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IE 313 Time Series Analysis

# 12. Time Series Models of Heteroscedasticity

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# Heteroscedasticity

- The models discussed so far concern the **conditional mean** structure of time series data

- However, more recently, there has been much work on modeling the **conditional variance** structure of time series data
  – Mainly motivated by the needs for financial modeling

# Heteroscedasticity

- ARMA models were used to model the conditional mean of a process when the conditional variance was constant
  - Using an AR(1) as an example, we assumed

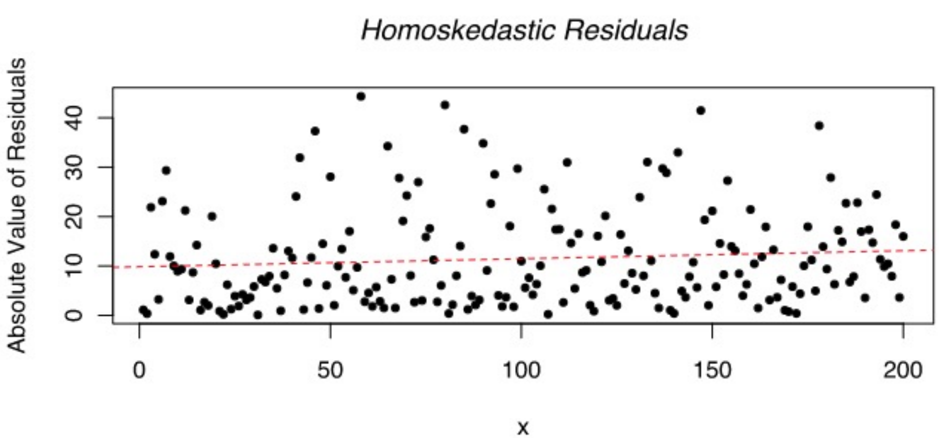
$$E(Y_t \mid Y_{t-1}, Y_{t-2}, \ldots) = \phi Y_{t-1},$$
$$Var(Y_t \mid Y_{t-1}, Y_{t-2}, \ldots) = Var(e_t) = \sigma_e^2$$

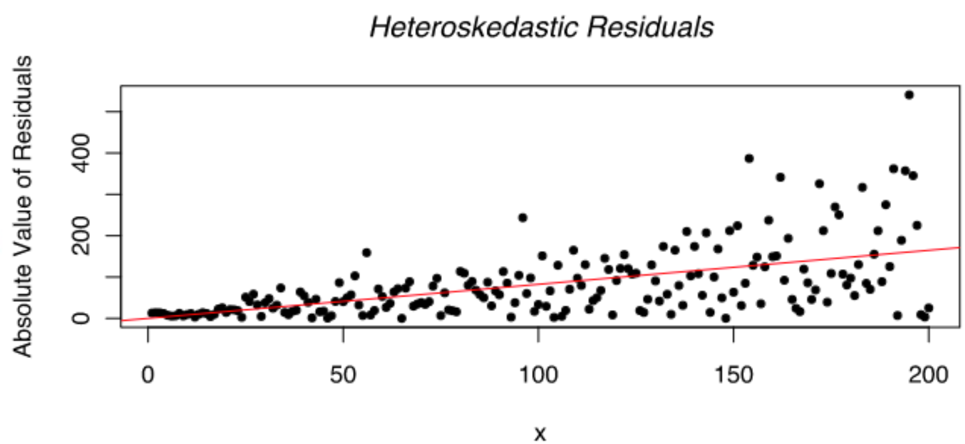
- In many problems, however, the **assumption of a constant conditional variance will be violated**
  - For example, option pricing in finance have motivated the study of the volatility, or variability, of a time series
  - Models such as **autoregressive conditionally heteroscedastic (ARCH) model**, first introduced by Engle (1982), were developed to model changes in volatility
  - These models were later extended to **generalized ARCH (GARCH) models** by Bollerslev (1986)

