2022 Fall

IE 313 Time Series Analysis

10. Seasonal Models

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Seasonal models

- In Chapter 3, we saw how seasonal deterministic trends might be modeled

Cyclical or seasonal trends

$Y_t = \mu_t + X_t$
$(E(X_t) = 0 \text{ for all } t)$

- The most general assumption for μ_t with **monthly seasonal data** is that there are 12 constants (parameters),

$$\beta_1, \beta_2, \dots, \beta_{12}$$

giving the expected average temperature for each of the 12 months
  - That is,

$$\mu_t = \begin{cases} \beta_1 & \text{for } t = 1, 13, 25, \dots & \text{January} \\ \beta_2 & \text{for } t = 2, 14, 26, \dots & \text{February} \\ \vdots & \vdots & \vdots \\ \beta_{12} & \text{for } t = 12, 24, 36, \dots & \text{December} \end{cases}$$

  - This is sometimes called a **seasonal means** model

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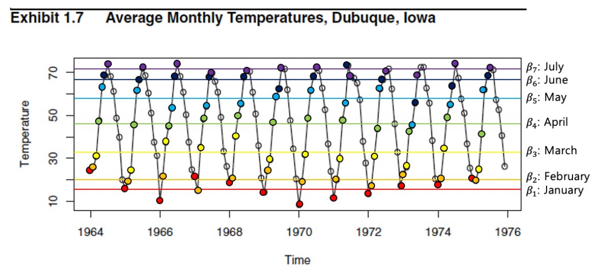
Cyclical or seasonal trends



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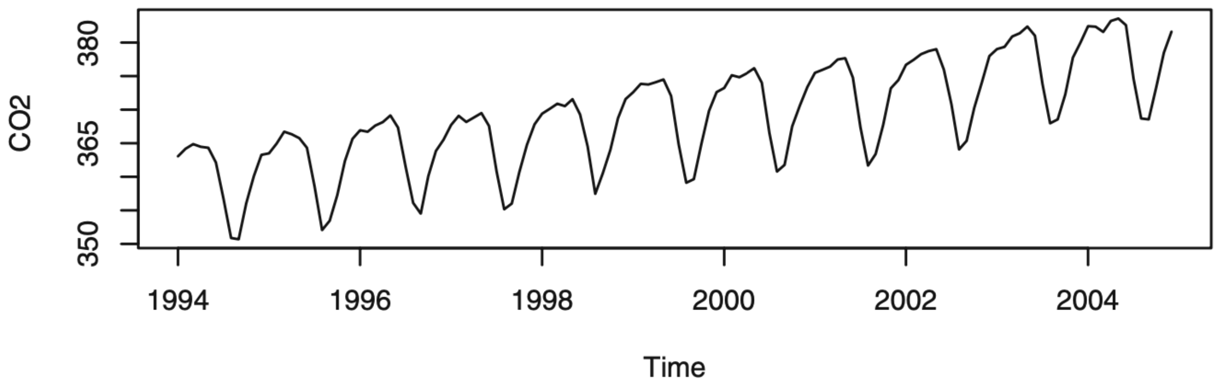
- However, the assumption of any deterministic trend is quite suspect even though cyclical tendencies are very common in such series

Seasonal models

- Example: Levels of CO_2 from Jan 1994 to Dec 2004

Exhibit 10.1 Monthly Carbon Dioxide Levels at Alert, NWT, Canada



  – It might be plausible to fit this into deterministic models such as
    - Seasonal means plus linear time trend
    - Sums of cosine curves at various frequencies plus linear time trend
  – But all the above models have residuals with high autocorrelation at many lags
  – Hence, we would consider ***stochastic* seasonal models**

Chapter 10.1

Seasonal ARIMA models

Seasonal ARIMA models

- Let s denote the known seasonal period
  - For monthly series $s = 12$
  - For quarterly series $s = 4$

