Multivariate Data Analysis

(MGT513, BAT531, TIM711)

Lecture 4

Chapter 6

Confirmatory Factor Analysis

Types of Factor Analysis

1. Exploratory Factor Analysis (EFA)

- used to discover the factor structure of a construct and examine its reliability. It is data driven.

2. Confirmatory Factor Analysis (CFA)

- used to confirm the fit of the hypothesized factor structure to the observed (sample) data. It is theory driven.

Issues

- *Confirmatory Factor Analysis* has **a prior information** about the structure of the factor solution, while *Exploratory Factor Analysis* has **no** prior information about that
- A single unique solution with no rotational indeterminacy: Confirmation
- To test our prior information to check if it is consistent with the patterns of data
- Estimation of parameters based on MLE(Maximum Likelihood Estimator) of the model

Psychological Testing of Children

- Five tests with 7th & 8th-graded children(n=145)
  1. X1 = Paragraph Comprehension (PARA)
  2. X2 = Sentence Completion (SENT)
  3. X3 = Word Meaning (WORD)
  4. X4 = Addition (ADD)
  5. X5 = Counting Dots (DOTS)

Psychological Testing of Children

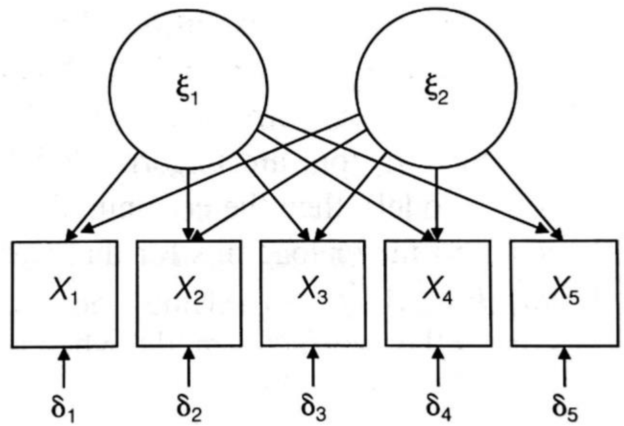
Correlation matrix of five test scores (X1-X5)

$$\mathbf{R} = \begin{bmatrix} 1.000 & .722 & .714 & .203 & .095 \\ .722 & 1.000 & .685 & .246 & .181 \\ .714 & .685 & 1.000 & .170 & .113 \\ .203 & .246 & .170 & 1.000 & .585 \\ .095 & .181 & .113 & .585 & 1.000 \end{bmatrix}$$

EFA: Psychological Testing of Children

- Exploratory Factor Analysis – Two Factor Model

FIGURE 5.3
Path diagram of two-factor model with five variables



$$X_i = \lambda_{i1}\xi_1 + \lambda_{i2}\xi_2 + \delta_i, \qquad i = 1,2,3,4,5$$

where $\mathrm{cor}(\xi_i, \xi_j) = 0$, $\mathrm{cor}(\delta_i, \delta_j) = 0$, and $\mathrm{cor}(\delta_i, \xi_j) = 0$ for $i \neq j$

where ξ_1 = Verbal Aptitude Factor

ξ_2 = Quantitative Aptitude Factor

EFA: Psychological Testing of Children

- Exploratory Factor Analysis – Two Factor Model

$$
\begin{aligned}
X_1 &= \lambda_{11}\xi_1 + \lambda_{12}\xi_2 + \delta_1 \\
X_2 &= \lambda_{21}\xi_1 + \lambda_{22}\xi_2 + \delta_2 \\
X_3 &= \lambda_{31}\xi_1 + \lambda_{32}\xi_2 + \delta_3 \\
X_4 &= \lambda_{41}\xi_1 + \lambda_{42}\xi_2 + \delta_4 \\
X_5 &= \lambda_{51}\xi_1 + \lambda_{52}\xi_2 + \delta_5
\end{aligned}
$$

EFA: Psychological Testing of Children

- Exploratory Factor Analysis – Two Factor Model

TABLE 5.5 Factor analysis with SMCs as initial estimates

Prior Communality Estimates: SMC

PARA	*SENT*	*WORD*	*ADD*	*DOTS*
0.6158	0.5914	0.5701	0.3672	0.3493

Final Eigenvalues

	1	2	3	4	5
Eigenvalue	2.2826	1.0273	0.0252	−0.0010	−0.0247

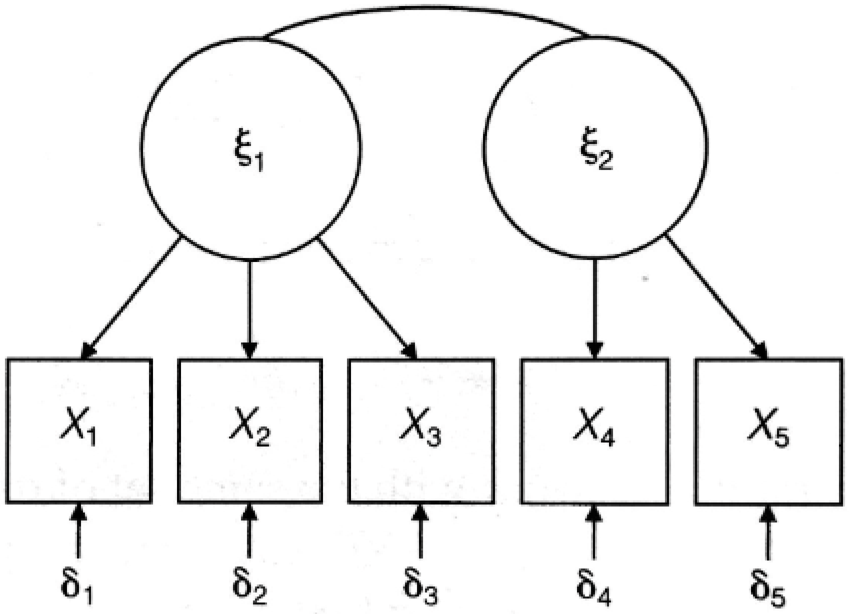
Factor Pattern

	Factor 1	Factor 2
PARA	0.8349	−0.2418
SENT	0.8253	−0.1398
WORD	0.7898	−0.2274
ADD	0.4146	0.6503
DOTS	0.3297	0.6890
Variance Explained by Each Factor		
	2.2826	1.0273

CFA: Psychological Testing of Children

- Confirmatory Factor Analysis - Two Factor Model

FIGURE 6.2
Path diagram of two-factor model of psychological test performance



Psychological Testing of Children

- Based on this result, confirmatory factor analysis assumes

$$
\begin{aligned}
X_1 &= \lambda_{11}\xi_1 & & + \delta_1 \\
X_2 &= \lambda_{21}\xi_1 & & + \delta_2 \\
X_3 &= \lambda_{31}\xi_1 & & + \delta_3 \\
X_4 &= & +\lambda_{42}\xi_2 & + \delta_4 \\
X_5 &= & +\lambda_{52}\xi_2 & + \delta_5
\end{aligned}
$$

where $\mathrm{cor}(\xi_1, \xi_2) = \phi_{12}$

- Verbal ability: X_1, X_2, X_3
- Quantitative ability: X_4, X_5

Confirmatory Factor Analysis

- In confirmatory factor analysis, instead of letting the data suggest the model structure (as in exploratory factor analysis), we begin with a strong prior notion regarding the structure of the model and proceed to test its adequacy.
  - By imposing sufficient structure on the factor loadings matrix (e.g., setting some parameters equal to zero), we are able to resolve the rotational indeterminacy of the exploratory factor model (where all variables load on all factors).
  - This additional structure increases the number of degrees of freedom available to estimate the model parameters and makes it possible to allow for correlated factors and correlated error terms.

