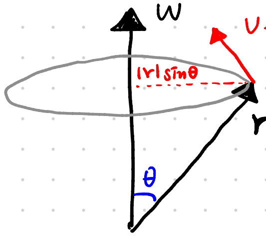
· Angular Velocities



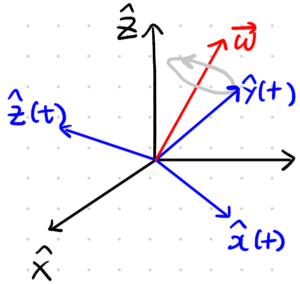
$v = w \times r$, $|v| = |w||r|\sin\theta$

↳ cross product

: $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{[a]:} b$$

⇒ Extend Concept to Frame?



$$\dot{\vec{P}}(t) = [\hat{X} \ \hat{Y} \ \hat{Z}] \begin{bmatrix} \dot{p}_x(t) \\ \dot{p}_y(t) \\ \dot{p}_z(t) \end{bmatrix}$$

$$[\hat{x}(t) \ \hat{y}(t) \ \hat{z}(t)] = [\hat{X} \ \hat{Y} \ \hat{Z}] \dot{R}(t)$$

$$\vec{w} = [\hat{X} \ \hat{Y} \ \hat{Z}] \underline{w_s} = [\hat{X} \ \hat{Y} \ \hat{Z}] \begin{bmatrix} w_{sx} \\ w_{sy} \\ w_{sz} \end{bmatrix}$$

Representation of $\vec{w}$ in fixed frame.

$$\Rightarrow [\dot{\hat{x}} \ \dot{\hat{y}} \ \dot{\hat{z}}] = [\vec{w} \times \hat{x} \mid \vec{w} \times \hat{y} \mid \vec{w} \times \hat{z}]$$

$$= \vec{w} \times [\hat{x} \ \hat{y} \ \hat{z}]$$

$$= [w_s][\hat{x} \ \hat{y} \ \hat{z}]$$

Consider $\vec{w}$ and $\hat{x}, \hat{y}, \hat{z}$ are represented in fixed frame.

skew-symmetric matrix,

$so(3) = \{A \in \mathbb{R}^{3\times3} \mid A^T = -A\}$ is Lie algebra of $SO(3)$

Proposition of so(3)

- for any $w \in \mathbb{R}^3$ and $R \in SO(3)$, "$R[w]R^T = [Rw]$" always holds
- given $a, b, c \in \mathbb{R}^3$, if $a \times b = c$, then "$[a][b] - [b][a] = [c]$"

$$\Rightarrow [\dot{\hat{x}} \ \dot{\hat{y}} \ \dot{\hat{z}}] = \vec{w} \times [\hat{x} \ \hat{y} \ \hat{z}] = [[\hat{X} \ \hat{Y} \ \hat{Z}] w_s][\hat{x} \ \hat{y} \ \hat{z}]$$

$$= [\hat{X} \ \hat{Y} \ \hat{Z}][w_s][\hat{X} \ \hat{Y} \ \hat{Z}]^T \underbrace{[\hat{x} \ \hat{y} \ \hat{z}]}_{[\hat{X} \ \hat{Y} \ \hat{Z}]R}$$

$$\Rightarrow [\hat{X} \ \hat{Y} \ \hat{Z}][w_s]R$$

$$w_s = R_{sb} w_b = R w_b$$

$$\therefore \dot{R} = [w_s]R, \quad [w_s] = \dot{R}R^{-1} = \dot{R}R^T = [Rw_b] = \underline{R[w_b]R^T}$$

$$[w_b] = R^{-1}\dot{R} = R^T\dot{R}$$

- Vector Linear Differential Equation

$\Rightarrow$ " $\dot{x}(t) = Ax(t)$ ", where $x(t) \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times3}$ is constant, $x(0)$ is given.

