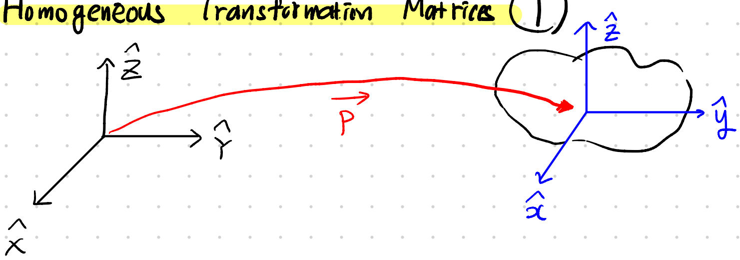
• **Homogeneous Transformation Matrices** (T)



{s} frame

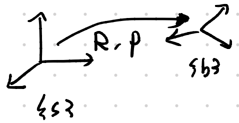
{b} frame

$$\left.\begin{aligned}\vec{p} &= p_1\hat{x} + p_2\hat{y} + p_3\hat{z}\\ \hat{x} &= r_{11}\hat{x} + r_{21}\hat{y} + r_{31}\hat{z}\\ \hat{y} &= r_{12}\hat{x} + r_{22}\hat{y} + r_{32}\hat{z}\\ \hat{z} &= r_{13}\hat{x} + r_{23}\hat{y} + r_{33}\hat{z}\end{aligned}\right\} R = \begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}$$

$$T = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & p_1\\ r_{21} & r_{22} & r_{23} & p_2\\ r_{31} & r_{32} & r_{33} & p_3\\ \hline 0 & 0 & 0 & 1\end{array}\right] = \begin{bmatrix} R & p\\ 0 & 1\end{bmatrix}$$

" SE(3) : Special Euclidean Group ".

⇒ Q1) $v \to$ rotate with R and move p ?

Q2) 

what if observing v in {b} from {s} ?

$\underline{Rv + p}$ is answer.

→ To make this operation easier, we represent $v \xrightarrow[\text{homogeneous coordinate}]{} \begin{bmatrix} v\\ 1\end{bmatrix}$, $\begin{bmatrix} R & p\\ 0 & 1\end{bmatrix}\begin{bmatrix} v\\ 1\end{bmatrix} = \begin{bmatrix} Rv+p\\ 1\end{bmatrix}$

• Properties))

- $T^{-1} = \begin{bmatrix} R^T & -R^Tp\\ 0 & 1\end{bmatrix}$
- $T_1, T_2, T_3 \in SE(3)$, $(T_1T_2)T_3 = T_1(T_2T_3)$, $T_1T_2 \neq T_2T_1$
- for $x, y \in \mathbb{R}^3$, $\|Tx - Ty\| = \|x - y\|$. $\left(Ta = T\begin{bmatrix} a\\ 1\end{bmatrix}\right)$
- $T_{ac} = T_{ab}T_{bc}$
- If Rotation is made with $\hat{\omega}$ and θ,

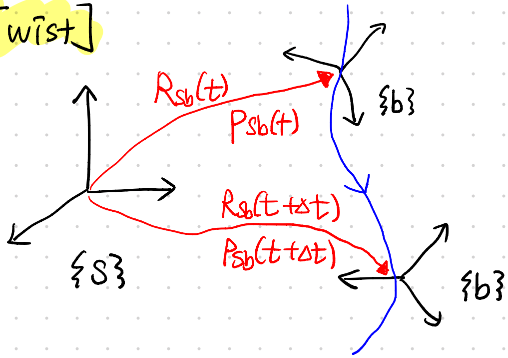
$$\left.\begin{aligned} \to\ Rot(\hat{\omega}, \theta) &= \begin{bmatrix} e^{[\hat{\omega}]\theta} & 0\\ 0 & 1\end{bmatrix}\\ Trans(p) &= \begin{bmatrix} I & p\\ 0 & 1\end{bmatrix}\end{aligned}\right\} Tv = Trans(p)\,Rot(\hat{\omega}, \theta)\,v$$

- $T_{sb'} = \underbrace{\overbrace{Trans(p)\,Rot(\hat{\omega}, \theta)}^{w.r.t\ \{s\}}\,T_{sb}}_{\text{rotate frame } \{b\}\text{, then translate.}}$ (vs) $T_{sb''} = T_{sb}\,\overbrace{Trans(p)}^{w.r.t\ \{b\}}\,\overbrace{Rot(\hat{\omega}, \theta)}^{w.r.t\ \{c\}}$?