- Then, consider the time series generated according to

$$Y_t = e_t - \Theta e_{t-12}$$

  - Notice that
    - $Cov(Y_t, Y_{t-1}) = Cov(e_t - \Theta e_{t-12}, e_{t-1} - \Theta e_{t-13}) = 0$
    - $Cov(Y_t, Y_{t-12}) = Cov(e_t - \Theta e_{t-12}, e_{t-12} - \Theta e_{t-24}) = -\Theta \sigma_e^2$

Seasonal ARIMA models

- Generalizing these ideas, we define a **seasonal MA(Q) model of order Q with seasonal period s** by

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \cdots - \Theta_Q e_{t-Qs}$$

  – With seasonal MA characteristic polynomial

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}$$

  – It is always stationary and its ACF will be nonzero only at the seasonal lags of s, 2s, 3s, …, Qs

  – For the model to be invertible, the roots of $\Theta(x) = 0$ must all exceed 1 in absolute value

Seasonal ARIMA models

- Similarly, we define a **seasonal AR(P) model of order P with seasonal period s** by

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \cdots + \Phi_P Y_{t-Ps} + e_t$$

  – With seasonal AR characteristic polynomial

$$\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps}$$

  – As always, e_t should be independent of $Y_{t-1}, Y_{t-2}, \ldots$

  – For stationarity, the roots of $\Phi(x) = 0$ must be greater than 1 in absolute value

  – Its ACF is nonzero only at lags s, 2s, 3s, …, where it behaves like a combination of decaying exponentials and damped sine functions

Chapter 10.2

Multiplcative Seasonal ARMA models

Multiplicative seasonal ARMA models

- Rarely shall we need models that incorporate autocorrelation *only* at the seasonal lags

- By combining the ideas of seasonal and nonseasonal ARMA models, we can develop parsimonious models that contain autocorrelation for the seasonal lags but also for low lags of neighboring series values

Multiplicative seasonal ARMA models

- Consider a model whose MA characteristic polynomial is given by

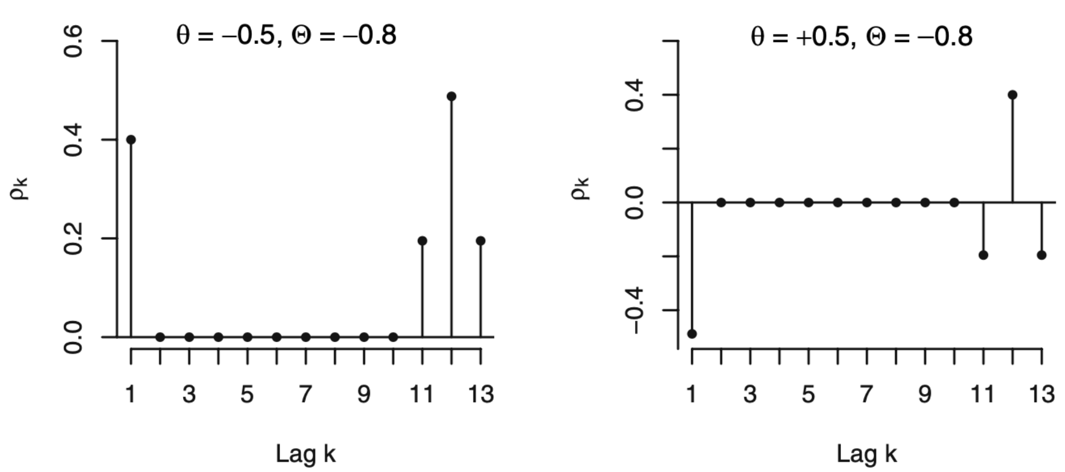
$$(1-\theta x)(1-\Theta x^{12}) = 1-\theta x-\Theta x^{12}+\theta\Theta x^{13}$$

  – Thus, the corresponding time series satisfies

$$Y_t = e_t-\theta e_{t-1}-\Theta e_{t-12}+\theta\Theta e_{t-13}$$

  – Then, we can check that its ACF is nonzero only at lags 1, 11, 12, and 13

Exhibit 10.3 Autocorrelations from Equations (10.2.2)-(10.2.5)



