AI70601/CSE61101

3D Vision and Machine Perception

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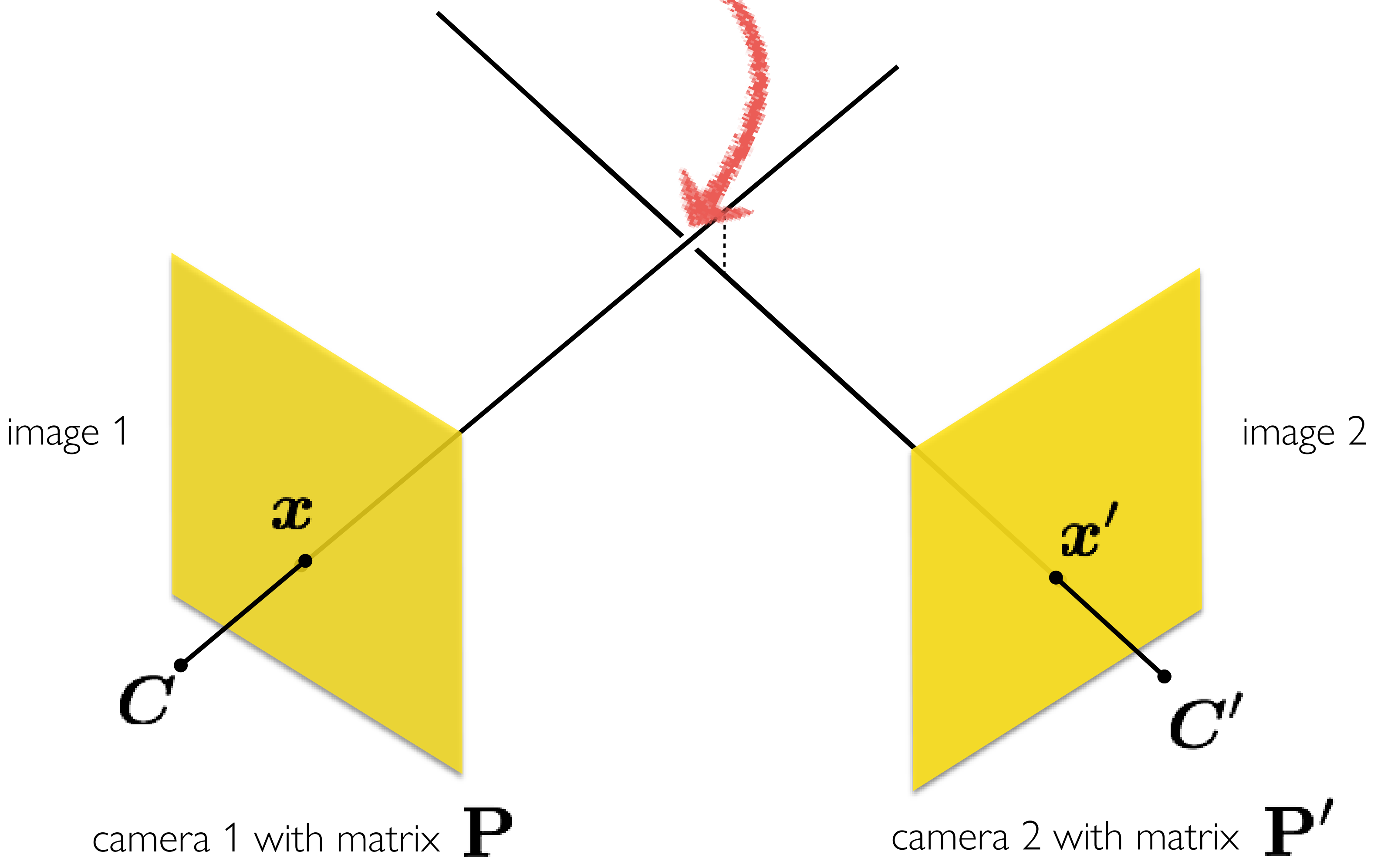
Some materials, figures, and slides (used for this course) are from textbooks, published papers, and other open lectures

Recap: Triangulation

- Find 3D object point
  (no single solution due to noise)



Recap: Triangulation

- Concatenate the 2D points from both images

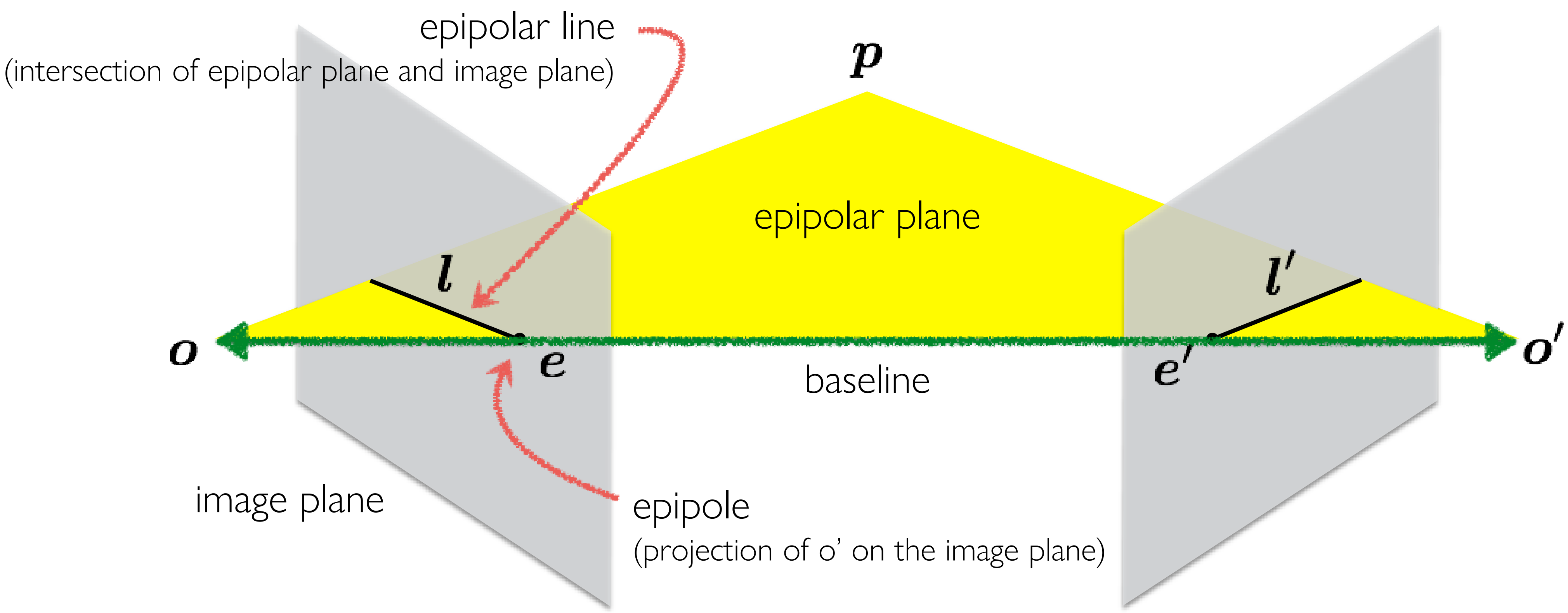
$$\begin{array}{c} \text{two rows from camera one} \\[2em] \text{two rows from camera two} \end{array} \quad \begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sanity check! dimensions?