# Heteroscedasticity

- **Homoscedasticity**
  - A collection of random variable is homoscedastic if all its random variables have the same finite variance

- **Heteroscedasticity**
  - A collection of random variable is heteroscedastic if there are subpopulations that have different variabilities from others
  - Hence, heteroscedasticity is the absence of homoscedasticity



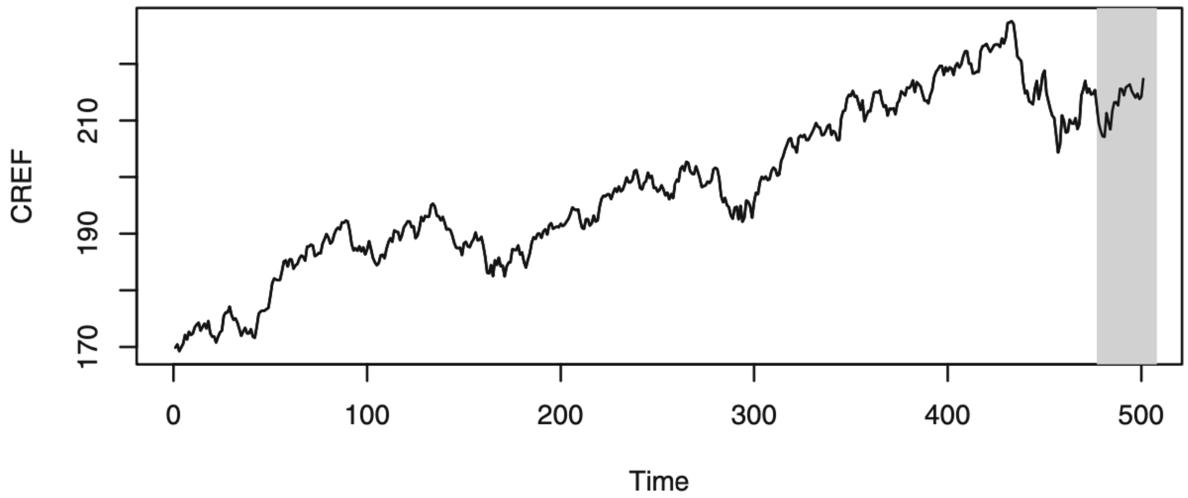
Chapter 12.1

# Some Common Features of Financial Time Series

# Price (or value) series

- Daily CREF (College Retirement Equities Fund) growth

**Exhibit 12.1 Daily CREF Stock Values: August 26, 2004 to August 15, 2006**




– Shows a generally increasing trend with a hint of higher variability with higher level of the fund value

# Return series

- Daily CREF (College Retirement Equities Fund) log return

  – If we let $\{p_t\}$ be the time series, then the log return (or continuously compounded return) on the $t$ th day is defined as

