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Homework Assignment 1

1) i) $E[\nabla Y_t] = E[Y_t - Y_{t-1}] = E(Y_t) - E(Y_{t-1}) \ominus$

Since $\{Y_t\}$ is stationary, $E(Y_t) = E(Y_{t-1}) = \mu \Rightarrow$

$\ominus\ \mu - \mu = 0$ (constant over time)

$Cov[\nabla Y_t, \nabla Y_{t-k}] = Cov[Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}] =$

$= Cov(Y_t, Y_{t-k}) - Cov(Y_{t-1}, Y_{t-k}) -$
$- Cov(Y_t, Y_{t-k-1}) + Cov(Y_{t-1}, Y_{t-k-1}) \ominus$

Here, we can make use of the covariance γ_k of our $\underbrace{\text{stochastic process}}_{\{Y_t\}}$

$\gamma_k = Cov(Y_t, Y_{t-k})$, $Cov(Y_{t-1}, Y_{t-k}) =$

$= Cov(Y_{t-1}, Y_{t-1-(k-1)}) = \gamma_{k-1}$

$Cov(Y_{t-1}, Y_{t-k-1}) = Cov(Y_{t-1}, Y_{t-(k+1)}) = \gamma_{k+1}$

$Cov(Y_{t-1}, Y_{t-k-1}) = Cov(Y_{(t-1)}, Y_{(t-1)-k}) = \gamma_k$

$\ominus\ \gamma_k - \gamma_{k-1} - \gamma_{k+1} + \gamma_k = 2\gamma_k - \gamma_{k-1} - \gamma_{k+1}$

We can say that the covariance function for $\{\nabla Y_t\}$ does not depend on time since $\gamma_k, \gamma_{k-1}, \gamma_{k+1}$ don't depend on time, which is the result of stationarity of $\{Y_t\}$

b) In i), we showed that if a process $\{Y_t\}$ is stationary, then its first difference $\{\nabla Y_t\}$ is also stationary. Since we now know that $\nabla Y_t = Y_t - Y_{t-1}$ is stationary, then its first difference $\nabla[Y_t - Y_{t-1}]$ will also be stationary.

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② $Y_t = 3 + e_t - \frac{1}{3} e_{t-1} + \frac{1}{2} e_{t-2}$

$Var(Y_t) = Var(3 + e_t - \frac{1}{3} e_{t-1} + \frac{1}{2} e_{t-2}) =$

$\sigma_e^2 + \frac{1}{9}\sigma_e^2 + \frac{1}{4}\sigma_e^2 = \frac{49}{36}\sigma_e^2 = \gamma_0$

$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(3 + e_t - \frac{1}{3} e_{t-1} + \frac{1}{2} e_{t-2},$
$3 + e_{t-1} - \frac{1}{3} e_{t-2} + \frac{1}{2} e_{t-3}) = Cov(e_t, e_{t-1}) -$
$- \frac{1}{3} Cov(e_{t-1}, e_{t-1}) + \frac{1}{2} Cov(e_{t-2}, e_{t-1}) -$
$- \frac{1}{3} Cov(e_t, e_{t-2}) + \frac{1}{9} Cov(e_{t-1}, e_{t-2}) - \frac{1}{6} Cov(e_{t-2}, e_{t-2})$
$+ \frac{1}{2} Cov(e_t, e_{t-3}) - \frac{1}{6} Cov(e_{t-1}, e_{t-3}) + \frac{1}{4} Cov(e_{t-2}, e_{t-3}) =$

$= -\frac{1}{3} Cov(e_{t-1}, e_{t-1}) - \frac{1}{6} Cov(e_{t-2}, e_{t-2}) =$
$= -\frac{1}{3}\sigma_e^2 - \frac{1}{6}\sigma_e^2 = -\frac{1}{2}\sigma_e^2$

$\gamma_2 = Cov(Y_t, Y_{t-2}) = Cov(3 + e_t - \frac{1}{3} e_{t-1} + \frac{1}{2} e_{t-2},$
$3 + e_{t-2} - \frac{1}{3} e_{t-3} + \frac{1}{2} e_{t-4})$ ⊜ we can see that only one term (e_{t-2}) exists in both Y_t and Y_{t-2}, so