$$\mathbf{A}_i \boldsymbol{X} = \mathbf{0}$$

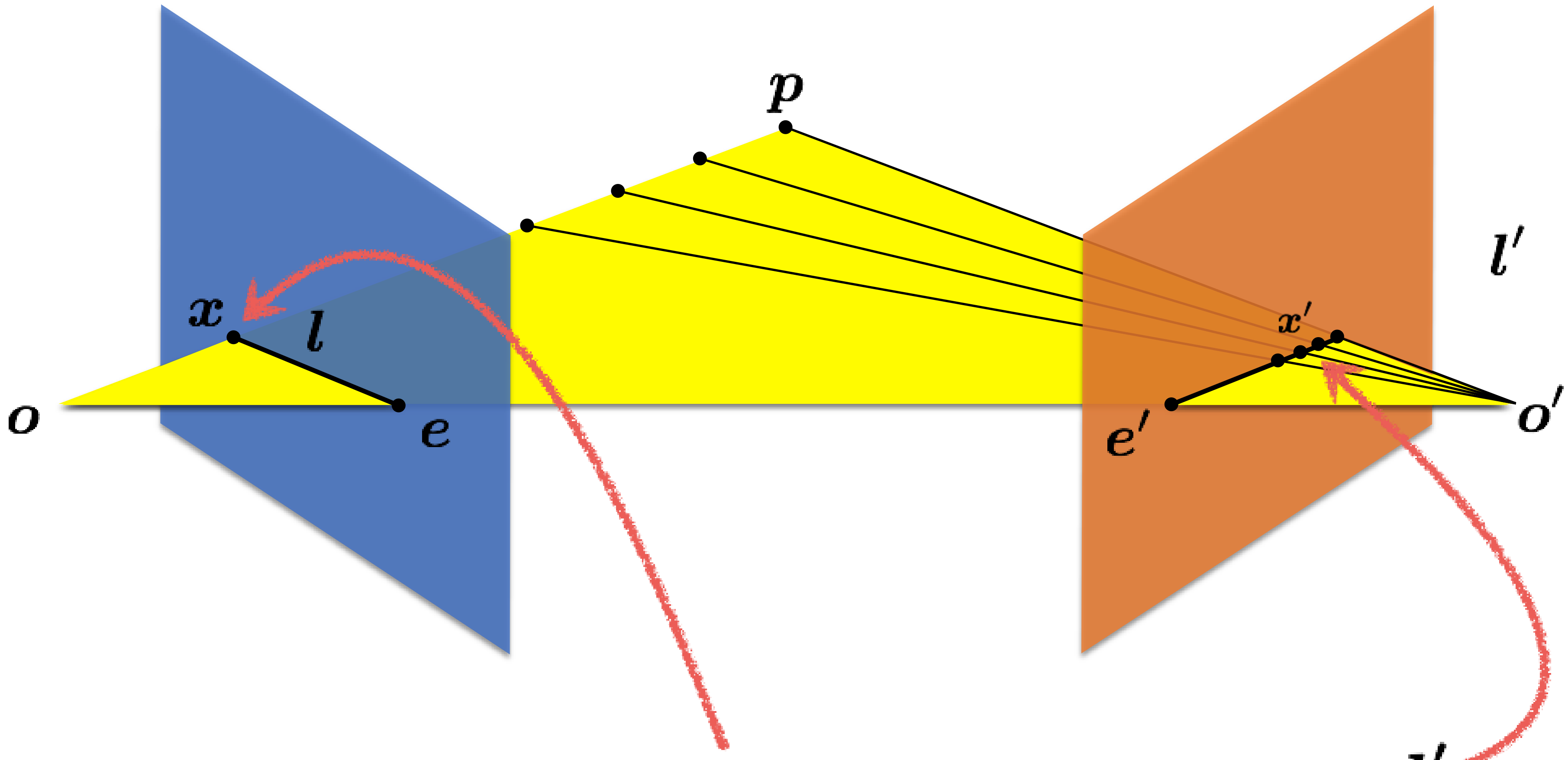
- How do we solve homogeneous linear system?

Recap: Epipolar geometry



The essential matrix

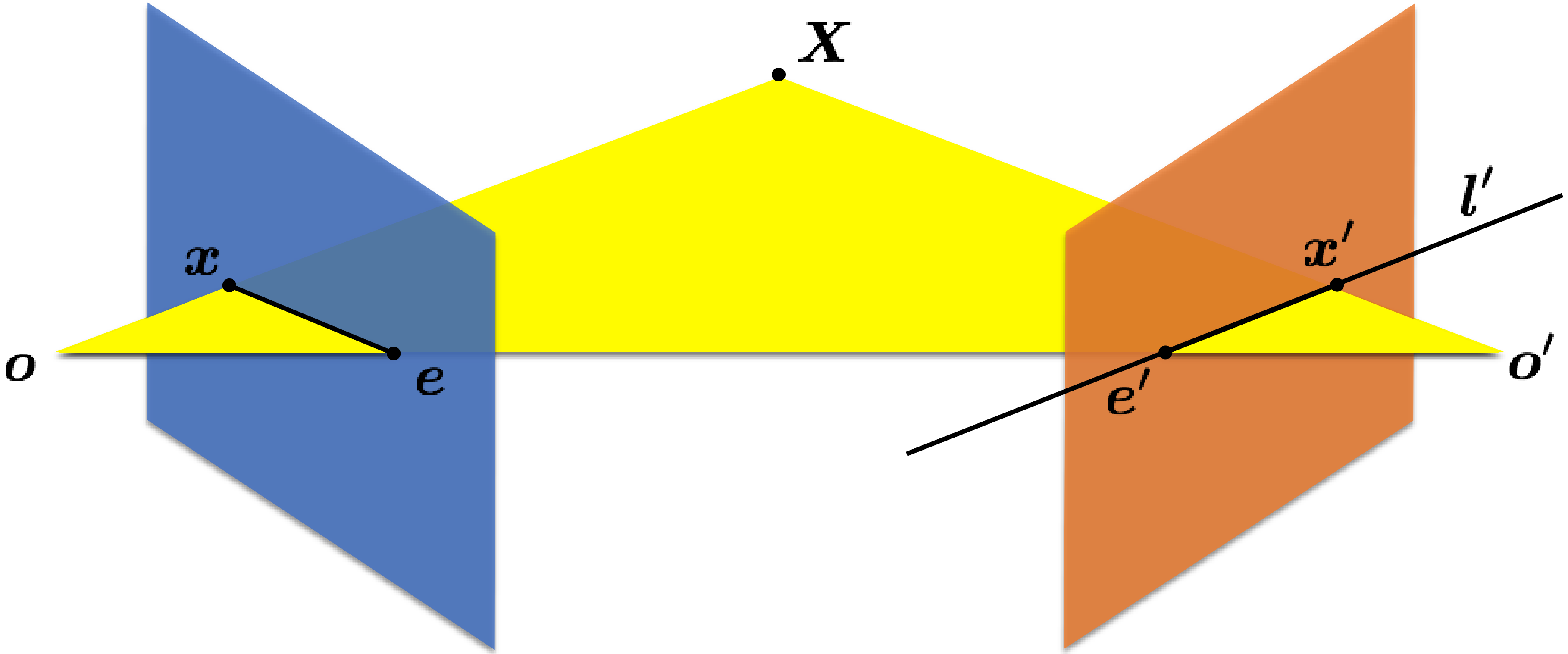
Recall : epipolar constraint



potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l'}$

- Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$



Motivation

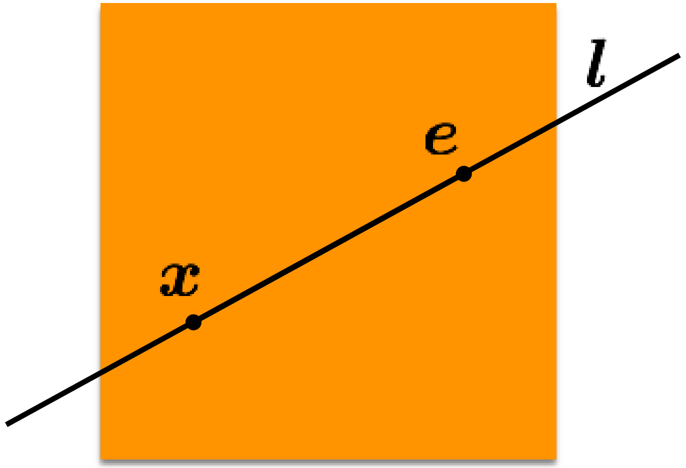
- The essential matrix is a 3 x 3 matrix that encodes **epipolar geometry**

- Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second image.