↳ assume $x(t) = e^{At}x(0)$ is solution,

$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots \quad \left(\text{like } e^{x} = 1 + x + \frac{x^2}{2!} + \cdots \right)$$

[It works!]

$$\dot{x}(t) = \left(A + A^2 t + \frac{A^3 t^2}{2!} + \cdots\right)x(0)$$

$$= A\left(I + At + \frac{(At)^2}{2!} + \cdots\right)x(0) = Ae^{At}x(0) = Ax(t)$$

Q) But How we exactly get e^{At}?

↳ 1) If A is diagonal, $A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & d_n \end{bmatrix}$

$$\Rightarrow e^{At} = I + \begin{bmatrix} d_1 t & & 0 \\ & d_2 t & \\ 0 & & \ddots \;\; d_n t \end{bmatrix} + \cdots = \begin{bmatrix} e^{d_1 t} & & 0 \\ & e^{d_2 t} & \\ 0 & & \ddots \;\; e^{d_n t} \end{bmatrix}$$

↳ 2) If A is not diagonal,

$\rightarrow$ eigen decomposition $\Rightarrow A = P\overset{\text{diagonal}}{D}P^{-1}$

$$\Rightarrow e^{At} = e^{PDP^{-1}t} = I + (PDP^{-1})t + \underbrace{(PDP^{-1})(PDP^{-1})}_{PD^2P^{-1}}\frac{t^2}{2!} + \cdots$$

$$= P\left(I + Dt + \frac{D^2}{2!}t + \cdots\right)P^{-1} = Pe^{Dt}P^{-1}$$

- Exponential Coordinate of Rotations

$\Rightarrow$ $\underbrace{\dot{R}(t) = [\omega_s]R(t)}_{} = R[\omega_b]$, $R = R_{sb}$, {b} frame keeps rotating.

↳ Solution for R?

$\Rightarrow R(t) = [\, r_1(t) \mid r_2(t) \mid r_3(t) \,]$

$\Rightarrow \dot{r}_i(t) = [\omega_s]\, r_i(t)$, $r_i(t) = e^{[\omega_s]t} r_i(0)$

$$R(t) = e^{[\omega_s]t}R(0)$$

if, $R(0) = I$

$\Rightarrow R(t) = e^{[\omega_s]t}$

- So, $R(t) = e^{[\omega_s]t}$ .. then ?

① $\Rightarrow$ If $\vec{\omega}$ is represented from {s} frame, it is ω_s.

Let $\omega_s = \underbrace{\hat{\omega}_s}_{\text{direction}} \underbrace{\|\omega_s\|}_{\text{magnitude}}$,

$\hookrightarrow [\omega_s]t = [\hat{\omega}_s \|\omega_s\|]t = [\hat{\omega}_s]\boxed{\|\omega_s\| t}$

Let this angular movement as θ.

$\Rightarrow [\omega_s]t = [\hat{\omega}_s]\theta$.

② $\Rightarrow$ characteristic equation of $[\omega]$, $P(s)$?

$$\because P(s) = \det\left(Is - \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\right) = s^3 + (\omega_1^2 + \omega_2^2 + \omega_3^2)s.$$

& If $\|\omega\| = 1$, $P(s) = s^3 + s$.

$P([\omega]) = [\omega]^3 + [\omega] = 0$

Cayley-Hamilton theorem

$\therefore [\omega]^3 = -[\omega]$.

$$\Rightarrow R(t) = e^{[\omega_s]t} = I + [\omega_s]t + \frac{[\omega_s]^2 t^2}{2!} + \cdots$$

$$= e^{[\hat{\omega}_s]\theta} = I + [\hat{\omega}_s]\theta + \frac{[\hat{\omega}_s]^2\theta^2}{2!} + \cdots$$

$$= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}_s] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)[\hat{\omega}_s]^2$$

$$= I + \sin\theta\,[\hat{\omega}_s] + (1-\cos\theta)[\hat{\omega}_s]^2$$

So, if frame {b} is rotated with $[\hat{\omega}_s]$ direction & θ magnitude,

$$R = e^{[\hat{\omega}_s]\theta} = I + \sin\theta\,[\hat{\omega}_s] + (1-\cos\theta)[\hat{\omega}_s]^2,\quad [\hat{\omega}_s] = \frac{R - R^T}{2\sin\theta}$$