$$r_t = \log\left(\frac{p_t}{p_{t-1}}\right) = \log(p_t) - \log(p_{t-1})$$

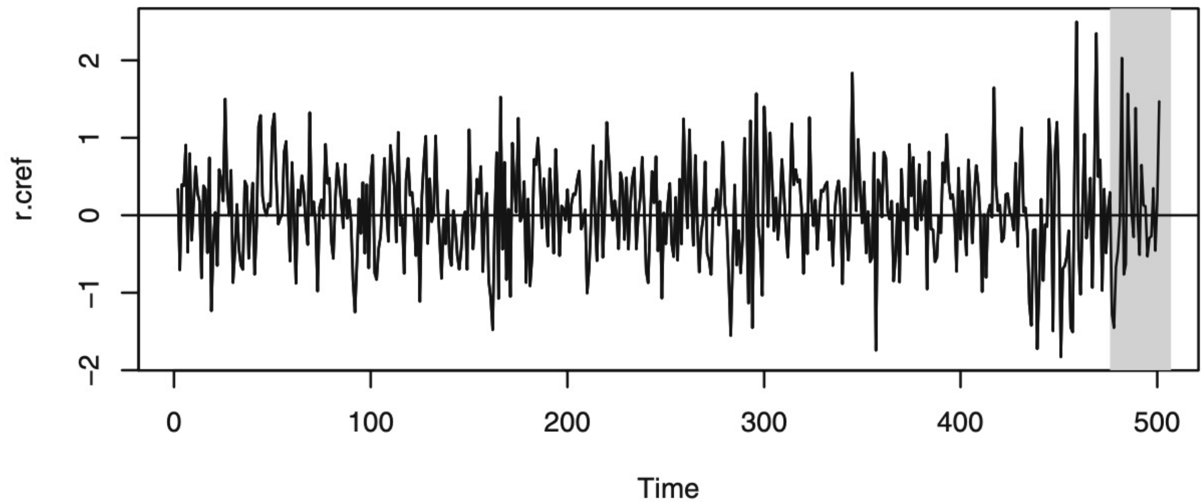
  – Sometimes the returns are the multiplied by 100 so that they can be interpreted as percentage changes in the price

    - This may also reduce numerical errors as the raw returns could be very small numbers and render large rounding errors in some calculations

# Volatility clustering

- Daily CREF (College Retirement Equities Fund) log return

**Exhibit 12.2 Daily CREF Stock Returns: August 26, 2004 to August 15, 2006**




- Returns were more volatile over some time period and became very volatile toward the end of the period
- This pattern of alternating quiet and volatile periods of substantial duration is referred to as ***volatility clustering***

# Volatility clustering

- CBOE VIX (Chicago Board Options Exchange's Volatility Index)
  – It measures stock market's expectation of volatility
  – Also known as 'fear index' or 'fear gauge'

# Low correlation

- ACF and PACF of CREF (College Retirement Equities Fund) returns

**Exhibit 12.3 Sample ACF of Daily CREF Returns: 8/26/04 to 8/15/06**

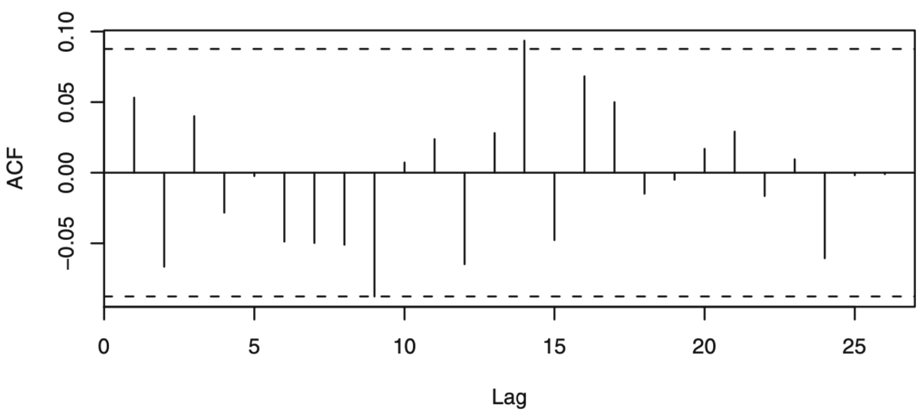
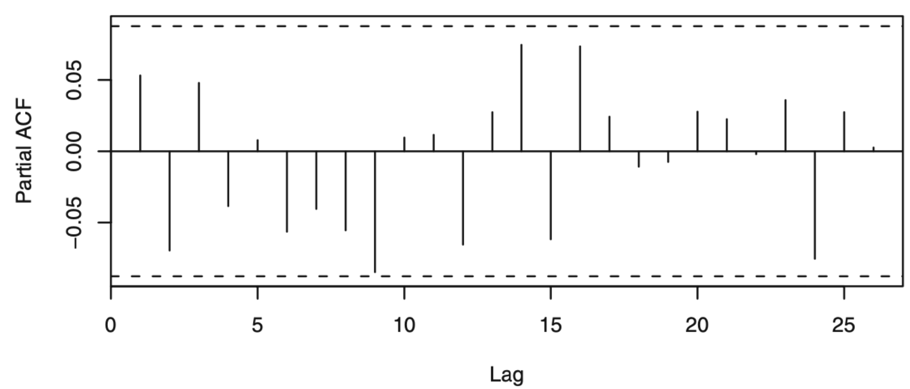



