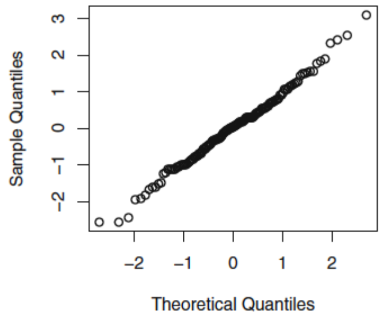


Exhibit 3.12 Q-Q Plot: Standardized Residuals of Seasonal Means Model



Looks like a normal distribution

Looks like a normal distribution (2)

- More precise test of normality can be done by the Shapiro-Wilk test

Residual analysis

- **Independence** in the stochastic component
  - One of the simplest ways to test this is **the runs test**
    - Runs above or below their median are counted
    - A small number of runs would indicate that neighboring residuals are positively dependent
      - Positively correlated
    - Too many runs would indicate that residuals oscillate back and forth across their mean
      - Negatively correlated
    - Hence, either too few or too many runs would reject independence
  - In the seasonal means model,
    - Observed runs = 65, expected runs = 72.875, p-value = 0.216
    - Independence couldn't be rejected

Sample autocorrelation function

- Sample autocorrelation function
  - Another very important diagnostic tool for examining dependence
  - Tentatively assuming stationarity, we would like to estimate the autocorrelation function ρ_k for a variety of lags $k = 1,2,\ldots$
    - More specifically, we will use the sample mean $\bar{Y}$ as a common mean for all Y_t, and the variance around the sample mean as a common variance for all Y_t
  - Then, the sample autocorrelation function r_k at lag k is

Note that if we do not have assumptions above, these two should be μ_t and μ_{t-k}, respectively

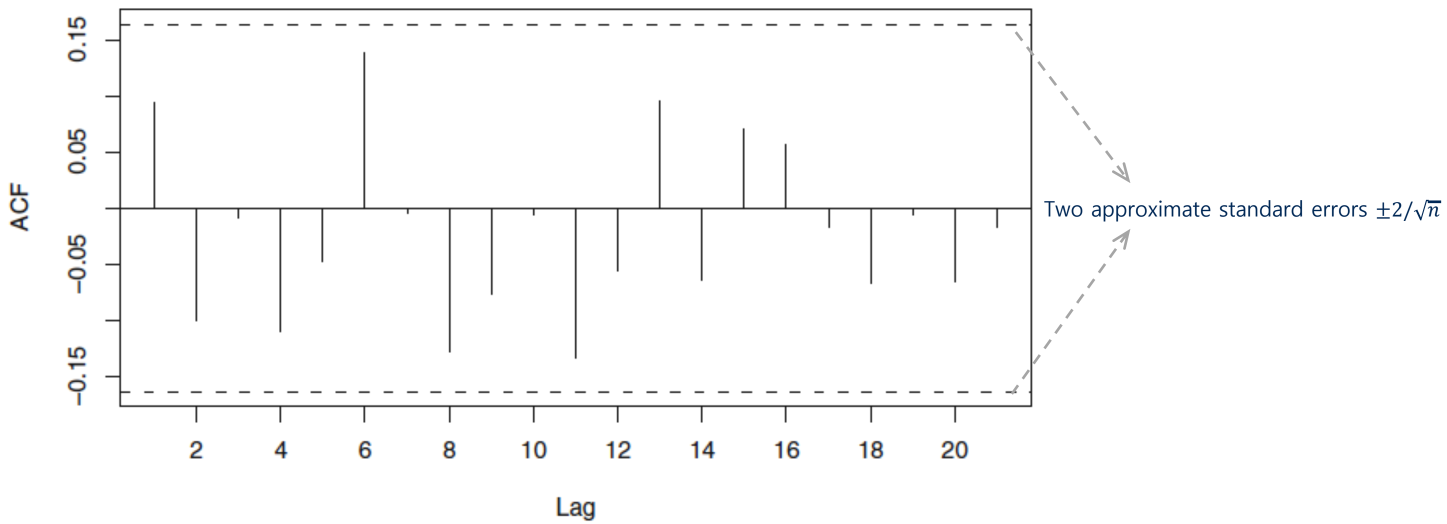
$$r_k = \frac{\sum_{t=k+1}^{n}(Y_t - \boxed{\bar{Y}})(Y_{t-k} - \boxed{\bar{Y}})}{\boxed{\sum_{t=1}^{n}(Y_t - \bar{Y})^2}}, \qquad \text{for } k = 1,2,\ldots$$

Note that if we do not have assumptions above, this should be $\sqrt{Var(Y_t)Var(Y_{t-k})}$

