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Forward and Inverse Kinematics for the Nao Humanoid Robot

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Abstract

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Contents

1 Introduction **1**
1.1 Thesis Constribution 1
1.2 Thesis Outline 2

2 Background **3**
2.1 RoboCup 3
2.1.1 The Standard Platform League 3
2.2 Kouretes Robocup SPL Team 4
2.3 Affine Transformations 6
2.4 Robot Kinematics 10
2.4.1 Forward Kinematics 10
2.4.2 Inverse Kinematics 10
2.5 DH (Denavit Hartenberg) Parameters 11
2.6 Mathematica 12

3 Problem statement **15**
3.1 The NAO Robot 15
3.1.1 NAO Cartesian Restrictions 16
3.2 Kinematics for NAO 21
3.2.1 The Forward Kinematics problem for NAO 21
3.2.2 The Inverse Kinematics problem for NAO 21

4 Related Work **23**
4.1 Aldebaran Forward and Inverse Kinematics Solution 23
4.2 BHuman Inverse Kinematics Solution 24
4.3 QIAU Inverse Kinematics Solution 24

5 Kinematics for NAO: Our approach **25**
5.1 Forward Kinematics for the NAO Robot 25
5.1.1 Forward Kinematics for the Head 28
5.1.2 Forward Kinematics for the Left Arm 28
5.1.3 Forward Kinematics for the Right Arm 29
5.1.4 Forward Kinematics for the Left Leg 30
5.1.5 Forward Kinematics for the Right Leg 30
5.1.6 Forward Kinematics for merged chains 31
5.1.7 Calculation of Center Of Mass with Forward Kinematics . 32
5.2 Inverse Kinematics for NAO 33
5.2.1 Inverse Kinematics for the Head 34
5.2.2 Inverse Kinematics for the Left Hand 35
5.2.3 Inverse Kinematics for the Right Hand 39
5.2.4 Inverse Kinematics for the Left Leg 40
5.2.5 Inverse Kinematics for the Right Leg 45
5.2.6 Undefined Target Points For Legs 46

6 Implementation **49**
6.1 KMat: Kouretes Math Library 49
6.2 Nao Kinematics in c++ 49

7 Results **51**
7.1 Demo Program 51

8 Conclusion **53**
8.1 Future Work 53

References **55**

List of Figures

2.1 RoboCup Standar Platform League 4
2.2 Kouretes team in RoboCup 2012 at Mexico City 5

3.1 Aldebaran NAO V3.3 DOF of Academic Edition 16
3.2 NAO links sizes 17
3.3 Head joints range 18
3.4 Arms joints range 18
3.5 Legs joints range 19

5.1 Torso Frame 26
5.2 Undefined Locus For Legs 42
5.3 Undefined Target Points For Legs 47

Chapter 1

Introduction

NAO is the robot that is using for the soccer games in RoboCup SPL league. In this league all the teams are using the same robot, so the same hardware. Thus, the teams need to implement there own walk, motions and balancing system if they want to be more competitive. The creation of dynamical movements, as the above, are not feasible without an inverse kinematic mechanism. Also, the balancing is impossible without the calculation of the center of mass with forward kinematics. This mechanism must be fast, because the RoboCup is a real-time environment. This thesis describes a forward kinematics solution for every chain of the NAO and an analytical solution for the problem of inverse kinematics without any approximations.

1.1 Thesis Constribution

As it mentioned before, there is a need to know the position of an end effector and the execution of dynamic trajectories. With this thesis we succeed to make a the mechanism that translate the joints of a chain to a Cartesian position for the end effector. Also we created a mechanism that translate dynamic trajectories to joint values in real-time. The contribution of this thesis to our SPL team, Kouretes, is the mechanism of inverse kinematics that make possible the creation of our walk and our kick engine.