Epipolar line

- Representing the …

$$ax + by + c = 0 \quad \text{in vector form} \quad \boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

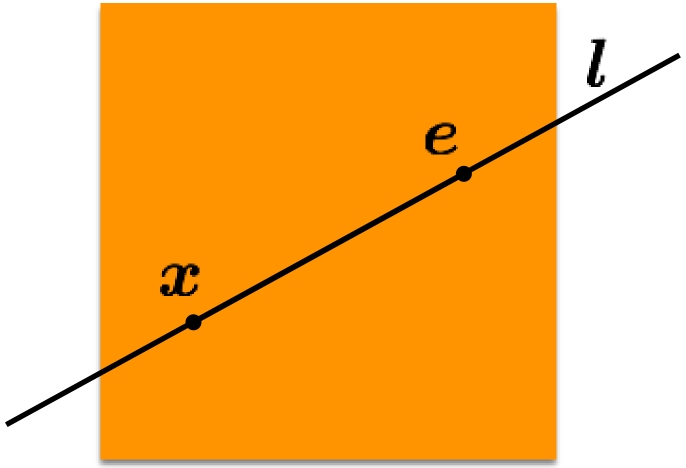


if the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = \ ?$$

Epipolar line

$$ax + by + c = 0$$ in vector form $$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

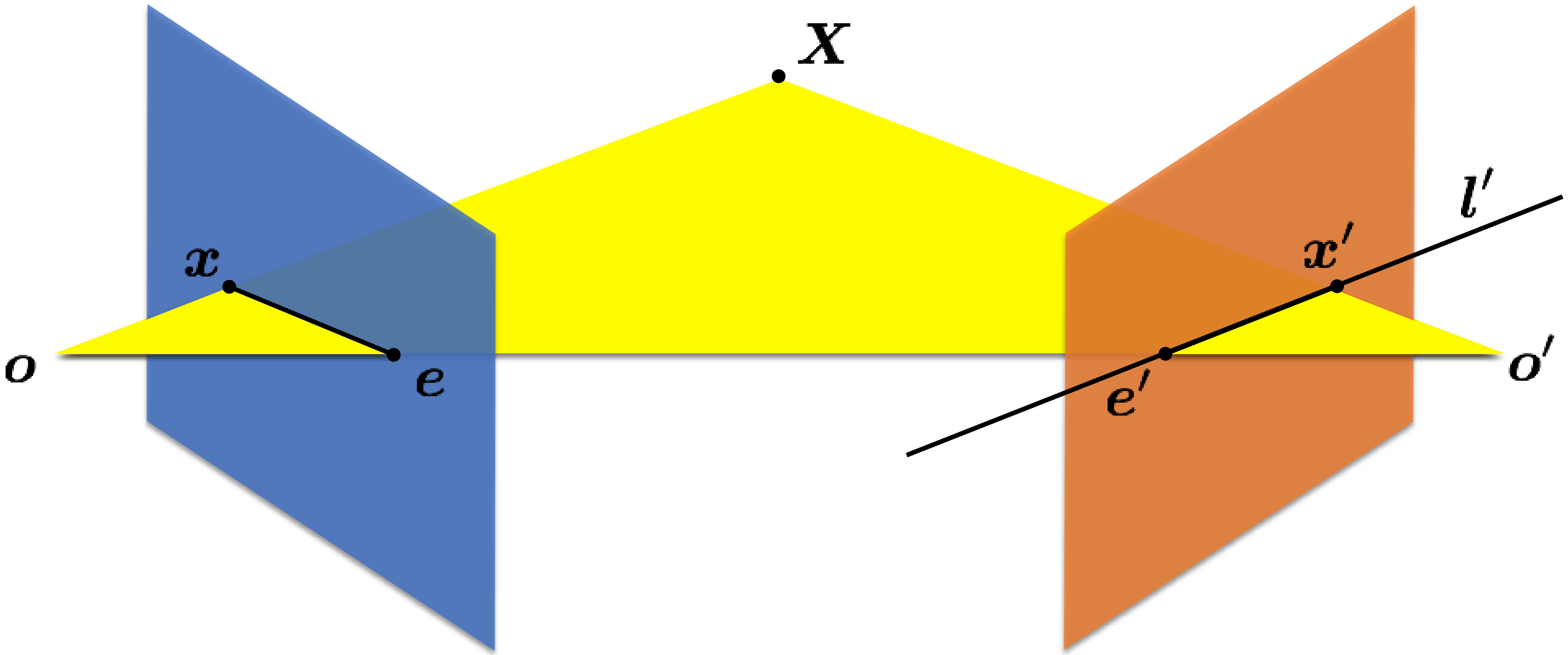


if the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = 0$$

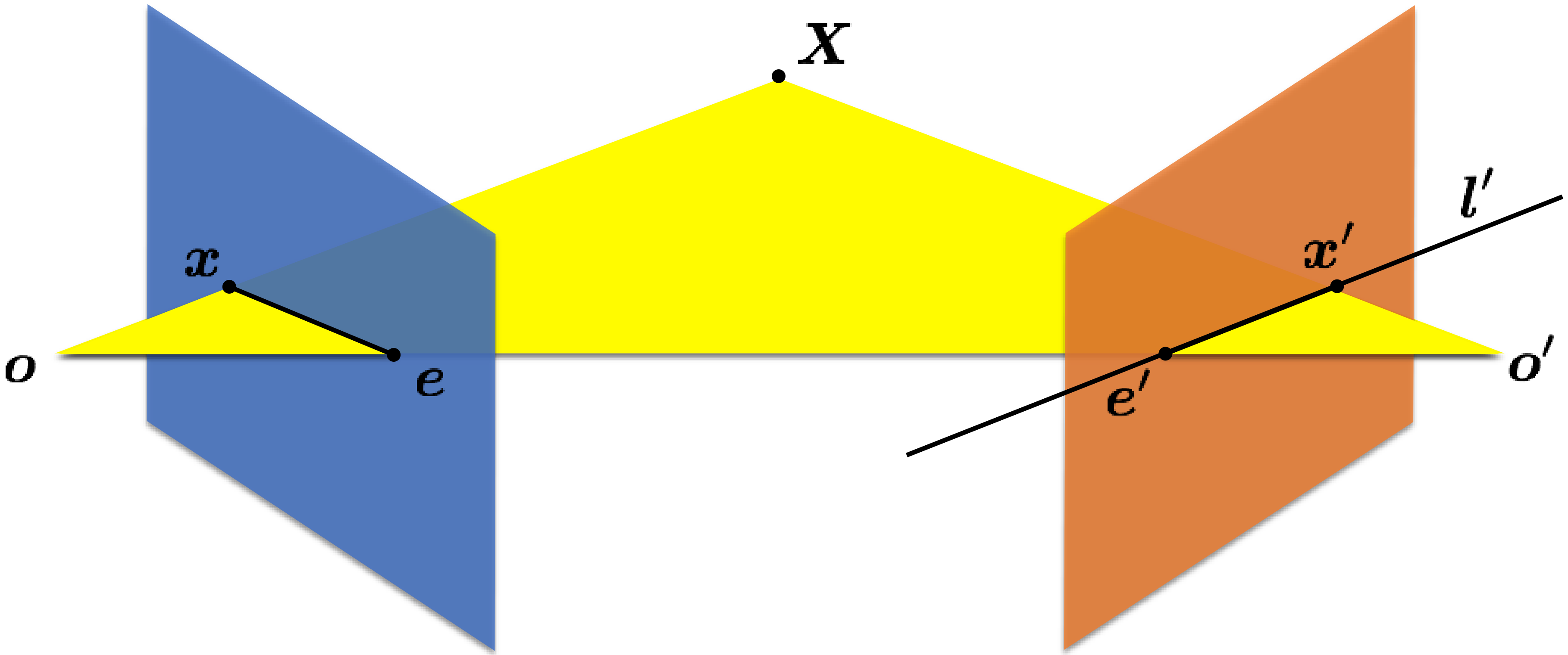
- So if $\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = \ ?$$



- So if $\boldsymbol{x}'^\top \boldsymbol{l}' = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boxed{\boldsymbol{x}'^\top \mathbf{E} \boldsymbol{x} = 0}$$



Essential matrix vs homography

- What's the difference between the essential matrix and a homography?