⊜ $\frac{1}{2} Cov(e_{t-2}, e_{t-2}) = \frac{1}{2}\sigma_e^2$

For time lag $s \geq 3$, $\gamma_s = 0$ since there will be no overlap

Now, we can find ACF:

$\rho_0 = 1$

$\rho_1 = \frac{Cov(Y_t, Y_{t-1})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-1})}} = \frac{-\frac{1}{2}\sigma_e^2}{\gamma_0} = \frac{-\frac{1}{2}\sigma_e^2}{\frac{49}{36}\sigma_e^2} = -\frac{18}{49}$

$\rho_2 = \frac{Cov(Y_t, Y_{t-2})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-2})}} = \frac{\frac{1}{2}\sigma_e^2}{\gamma_0} = \frac{\frac{1}{2}\sigma_e^2}{\frac{49}{36}\sigma_e^2} = \frac{18}{49}$

$\rho_s = 0$, when $s \geq 3$

(3) We know that for MA(1) process $Y_t = e_t - \theta e_{t-1}$,

$$\rho_k = \begin{cases} 1 & \text{if } k=0 \\ -\theta/(1+\theta^2) & \text{if } k=1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

If θ is changed to $1/\theta$, then $Y_t = e_t - \frac{1}{\theta} e_{t-1}$, and

$\gamma_0 = Var(Y_t) = Var(e_t - \frac{1}{\theta} e_{t-1}) = \sigma_e^2 + \frac{1}{\theta^2}\sigma_e^2 =$

$= \sigma_e^2 \left(\frac{1+\theta^2}{\theta^2}\right)$

$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(e_t - \frac{1}{\theta} e_{t-1}, e_{t-1} - \frac{1}{\theta} e_{t-2}) =$

$= Cov(-\frac{1}{\theta} e_{t-1}, e_{t-1}) = -\frac{1}{\theta}\sigma_e^2$

$\gamma_2 = Cov(Y_t, Y_{t-2}) = Cov(e_t - \frac{1}{\theta} e_{t-1}, e_{t-2} - \frac{1}{\theta} e_{t-3}) =$

$= 0$ (since there is no overlap)

Similarly, $Cov(Y_t, Y_{t-k}) = 0$ for $k \geq 2$

So, $\rho_1 = Cov(Y_t, Y_{t-1}) / Var(Y_t) = -\frac{1}{\theta}\sigma_e^2 / \left(\sigma_e^2\left(\frac{1+\theta^2}{\theta^2}\right)\right) =$

$= -\frac{1}{\theta} \times \frac{\theta^2}{1+\theta^2} = -\frac{\theta}{1+\theta^2}$, $\rho_0 = 1$ and $\rho_k = 0$ for $k \geq 2$

We can see that the autocorrelation function for an MA(1) does not change.

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(4) $Y_t = 3Y_{t-1} + e_t$ (nonstationary AR(1) model)

i) We will show that $Y_t = -\sum_{j=1}^{\infty} (\frac{1}{3})^j e_{t+j}$

satisfies the above AR(1) equation (or its equivalent $Y_t - 3Y_{t-1} = e_t$)

$$-\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j} - 3\left(-\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t-1+j}\right) = -\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j} +$$

$$+ 3\left(\tfrac{1}{3}e_t + \sum_{j=2}^{\infty}(\tfrac{1}{3})^j e_{t-1+j}\right) = -\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j} + e_t +$$

$$+ 3\left(\sum_{j=1}^{\infty}(\tfrac{1}{3})^{j+1} e_{t+j}\right) = -\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j} + e_t + 3\left(\tfrac{1}{3}\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j}\right) =$$

$$= -\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j} + \sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j} + e_t = e_t$$

ii) $Y_t = -\sum_{j=1}^{\infty}(\frac{1}{3})^j e_{t+j}$

$$E(Y_t) = E\left(-\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j}\right) = -\sum_{j=1}^{\infty}(\tfrac{1}{3})^j E(e_{t+j}) = 0 \text{ (constant over time)}$$

$$\gamma_0 = Var(Y_t) = Var\left(-\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j}\right) = \sum_{j=1}^{\infty}(\tfrac{1}{3})^{2j} Var(e_{t+j}) =$$