1.2 Thesis Outline

Chapter 2 provides a background for the RoboCup competition and the SPL league as long as brief description of the Kouretes SPL team. Furthermore, it describes the affine transformation matrix, DH parameters and robot kinematics. In Chapter 3 we provide a complete description of the hardware of NAO as long as the problem of kinematics for NAO. Next in Chapter 4 we will discuss the related work about kinematics. In Chapter 5 we will describe analytical our solutions to the problem of forward and inverse kinematics. Chapter 6 explains the implementation of kinematics to the team code. In Chapter 7 we will show demos about the implementation of kinematics. Chapter **??** has some interesting ideas for the use of kinematics for the future. Finally in Chapter 8 we will discuss the results of this thesis and we will compare them with other, similar, works

Chapter 2

Background

2.1 RoboCup

The RoboCup competition was initially inspired by Hiroaki Kitano [1] in 1993 and then his idea led to the foundation of the RoboCup Federation. The RoboCup competition has a bold vision:“By the year 2050, to develop a team of fully autonomous humanoid robots that can win against the human world soccer champions”. All the teams that participate in RoboCup have to find solutions to some of the most difficult problems, that the robot community has to solve, in real time (perception, cognition, action, co-ordination). All the competitions in RoboCup (soccer, RoboCup@Home, RoboRescue etc.) are testing the solutions that teams find for the problems above. So far, the researchers that participate in RoboCup have made a lot of progress to solve real-world problems that are presented through the RoboCups competitions.

2.1.1 The Standard Platform League

The Standard Platform League (SPL) is one of the soccer leagues of RoboCup. In this league all the teams use the same robot, Aldebaran-NAO, and they focus only in the software. The teams are prohibited to make any changes to the hardware of the robot, so that everyone use the same platform and compete only in software. The robots are complete autonomous and no human interaction, from the team members, is allowed during the games. The only interaction with the outer world, that robots have, is the data that Game Controller sends, which declare the state of the game. Current the games are conducted in a field with dimensions $4m \times 8m$ [2]. The field is consists from a green carpet with white lines and 2 yellow goals. The appearance of the field is similar with the real soccer field but it is scaled. The ball is a polished orange street hokey ball. Each team consists of four robots and each robot has a band, the color of the band defines the team (blue or pink). The total game time is twenty minutes and it is broken in two half, its of them has a duration of 10 minutes.

Figure 2.1: RoboCup Standar Platform League

2.2 Kouretes Robocup SPL Team

Kouretes is the first RoboCup team founded in Greece at the Tech-nical University of Crete in February 2006. The first participation of the team was in the Technical Challenges of Robocup 2006 in Bremen, Germany, when still the Four-Legged Sony AIBO robots were used. Next year the team participated also in the Four-Legged league of the RoboCup German Open 2007 competition in

Hannover, Germany and ranked in the 7th/8th place. Months later the team's participation in the MSRS Simulation Challenge at RoboCup 2007 in Atlanta led to the placement of the team at the 2nd place worldwide. In RoboCup 2008 in Suzhou, China, Kouretes team participated and won two trophies: 1st place in the SPL-MSRS league and 3rd place in the SPL-NAO league. In 2009 the team participated both at the RoboCup German Open 2009 competition in Hannover and in RoboCup 2009 in Graz, Austria, in the SPL. In 2010 the team participated in the 1st ever RoboCup Mediterranean Open competition in Rome and in RoboCup 2010 in Singapore. In 2011, the team participated in the RoboCup German Open 2011 competition in Megdeburg, in Iran Open 2011 in Tehran and in RoboCup 2011 in Istanbul. Team Kouretes joined Team Noxious from UK to form a joint team for RoboCup 2011. The joint team won the 2nd place in the SPL Open Challenge competition with 139 points, only 3 points behind the top team. In 2012 the team Kouretes participated in the RoboCup German Open 2012 competion in Magdebug, in Iran Open 2011 in Tehran and in RoboCup 2012 in Mexico. The team, in Mexico, has succeed to qualify to the top 16 teams.

Figure 2.2: Kouretes team in RoboCup 2012 at Mexico City

2.3 Affine Transformations

An affine transformation is a map transferring points and vectors from one space to another, being able to preserve ratios of distances. The space can be n-dimensional with $n \geq 2$. The following are affine transformations: Geometric contraction, expansion, dilation, reflection, rotation, shear, similarity transformations, spiral similarities and translation. All the possible combinations of the above produce an affine transformation. Its flexibility concerning object manipulation, makes it a very useful tool in computer graphics.
For the purpose of this thesis we only used rotation and translation, so we will analyze these two types of affine transformation. Also we are working in a $3-\mathit{dimensional}$ space and all the examples from now on will be in this space.