Essential matrix vs homography

- What's the difference between the essential matrix and a homography?

- They are both 3 x 3 matrices but …

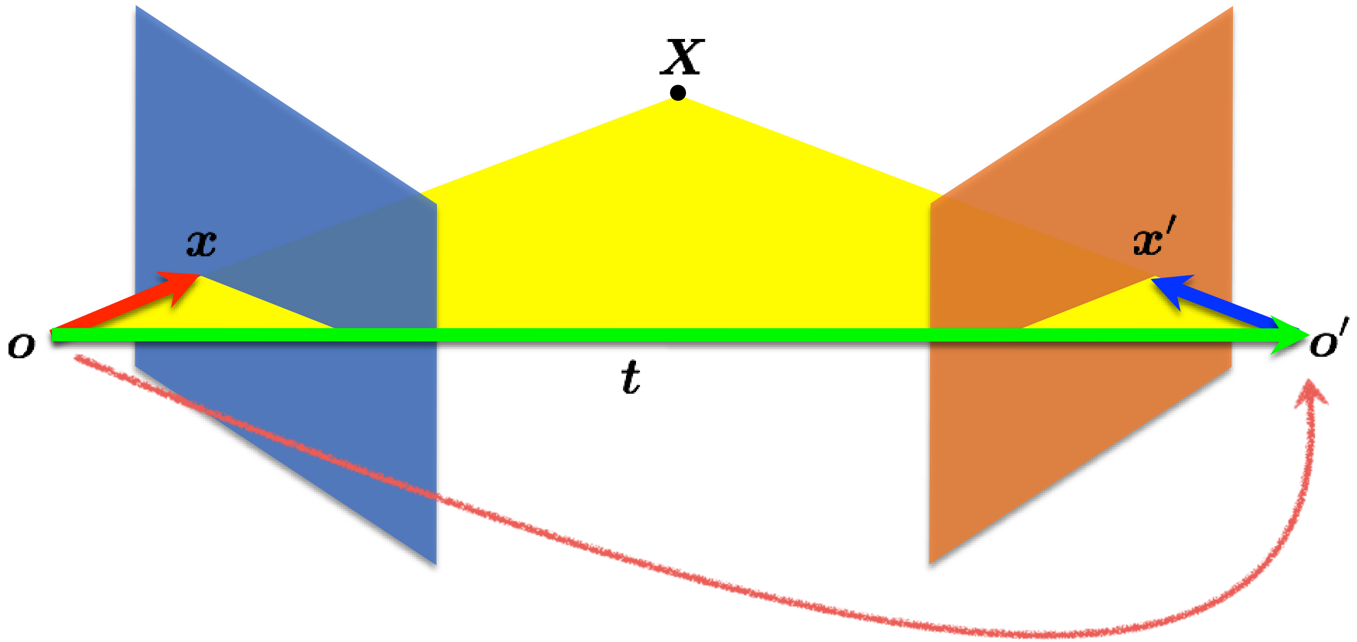
$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$$

essential matrix maps
a **point** to a **line**

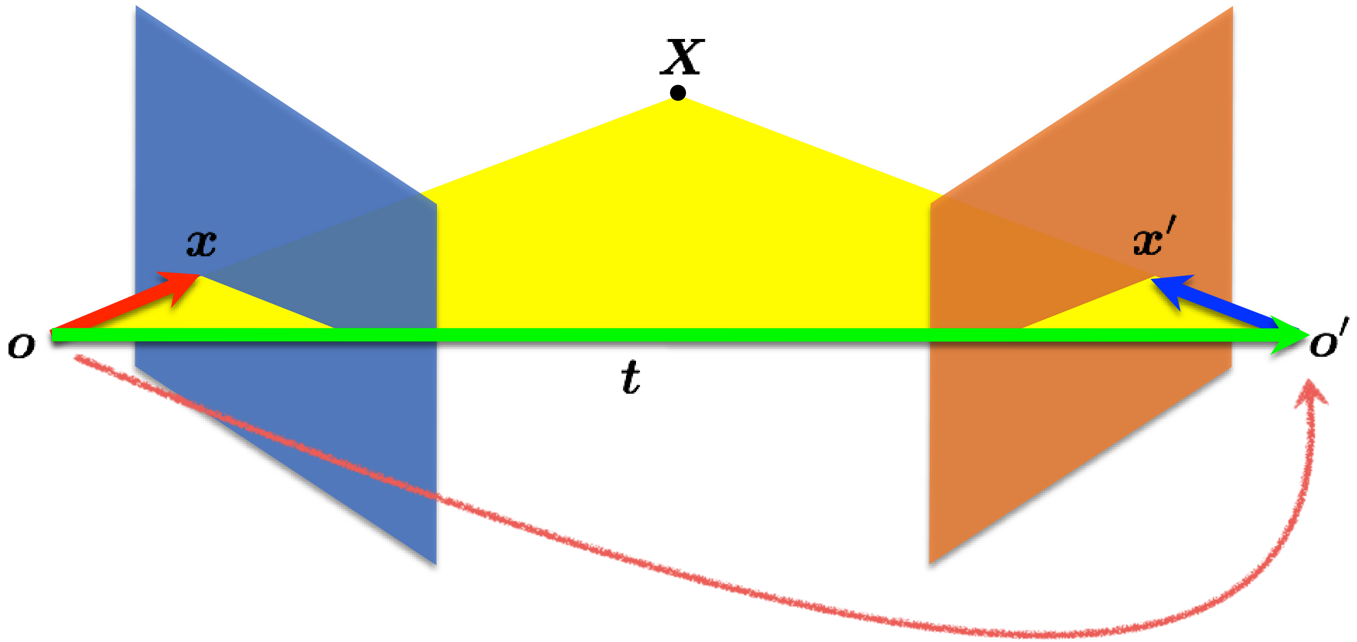
$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

homography maps a
point to a **point**

- Where does the essential matrix come from?