Sample autocorrelation function

- A plot of r_k versus lag k is often called a correlogram

Exhibit 3.13 Sample Autocorrelation of Residuals of Seasonal Means Model



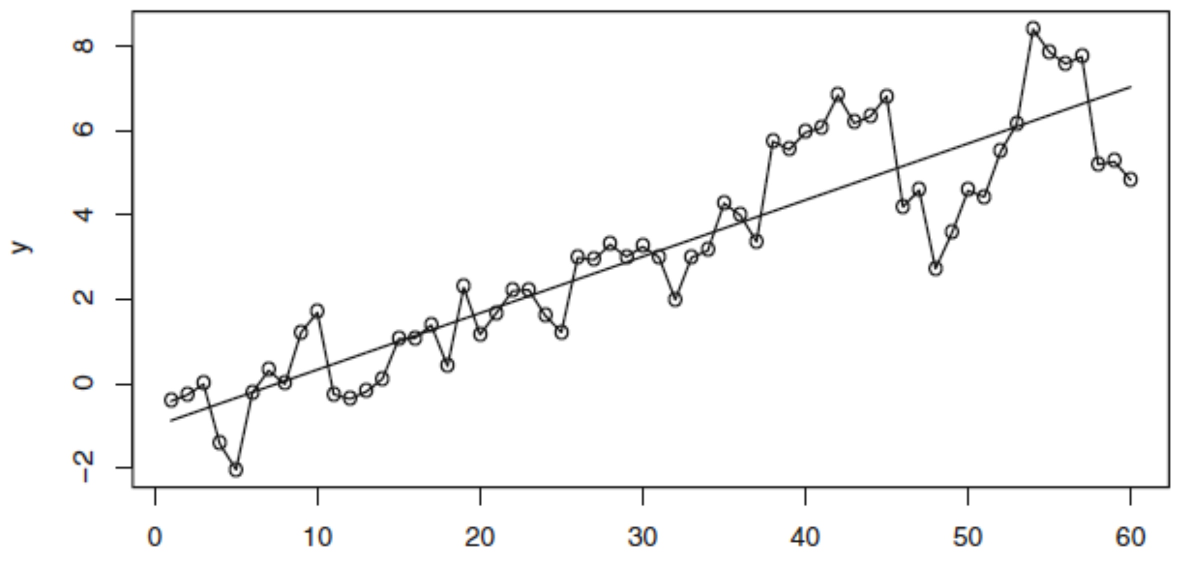
– For the residual $\{X_t\}$ of our seasonal means model,
  - All values are within two standard errors
  - According to this, non of the hypothesis $\rho_k = 0$ can be rejected at the usual significance level (Hence, X_t would be white noise)