Affine Transformation Matrix

The affine transformation matrix is a $(n+1 \times n+1)$ matrix where n is the number of dimensions. In the general form the affine transformation matrix consists of 2 parts:

$$T = \begin{bmatrix} X & Y \\ \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} & 1 \end{bmatrix}$$

X is a $(n \times n)$ matrix and Y is a $(n \times 1)$ vector and the last line has $n-1$ zeros followed by a 1. The matrix is invertible if and only if X is invertible and the representation of the inverse is:

$$T = \begin{bmatrix} X^{-1} & -X^{-1}Y \\ \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} & 1 \end{bmatrix}$$

Now if we want to apply the transformation, to a given column vector v, we multiply the affine transformation matrix with the given vector:

$$v' = Tv = \begin{bmatrix} X & Y \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

A matrix that is the result of n multiplications between affine transformation matrices is still an affine transformation. So in the following example, given that all the matrices in the right part of the equation are affine transformation matrices then the resulting matrix T will be an affine transformation matrix.

$$T = T_1 T_2 T_3 \cdots T_n$$

Translation Matrix

Translation, in the Euclidean space, is a function that moves every point by constant distance in a specified direction. We can describe the translation in the $3-\textit{dimensional}$ space with a (4×4) table that has the following form:

$$A = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The d_x, d_y, d_z defines the distance that we will move all of our points through the x, y, z dimension. Obviously, this is an affine transformation matrix with $X = I$. So now if we want to apply the translation on a given column vector v, we apply it in same way as the transformation:

$$v' = Tv = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Rotation Matrix

Rotation matrix is a matrix that is used to perform rotation in Euclidean space. The rotation matrices are always orthogonal matrices with determinant 1.

$$R^T = R^{-1}, det(R) = 1$$

In the $3-\textit{dimensional}$ Euclidean space there are three rotation matrices, each one of them makes a rotation over the x or y or z axis. The size of the rotation matrices, for this space, is (3×3) and they have the following form:

$$R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$$

Now if we want to rotate an object point, defined by the (3×1) column vector v, by the x axis and then by the y axis we will do the following:

$$v' = R_x R_y v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

We can use combinations of those 3 matrices and make other useful rotation matrices. For example we can construct the rotation matrix that rotates all the axes with the following order, first the z axis then the y axis and finally the x axis.

$$R = R_z R_y R_x$$

The analytical form of the above rotation matrix:

$$R = \begin{bmatrix} \cos\theta_y \cos\theta_z & -\cos\theta_x \sin\theta_z + \sin\theta_x \sin\theta_y \cos\theta_z & \sin\theta_x \sin\theta_z + \cos\theta_x \sin\theta_y \cos\theta_z \\ \cos\theta_y \sin\theta_z & \cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_y \sin\theta_z & -\sin\theta_x \cos\theta_z + \cos\theta_x \sin\theta_y \sin\theta_z \\ -\sin\theta_y & \sin\theta_x \cos\theta_y & \cos\theta_x \cos\theta_y \end{bmatrix}$$

Finally we can easily transform a rotation matrix to an affine transformation matrix just by padding with zeros and an one in the corner. So from now on whenever we use a rotation matrix, this matrix will have the following form:

$$R' = \begin{bmatrix} R & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

Affine Transformation Matrix in this Thesis

In this thesis we are using the rotation and the translation matrix so we can move points in the $3-dimensional$ space. We can assume that our affine transformation matrix is composed by a rotation and a translation matrix. The affine transformation matrix, that we will work with, has $X = R$ and $Y = v$ where v is the vector with the distance points from the translation matrix.

$$T = \begin{bmatrix} & R & & \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix}$$

Also we can decompose this transformation matrix to a rotation matrix followed by a translation matrix. Given a rotation R and a translation matrix A:

$$R = \begin{bmatrix} & R & & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} & I & & \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix}$$

The result from the multiplication of R with A is:

$$T = \begin{bmatrix} & R & & R \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix}$$

So we can assume that the translation matrix A' is:

$$A' = \begin{bmatrix} & I & & R \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix}$$

Now if we multiply A' with the R we will have the same transformation matrix as before:

$$A'R = \begin{bmatrix} & I & & R \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix} \begin{bmatrix} & R & & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix} = \begin{bmatrix} & R & & R \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & & & 1 \end{bmatrix} = RA$$

2.4 Robot Kinematics

Robot kinematics is the way to apply the geometry to the study of the multi-degree of freedom kinematic chains. Kinematic chain is the assembly of links connected by joints. The degree of freedom (DOF) refers to the number of joints in the kinematic chain. Robot kinematics is the way to go from the *joint* space, where the kinematic chains are defined, to the Euclidean space and vice versa. They are, also, very useful because we can plan and control movement and calculate actuator forces and torques.