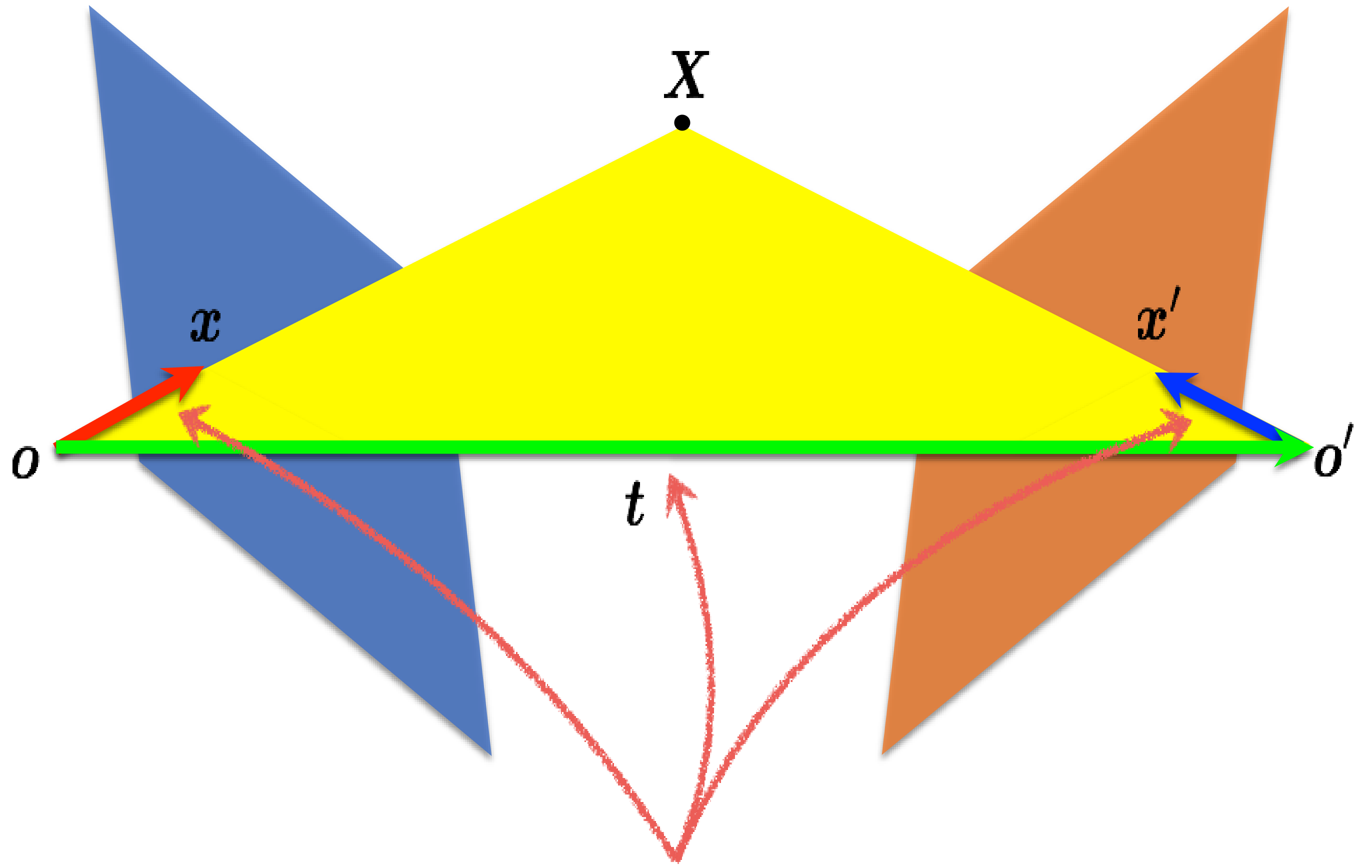


$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$



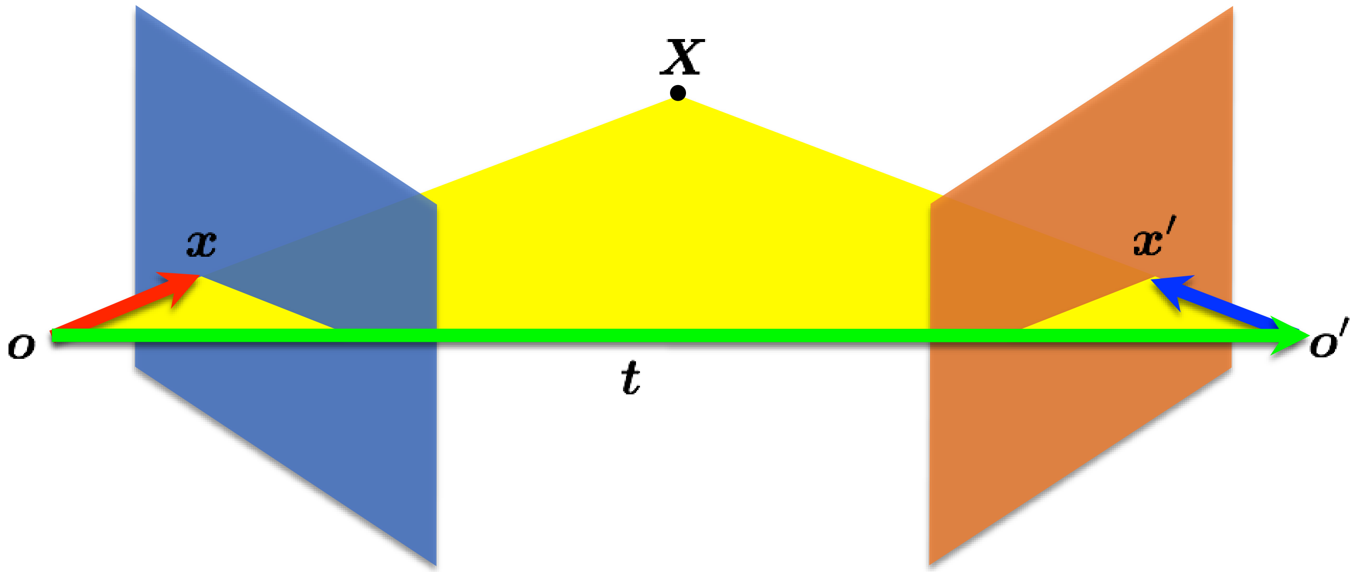
$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

- **Camera-camera** transform just like **world-camera** transform



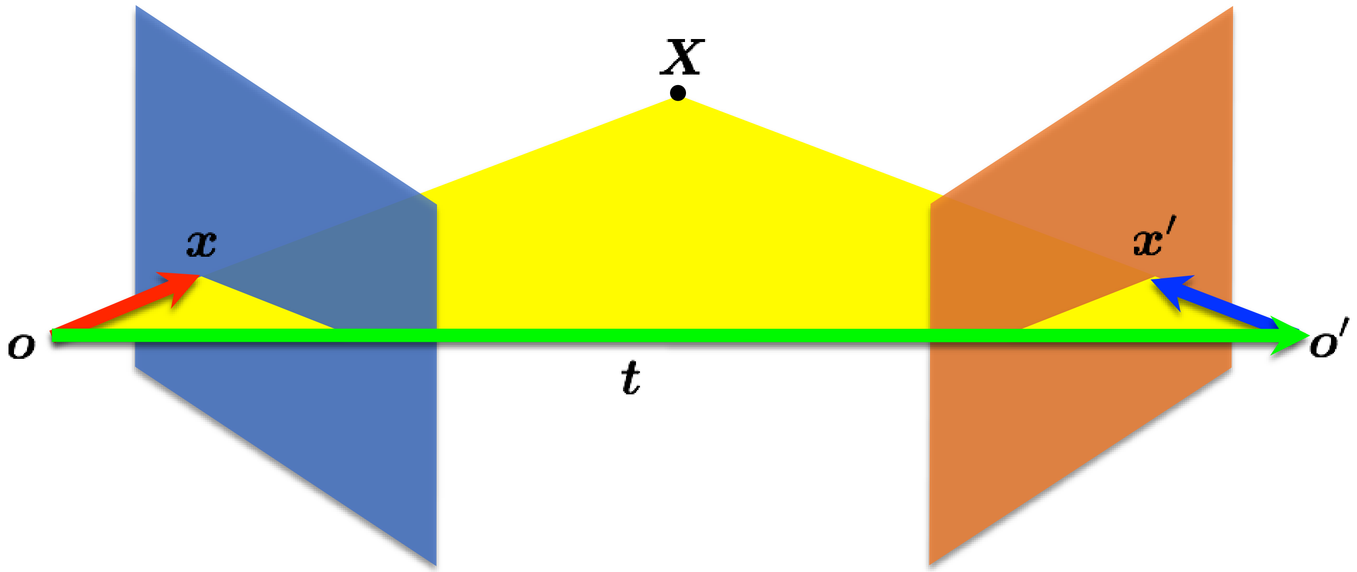
these three vectors are coplanar

$$\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$$



- If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^\top(\boldsymbol{t} \times \boldsymbol{x}) = \ ?$$