Confirmatory Factor Model

- Confirmatory Factor Model

$$\boldsymbol{X} = \boldsymbol{\Xi}\boldsymbol{\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Delta}$$

  - $\boldsymbol{\Lambda}$: factor loading matrix
  - $\boldsymbol{\Xi}$: factor score matrix

- The population covariance matrix for $\boldsymbol{X}$ is

$$\boldsymbol{\Sigma} = \mathrm{Var}(\boldsymbol{X}) = \mathrm{Var}\left(\boldsymbol{\Xi}\boldsymbol{\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Delta}\right) = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Theta}$$

  - $\boldsymbol{\Phi}$: factor correlation matrix
  - $\boldsymbol{\Theta}$: covariance matrix for specific factors

Confirmatory Factor Model

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Theta}$$

- Exploratory Factor Analysis
  - $\boldsymbol{\Lambda}$: unconstrained
  - $\boldsymbol{\Phi}$ is set equal to the identity matrix $\boldsymbol{I}$ (no correlated factors)
  - $\boldsymbol{\Theta}$ is a diagonal matrix (no correlated errors)
- Confirmatory Factor Analysis
  - $\boldsymbol{\Lambda}$: by imposing constraints, zero elements can exist
  - $\boldsymbol{\Phi}$: nonzero off-diagonal elements can exist (correlated factors)
  - $\boldsymbol{\Theta}$: nonzero off-diagonal elements can exist (correlated error terms)

Potential Applications

Example 1 : Measuring status of different wine appellations

TABLE 6.1 Correlation matrix for five experts rating the status of 59 different wine appellations

	X_1	X_2	X_3	X_4	X_5
X_1	1.00000	0.76490	0.67821	0.67515	0.68186
X_2	0.76490	1.00000	0.73522	0.62564	0.76585
X_3	0.67821	0.73522	1.00000	0.63170	0.71356
X_4	0.67515	0.62564	0.63170	1.00000	0.51748
X_5	0.68186	0.76585	0.71356	0.51748	1.00000

Potential Applications

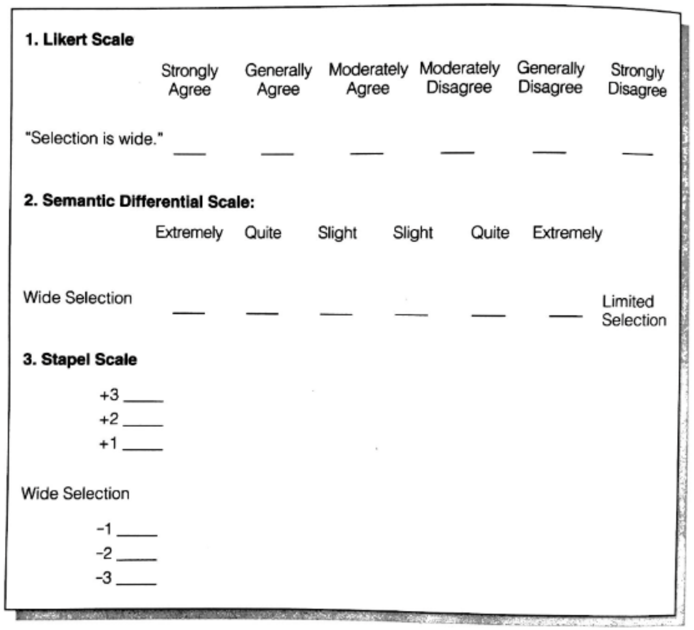
Example 1 : Measuring status of different wine appellations

- Testing the goodness of fit of different factor models
  - Assessing the reliability of measures: if observed data are consistent with a single-factor model of status (*convergent validity*)
    1. To determine whether a particular factor structure is consistent with the observed data
    2. Test if there are any differences across experts in their reliability

Potential Applications

Example 2: Studying traits (Store Appearance & Product Assortment) of different retail chains

FIGURE 6.1 Examples of Likert, semantic differential, and Stapel scales used by Menezes and Elbert (1979)


1. Likert Scale

	Strongly Agree	Generally Agree	Moderately Agree	Moderately Disagree	Generally Disagree	Strongly Disagree
"Selection is wide."	___	___	___	___	___	___

2. Semantic Differential Scale:

	Extremely	Quite	Slight	Slight	Quite	Extremely	
Wide Selection	___	___	___	___	___	___	Limited Selection

3. Stapel Scale

+3 ___
+2 ___
+1 ___

Wide Selection

-1 ___
-2 ___
-3 ___

TABLE 6.2 Correlation matrix for two traits of grocery stores (A and P) rated using three different methods (L, D, and S)

	AL	*AD*	*AS*	*PL*	*PD*	*PS*
AL	1.000					
AD	0.776	1.000				
AS	0.676	0.739	1.000			
PL	0.638	0.600	0.539	1.000		
PD	0.561	0.635	0.527	0.713	1.000	
PS	0.522	0.559	0.589	0.720	0.698	1.000

Potential Applications

Example 2: Studying traits of different retail chains

- Evaluating Construct Validity
  - Check whether two different constructs are in fact different from one another (*divergent validity*)
    1. To test whether store appearance and product assortment are separable constructs
    2. To assess the adequacy that they are sufficiently reliable to achieve convergent validity
    3. To capture correlated errors in measurement that might occur when using the same method to measure two different constructs

Intuition: Psychological Testing of Children

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{T} + \boldsymbol{\Theta}$$

- We have to estimate parameters Two Different Things:

1) Factor loadings matrix ($\mathbf{\Lambda}$)
2) Factor correlation matrix ($\mathbf{\Phi}$)

However, we can not simultaneously estimate both $\mathrm{Var}(\xi)$ and λ; As λ increases, $\mathrm{Var}(\xi)$ decreases and $\mathrm{Var}(X)$ remains constant $\Rightarrow$ **Two suggestions**

Intuition: Psychological Testing of Children

1. Standardization to make each variance of factors “one”:

- Factor loadings matrix ($\mathbf{\Lambda}$)

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & \lambda_{42} \\ 0 & \lambda_{52} \end{bmatrix}$$

- Factor correlation matrix ($\mathbf{\Phi}$)

$$\mathbf{\Phi} = \begin{bmatrix} 1 & \phi_{12} \\ \phi_{21} & 1 \end{bmatrix}$$

Intuition: Psychological Testing of Children

2. Estimate Var(ξ) as a parameter, but to fix λ to some arbitrary value (usually 1):

- Factor loadings matrix ($\mathbf{\Lambda}$)

$$\mathbf{\Lambda} = \begin{bmatrix} 1.0 & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & 1.0 \\ 0 & \lambda_{52} \end{bmatrix}$$