Multiplicative seasonal ARMA models

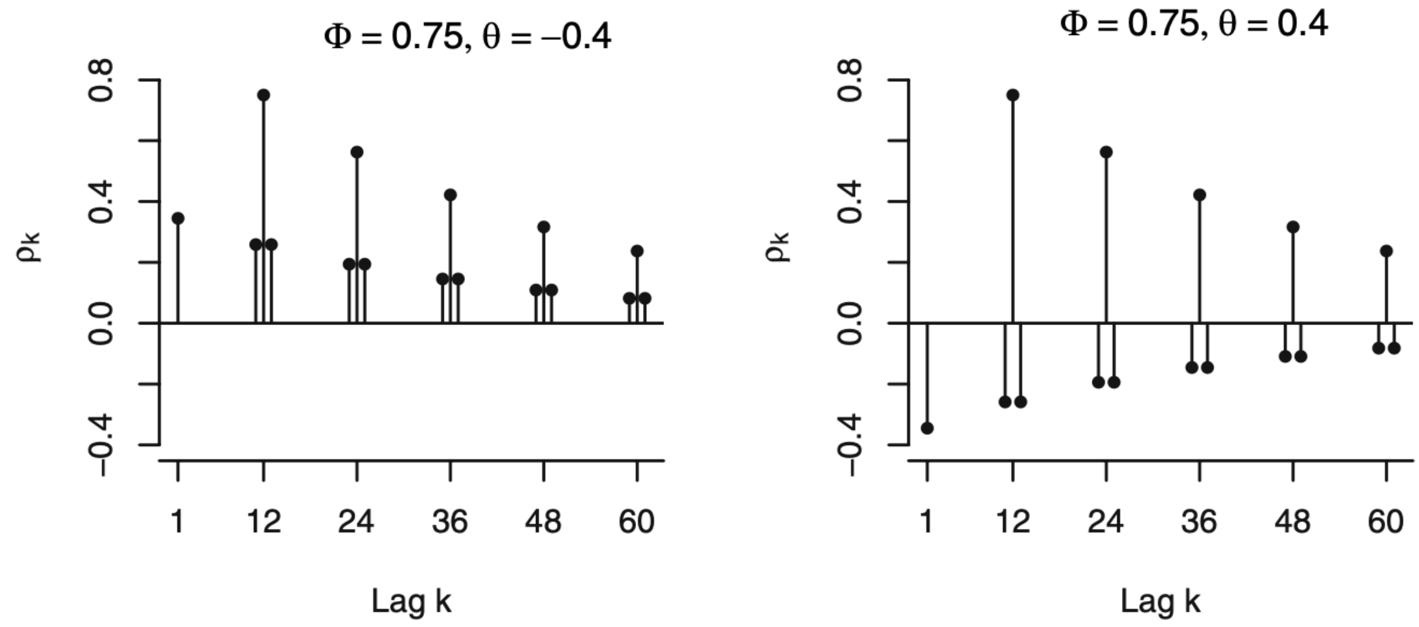
- In general, we define a **multiplicative seasonal ARMA(p,q)x(P,Q)$_s$ model with seasonal period s** as a model
  - with AR characteristic polynomial $\phi(x)\Phi(x)$ and MA characteristic polynomial $\theta(x)\Theta(x)$, where
    - $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$
    - $\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps}$
    - $\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$
    - $\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}$
  - The model may also contain a constant term θ_0

Multiplicative seasonal ARMA models

- Example: ARMA(0,1)x(1,0)$_{12}$ model

$$Y_t = \Phi Y_{t-12} + e_t - \theta e_{t-1}$$

Exhibit 10.4 Autocorrelation Functions from Equation (10.2.11)



Chapter 10.3

Nonstationary Seasonal ARIMA models

Multiplicative seasonal ARIMA models

- An important tool in modeling nonstationary seasonal processes is the seasonal difference