Joint Space

Kinematic chain typically is a manipulator that interacts with the environment. The joints are controlling the manipulator. Not all the combinations of all joints positions in the chain are valid, because some combinations lead to collisions between the links of the chain or with some item of the environment. All the valid combinations create the *joint* space.

2.4.1 Forward Kinematics

The *joint* space reveals very little information about the position of the end effector of the kinematic chain. With the forward kinematics we can move from the *joint* space to the $3-dimensional$ space. Given a kinematic chain and all the current joint values, forward kinematics can find the point p_x, p_y, p_z and the orientation a_x, a_y, a_z of the end effector of the kinematic chain. Forward kinematics is not an application specific problem and can be addressed to any kinematic chain.

2.4.2 Inverse Kinematics

Generally it is very easy for humans to give a target point or design a trajectory in $3-dimensional$ space, but if we want the end effector to follow the trajectory, we must assign the right values to the joints of the kinematic chain. So, we need a way to go from $3-dimensional$ space back to *joint* space. The problem of inverse kinematics is application specific and every kinematic chain has a different solution. The solution of the problem can be analytical or it can be approximate (e.g with Jacobian approximation method). As the DOF increases, each point in the $3-\textit{dimensional}$ space may have more than one matching point in the *joint* space, so we may have more than one solution in the *joint* space for a given point in $3-\textit{dimensional}$ space.

2.5 DH (Denavit Hartenberg) Parameters

Denavit and Hartenberg found a way to create a transformation matrix that describes points in one end of the joint to a coordinate system that is fixed to the other end, as a function of the joint state [3] [4]. They found that we can construct this transformation matrix using only 4 parameters, the DH (Denavit Hartenberg) Parameters. These parameters are:

$$a, \alpha, d \text{ and } \theta$$

Before we can explain these parameters we must first establish the reference frame of each joint respectively to the previous joint reference frame.

- The z_n-axis is in the direction of the joint axis.
- The x_n-axis is parallel to the common normal: $x_n = z_n \times z_{n-1}$. The direction of x_n is from z_{n-1} to z_n.
- The y_n-axis follows from the x_n and z_n-axis by choosing it to be a right-handed coordinate system.

Now we can describe the DH parameters as following:

- a: length of the common normal.
- $alpha$: angle about common normal, from z_{n-1}-axis to z_n-axis.
- d: offset along z_{n-1} to the common normal.
- θ: angle about z_{n-1}, from x_{n-1} to x_n.

There is a great video that visualizes how to calculate all the above parameters (http://www.youtube.com/watch?v=rA9tm0gTln8). Now we can go from one reference frame to the other by composing the transformation matrix:

$$DH = R_x(\alpha)T_x(a)R_z(\theta)T_z(d)$$

The analytical form of the resulting matrix from the above composition is:

$$DH = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & a \\ \sin\theta\cos\alpha & \cos\theta\cos\alpha & -\sin\alpha & -d\sin\alpha \\ \sin\theta\cos\alpha & \cos\theta\sin\alpha & \cos\alpha & d\cos\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can notice that the above transformation matrix is an affine transformation matrix because it is the product of the multiplication of affine transformation matrices.

2.6 Mathematica

Mathematica© is a software tool for mathematic computations created by the Wolfram company [link]. It is very useful, because it can find solution to differential equations and can very easily execute symbolic computations. For the purpose of this thesis, we want a software with the capability to perform fast and large symbolic computations with matrices. Also it has the capability to simplify the symbolic results really fast.
The following code is a small example for the symbolic computation. We construct two matrices with cosines and sines and we have two symbols, theta1 and theta2. Next we just do the multiplication between those to matrices and we simplify the result:

```
Matrix1 = {{Cos[theta1], -Sin[theta1]}, {Cos[theta1], -Cos[theta1]}};
Matrix2 = {{Cos[theta2], -Sin[theta2]}, {Cos[theta2], -Cos[theta2]}};
T = Matrix1.Matrix2;
Simplify[T];
MatrixForm[%]
```