Residual analysis

- Residuals in random walk with linear time trend

$$\mu_t = \beta_0 + \beta_1 t$$

Exhibit 3.2 Random Walk with Linear Time Trend

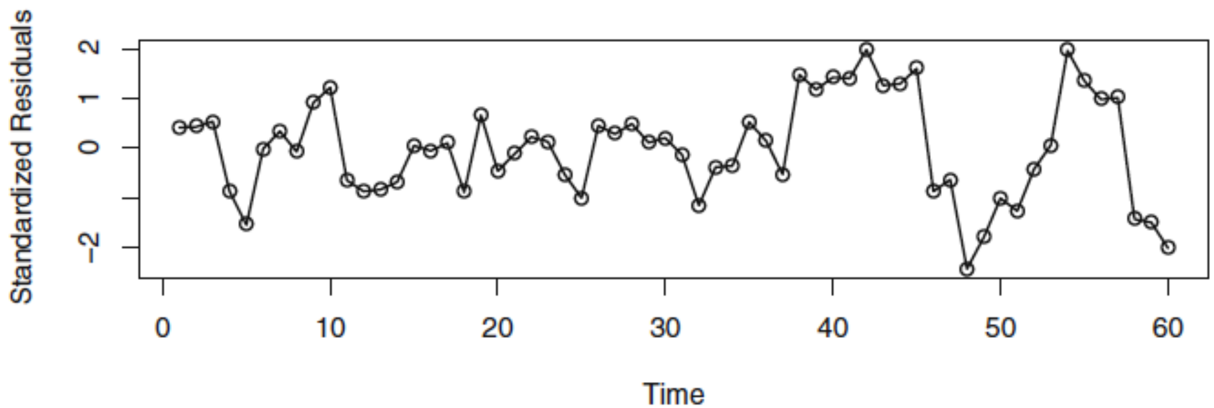


Residual analysis

- Residuals in random walk with linear time trend

$$\mu_t = \beta_0 + \beta_1 t$$

Exhibit 3.14 Residuals from Straight Line Fit of the Random Walk



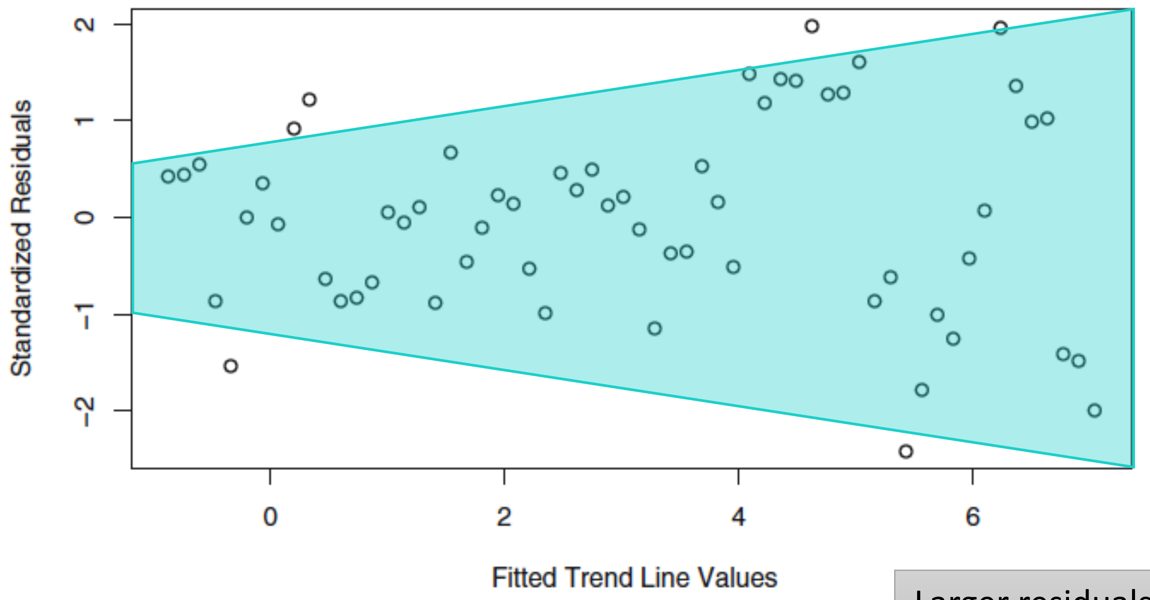
Residuals "hang together" too much to be regarded as white noise.
The plot is too smooth

Residual analysis

- Residuals in random walk with linear time trend

$$\mu_t = \beta_0 + \beta_1 t$$

Exhibit 3.15 Residuals versus Fitted Values from Straight Line Fit



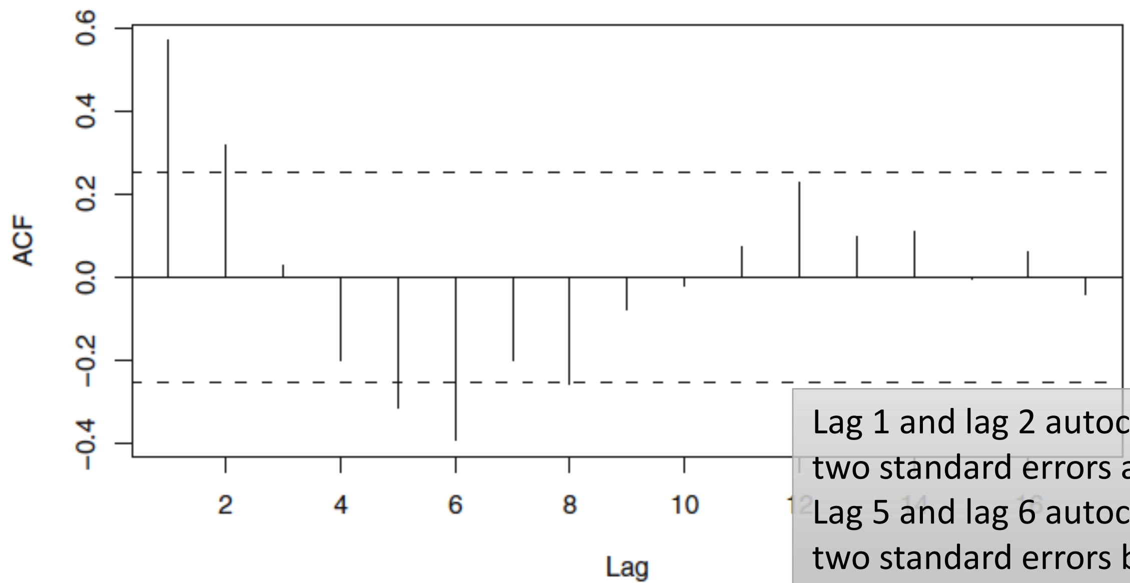
Larger residuals are associated with larger fitted values

Residual analysis

- Residuals in random walk with linear time trend

$$\mu_t = \beta_0 + \beta_1 t$$

Exhibit 3.16 Sample Autocorrelation of Residuals from Straight Line Model



Lag 1 and lag 2 autocorrelations exceed two standard errors above zero
Lag 5 and lag 6 autocorrelations are below two standard errors below zero

This is not what we expect from a white noise process

Summary of Chapter 3

- This chapter is concerned with **describing, modeling, and estimating** deterministic trends in time series

- Regression **methods were pursued to estimate various trends**
  - Linear
  - Quadratic
  - Seasonal
  - Cosine

- Residual analysis **can investigate the quality of the fitted model**
  - **Sample autocorrelation function** will be revisited throughout the remainder of this course