- The **seasonal difference of period s** for the series $\{Y_t\}$ is denoted $\nabla_s Y_t$ and is defined as

$$\nabla_s Y_t = Y_t - Y_{t-s}$$

Multiplicative seasonal ARIMA models

- A process $\{Y_t\}$ is said to be a **multiplicative seasonal ARIMA model** with nonseasonal (regular) orders p, d, and q, seasonal orders P, D, and Q, and seasonal period s if the differenced series

$$W_t = \nabla^d \nabla_s^D Y_t$$

  satisfies an ARMA(p,q)x$(P,Q)_s$ model with seasonal period s

- We say that $\{Y_t\}$ is an ARIMA(p,d,q)x$(P,D,Q)_s$ model with seasonal period s

Chapter 10.4

Model specification, fitting, and checking

Model specification, fitting, and checking

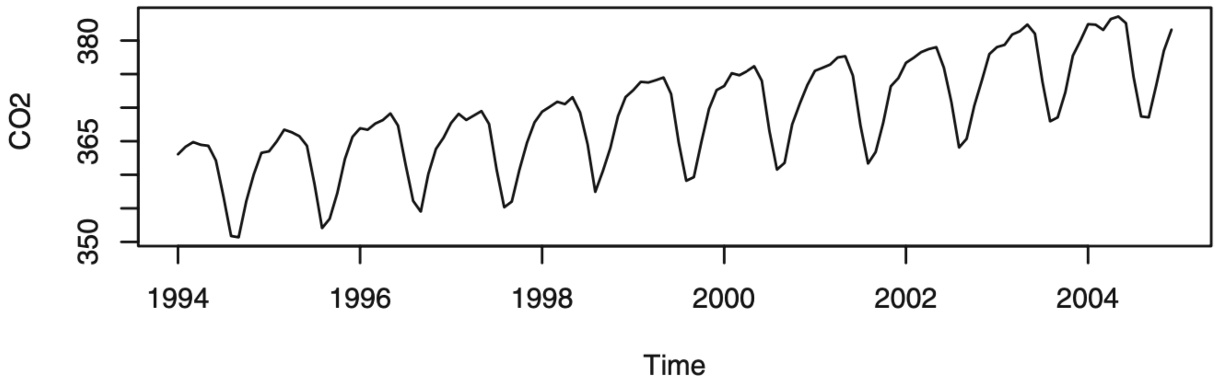
- Model specification, fitting, and diagnostic checking for seasonal models follow the same general techniques developed in Chapters 6, 7, and 8

- Here, we shall simply highlight the application of these ideas specifically to seasonal models and pay special attention to the seasonal lags

Model specification, fitting, and checking

- Original series

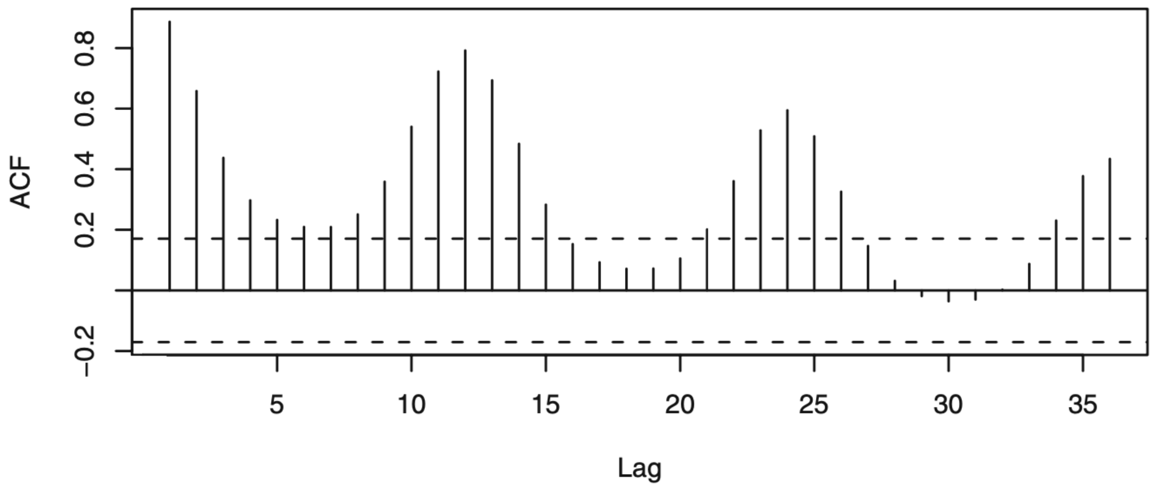
Exhibit 10.1 Monthly Carbon Dioxide Levels at Alert, NWT, Canada