The symbolic computations are the following:

$$\text{Matrix1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \cos\theta_1 & -\cos\theta_1 \end{bmatrix}$$

$$\text{Matrix2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \cos\theta_2 & -\cos\theta_2 \end{bmatrix}$$

$$\mathrm{T} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \cos\theta_2\sin\theta_1 & \cos\theta_2\sin\theta_1 - \cos\theta_1\sin\theta_2 \\ 0 & \cos\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 \end{bmatrix}$$

$$\mathrm{T}_{\text{simplified}} = \begin{bmatrix} \cos\theta_2\left(\cos\theta_1 - \sin\theta_1\right) & \sin\left(\theta_1 - \theta_2\right) \\ 0 & \cos\theta_1\left(\cos\theta_2 - \sin\theta_2\right) \end{bmatrix}$$

The example above is very simple, but in our work we worked with larger symbolic matrix and we couldn't work with the tables before the simplified step.

Chapter 3

Problem statement

3.1 The NAO Robot

Aldebaran NAO is a humanoid robot. It is 58 cm tall and it has 5kg approximate mass. The version we are working on is the RoboCup version 3.3 with 21 DOF. It has 2 DOF on the head, 4 on each arm and 5 on each leg and 1 DOF that is common between the two legs. NAO has five kinematic chains and they are the following:

- Head chain: HeadYaw, HeadPitch.
- Left Arm chain: LShoulderPitch, LShoulderRoll, LElbowYaw, LElbowRoll.
- Right Hand chain: RShoulderPitch, RShoulderRoll, RElbowYaw, RElbowRoll.
- Left Leg chain: LHipYawPitch, LHipRoll, LHipPitch, LKneePitch, LAnklePitch, LAnkleRoll.
- Right Leg chain: RHipYawPitch, RHipRoll, RHipPitch, RKneePitch, RAnklePitch, RAnkleRoll.

The common joint is the HipYawPitch joint, so LHipYawPitch and RHipYawPitch are the same joint.
In the figure bellow we are presenting the academic edition of NAO. NAO RoboCup edition misses 4 DOF on the hand (LWristYaw, LHand, RWristYaw and RHand).

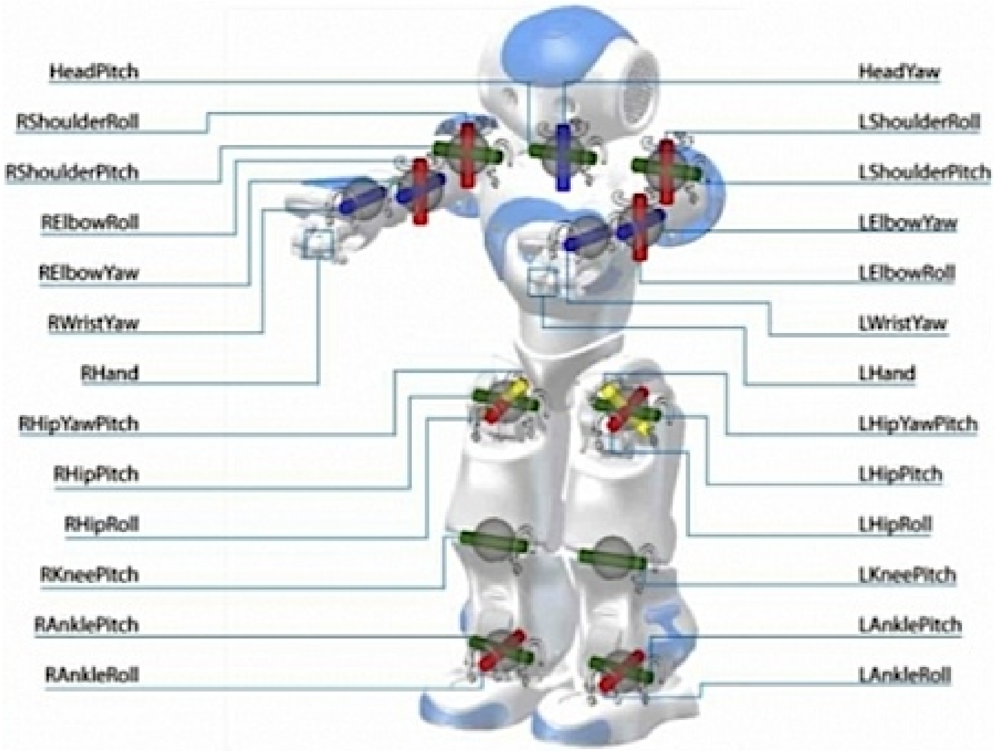
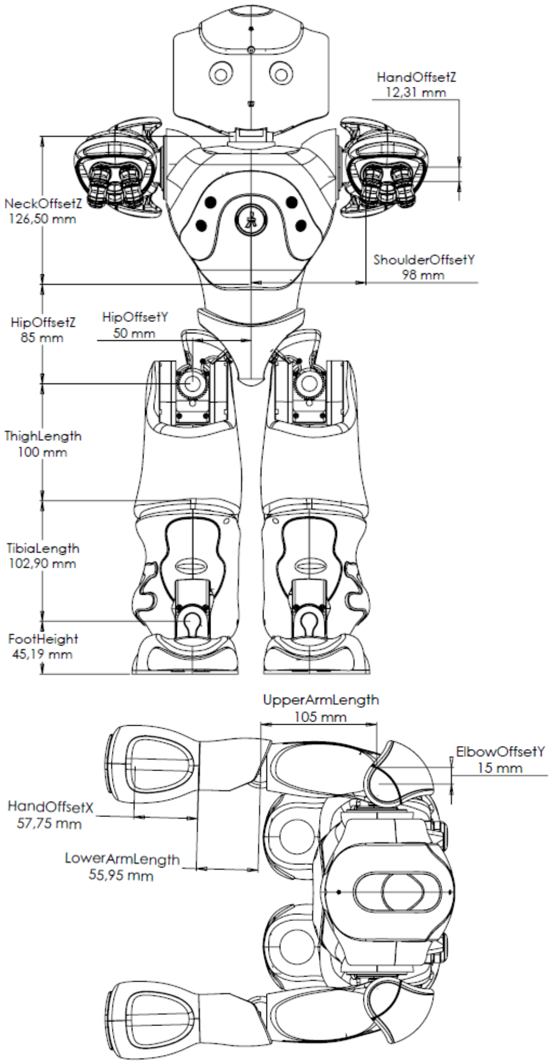