- Factor correlation matrix ($\mathbf{\Phi}$)

$$\mathbf{\Phi} = \begin{bmatrix} {\phi_{11}}^2 & \phi_{12} \\ \phi_{21} & {\phi_{22}}^2 \end{bmatrix}$$

where ${\phi_{11}}^2 \neq 1$ and ${\phi_{22}}^2 \neq 1$

Model identification: 11 unknown parameters

- Factor loadings - 5 parameters

  : $(\lambda_{11}, \lambda_{21}, \lambda_{31}, \lambda_{42}, \lambda_{52})$
- Error variances the specific factors – 5 parameters

  : $(\theta_{11}^2, \theta_{22}^2, \theta_{33}^2, \theta_{44}^2, \theta_{55}^2)$
- Correlation between the two factors – 1 parameter

  : ϕ_{12}

Model identification: 15 observations (1)

- $\theta_{11}^2 = 1.0 - \lambda_{11}^2$
- $\theta_{22}^2 = 1.0 - \lambda_{22}^2$
- $\theta_{33}^2 = 1.0 - \lambda_{33}^2$
- $\theta_{44}^2 = 1.0 - \lambda_{44}^2$
- $\theta_{55}^2 = 1.0 - \lambda_{55}^2$
- $\mathrm{corr}(X_1, X_2) = \mathrm{corr}(\lambda_{11}\xi_1 + \delta_1, \lambda_{21}\xi_1 + \delta_2) = \lambda_{11}\lambda_{21} = 0.722$
- $\mathrm{corr}(X_1, X_3) = \mathrm{corr}(\lambda_{11}\xi_1 + \delta_1, \lambda_{31}\xi_1 + \delta_3) = \lambda_{11}\lambda_{31} = 0.714$
- $\mathrm{corr}(X_1, X_4) = \mathrm{corr}(\lambda_{11}\xi_1 + \delta_1, \lambda_{42}\xi_2 + \delta_4) = \lambda_{11}\mathrm{corr}(\xi_1, \xi_2)\lambda_{42} = \lambda_{11}\phi_{12}\lambda_{42} = 0.203$
- $\mathrm{corr}(X_1, X_5) = \mathrm{corr}(\lambda_{11}\xi_1 + \delta_1, \lambda_{52}\xi_2 + \delta_5) = \lambda_{11}\mathrm{corr}(\xi_1, \xi_2)\lambda_{52} = \lambda_{11}\phi_{12}\lambda_{52} = 0.095$

Model identification: 15 observations (2)

- $\text{corr}(X_2, X_3) = \text{corr}(\lambda_{21}\xi_1 + \delta_2, \lambda_{31}\xi_1 + \delta_3) = \lambda_{21}\lambda_{31} = 0.685$
- $\text{corr}(X_2, X_4) = \text{corr}(\lambda_{21}\xi_1 + \delta_2, \lambda_{42}\xi_2 + \delta_4) =$
  $\lambda_{21}\text{corr}(\xi_1, \xi_2)\lambda_{42} = \lambda_{21}\phi_{12}\lambda_{42} = 0.246$
- $\text{corr}(X_2, X_5) = \text{corr}(\lambda_{21}\xi_1 + \delta_2, \lambda_{52}\xi_2 + \delta_5) =$
  $\lambda_{21}\text{corr}(\xi_1, \xi_2)\lambda_{52} = \lambda_{21}\phi_{12}\lambda_{52} = 0.181$
- $\text{corr}(X_3, X_4) = \text{corr}(\lambda_{31}\xi_1 + \delta_3, \lambda_{42}\xi_2 + \delta_4) =$
  $\lambda_{31}\text{corr}(\xi_1, \xi_2)\lambda_{42} = \lambda_{31}\phi_{12}\lambda_{42} = 0.170$
- $\text{corr}(X_3, X_5) = \text{corr}(\lambda_{31}\xi_1 + \delta_3, \lambda_{52}\xi_2 + \delta_5) =$
  $\lambda_{31}\text{corr}(\xi_1, \xi_2)\lambda_{52} = \lambda_{31}\phi_{12}\lambda_{52} = 0.113$
- $\text{corr}(X_4, X_5) = \text{corr}(\lambda_{42}\xi_2 + \delta_4, \lambda_{42}\xi_2 + \delta_5) = \lambda_{42}\lambda_{52} = 0.585$

Model identification

- For a confirmatory factor model to be identified:

  # of parameters to be estimated (unknown parameters) $<$ # of independent observations

- Degrees of freedom : the difference between these two quantities
- *Over-identified model*: not unique solution $\Longrightarrow$ Need optimization (e.g. **M**aximum **L**ikelihood **E**stimation)
  - *Just- identified model* (saturated model): perfect fit, # of parameters to be estimated (unknown parameters) $=$ # of independent observations
  - *Under-identified model*: # of parameters to be estimated (unknown parameters) $>$ # of independent observations

Maximum Likelihood Estimation of CFA Models

- Essential difference between exploratory and confirmatory factor model is the solution procedure
  - EFA: Matrix decomposition – Principal factor method
  - CFA: Maximum Likelihood Estimation (MLE)

- Confirmatory Factor Model

$$\boldsymbol{X} = \boldsymbol{\Xi}\boldsymbol{\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Delta}$$

  - $\boldsymbol{\Lambda}$: factor loading matrix (zeros exist – by restriction)
  - $\boldsymbol{\Xi}$: factor score matrix

Maximum Likelihood Estimation of CFA Models

- The population covariance matrix for $\boldsymbol{X}$ is
$$\boldsymbol{\Sigma} = \mathrm{Var}(\boldsymbol{X}) = \mathrm{Var}\left(\boldsymbol{\Xi\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Delta}\right) = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\boldsymbol{T}} + \boldsymbol{\Theta}$$
  - $\boldsymbol{\Phi}$: factor correlation matrix
  - $\boldsymbol{\Theta}$: covariance matrix (can be non-diagonal)

- The sample covariance matrix for $\boldsymbol{X}$ is
$$\boldsymbol{S} = \frac{1}{n}\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X} = \left(s_{ij}\right) = \left(\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}(x_{ki} - \overline{x_i})(x_{kj} - \overline{x_j})^{\mathrm{T}}\right)$$

- Claim: Estimate $\boldsymbol{\Lambda}, \boldsymbol{\Phi}, \boldsymbol{\Theta}$ so that the fitted values of the population covariance matrix $\boldsymbol{\Sigma}$ are as close as possible to the observed values in the sample covariance matrix $\boldsymbol{S}$

Maximum Likelihood Estimation of CFA Models

- Assume for $i = 1, \dots n$,

$$\mathbf{x_i} = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ip} \end{pmatrix} \sim N_p(\mathbf{0}, \mathbf{\Sigma})$$

where the density function of $\mathbf{x_i}$ is given by

$$\mathrm{f}(\mathbf{x_i}) = (2\pi|\mathbf{\Sigma}|)^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{x_i^T}\mathbf{\Sigma^{-1}}\mathbf{x_i})$$

Maximum Likelihood Estimation of CFA Models

- The joint density (likelihood function) of $(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ is

$$L = \prod_{i=1}^{n} \mathrm{f}(\mathbf{x_i}) = \prod_{i=1}^{n} (2\pi|\mathbf{\Sigma}|)^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{x_i^T}\mathbf{\Sigma^{-1}}\mathbf{x_i})$$