Let $\underbrace{Trans(p)}_{T_{bc}}$ $\underbrace{Rot(\hat{\omega}, \theta)}_{T_{cd}}$.

$\Rightarrow T_{sb''} = T_{sb}T_{bc}T_{cd} = T_{sd}$.

[Twist]



- $T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$
- $[w_b] = R^{-1}\dot{R}$
- $T^{-1}\dot{T} = \begin{bmatrix} R^T & -R^Tp \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [w_b] & V_b \\ 0 & 0 \end{bmatrix}$$

→ Meaning of $R^T\dot{p}$?

$$\Rightarrow R^T\dot{p} = R_{sb}^T\,\dot{p} = R_{sb}^{-1}\,\dot{p} = R_{bs}\,\dot{p}$$

velocity of {b} frame's center, from {s}' perspective.

↳ Is converted to {b}'s perspective!

[Definition]

- given $w_b, v_b \in \mathbb{R}^3$, the spatial velocity in the body frame, or simply the body twist, is defined as : $\mathcal{V}_b = \begin{bmatrix} w_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$.
- given spatial velocity $\mathcal{V}_b \in \mathbb{R}^6$, define $[\cdot]$ notation as $\Big]$ $T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [w_b] & v_b \\ 0 & 0 \end{bmatrix}$

⇒ The set of all 4x4 matrices of the form is called $se(3)$, a Lie algebra of the Lie group $SE(3)$

→ what about $\dot{T}T^{-1}$? like $[w_s] = \dot{R}R^T$?

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R^T & -R^Tp \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^Tp + \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [w_s] & V_s \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow V_s = \dot{p} - \dot{R}R^Tp = \dot{p} - [w_s]p = \underbrace{\dot{p}}_{\text{\{b\}'s transition velocity}} + \underbrace{w_s}_{\text{\{b\}'s angular velocity}} \times (\underbrace{-p}_{\text{vector } \{b\} \mapsto \{s\}})$$

$$[V_b] = T^{-1}\dot{T}, \quad \dot{T} = T[V_b]$$

$$\Rightarrow [V_s] = \dot{T}T^{-1} = T[V_b]T^{-1} = \begin{bmatrix} R[w_b]R^T & -R[w_b]R^T p + R v_b \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \underset{\begin{bmatrix} w_s \\ v_s \end{bmatrix}}{V_s} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \underset{\begin{bmatrix} w_b \\ v_b \end{bmatrix}}{V_b}.$$

[Definition]

Given $T = (R, p) \in SE(3)$ its Adjoint representation $[Ad_T]$ is

$$[Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Adjoint Mapping is represented as: $V' = [Ad_T]V$ or $V' = Ad_T(V)$

[Properties]

- Let $T_1, T_2 \in SE(3)$, and $V = (w, v)$. then

  $$Ad_{T_1}(Ad_{T_2}(V)) = Ad_{T_1 T_2}(V), \quad [Ad_{T_1}][Ad_{T_2}]V = [Ad_{T_1 T_2}]V$$