Model specification, fitting, and checking

- Sample ACF

Exhibit 10.5 Sample ACF of CO_2 Levels

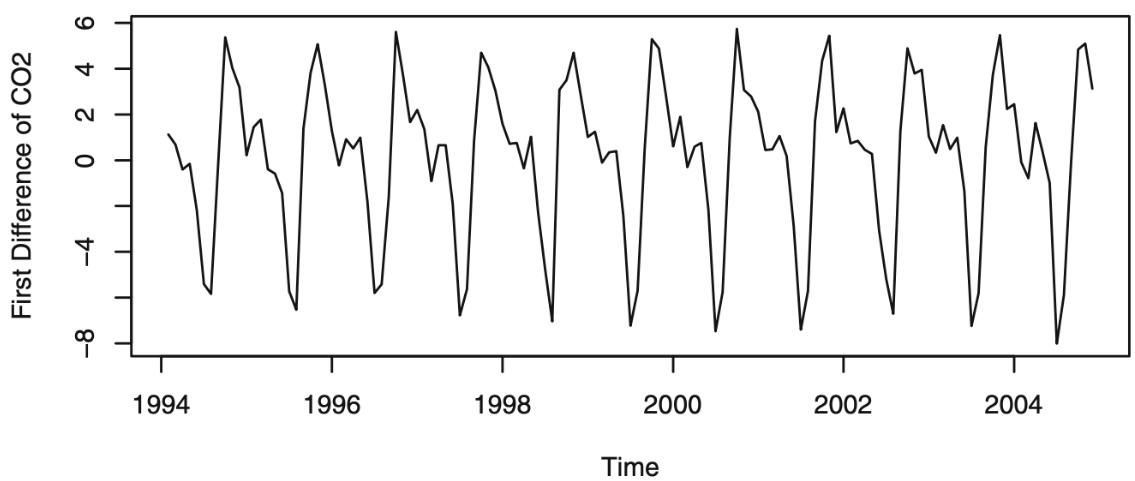


– Notice the strong correlations at lags 12, 24, 36, and so on
– In addition, there is substatial other correlation that needs to be modeled