- The *log-likelihood* is

$$\ln(L) = \sum_{i=1}^{n} [-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{\Sigma}| - \frac{1}{2}\mathbf{x_i^T}\mathbf{\Sigma^{-1}}\mathbf{x_i}]$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln|\mathbf{\Sigma}| - \frac{1}{2}\sum_{i=1}^{n} \mathbf{x_i^T}\mathbf{\Sigma^{-1}}\mathbf{x_i}$$

$$= -\frac{n}{2}[\ln(2\pi) + \ln|\mathbf{\Sigma}| + \mathrm{tr}\left(\frac{1}{n}\boldsymbol{X}\mathbf{\Sigma^{-1}}\boldsymbol{X}^{\boldsymbol{T}}\right)]$$

where $\boldsymbol{X}^{\boldsymbol{T}} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\mathrm{n})$: p by n matrix

Maximum Likelihood Estimation of CFA Models

- Since the MLE is not affected by $-\frac{n}{2}\ln(2\pi)$, $\ln(L)$ is re-written as

$$\ln(L) = -\frac{n}{2}\left[\ln|\boldsymbol{\Sigma}| + \mathrm{tr}\left(\boldsymbol{S}\boldsymbol{\Sigma}^{-\mathbf{1}}\right)\right]$$

because $\mathrm{tr}\left(\frac{1}{n}\boldsymbol{X}\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{X}^{\boldsymbol{T}}\right) = \mathrm{tr}\left(\frac{1}{n}\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}\boldsymbol{\Sigma}^{-\mathbf{1}}\right) = \mathrm{tr}\left(\frac{1}{n}\mathbf{S}\boldsymbol{\Sigma}^{-\mathbf{1}}\right)$

- Estimation problem can be solved via numerical methods.
- Infeasible solutions (i.e. factor loading >1, or error variance<0) might be the result of a mis-specified model

Two Advantages of MLE solution

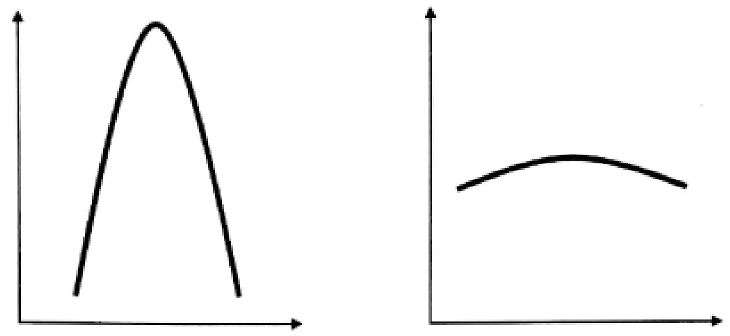
1. Asymptotic Standard Errors
   – Assess the stability of the parameter estimates

2. Goodness-of-fit tests
   – Test nested models (for model comparisons)
   – Test general model fits

Advantages of MLE solution: 1. Asymptotic Standard Errors

- The asymptotic standard errors are a by-product of the numerical optimization routine - the inverse of the matrix of second derivatives from the maximum likelihood estimation
  - Able to assess the stability of the parameter estimates
  - The small standard error ⇒ the optimal estimate of parameter with reasonable precision

FIGURE 6.3 Diagram showing relationship between curvature of objective function (i.e., second derivative) and standard error of a parameter estimate

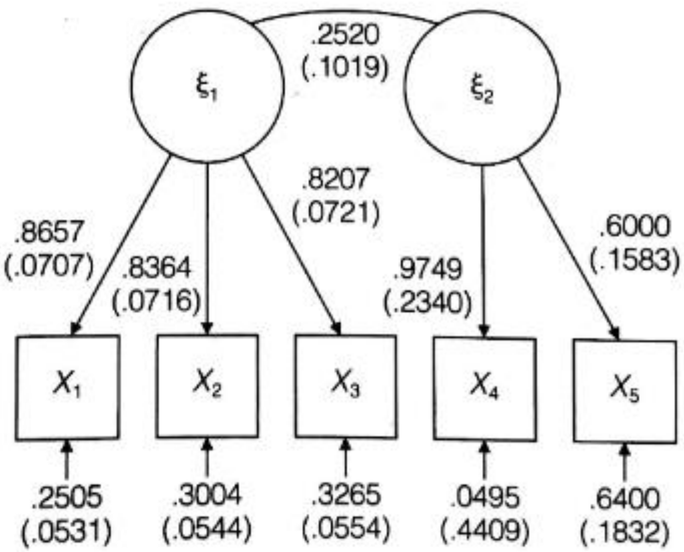


(a) Objective function with high curvature/low standard error

(b) Objective function with low curvature/high standard error

Advantages of MLE solution:
1. Asymptotic Standard Errors

FIGURE 6.4
Path diagram for model of psychological test performance with parameter estimates and standard errors



[Figure 6.4: Similar to the result from oblique rotation of exploratory factor analysis]

Advantages of MLE solution: 2. Goodness-of-fit tests

1. Test nested models [Likelihood Ratio Test]

- Likelihood Ratio Test
  - We can test the adequacy of any confirmatory factor model (relative to some alternative nested model) using the likelihood ratio. The log of likelihood ratio follows:

$$-2[\ln(L_R) - \ln(L_F)] \sim \chi^2(df_R - df_F)$$

L_R: likelihood of the restricted model

L_F : likelihood of full (less restrictive) model

df_R: degrees of freedom of the restricted model

df_F:degrees of freedom of full model

Advantages of MLE solution:
2. Goodness-of-fit tests

2-1. Test general model fits [Likelihood Ratio Test]

Consider the hypotheses

$$H_0: \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_R \qquad H_1: \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_F$$

- Restricted model (overdetermined model: # of parameters < # of independent observations) - some differences between the fitted matrix $\boldsymbol{\Sigma}$ and the observed matrix $\boldsymbol{S}$
- Instead of testing the proposed model against a highly restricted null model and looking for big differences in fit, we test the *proposed model* versus a *completely general model* that fits the observed data perfectly and look for relatively *small* differences in fit.

Advantages of MLE solution: 2. Goodness-of-fit tests

2-1. Test general model fits [Likelihood Ratio Test]

- In other words, we begin with the null hypothesis that the proposed model fits as well as a perfect model. If the difference in fit between models is relatively small, then we cannot reject the null hypothesis and so must accept the proposed model as no different from a perfectly fitting model.
- Of course, when our goal is to not reject the null hypothesis (rather than reject it), a different standard for comparison is required. Rather than setting a high hurdle (e.g., a p-value of 0.01) and looking for evidence to exceed that threshold, we set a low hurdle (e.g., a p-value of 0.20) and look for evidence that does not exceed the threshold.