"Exponential Coordinate Representation of R"

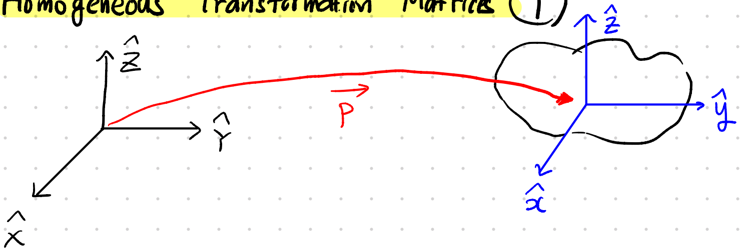
$\hookrightarrow$ Log (?)

$\begin{bmatrix} \exp: x \in \mathbb{R} \rightarrow e^x \in \mathbb{R} \\ \log: e^x \in \mathbb{R} \rightarrow x \in \mathbb{R} \end{bmatrix} \Rightarrow$

exp : $[\hat{\omega}]\theta \in so(3) \rightarrow e^{[\hat{\omega}]\theta} = R \in SO(3)$

log : $R = e^{[\hat{\omega}]\theta} \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$

- **Homogeneous Transformation Matrices** (T)



{s} frame

{b} frame

$$\vec{P} = p_1 \hat{X} + p_2 \hat{Y} + p_3 \hat{Z}$$

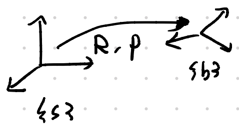
$$\left.\begin{aligned} \hat{x} &= r_{11}\hat{X} + r_{21}\hat{Y} + r_{31}\hat{Z} \\ \hat{y} &= r_{12}\hat{X} + r_{22}\hat{Y} + r_{32}\hat{Z} \\ \hat{z} &= r_{13}\hat{X} + r_{23}\hat{Y} + r_{33}\hat{Z} \end{aligned}\right\} R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$T = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

" SE(3) : Special Euclidean Group ".

⟹ Q1) $v \to$ rotate with R and move p ?

Q2)



what if observing v in {b} from {s} ?

$Rv + p$ is answer.

→ To make this operation easier,

we represent $v \xrightarrow{\text{homogeneous coordinate}} \begin{bmatrix} v \\ 1 \end{bmatrix}$, $\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ 1 \end{bmatrix} = \begin{bmatrix} Rv + p \\ 1 \end{bmatrix}$

- Properties))
  - $T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
  - $T_1, T_2, T_3 \in SE(3)$, $(T_1 T_2) T_3 = T_1 (T_2 T_3)$, $T_1 T_2 \neq T_2 T_1$
  - for $x, y \in \mathbb{R}^3$, $\|Tx - Ty\| = \|x - y\|$. $(Ta = T\begin{bmatrix} a \\ 1 \end{bmatrix})$
  - $T_{ac} = T_{ab} T_{bc}$
  - If Rotation is made with $\hat{\omega}$ and θ,

$$\left.\begin{aligned} \to Rot(\hat{\omega}, \theta) &= \begin{bmatrix} e^{[\hat{\omega}]\theta} & 0 \\ 0 & 1 \end{bmatrix} \\ Trans(p) &= \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \end{aligned}\right\} Tv = Trans(p)\, Rot(\hat{\omega}, \theta)\, v$$

- $\underbrace{T_{sb'} = Trans(p)\, Rot(\hat{\omega}, \theta)\, T_{sb}}_{\text{rotate frame \{b\}, then translate.}}$ (w.r.t {s}) vs $T_{sb''} = T_{sb}\, \underbrace{Trans(p)}_{T_{bc}}\, \underbrace{Rot(\hat{\omega}, \theta)}_{T_{cd}}$? (w.r.t {b}) (w.r.t {c})

Let T_{bc}, T_{cd}.

$\Rightarrow T_{sb''} = T_{sb} T_{bc} T_{cd} = T_{sd}$.