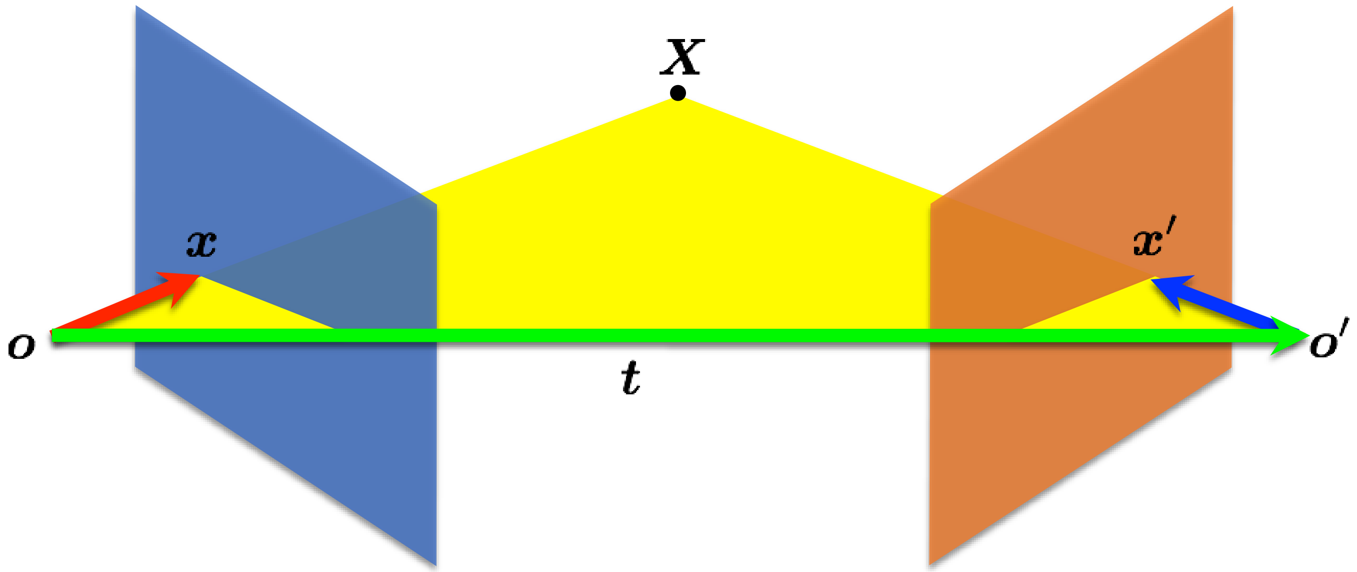


- If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

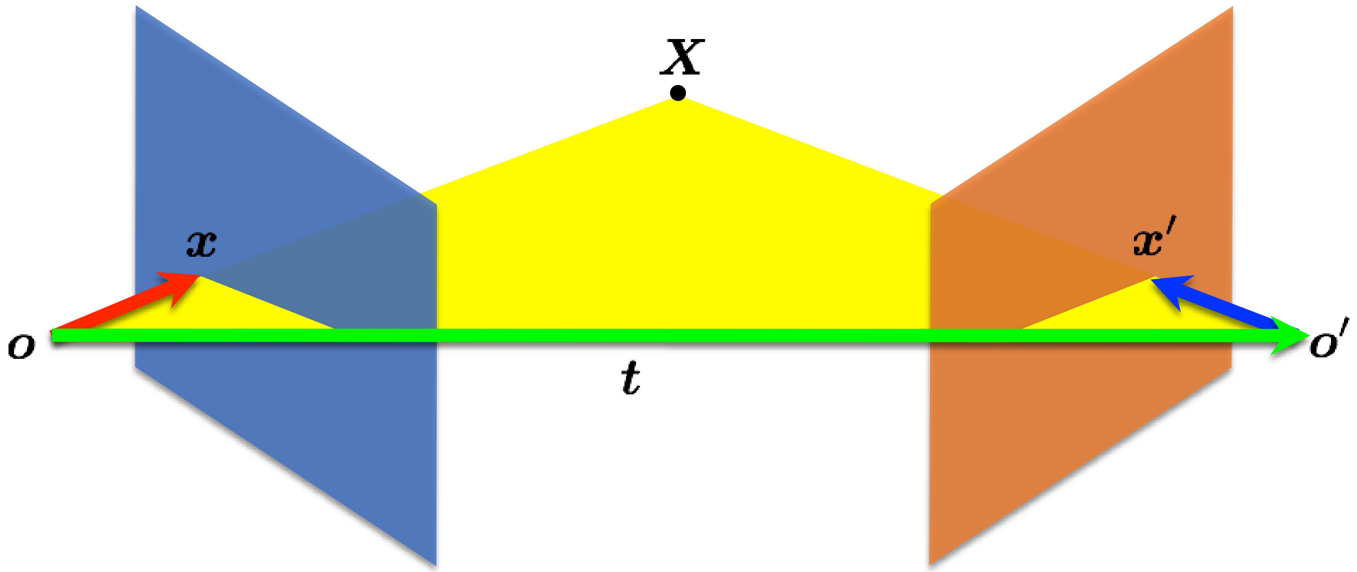
dot product of orthogonal vectors

cross-product: vector orthogonal to plane



- If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = \ ?$$



- If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^{\top}(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

- Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

- Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

- Use skew-symmetric matrix to represent cross product

- Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

- Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

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$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top(\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0$$

- Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

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$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boxed{\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0}$$

essential matrix

[Longuet-Higgins 1981]

Properties of the E matrix

- 2D points expressed in <u>camera</u> coordinate system

Longuet-Higgins equation $$\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$$

Properties of the E matrix

- 2D points expressed in camera coordinate system

Longuet-Higgins equation $\quad \boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$

Epipolar lines

$$\boldsymbol{x}^{\top}\boldsymbol{l} = 0 \qquad \boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$$

$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x} \qquad \boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$$