Advantages of MLE solution: 2. Goodness-of-fit tests

2-1. Test general model fits [Likelihood Ratio Test]

Consider the hypotheses

$$H_0: \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_R \qquad H_1: \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_F$$

- Restricted model (overdetermined model: # of parameters < # of independent observations) - some differences between the fitted matrix $\boldsymbol{\Sigma}$ and the observed matrix $\boldsymbol{S}$

$$\ln(L_R) = -\frac{n}{2}[\ln|\boldsymbol{\Sigma}| + \mathrm{tr}(\boldsymbol{S}\boldsymbol{\Sigma}^{-1})]$$

- Completely general model or full model (just identified model: # of parameters = # of independent observations) – perfect fit between the fitted matrix $\boldsymbol{\Sigma}$ and the observed matrix $\boldsymbol{S}$

$$\ln(L_F) = -\frac{n}{2}[\ln|\boldsymbol{S}| + \mathrm{tr}(\boldsymbol{S}\boldsymbol{S}^{-1})] = -\frac{n}{2}[ln|\boldsymbol{S}| + p]$$

where p is the number observed variables in the model

Advantages of MLE solution: 2. Goodness-of-fit tests

2-1. Test general model fits [Likelihood Ratio Test]

Thus, for a sufficiently large n, the χ^2 goodness-of-fit test is

$$
\begin{aligned}
&-2[\ln(L_R) - \ln(L_F)] \\
&= n\left[\ln|\boldsymbol{\Sigma}| + \mathrm{tr}\left(\boldsymbol{S}\boldsymbol{\Sigma}^{-\mathbf{1}}\right) - \ln|\boldsymbol{S}| - p\right] \sim \chi^2(df_R - df_F)
\end{aligned}
$$

- χ^2 test statistic is proportional to the sample size n: When the sample size is large, even the smallest discrepancies between the fitted matrix $\boldsymbol{\Sigma}$ and the observed matrix $\boldsymbol{S}$ will be judged significant, leading to rejection of the null hypothesis and the proposed model.

Advantages of MLE solution: 2. Goodness-of-fit tests

2-2. Test general model fits [Sample size-free model fit indices]

- Goodness-of-fit index (*GFI*):

$$GFI = 1 - \frac{\mathrm{tr}\left[\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{S} - \boldsymbol{I}\right)^2\right]}{\mathrm{tr}\left[\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)^2\right]}$$

- Adjusted goodness-of-fit index (*AGFI*):

$$AGFI = 1 - \frac{p(p+1)}{2\mathrm{df}} \frac{\mathrm{tr}\left[\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{S} - \boldsymbol{I}\right)^2\right]}{\mathrm{tr}\left[\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)^2\right]}$$

- When $\boldsymbol{\Sigma} = \boldsymbol{S}$, *GFI*=1
- *GFI* is analogous to R^2 in regression analysis
- *AGFI* is analogous to $\bar{R}^2$ in regression analysis

Advantages of MLE solution: 2. Goodness-of-fit tests

2-2. Test general model fits [Sample size-free model fit indices]

- Two measures for a small n

  (1) *GFI*: Goodness-of-fit index

  (2) *AGFI*: Adjusted Goodness-of-fit index

  $0 \leq GFI \leq 1$ and $0 \leq AGFI \leq 1$

- Decision rule: Rule of Thumb such as
  - $GFI > 0.95$ for a good fit
  - $GFI > 0.90$ for an acceptable fit
  - $AGFI > 0.90$ for a good fit
  - $AGFI > 0.80$ for an acceptable fit

Reliability of (Single) Measure

- Reliability: whether or not a particular variable X does a good job of measuring the true underlying factor of construct ξ that it purports to measure
- The squared correlation between the observed score (X) and the true score (ξ)

$$\rho_{X\xi}^2 = \frac{\sigma_\xi^2}{\sigma_X^2} = 1 - \frac{\sigma_\delta^2}{\sigma_X^2}$$

- As $\rho_{X\xi}^2 \to 1, \sigma_\delta^2 \to 0$ (the closer the correspondence between the measure and the true construct)
- In CFA, λ describes the correlation between X and ξ
- λ^2=squared factor loading = the reliability
- A general rule of thumb: $\lambda > 0.7$ is good

Reliability of an Index: Cronbach's alpha

- When there are multiple measurements for a single construct
- To examine the internal consistency of the index

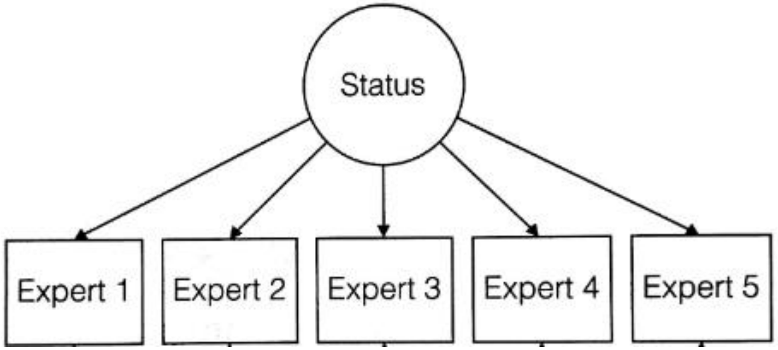
$$0 \leq \alpha = \frac{k\bar{r}}{[1 + (k-1)\bar{r}]} \leq 1$$

where k is the # of variables (items) in the index and $\bar{r}$ is the average inter-item correlation among the k items.

- In general, if α > 0.9 is good, if α > 0.7 is acceptable

Reliability of an Index: Equal Weights

- Wine appellation status example
  - In the original paper: equally weighted sum of expert judgments is used for an index to reflect wine appellation status
  - Internal consistent: $\text{Cronbach's } \alpha = 0.91$
  - Two (Implicit) assumptions
    - Status is unidimensional construct
    - All experts are equally reliable



Reliability of an Index: Factor Analysis

- One-factor model of status: good fit

FIGURE 6.5 Path diagram of one-factor model of wine appellation status

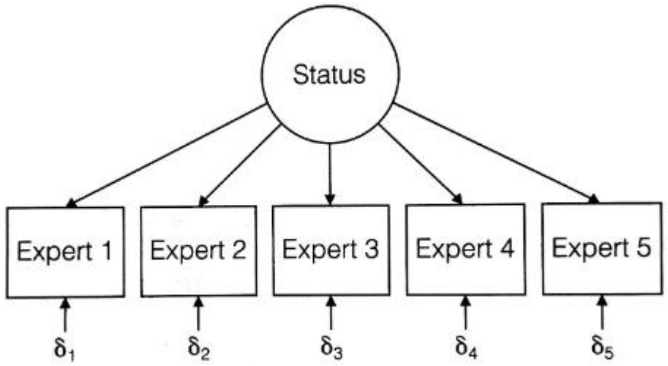


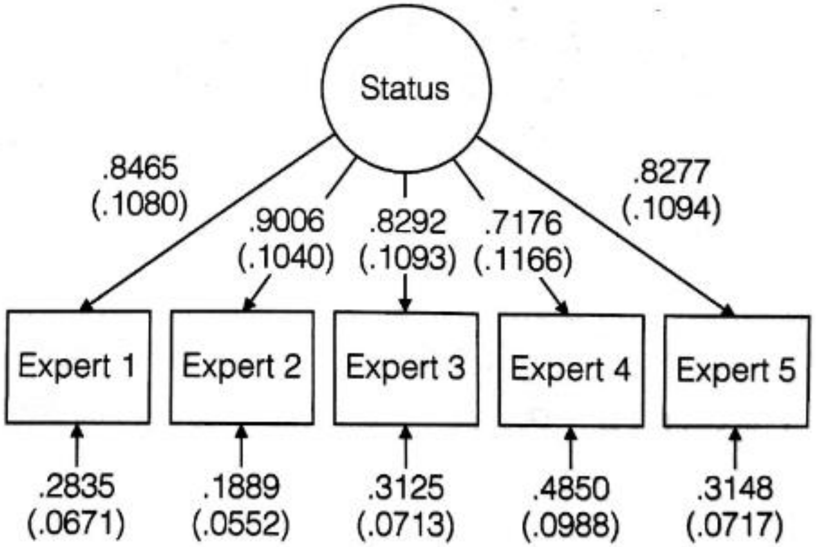
TABLE 6.4 Goodness-of-fit statistics for one-factor model of wine status