**Exhibit 12.4 Sample PACF of Daily CREF Returns: 8/26/04 to 8/15/06**




  - These results suggest that the returns have almost no serial correlation at all
  - The sample EACF (not shown here) also suggests that a white noise model is appropriate for these data
    - This is quite common in finance. Stock returns are often modeled as some variants of random walk process

# Higher-order dependencies

- Although ACF and PACF shows no (or very little) correlation, the volatility clustering phenomenon gives us a hint that CREF returns may not be i.i.d.
  - Otherwise, the variance would be constant over time

- This is the first occasion in our study of time series models where we need to distinguish between ‘uncorrelated’ and ‘independent’
  - Recall that correlation only measures linear dependencies
  - Hence, no correlation does not mean that there is no dependencies

# Higher-order dependencies

- Higher-order serial dependence structure in data can be explored by studying the autocorrelation structure of absolute returns or squared returns
  - If the returns are i.i.d., then so are the absolute returns and squared returns
  - Any significant autocorrelations in absolute or squared returns would be the evidence against the i.i.d. assumption

# Higher-order dependencies

**Exhibit 12.5 Sample ACF of the Absolute Daily CREF Returns**

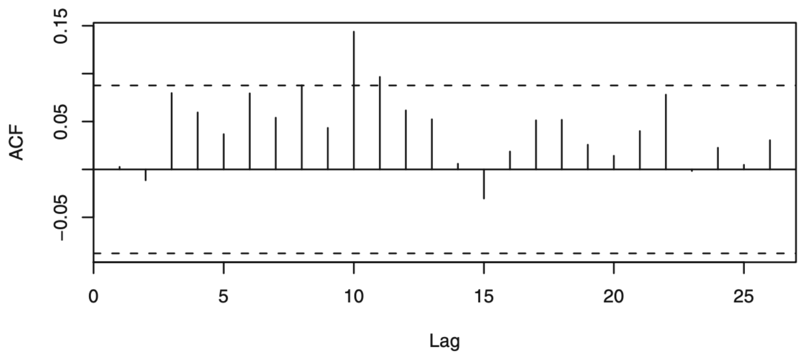



**Exhibit 12.6 Sample PACF of the Absolute Daily CREF Returns**

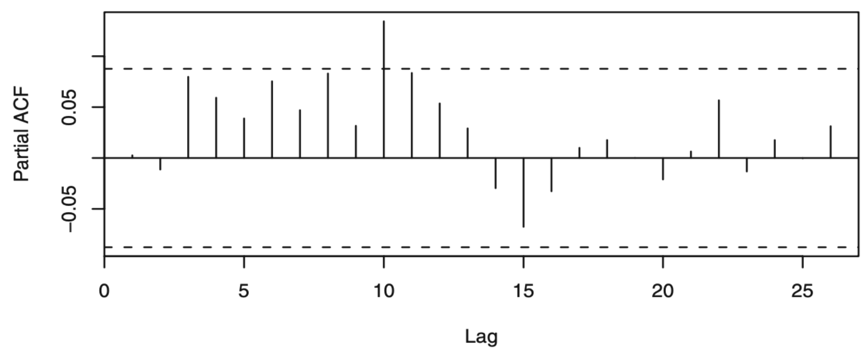