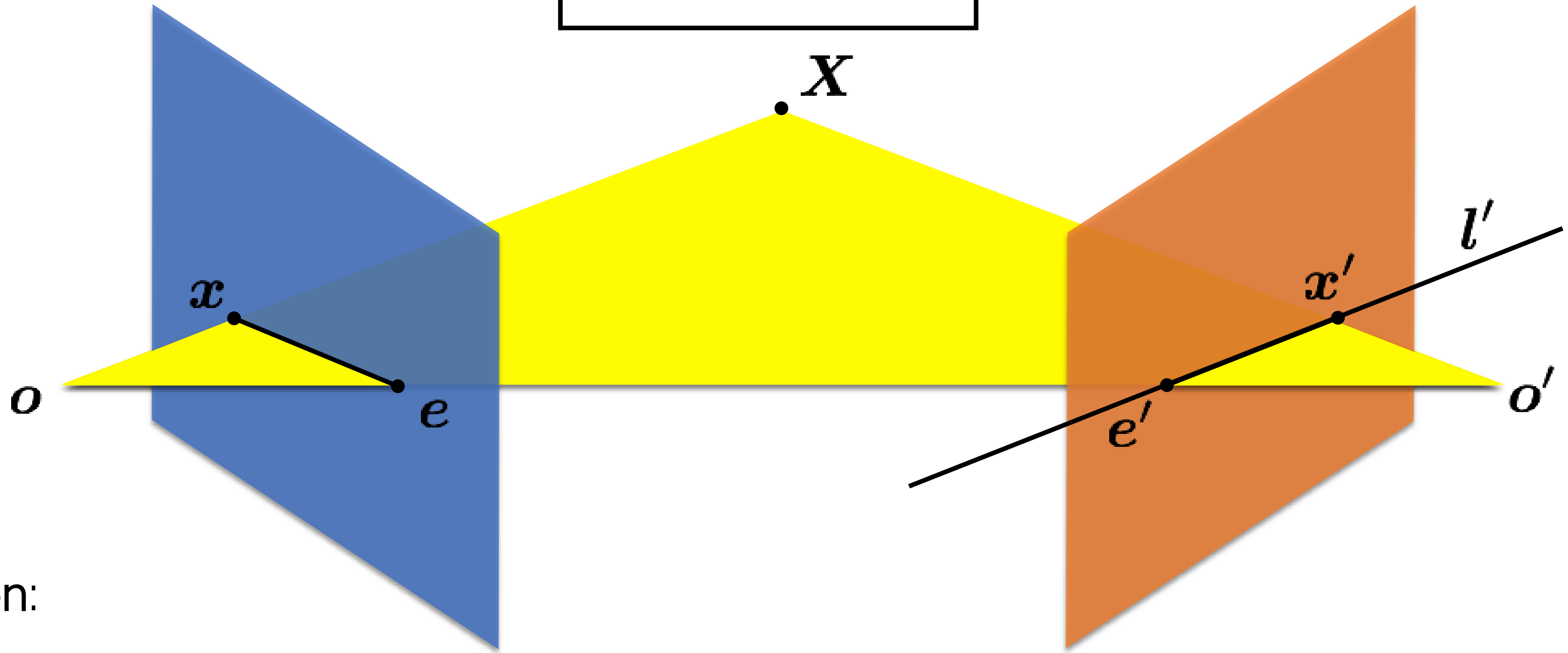
Properties of the E matrix

- 2D points expressed in camera coordinate system

Longuet-Higgins equation	$\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$	
Epipolar lines	$\boldsymbol{x}^{\top}\boldsymbol{l} = 0$ $\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$	$\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ $\boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$
Epipoles	$\boldsymbol{e}'^{\top}\mathbf{E} = \mathbf{0}$	$\mathbf{E}\boldsymbol{e} = \mathbf{0}$

- Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$



- Assumption:

  2D points expressed in camera coordinate system (i.e., intrinsic matrices are identities)

- How do you generalize to non-identity intrinsic matrices?

The fundamental matrix

- The **fundamental matrix** is a **generalization** of the **essential matrix,** where the assumption of **Identity matrices** is removed

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

- The essential matrix operates on image points expressed in 2D coordinates expressed in the camera coordinate system

$$\hat{\boldsymbol{x}}' = \mathbf{K}'^{-1}\boldsymbol{x}' \qquad \underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

- The essential matrix operates on image points expressed in 2D coordinates expressed in the camera coordinate system

$$\hat{\boldsymbol{x}}' = \mathbf{K}'^{-1}\boldsymbol{x}' \qquad\qquad \underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

- Writing out the epipolar constraint in terms of image coordinates

$$\boldsymbol{x}'^{\top}(\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1})\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

- Same equation works in image coordinates!

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

- It maps pixels to epipolar lines

Properties of the ~~E~~ F matrix

Longuet-Higgins equation	$\boldsymbol{x}'^{\top}\cancel{\mathbf{E}}\,\mathbf{F}\boldsymbol{x} = 0$	
Epipolar lines	$\boldsymbol{x}^{\top}\boldsymbol{l} = 0$	$\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$
	$\boldsymbol{l}' = \cancel{\mathbf{E}}\,\mathbf{F}\boldsymbol{x}$	$\boldsymbol{l} = \cancel{\mathbf{E}}\,\mathbf{F}^{T}\boldsymbol{x}'$
Epipoles	$\boldsymbol{e}'^{\top}\cancel{\mathbf{E}}\,\mathbf{F} = \mathbf{0}$	$\cancel{\mathbf{E}}\,\mathbf{F}\boldsymbol{e} = \mathbf{0}$

(points in **image** coordinates)

- Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

- Depends on both intrinsic and extrinsic parameters

- Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

- Depends on both intrinsic and extrinsic parameters

- How would you solve for F?

$$\boldsymbol{x}_m'^{\top}\mathbf{F}\boldsymbol{x}_m = 0$$

The 8-point algorithm

- Assume you have M matched image points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \quad m = 1, \ldots, M$$

- Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

- How would you solve for the 3 x 3 **F** matrix?

- Assume you have M matched image points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \quad m = 1, \ldots, M$$

- Each correspondence should satisfy

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$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \quad m = 1, \ldots, M$$

- Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

- How would you solve for the 3 x 3 **F** matrix?
- Set up a homogeneous linear system with 9 unknowns

$$\boldsymbol{x}_m'^\top \mathbf{F} \boldsymbol{x}_m = 0$$

$$\left[\begin{array}{ccc} x_m' & y_m' & 1 \end{array}\right] \left[\begin{array}{ccc} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{array}\right] \left[\begin{array}{c} x_m \\ y_m \\ 1 \end{array}\right] = 0$$

- How many equations do you get from one correspondence?

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

- **One** correspondence gives you **one** equation

$$\begin{aligned} x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 + \\ y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 + \\ x'_m f_7 + y'_m f_8 + f_9 = 0 \end{aligned}$$

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

- Set up a homogeneous linear system with 9 unknowns

$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_Mx'_M & x_My'_M & x_M & y_Mx'_M & y_My'_M & y_M & x'_M & y'_M & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix} = \mathbf{0}$$

- How many equations do you need?

- Each point pair (according to epipolar constraint) contributes only one scalar equation

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

note: this is different from the homography estimation where each point pair contributes 2 equations.

- We need at least 8 points
- Hence, the 8 point algorithm!

- How do you solve a homogeneous linear system?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

- How do you solve a homogeneous linear system?

$$\mathbf{AX} = \mathbf{0}$$

- Total least squares

$$\begin{aligned} &\text{minimize} \quad \|\mathbf{A}\boldsymbol{x}\|^2 \\ &\text{subject to} \quad \|\boldsymbol{x}\|^2 = 1 \end{aligned}$$

- How do you solve a homogeneous linear system?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

- Total least squares

$$\text{minimize} \quad \|\mathbf{A}\boldsymbol{x}\|^2$$

$$\text{subject to} \quad \|\boldsymbol{x}\|^2 = 1$$

S V D !

Eight-point algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

Eight-point algorithm

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see Hartley-Zisserman for
why we do this

Eight-point algorithm

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how do we do this?

Eight-point algorithm

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5. (Un-normalize F)

how do we do this?

S V D !