Goodness-of-fit index (GFI)		0.9570
GFI adjusted for degrees of freedom (AGFI)		0.8709
Root mean square residual (RMR)		0.0310
$\chi^2 = 6.7550$	$df = 5$	$p = 0.2395$
Null model chi-square	$df = 10$	203.0803

Reliability of an Index: Factor Analysis

- One-factor model of status: Two assumptions may not be appropriate
  - Expert 4 is the least reliable
  - Composite index: able to put lower weight on the evaluation of the least reliable expert

FIGURE 6.6
Path diagram of one-factor model of wine status with parameters and standard errors



Reliability of an Index: Factor Analysis

- Factor scores coefficients ($\boldsymbol{B} = \boldsymbol{R}^{-1}\boldsymbol{\Lambda}_c$) can reflect unequal weight where $\boldsymbol{\Xi} = \boldsymbol{X}_s\boldsymbol{B}$
- Factor score equation:

$$0.23X_1 + 0.36X_2 + 0.20X_3 + 0.11X_4 + 0.20X_5$$

Expert 2 receives more than three times the weight of expert 4 due to their differences in reliability

Reliability of an Index: Factor Analysis

- Unequally weighted composite index by Werts, Linn, and Joreskog (1974)

- Reliability of a composite index *C* (made up of several measure $X_1, X_1, \dots, X_k$) is

$$\rho_c^2 = \frac{(\sum_{i=1}^{k} \lambda_i)^2}{(\sum_{i=1}^{k} \lambda_i)^2 + (\sum_{i=1}^{k} \theta_{ii})^2}$$

where k is the # of items measuring the underlying construct ξ

- For wine data, $\rho_c^2 = 0.91 \approx \text{Cronbach's } \alpha = 0.91$

Preview: Model fits

Measure	Name	Description	Cut-off for good fit
X^2	Model Chi-Square	Assess overall fit and the discrepancy between the sample and fitted covariance matrices. Sensitive to sample size. H_0: The model fits perfectly.	p-value> 0.05
(A)GFI	(Adjusted) Goodness of Fit	GFI is the proportion of variance accounted for by the estimated population covariance. Analogous to R^2. AGFI favors parsimony.	GFI ≥ 0.95 AGFI ≥0.90
(N)NFI TLI	(Non) Normed-Fit Index Tucker Lewis index	An NFI of .95, indicates the model of interest improves the fit by 95% relative to the null model. NNFI is preferable for smaller samples. Sometimes the NNFI is called the Tucker Lewis index (TLI)	NFI ≥ 0.95 NNFI ≥ 0.95
CFI	Comparative Fit Index	A revised form of NFI. Not very sensitive to sample size. Compares the fit of a target model to the fit of an independent, or null, model.	CFI ≥.90
RMSEA	Root Mean Square Error of Approximation	A parsimony-adjusted index. Values closer to 0 represent a good fit.	RMSEA < 0.08
(S)RMR	(Standardized) Root Mean Square Residual	The square-root of the difference between the residuals of the sample covariance matrix and the hypothesized model. If items vary in range (i.e. some items are 1-5, others 1-7) then RMR is hard to interpret, better to use SRMR.	SRMR <0.08
AVE (CFA only)	Average Value Explained	The average of the R^2s for items within a factor	AVE >.5

Source:
https://www.cscu.cornell.edu/news/Handouts/SEM_fit.pdf

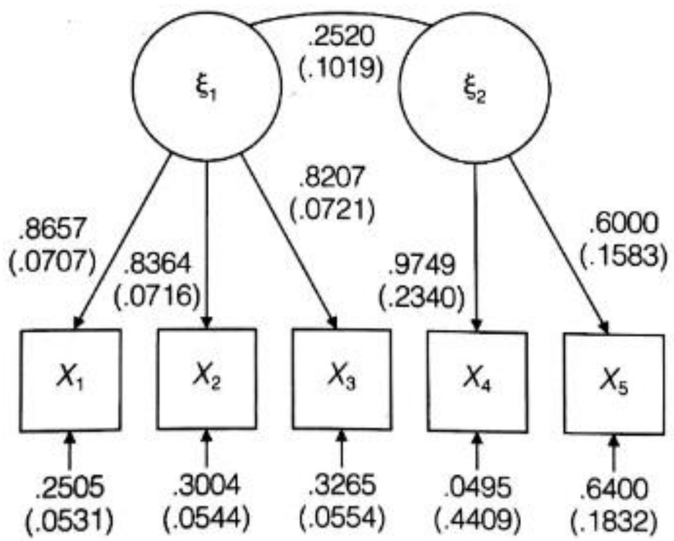
Model Comparison: 1. Testing Model Parameters

- Is the correlation between the two common factors significantly different from zero?

$$H_0: \phi_{12} = 0 \quad \text{vs.} \quad H_1: \phi_{12} \neq 0$$

- Nested model (fix $\phi_{12} = 0$ vs. not fix ϕ_{12})
- LRT result: χ^2 statistics: 10.2 with 1 df - significant (α=5%)

FIGURE 6.4
Path diagram for model of psychological test performance with parameter estimates and standard errors



Model Comparison:
2. Testing More than One Factor

- Nested model (fix $\phi_{12} = 1$ vs. not fix ϕ_{12})

FIGURE 6.7 One-factor model of psychological test performance: Parameter estimates and standard errors

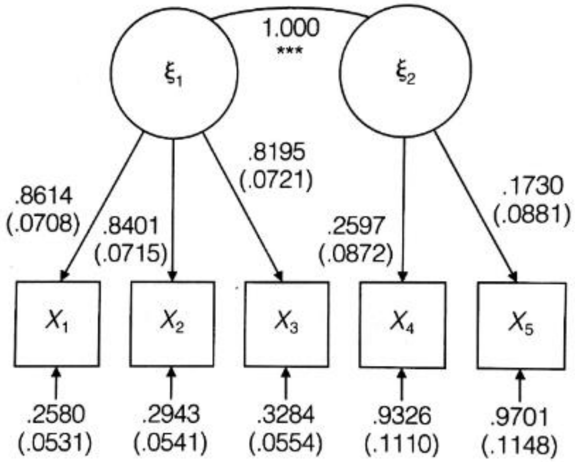


TABLE 6.5 Goodness-of-fit statistics for one-factor model of student test performance

Goodness-of-fit index (GFI)		0.8764
GFI adjusted for degrees of freedom (AGFI)		0.6291
Root mean square residual (RMR)		0.1414
$\chi^2 = 59.4732$	$df = 5$	$p = 0.0001$
Null model chi-square	$df = 10$	298.6480

TABLE 6.3 Goodness-of-fit statistics for two-factor model of student test performance

Goodness-of-fit index (GFI)		0.9919
GFI adjusted for degrees of freedom (AGFI)		0.9697
Root mean square residual (RMR)		0.0218
$\chi^2 = 2.9306$	$df = 4$	$p = 0.5695$
Null model chi-square	$df = 10$	298.6480

- LRT: One-factor model is significantly different from Two-factor model (α=5%)

$\Rightarrow$ Choose Two-factor model

- Discriminant validity (weak correlation: 0.252)

Model Comparison:
3. Restricted or General Model?

