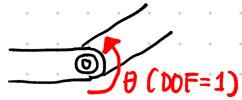
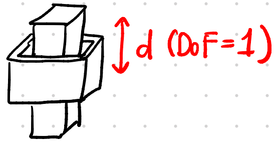
2024. 10. 02.

- **Kinematics** : To describe the motion of manipulator without consideration of the forces and torques causing the motion.

- **Links and Joints** → Links: Rigid Body; Joints: Constraints on two links.



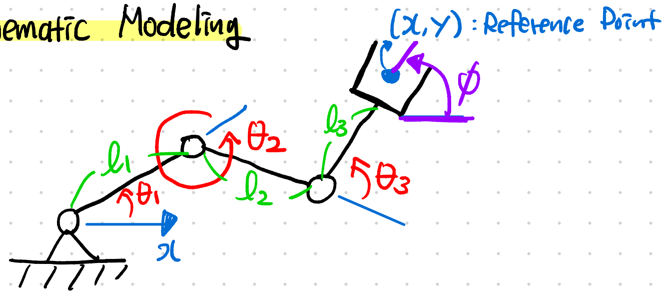
Revolute Joint



Prismatic Joint

⇓ (open kinematic chain)

- **Kinematic Modeling**



- Link ?
- Actuated Joint ?
- End Effector ?
- $(\theta_1, \theta_2, \theta_3)$?
- (x, y, ϕ) ?
- (l_1, l_2, l_3) ?

- **Forward & Inverse Kinematics**

$$(\theta_1, \theta_2, \theta_3) \xrightarrow{\text{Forward}} (x, y, \phi)$$

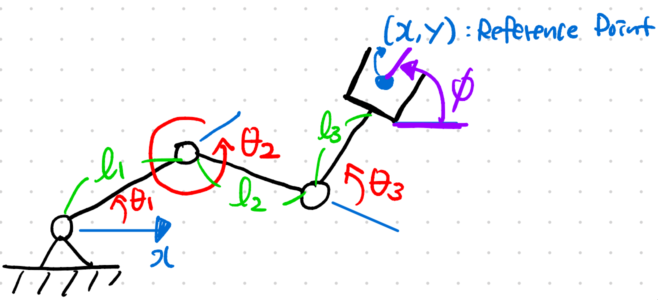
$$(\theta_1, \theta_2, \theta_3) \xleftarrow{\text{Inverse}} (x, y, \phi)$$

$(\theta_1, \theta_2, \theta_3)$: Joint Configuration

(x, y): position; ϕ: orientation

Inverse → Has multiple solutions!

- **FK (forward Kinematics) example**

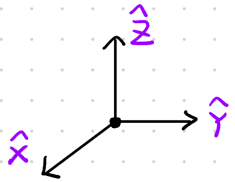


$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

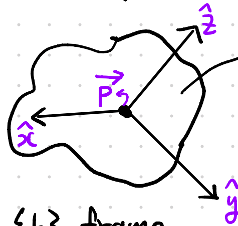
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

- We need to talk about "Rigid Body" first.



{s} frame
fixed or space frame.



{b} frame
moving or body frame

→ Rigid Body : one coordinate system is enough.

[In general, robot consists of rigid bodies]

⇒ Let's represent {b} from {s}!

- $\vec{P} = P_1\vec{X} + P_2\vec{Y} + P_3\vec{Z}$, let $P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ ⇒ $\vec{P} = [\vec{X}\ \vec{Y}\ \vec{Z}]P$

- $\hat{x} = r_{11}\vec{X} + r_{21}\vec{Y} + r_{31}\vec{Z}$
  $\hat{y} = r_{12}\vec{X} + r_{22}\vec{Y} + r_{32}\vec{Z}$,
  $\hat{z} = r_{13}\vec{X} + r_{23}\vec{Y} + r_{33}\vec{Z}$

  let $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = R_{sb}$

  $[\hat{x}\ \hat{y}\ \hat{z}] = [\vec{X}\ \vec{Y}\ \vec{Z}]R$

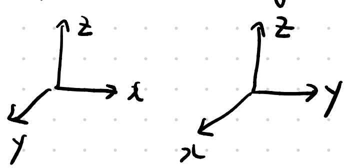
Q) rigid body frame has 6 independent parameters! $\left(\begin{smallmatrix} pos : 3 \\ rot : 3 \end{smallmatrix}\right)$
But there are 12 parameters! $\left(\begin{smallmatrix} pos = 3 \\ rot = 9 \end{smallmatrix}\right)$

A) These 12 params are not independent! there are conditions :

(i) $\|\hat{x}\|^2 = 1$: $r_{11}^2 + r_{21}^2 + r_{31}^2 = 1$
(ii) $\|\hat{y}\|^2 = 1$: $r_{12}^2 + r_{22}^2 + r_{32}^2 = 1$
(iii) $\|\hat{z}\|^2 = 1$: $r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$
(iv) $\hat{x} \perp \hat{y}$: $r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 1$
(v) $\hat{y} \perp \hat{z}$: $r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} = 1$
(vi) $\hat{z} \perp \hat{x}$: $r_{13}r_{11} + r_{23}r_{21} + r_{33}r_{31} = 1$

} $R^TR = 1$
(orthogonal matrix)

↳ But there are still vague points!