**Exhibit 12.7 Sample ACF of the Squared Daily CREF Returns**

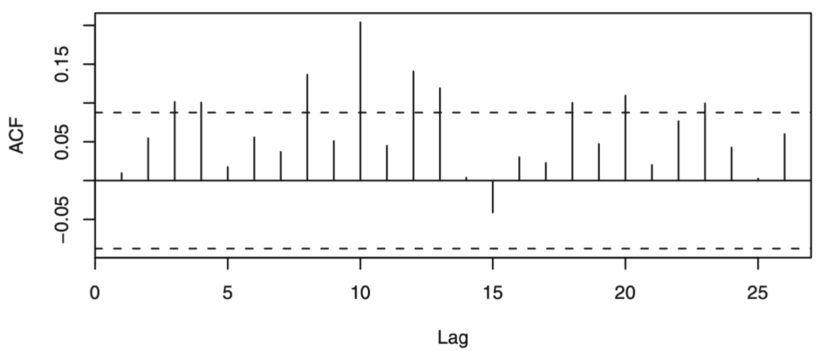
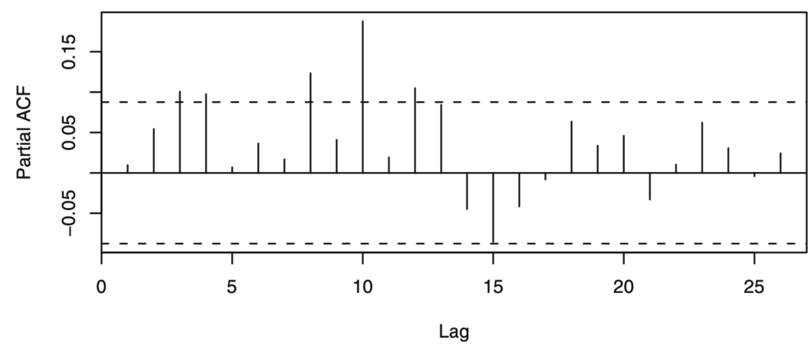



**Exhibit 12.8 Sample PACF of the Squared Daily CREF Returns**




– Indeed, there are some significant autocorrelations in the above plots, which provide some evidence that the CREF returns are not i.i.d.

Chapter 12.2

# The ARCH(1) Model

# ARCH model

- **Autoregressive conditional heteroscedasticity (ARCH)** model
  - Engel (1982) first proposed it for modeling the changing variance of a time series
  - As discussed before, the return series of a financial asset $\{r_t\}$ is often a serially uncorrelated sequence with zero mean, even as it exhibits volatility clustering
    - This suggests that the conditional variance $r_t$ given past returns is not constant
    - **Conditional variance** (or **conditional volatility**) of $r_t$ given returns through time $t-1$ is denoted by $\sigma^2_{t|t-1}$

# ARCH model

- The ARCH model is formally a **regression model** with
  - Conditional volatility $\sigma^2_{t|t-1}$ as the **dependent** variable
  - Past lags of the squared returns $r^2_{t-1}, r^2_{t-2}, \ldots$ as the **independent** variables

# ARCH(1)

- **ARCH(1) model** assumes that the return series $\{r_t\}$ is generated as

$$r_t = \sigma_{t|t-1}\varepsilon_t$$
$$\sigma_{t|t-1}^2 = \omega + \alpha r_{t-1}^2$$

  – $\alpha$ and $\omega$ are unknown parameters

  – $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with zero mean and unit variance (also known as *innovations*)