- Non-nested model
- Restricted model – parsimonious /still quite good model fit

FIGURE 6.8 Restricted one-factor model of wine status: Equal expert reliabilities

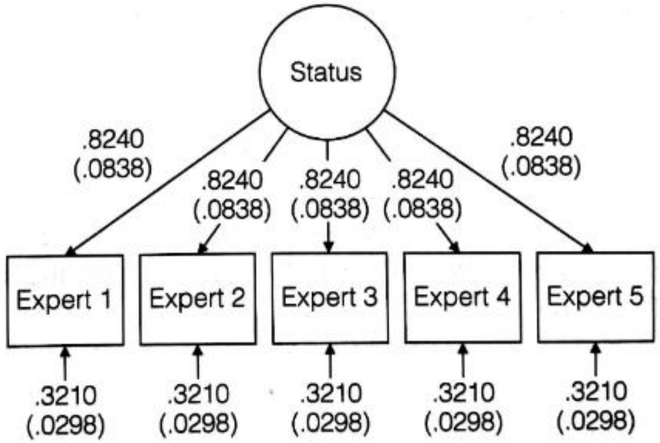


TABLE 6.6 Goodness-of-fit statistics for restricted models of wine status

Goodness-of-fit index (GFI)		0.8956
GFI adjusted for degrees of freedom (AGFI)		0.8796
Root mean square residual (RMR)		0.0580
$\chi^2 = 15.6181$	$df = 13$	$p = 0.2704$
Null model chi-square	$df = 10$	203.0803

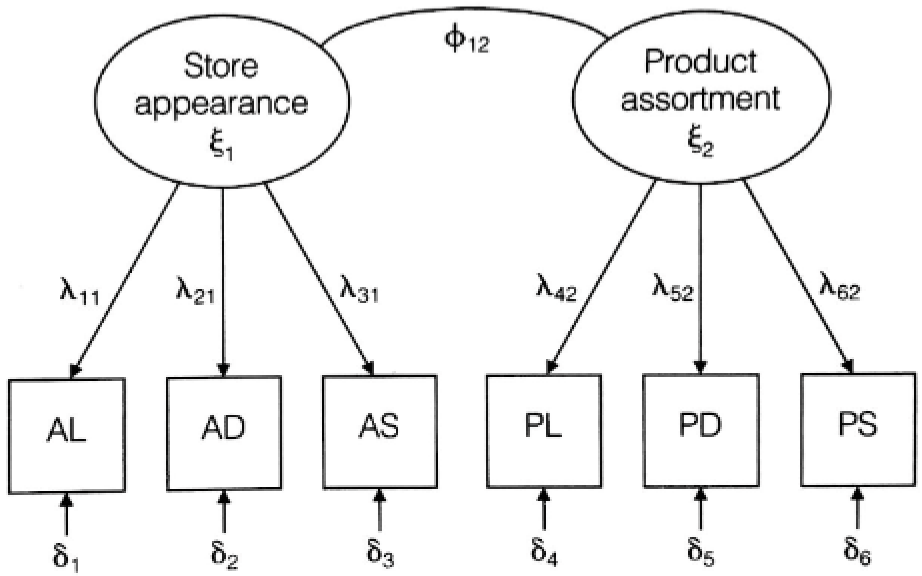
General model

TABLE 6.4 Goodness-of-fit statistics for one-factor model of wine status

Goodness-of-fit index (GFI)		0.9570
GFI adjusted for degrees of freedom (AGFI)		0.8709
Root mean square residual (RMR)		0.0310
$\chi^2 = 6.7550$	$df = 5$	$p = 0.2395$
Null model chi-square	$df = 10$	203.0803

Going Beyond Simple Factor Structure: 1. Two-factor Model

FIGURE 6.9
Two-factor model of store traits: No method factors or correlated error



Going Beyond Simple Factor Structure: Standardized Residual Matrix

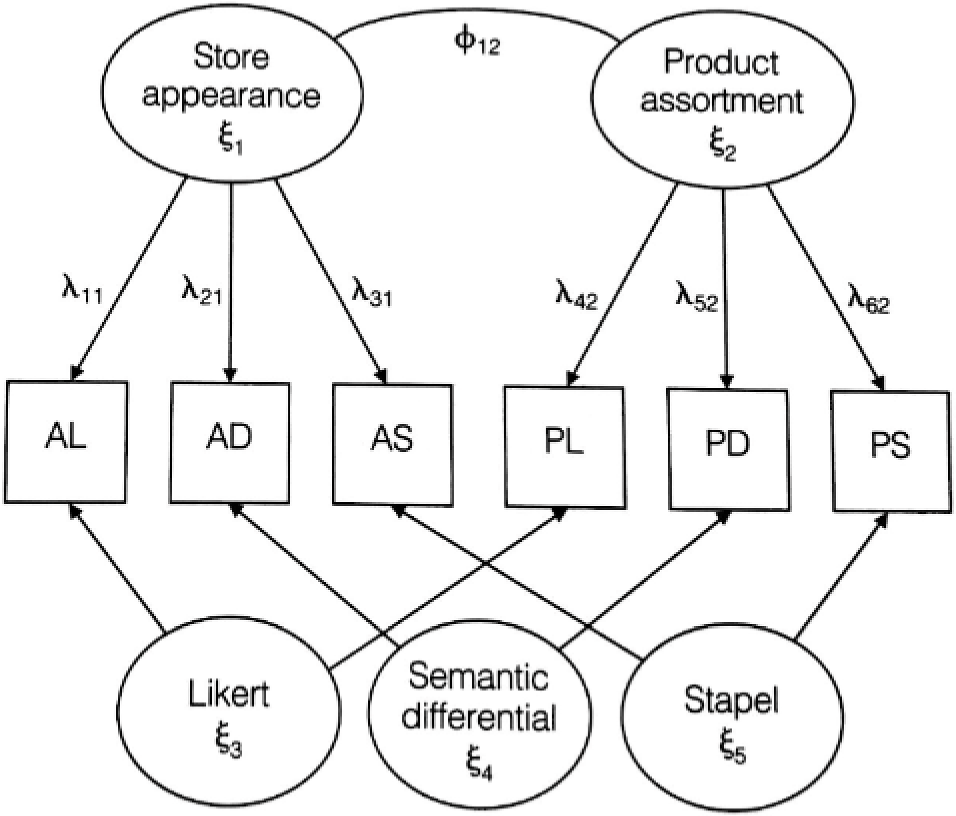
- The residuals are positive between measures of different traits using the same method (e.g., between AL and PL, both of which use the Likert scale) and negative elsewhere.
- What this suggests is that when respondents use the same scale, their ratings are correlated even when different underlying traits are being measured.