Figure 3.1: Aldebaran NAO V3.3 DOF of Academic Edition

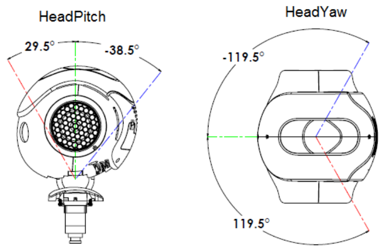
3.1.1 NAO Cartesian Restrictions

In the tables bellow we will present the length of all the links of the robot, the range that each joint is working in radians and in degrees and finally the masses of the NAO. Each joint has its own mass and center of mass. The center of mass for each joint is represented by a point in the $three - dimensional$ space of the joint. The robot theoretically is symmetric but we will see, in the joints ranges, that some left joints have different range than the right ones. The values are extracted from the user manual of the robot that Aldebaran provides.
The user manual has only masses for the right part of the robot but we assume that the robot is fully symmetrical on the masses.
Although some joints appear to be able to go to a position, the hardware controller of the robot will prohibit this movement because of the possible collisions with NAO shell.



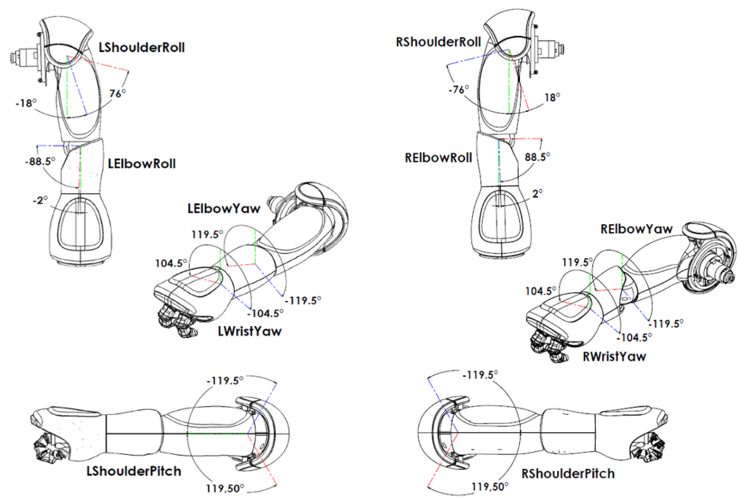
Name	Size (mm)
NeckOffsetZ	126.50
ShoulderOffsetY	98.00
ElbowOffsetY	15.00
UpperArmLength	105.00
LowerArmLength	55.95
ShoulderOffsetZ	100.00
HandOffsetX	57.75
HipOffsetZ	85.00
HipOffsetY	50.00
ThighLength	100.00
TibiaLength	102.90
FootHeight	45.19
HandOffsetZ	12.31