Enforcing rank constraints

- **Problem:** given a matrix F, find the matrix F' of rank k that is closest to F,

$$\min_{\substack{F' \\ \mathrm{rank}(F')=k}} \quad ||F - F'||^2$$

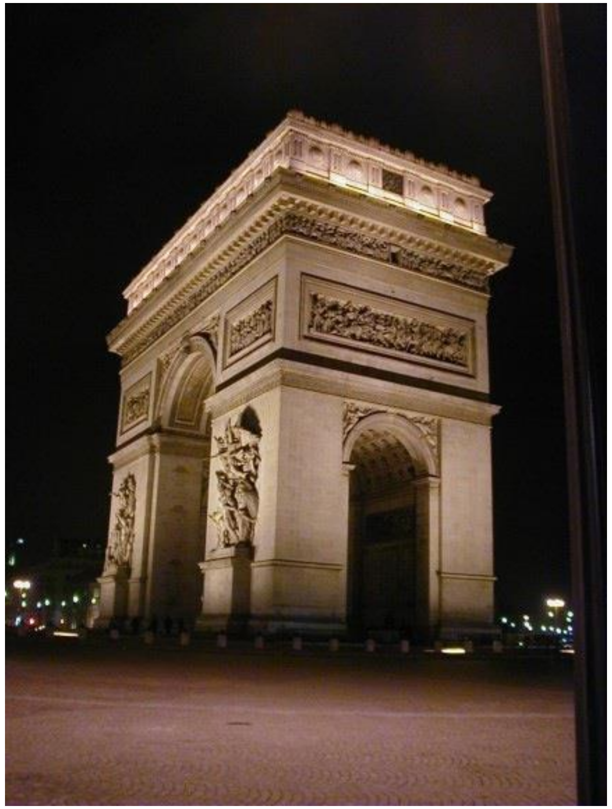
- **Solution:** compute the singular value decomposition of F,

$$F = U\Sigma V^T$$

- Form a matrix Σ' by replacing all, but the k largest singular values in Σ with 0.
- Then the problem solution is the matrix F' formed as,

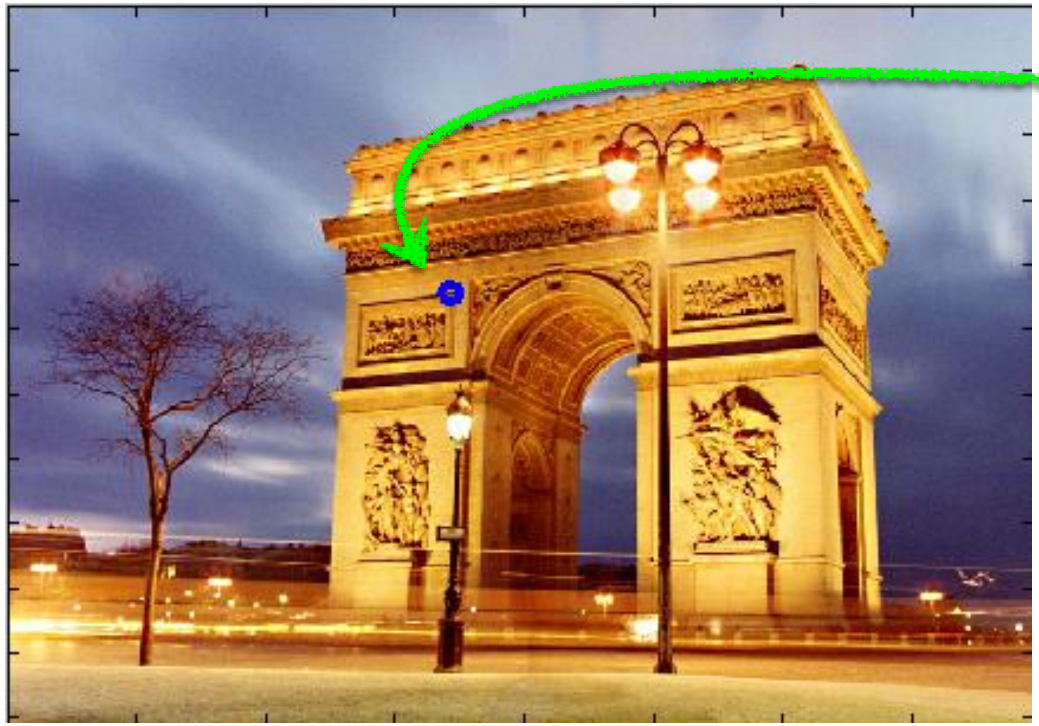
$$F' = U\Sigma' V^T$$

Example



Epipolar lines

$$\mathbf{F} = \begin{bmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{bmatrix}$$



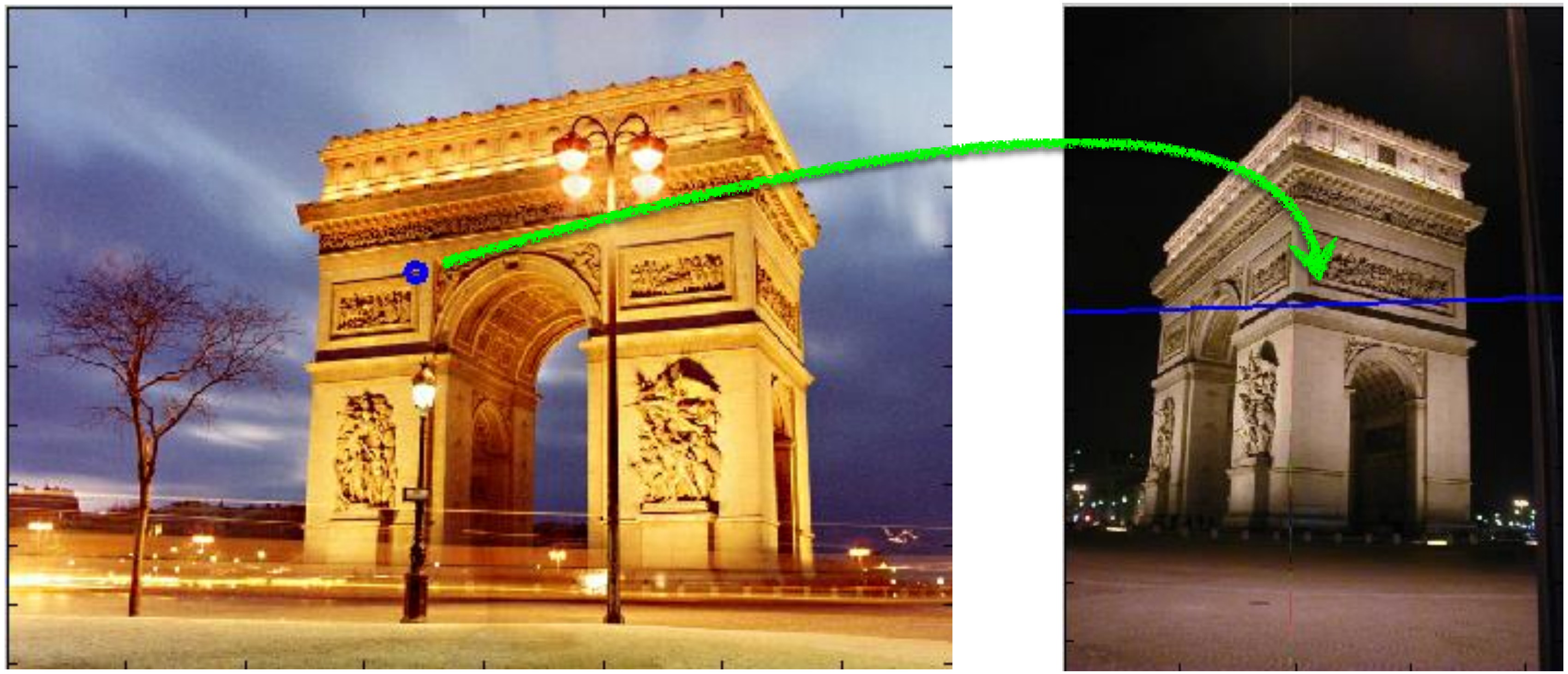
$$\boldsymbol{x} = \begin{bmatrix} 343.53 \\ 221.70 \\ 1.0 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$



- Where is the epipole?

- How would you compute it?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

- The epipole is in the right null space of F
- How would you solve for the epipole?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

- The epipole is in the right null space of **F**
- How would you solve for the epipole?

S V D !