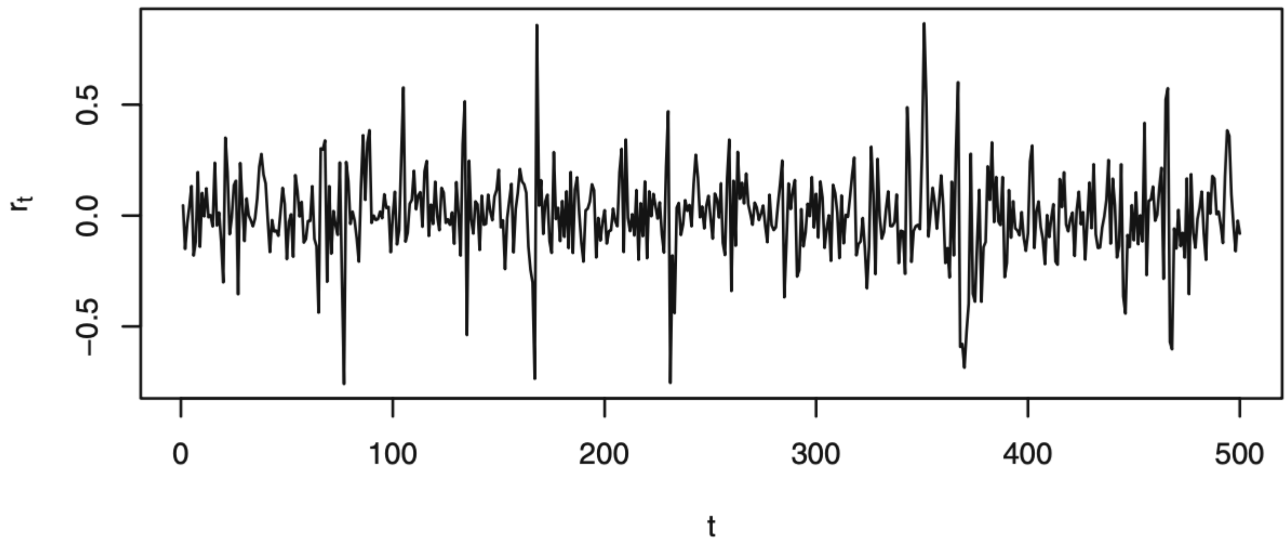
  – If we let $\eta_t = r_t^2 - \sigma_{t|t-1}^2$, we have

$$r_t^2 = \omega + \alpha r_{t-1}^2 + \eta_t$$

  – $\{r_t^2\}$ satisfies an **AR(1) model** under the assumption of an ARCH(1) model for $\{r_t\}$

# ARCH(1)

**Exhibit 12.11 Simulated ARCH(1) Model with $\omega = 0.01$ and $\alpha_1 = 0.9$**



# ARCH(q)

- **ARCH(q) model** assumes that the return series $\{r_t\}$ is generated as

$$r_t = \sigma_{t|t-1}\varepsilon_t$$
$$\sigma^2_{t|t-1} = \omega + \alpha_1 r^2_{t-1} + \alpha_2 r^2_{t-2} + \cdots + \alpha_q r^2_{t-q}$$

  – And $\{r_t^2\}$ satisfies an AR(q) model

Chapter 12.3

# GARCH Models

# GARCH(p,q)

- **GARCH(p,q) model** assumes that the return series $\{r_t\}$ is generated as

$$r_t = \sigma_{t|t-1}\varepsilon_t$$
$$\sigma^2_{t|t-1} = \omega + \beta_1\sigma^2_{t-1|t-2} + \beta_2\sigma^2_{t-2|t-3} + \cdots + \beta_p\sigma^2_{t-p|t-p-1}$$
$$+\alpha_1 r^2_{t-1} + \alpha_2 r^2_{t-2} + \cdots + \alpha_q r^2_{t-q}$$