Model specification, fitting, and checking

- First differencing

Exhibit 10.6 Time Series Plot of the First Differences of CO_2 Levels

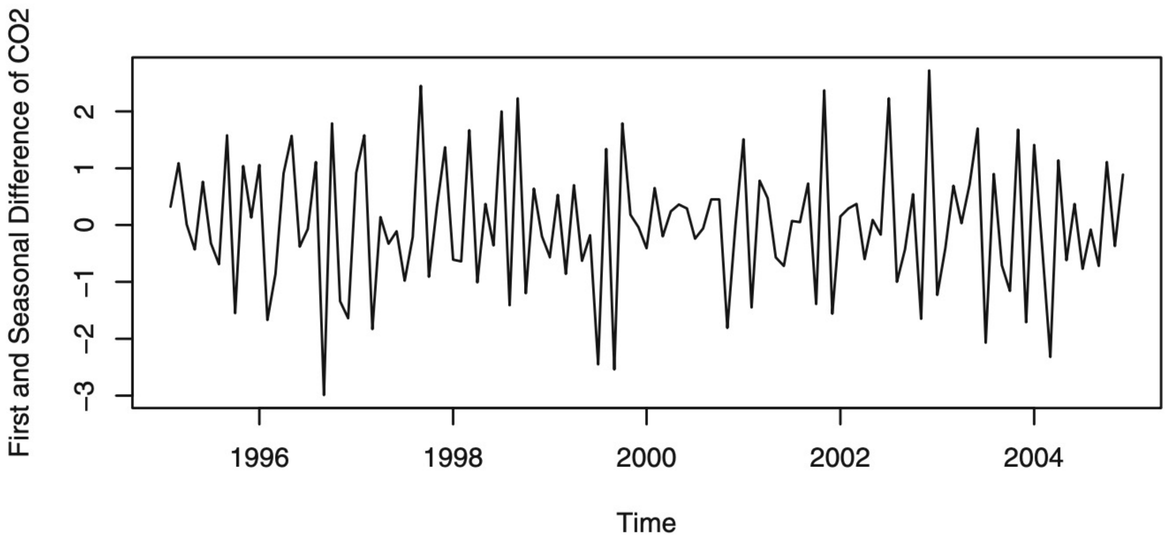


  – The general upward trend has now disappeared
  – But the strong seasonality is still present

Model specification, fitting, and checking

- First differencing + seasonal differencing

Exhibit 10.8 Time Series Plot of First and Seasonal Differences of CO_2

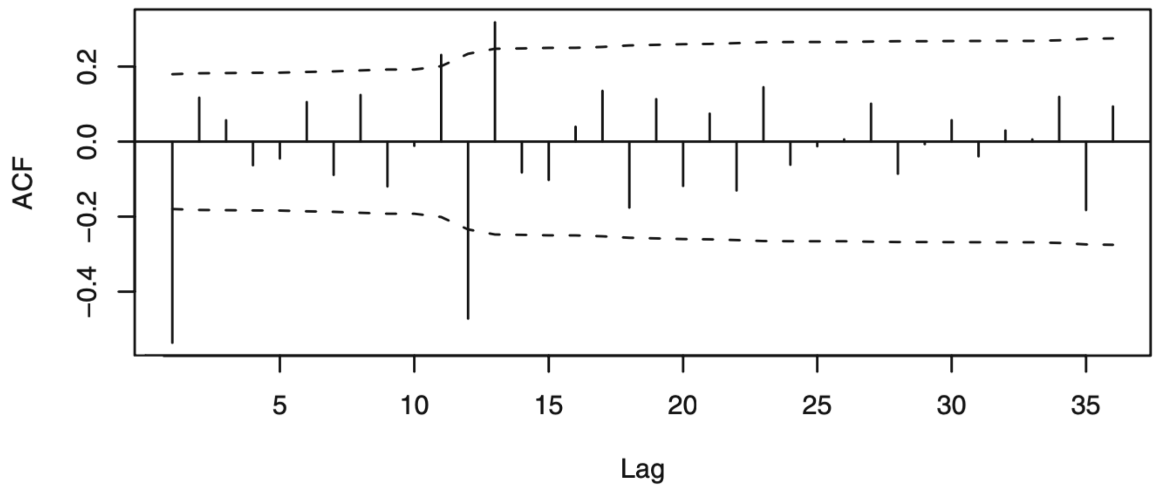


– It appears that most, if not all, of the seasonality is gone now

Model specification, fitting, and checking

- ACF of first differencing + seasonal differencing

Exhibit 10.9 Sample ACF of First and Seasonal Differences of CO_2



– It confirms that very little autocorrelation remains in the series after these two differences have been taken

– And it also suggests that a simple model which incorporates the lag 1 and lag 12 autocorrelations might be adequate

Model specification, fitting, and checking

- Fit ARIMA(0,1,1)x$(0,1,1)_{12}$ model:

$$\nabla_{12}\nabla Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$$

Exhibit 10.10 Parameter Estimates for the CO_2 Model

Coefficient	θ	Θ
Estimate	0.5792	0.8206
Standard error	0.0791	0.1137

$\hat{\sigma}_e^2 = 0.5446$: log-likelihood $= -139.54$, AIC $= 283.08$

– Coefficients are all highly significant

Model specification, fitting, and checking

- Residuals of ARIMA(0,1,1)x$(0,1,1)_{12}$ model:

$$\nabla_{12}\nabla Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$$

Exhibit 10.11 Residuals from the ARIMA(0,1,1)×$(0,1,1)_{12}$ Model

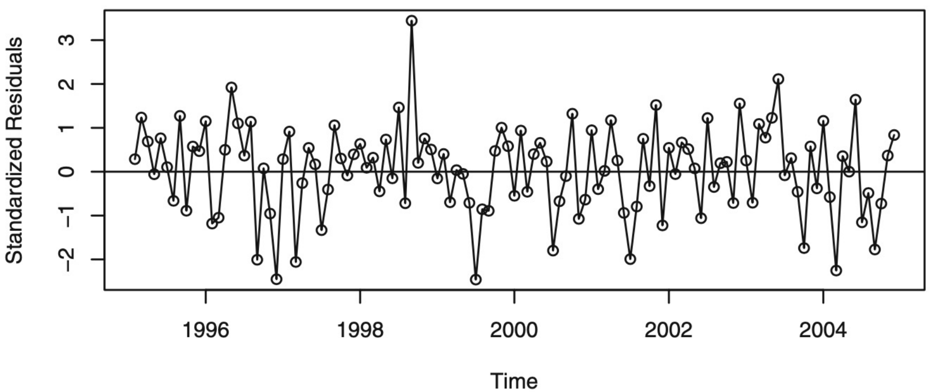


Exhibit 10.12 ACF of Residuals from the ARIMA(0,1,1)×$(0,1,1)_{12}$ Model

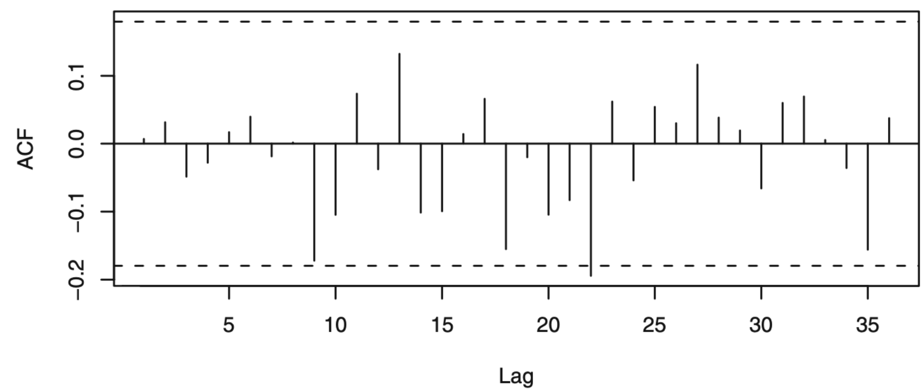


Exhibit 10.13 Residuals from the ARIMA(0,1,1)×$(0,1,1)_{12}$ Model

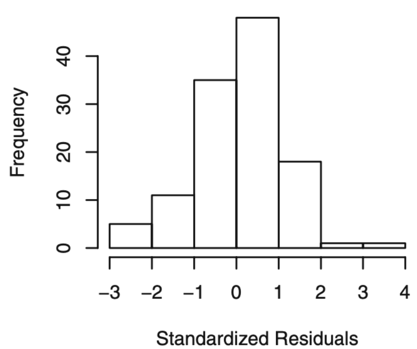
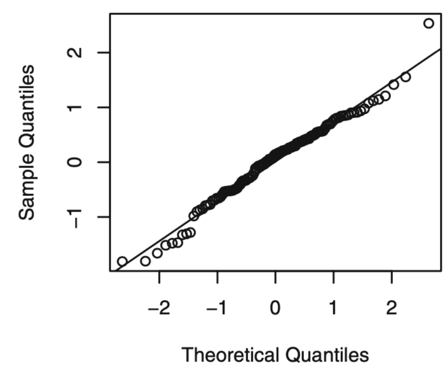


Exhibit 10.14 Residuals: ARIMA(0,1,1)×$(0,1,1)_{12}$ Model



Model specification, fitting, and checking

- Overfit with ARIMA(0,1,2)x(0,1,1)$_{12}$

Exhibit 10.10 Parameter Estimates for the CO_2 Model

Coefficient	θ	Θ
Estimate	0.5792	0.8206
Standard error	0.0791	0.1137

$\hat{\sigma}_e^2 = 0.5446$: log-likelihood $= -139.54$, AIC $= 283.08$

Exhibit 10.15 ARIMA(0,1,2)×(0,1,1)$_{12}$ Overfitted Model

Coefficient	θ_1	θ_2	Θ
Estimate	0.5714	0.0165	0.8274
Standard error	0.0897	0.0948	0.1224