Figure 3.2: NAO links sizes



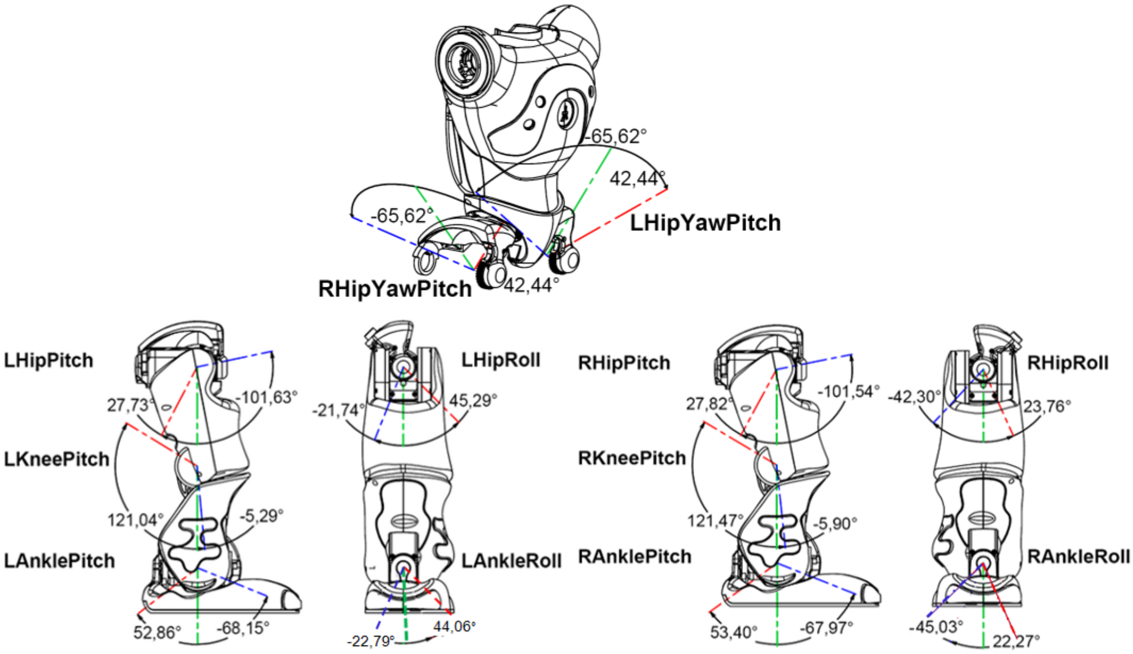
Joint Name	Range in Degrees°	Range in Radians
HeadYaw	-119.5° to 119.5°	-2.0857 to 2.0857
HeadPitch	-38.5° to 29.5°	-0.6720 to 0.5149

Figure 3.3: Head joints range



Joint Name	Range in Degrees°	Range in Radians
LShoulderPitch	-119.5° to 119.5°	-2.0857 to 2.0857
LShoulderRoll	-18° to 76°	-0.3142 to 1.3265
LElbowYaw	-119.5° to 119.5°	1.5446 to 0.0349
LElbowRoll	-88.5° to -2°	-0.6720 to 0.5149
RShoulderPitch	-119.5° to 119.5°	-2.0857 to 2.0857
RShoulderRoll	-38.5° to 29.5°	-1.3265 to 0.3142
RElbowYaw	-119.5° to 119.5°	-2.0857 to 2.0857
RElbowRoll	-38.5° to 29.5°	0.0349 to 1.5446
LWristYaw and RWristYaw	disabled	disabled