  – If we let $\eta_t = r_t^2 - \sigma^2_{t|t-1}$, we have

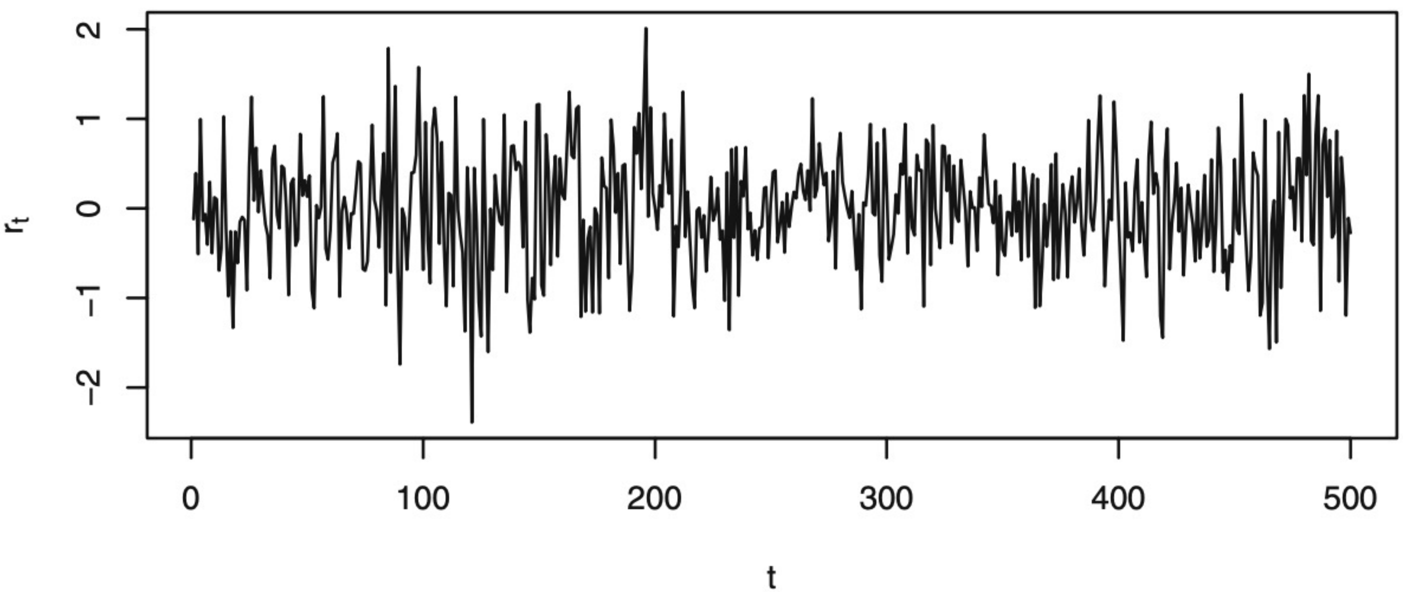
$$r_t^2 = \omega + (\beta_1 + \alpha_1)r^2_{t-1} + \cdots + \left(\beta_{\max(p,q)} + \alpha_{\max(p,q)}\right)r^2_{t-\max(p,q)}$$
$$+\eta_t - \beta_1\eta_{t-1} - \cdots - \beta_p\eta_{t-p}$$

    - $\beta_k$ for all $k > p$ and $\alpha_k$ for all $k > q$

  – $\{r_t^2\}$ satisfies an **ARMA(max(p,q),p) model** under the assumption of an GARCH(p,q) model for $\{r_t\}$