TABLE 6.8 Residuals from two-factor model of store appearance and product assortment

	Asymptotically Standardized Residual Matrix					
	AL	*AD*	*AS*	*PL*	*PD*	*PS*
AL	0.0000	0.5168	−1.0905	2.6465	−0.2193	−1.5575
AD	0.5168	0.0000	0.6291	−1.2022	1.6391	−1.9432
AS	−1.0905	0.6291	0.0000	−0.5218	−0.4097	2.1455
PL	**2.6465**	−1.2022	−0.5218	0.0000	−1.1330	0.8058
PD	−0.2193	**1.6391**	−0.4097	−1.1330	0.0000	0.3149
PS	−1.5575	−1.9432	**2.1455**	0.8058	0.3149	0.0000

Going Beyond Simple Factor Structure: 2. Multitrait, Multimethod Model (MTMM)

FIGURE 6.10
Five-factor model of store traits: Two trait factors and three method factors



Going Beyond Simple Factor Structure: 2. Multitrait, Multimethod Model (MTMM)

$$AL = \lambda_{11}\xi_1 + \lambda_{13}\xi_3 + \delta_1$$
$$PL = \lambda_{42}\xi_2 + \lambda_{43}\xi_3 + \delta_4$$

$$corr(AL, PL) = \lambda_{11}\phi_{12}\lambda_{42} + \lambda_{13}\lambda_{43}$$

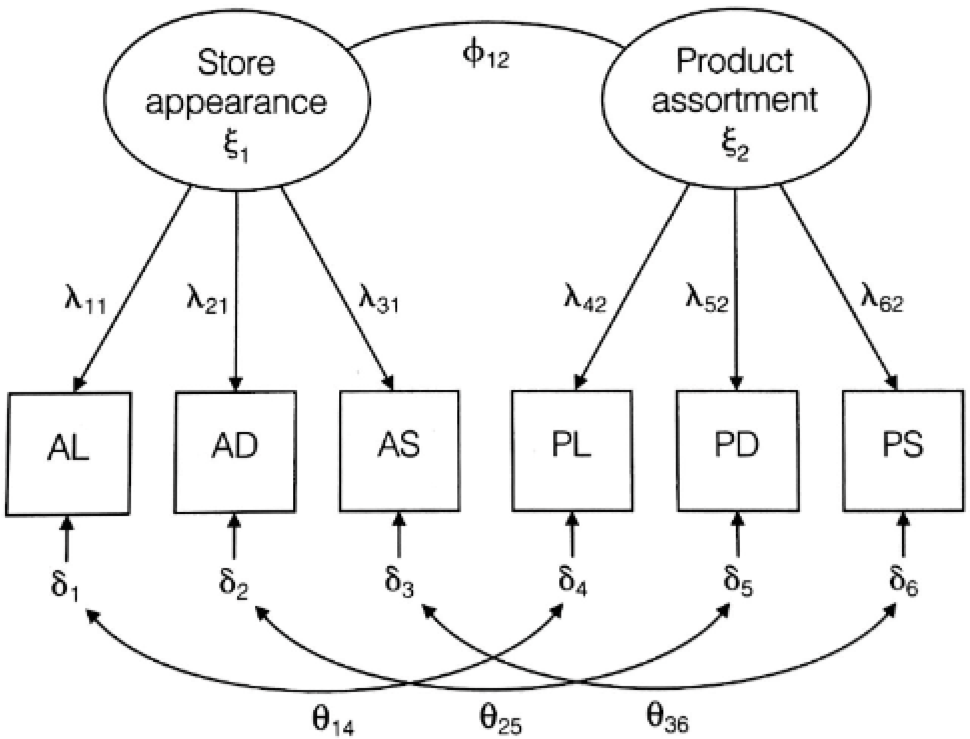
$$\lambda_{13} = \lambda_{43} = \lambda_3$$
$$\lambda_{24} = \lambda_{54} = \lambda_4$$
$$\lambda_{35} = \lambda_{65} = \lambda_5$$

Going Beyond Simple Factor Structure: 2. Multitrait, Multimethod Model (MTMM)

$$\Lambda = \begin{bmatrix} \lambda_{11} & 0 & \lambda_3 & 0 & 0 \\ \lambda_{21} & 0 & 0 & \lambda_4 & 0 \\ \lambda_{31} & 0 & 0 & 0 & \lambda_5 \\ 0 & \lambda_{42} & \lambda_3 & 0 & 0 \\ 0 & \lambda_{52} & 0 & \lambda_4 & 0 \\ 0 & \lambda_{62} & 0 & 0 & \lambda_5 \end{bmatrix}$$

Going Beyond Simple Factor Structure: 3. Correlated Errors

FIGURE 6.11
Two-factor model of store traits with correlated errors



Going Beyond Simple Factor Structure: 3. Correlated Errors

$$\Theta = \begin{bmatrix} \theta_{11}^2 & 0 & 0 & \theta_{14} & 0 & 0 \\ 0 & \theta_{22}^2 & 0 & 0 & \theta_{25} & 0 \\ 0 & 0 & \theta_{33}^2 & 0 & 0 & \theta_{36} \\ \theta_{41} & 0 & 0 & \theta_{44}^2 & 0 & 0 \\ 0 & \theta_{52} & 0 & 0 & \theta_{55}^2 & 0 \\ 0 & 0 & \theta_{63} & 0 & 0 & \theta_{66}^2 \end{bmatrix}$$

TABLE 6.7 Parameter estimates and fit statistics for three models of store appearance and product assortment

	Simple Model	Method Factors	Correlated Error
λ_{11}	.8512 (.0525)	.8443 (.0528)	.8443 (.0528)
λ_{21}	.9088 (.0506)	.9141 (.0508)	.9141 (.0508)
λ_{31}	.8087 (.0539)	.8106 (.0541)	.8106 (.0541)
λ_{42}	.8608 (.0527)	.8581 (.0529)	.8581 (.0529)
λ_{52}	.8387 (.0534)	.8373 (.0540)	.8373 (.0540)
λ_{62}	.8285 (.0537)	.8315 (.0537)	.8315 (.0537)
θ^2_{11}	.2754 (.0343)	.2089 (.0339)	.2848 (.0366)
θ^2_{22}	.1741 (.0302)	.1175 (.0317)	.1617 (.0333)
θ^2_{33}	.3461 (.0385)	.2638 (.0383)	.3504 (.0400)
θ^2_{44}	.2590 (.0356)	.1851 (.0353)	.2610 (.0378)
θ^2_{55}	.2966 (.0373)	.2563 (.0370)	.3005 (.0400)
θ^2_{66}	.3137 (.0382)	.2214 (.0382)	.3080 (.0395)
ϕ_{12}	.7929 (.0322)	.7634 (.0334)	.7634 (.0334)
$\lambda_{.3}$		.2756 (.0477)	
$\lambda_{.4}$		.2104 (.0599)	
$\lambda_{.5}$		.2942 (.0480)	
θ_{14}			.0760 (.0263)
θ_{25}			.0442 (.0252)
θ_{36}			.0660 (.0283)
χ^2 (prob)	33.91 (<.001)	1.74 (0.884)	1.74 (0.884)
GFI	.958	.998	.998
AGFI	.889	.990	.990

Model Validation

- Strictly speaking, a confirmatory factor model (one in which the structure of the model is decided on before looking al the data) does not require validation. However, if the model is changed or adjusted to improve its goodness of fit, then the final version should be retested on holdout data or with data from a new sample.