$$= \sigma_e^2 \sum_{j=1}^{\infty}(\tfrac{1}{3})^{2j} = \sigma_e^2\left(\tfrac{1}{9} + \tfrac{1}{81} + \dots\right) = \tfrac{1}{9}\sigma_e^2\left(1 + \tfrac{1}{9} + \dots\right) =$$

$$= \tfrac{1}{9}\sigma_e^2\left(1 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^4 + \dots\right) = \tfrac{1}{9}\sigma_e^2 / (1 - \tfrac{1}{9}) = \tfrac{1}{9}\sigma_e^2 \times \tfrac{9}{8} = \tfrac{1}{8}\sigma_e^2$$

$$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov\left(-\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t+j}, -\sum_{j=1}^{\infty}(\tfrac{1}{3})^j e_{t-1+j}\right) =$$

$$= Cov\left(-\tfrac{1}{3}e_{t+1}, -(\tfrac{1}{3})^2 e_{t+1}\right) + Cov\left(-(\tfrac{1}{3})^2 e_{t+2}, -(\tfrac{1}{3})^3 e_{t+2}\right) + \dots =$$

$$= \sigma_e^2\left((\tfrac{1}{3})^3 + (\tfrac{1}{3})^5 + \dots\right) = \tfrac{1}{27}\sigma_e^2\left(1 + (\tfrac{1}{3})^2 + \dots\right) = \tfrac{1}{27}\sigma_e^2 / (1 - \tfrac{1}{9}) =$$

$$= \tfrac{1}{27}\sigma_e^2 \times \tfrac{9}{8} = \frac{\sigma_e^2}{24} = \tfrac{1}{8}\sigma_e^2 \times (\tfrac{1}{3})^1$$

So, $\gamma_K = Cov(Y_t, Y_{t-K}) = \frac{1}{8}\sigma_e^2 \times (\frac{1}{3})^K$, which only on time lag

Since mean and covariance conditions are satisfied, $\{Y_t\}$ is stationary.

iii) If we look at the original representation ($Y_t = 3Y_{t-1} + e_t$), we can see that past values in the stochastic process determine future values. However, in the new formulation, $Y_t = -\sum_{j=1}^{\infty} (\frac{1}{3})^j e_{t+j}$, current values are calculated using future values, which is not actually possible in reality and thus we can consider that new representation as unsatisfactory.

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5) $Y_t = A + Bt + X_t$, where $\{X_t\}$ is a random walk

i) $E(Y_t) = E(A + Bt + X_t) = A + Bt + E(X_t) = A + Bt$

For the process to be stationary, its mean function should be constant over time. Since in our case this is not true, $\{Y_t\}$ is non-stationary.

Now we will check $\{\nabla Y_t\}$ for stationarity

$$\nabla Y_t = Y_t - Y_{t-1} = A + Bt + X_t - A - B(t-1) - X_{t-1} =$$

$$= B + X_t - X_{t-1} = B + e_1 + e_2 \dots + e_t - e_1 - e_2 - \dots - e_{t-1} =$$

$= B + e_t$, which is just white noise with mean equal to B, and thus $\{\nabla Y_t\}$ is stationary. (the white noise process was proved to be stationary in the lecture)

ii) If A and B are random variables, then

$E(Y_t) = E(A) + E(B)t + E(X_t) = E(A) + E(B)t$,

which still depends on time and thus $\{Y_t\}$ is non-stationary.

Now, let's check $\{\nabla Y_t\}$ for stationarity

$\nabla Y_t = Y_t - Y_{t-1} = B + e_t$

$E(\nabla Y_t) = E(B) + E(e_t) = E(B)$ (constant over time)

$$\gamma_k = Cov(\nabla Y_t, \nabla Y_{t-k}) = Cov(B + e_t, B + e_{t-k}) =$$

$$= Cov(B, B) + Cov(e_t, B) + Cov(B, e_{t-k}) + Cov(e_t, e_{t-k})$$

$\stackrel{=}{}$ since e_t and e_{t-k} are i.i.d r.v, $Cov(e_t, e_{t-k}) = 0$ and

$Cov(e, B) = Cov(e_t, B) = Cov(B, e_{t-k})$ $\stackrel{=}{}$ $\sigma_B^2 + 2\,Cov(e, B)$,

which does not depend on time. Thus $\{\nabla Y_t\}$ is stationary

we can remove the subscript t.