# GARCH(p,q)

**Exhibit 12.12 Simulated GARCH(1,1) Process**



Chapter 12.7

# Some Extensions of the GARCH Model

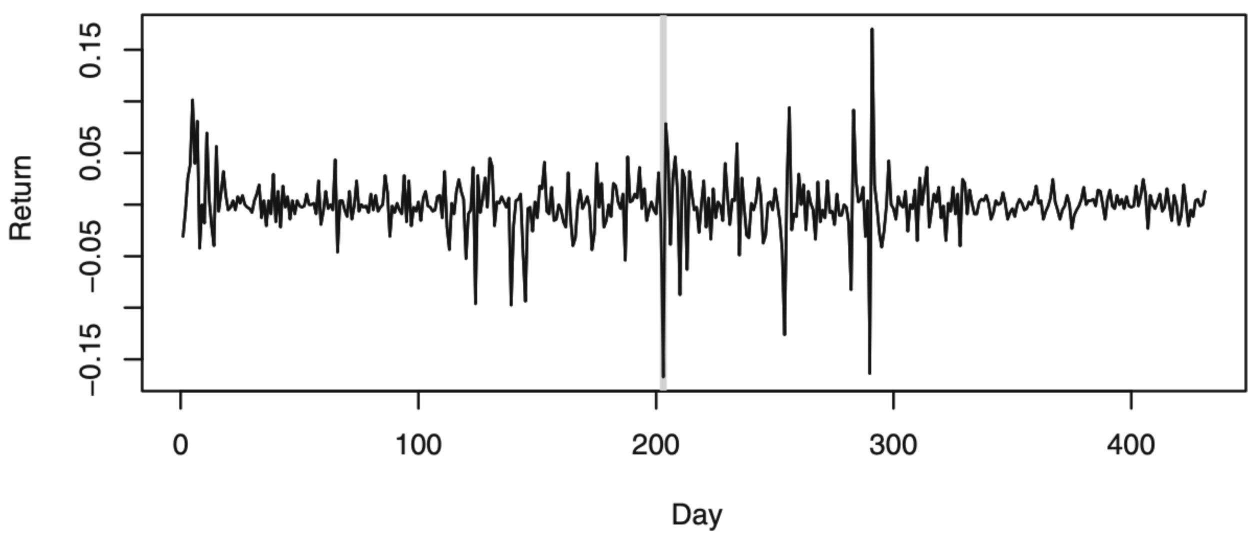
# ARMA + GARCH

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \cdots + \phi_u Y_{t-u} \theta_0 + e_t + \theta_1 e_{t-1} + \cdots + \theta_v e_{t-v} \\
e_t &= \sigma_{t|t-1} \varepsilon_t \\
\sigma^2_{t|t-1} &= \omega + \alpha_1 e^2_{t-1} + \cdots + \alpha_q e^2_{t-q} + \beta_1 \sigma^2_{t-1|t-2} + \cdots + \beta_p \sigma^2_{t-p|t-p-1}
\end{aligned}
$$

# ARMA + GARCH

- Example: daily USD/HKD exchange rates

**Exhibit 12.33 Daily Returns of USD/HKD Exchange Rate: 1/1/05–3/7/06**



# ARMA + GARCH

- Example: daily USD/HKD exchange rates

**Exhibit 12.35 AIC Values for Various Fitted Models for the Daily Returns of the USD/HKD Exchange Rate**

| AR order | GARCH order ($p$) | ARCH order ($q$) | AIC | Stationarity |
|---|---|---|---|---|
| 0 | 3 | 1 | −1915.3 | nonstationary |
| 1 | 1 | 1 | −2054.3 | nonstationary |
| 1 | 1 | 2 | −2072.5 | nonstationary |
| 1 | 1 | 3 | −2051.0 | nonstationary |
| 1 | 2 | 1 | −2062.2 | nonstationary |
| 1 | 2 | 2 | −2070.5 | nonstationary |
| 1 | 2 | 3 | −2059.2 | nonstationary |
| 1 | 3 | 1 | −2070.9 | stationary |
| 1 | 3 | 2 | −2064.8 | stationary |
| 1 | 3 | 3 | −2062.8 | stationary |
| 1 | 4 | 1 | −2061.7 | nonstationary |
| 1 | 4 | 2 | −2054.8 | stationary |
| 1 | 4 | 3 | −2062.4 | stationary |
| 2 | 3 | 1 | −2066.6 | stationary |

# ARMA + GARCH

- Example: daily USD/HKD exchange rates

**Exhibit 12.36 Estimated Conditional Variances of the Daily Returns of USD/HKD Exchange Rate from the Fitted AR(1) + GARCH(3,1) Model**