So we add new cond.
(vii) $\det R = 1$
$\Leftrightarrow \hat{x} \times \hat{y} = \hat{z}$.

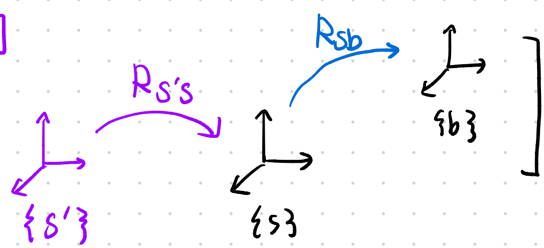
- **Special Orthogonal Group in $\mathbb{R}^{3\times3}$ (SO(3))**

$\Rightarrow$ Any $R \in \mathbb{R}^{3\times3}$ that satisfies $R^T R = I$ and $\det(R) = 1$ is a rotational matrix.
The set of all rotational matrices is called $SO(3)$.

- **Properties of SO(3)**
  - $\rightarrow$ $R^{-1} = R^T$
  - $\rightarrow$ If $R_1, R_2 \in SO(3)$, $R_3 = R_1 R_2 \in SO(3)$
  - $\rightarrow$ If $x \in \mathbb{R}^3$ and $R \in SO(3) \Rightarrow$ for $y = Rx$, $\|y\| = \|x\|$
  - $\rightarrow$ $R_{ab} R_{bc} = R_{ac}$.
  - $\rightarrow$ $R_{ab} R_{ba} = I$.

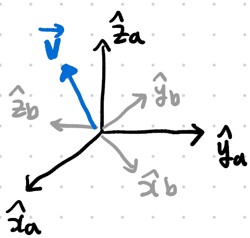
- **Use of SO(3)?**
  - $\rightarrow$ Represent Orientation.
  - $\rightarrow$ Change the reference frame in which a vector or a frame is represented.

[frame]



$R_{s'b} = R_{s's} \cdot R_{sb}$

[vector]



V_a : $\vec{v}$ from $\{a\}$, V_b = $\vec{v}$ from $\{b\}$

- $\vec{v} = V_{ax}\hat{x}_a + V_{ay}\hat{y}_a + V_{az}\hat{z}_a = [\hat{x}_a \ \hat{y}_a \ \hat{z}_a] \underbrace{\begin{bmatrix} V_{ax} \\ V_{ay} \\ V_{az} \end{bmatrix}}_{V_a}$

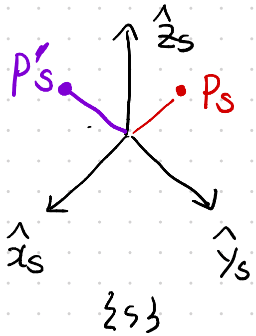
$\quad = V_{bx}\hat{x}_b + V_{by}\hat{y}_b + V_{bz}\hat{z}_b = [\hat{x}_b \ \hat{y}_b \ \hat{z}_b] \underbrace{\begin{bmatrix} V_{bx} \\ V_{by} \\ V_{bz} \end{bmatrix}}_{V_b}$

- $[\hat{x}_b \ \hat{y}_b \ \hat{z}_b] = [\hat{x}_a \ \hat{y}_a \ \hat{z}_a] R_{ab}$

$\Rightarrow \vec{v} = [\hat{x}_a \ \hat{y}_a \ \hat{z}_a] \begin{bmatrix} V_{ax} \\ V_{ay} \\ V_{az} \end{bmatrix} = [\hat{x}_a \ \hat{y}_a \ \hat{z}_a] R_{ab} \begin{bmatrix} V_{bx} \\ V_{by} \\ V_{bz} \end{bmatrix}$

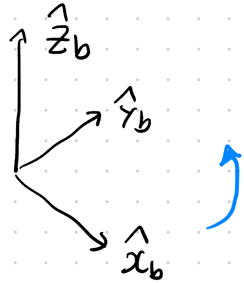
$\Rightarrow V_a = R_{ab} V_b$

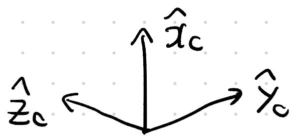
- Rotate a vector or a frame.



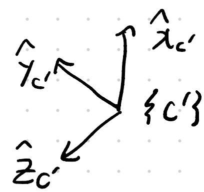
$$\xrightarrow{Rot(\hat{z}, 90^\circ) = R_{sb} = R}$$

$$P'_s = R_{sb} P_s$$



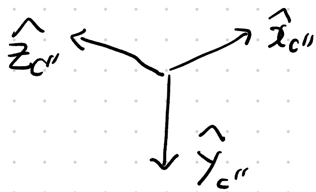


rotation about $\hat{z}_s$



$$R_{sc'} = R R_{sc}$$

rotation about $\hat{z}_c$



$$R_{sc''} = R_{sc} R$$