- For any $T \in SE(3)$, $\therefore [Ad_T]^{-1} = [Ad_{T^{-1}}]$

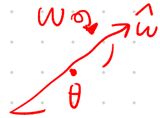
- For any rotating frame $\{b\}$, if twists are V_s, V_b,

  $$\therefore V_s = [Ad_{T_{sb}}]V_b, \quad V_b = [Ad_{T_{sb}^{-1}}]V_s.$$

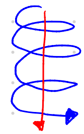
[Screw]

$$\underset{[\text{twist}]}{V} = \underset{[\text{Screw}]}{S}\,\dot{\theta}, \quad \text{where} \quad S = \begin{pmatrix} \hat{\omega} \\ \frac{V}{\|\omega\|} \end{pmatrix}, \quad \|\hat{\omega}\| = 1, \quad \dot{\theta} = \|\omega\|$$

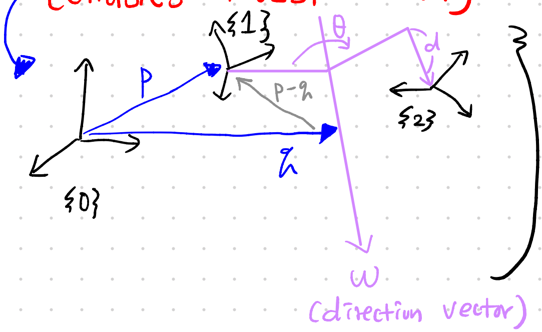
<normalize V with ω>



✱ Every rigid body motion can be expressed as a screw motion



(Chasles - Mozzi Theorem)



$T_{01} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \quad T_{02} = \begin{bmatrix} R' & p' \\ 0 & 1 \end{bmatrix}$

$R' = e^{[\omega]\theta} R$

$p' = q + e^{[\omega]\theta}(p - q) + \omega d.$

⇒ Screw Pitch " $h = \frac{d}{\theta}$ "

[Exponential Coordinate of Rigid-body Motions]

$$\Rightarrow \begin{bmatrix} R' & p' \\ 0 & 1 \end{bmatrix} = \underline{\begin{bmatrix} e^{[\omega]\theta} & (I - e^{[\omega]\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

↳ Let's represent it with S.

- when $S = (\omega, v)$, let $[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$.

$$\rightarrow e^{[S]\theta} = \begin{bmatrix} I + [\omega]\theta + [\omega]^2 \theta^2/2! + \cdots & (I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots)v \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \quad G(\theta) = I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2.$$

⇒ Can we make $e^{[S]\theta}$ to be a rigid-body motion by rotating θ with S?

↳ Setting $v = -\omega \times q + h\omega$ can solve!

[Joints as Screw Motion]

(Revolute) : $\|w\| = 1$

$V = -\omega \times q$

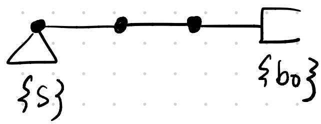
$(\because h = 0)$

(Prismatic) $\|w\| = 0$

$\|v\| = 1$

axis of movement defines v.

[product of Exponential Approach]

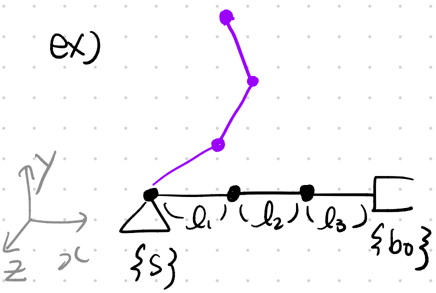


- Choose fixed frame {s}, {b}
- put all $\theta_i = 0$
- Let $M \in SE(3)$ be configuration of {b} in {s} when robot is in zero position

→ for each Joint, define the screw axis & motion

$$\Rightarrow T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \cdots e^{[S_n]\theta_n} M.$$

ex)



1) Find $M = \begin{bmatrix} R_0 & p_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l_1+l_2+l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2) find $w_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3) $V_i = -W_i \times q_{i,}$

ex) $V_3 = -W_3 \times q_3$, $q_3 = \begin{bmatrix} l_1+l_2 \\ 0 \\ 0 \end{bmatrix}$.

4) put to $e^{[S_i]\theta_i}$!