$\hat{\sigma}_e^2 = 0.5427$: log-likelihood $= -139.52$, AIC $= 285.05$

- The estimates of θ_1 and Θ have changed very little
- The estimate of θ_2 is not statistically different from zero
- Hence, ARIMA(0,1,2)x(0,1,1)$_{12}$ seems appropriate

Chapter 10.5

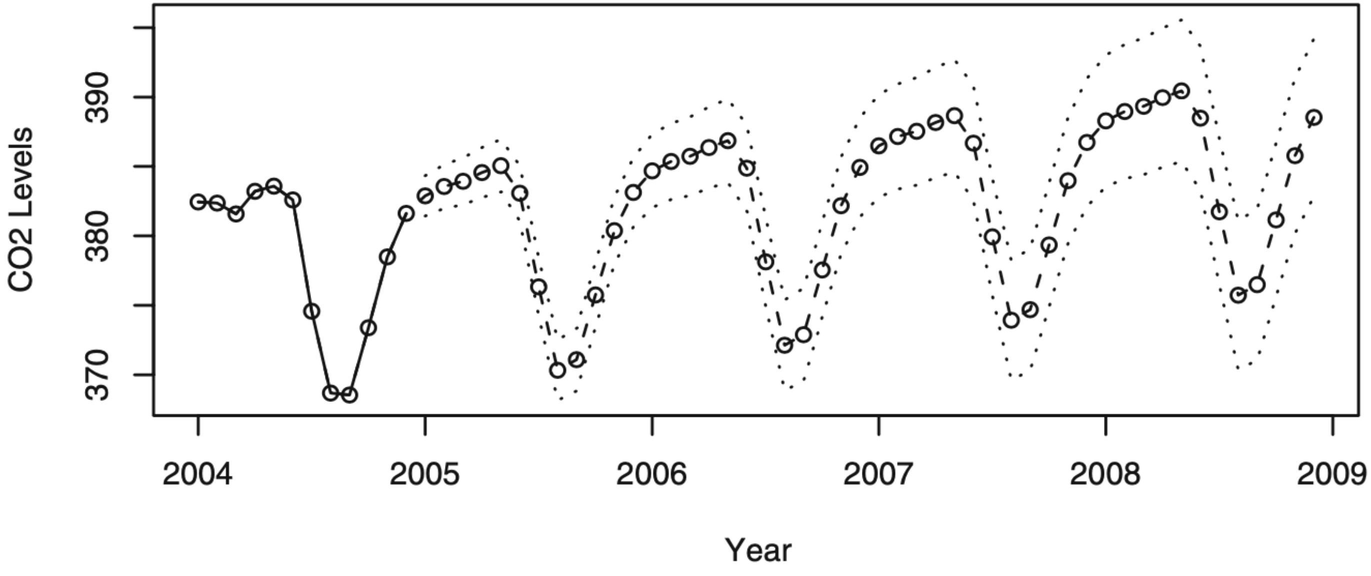
Forecasting Seasonal Models

Forecasting

- Forecast from ARIMA(0,1,1)x$(0,1,1)_{12}$ model:

$$\nabla_{12}\nabla Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$$

Exhibit 10.17 Long-Term Forecasts for the CO_2 Model



Announcement

- **Midterm exam**
  - **Date & time:** November 7 (Mon), 16:00 ~ 17:30
  - **Location:** 110-N101

  - **Coverage**
    - Everything we covered before the midterm
    - No programming related problems
    - Some problems will be almost the same as the examples in the slides and homework assignment problems

  - **Cheating sheet is allowed**
    - A4 1 page
    - Front and back
    - “Hand-written” only
      - Printed ones will be taken and removed

  - ***Any form of academic misconduct will not be tolerated***
  - ***You should explain your answers in detail (otherwise, no point will be given)***