Figure 3.4: Arms joints range



Joint Name	Range in Degrees°	Range in Radians
LHipYawPitch-RHipYawPitch	-65.62 to 42.44	-1.145303 to 0.740810
LHipRoll	-21.74° to 45.29°	-0.379472 to 0.790477
LHipPitch	-101.63° to 27.73°	-1.773912 to 0.484090
LKneePitch	-5.29° to 121.04°	-0.092346 to 2.112528
LAnklePitch	-68.15° to 52.86°	-1.189516 to 0.922747
LAnkleRoll	-22.79° to 44.06°	-0.397880 to 0.769001
RHipRoll	-42.30° to 23.76°	-0.738321 to 0.414754
RHipPitch	-101.54° to 27.82°	-1.772308 to 0.485624
RKneePitch	-5.90° to 121.47°	-0.103083 to 2.120198
RAnklePitch	-67.97° to 53.40°	-1.186448 to 0.932056
RAnkleRoll	-45.03° to 22.27°	-0.785875 to 0.388676

Figure 3.5: Legs joints range

Table 3.1: Masses of NAO

Masses for NAO v3.3 robocup edition (H21)				
Total Mass For NAO H21				4.879 kg
Frame Name	**Mass (Kg)**	**CoM$_x$ (mm)**	**CoM$_y$ (mm)**	**CoM$_z$ (mm)**
Torso	1.03948	-4.15	0.07	42.58
HeadYaw	0.05930	-0.02	0.17	25.56
HeadPitch	0.52065	1.2	-0.84	53.53
RShoulderPitch	0.06996	-1.78	24.96	0.18
RShoulderRoll	0.12309	18.85	-5.77	0.65
RElbowYaw	0.05971	-25.6	0.01	-0.19
RElbowRoll	0.185	65.36	-0.34	-0.02
LShoulderPitch	0.06996	-1.78	-24.96	0.18
LShoulderRoll	0.12309	18.85	5.77	0.65
LElbowYaw	0.05971	-25.6	-0.01	-0.19
LElbowRoll	0.185	65.36	0.34	-0.02
RHipYawPitch	0.07117	-7.66	12	27.17
RHipRoll	0.1353	-16.49	-0.29	-4.75
RHipPitch	0.39421	1.32	-2.35	-53.52
RKneePitch	0.29159	4.22	-2.52	-48.68
RAnklePitch	0.13892	1.42	-0.28	6.38
RAnkleRoll	0.16175	25.4	-3.32	-32.41
LHipYawPitch	0.07117	-7.66	-12	27.17
LHipRoll	0.1353	-16.49	0.29	-4.75
LHipPitch	0.39421	1.32	2.35	-53.52
LKneePitch	0.29159	4.22	2.52	-48.68
LAnklePitch	0.13892	1.42	0.28	6.38
LAnkleRoll	0.16175	25.4	3.32	-32.41

3.2 Kinematics for NAO

Because NAO has a lot of DOF, it can do several complex moves. Some examples of those complex moves are walking, kicking a ball, standing up etc. NAO has five kinematic chains, three of them completely independent and two of them with one common joint. Kinematics is very useful for the programmers of NAO because they can create dynamic movements with the use of inverse kinematics or they can find the horizon of the camera on the head of the robot using forward kinematics.

3.2.1 The Forward Kinematics problem for NAO

Forward kinematics for NAO can been seen as five individual solutions. NAO has five chains and because we will not manipulate the joints, but we will only take the current state of each joint, we can assume that the five chains are completely independent. Then, we can find five forward kinematics solutions, one for each kinematic chain and we will be able to combine these solutions to find a solution for a bigger kinematic chain (e.g. the kinematic chain from the right foot to the head). The reason for dealing with this problem is twofold:firstly it is of great importance (concerning many applications, such as, finding its horizon, as well as placing it correctly).Besides this, as it will be described below, solution to the inverse kinematics problem would be intractable without solving the forward first.

3.2.2 The Inverse Kinematics problem for NAO

The inverse kinematics problem is a more complex problem and although we have the common joint between two kinematic chains, we are working with the assumption that we have five completely independent kinematic chains, so we will find five solutions for this problem. The reasons that led us to solve it are many. More specifically, we need to create dynamic kicks and an omni-directional walk. Both problems need a mechanism to move trajectories from the $three-dimensional$ space to the $joint$ space and the inverse kinematics can provide this mechanism.

Analytic versus Approximate solutions

Inverse kinematics problem can be solved analytically or approximately. We have chosen to find the analytical solution rather than the approximate solution because the second one has some problems. We need real-time execution to the problem of inverse kinematics. The analytical solution is faster than the fastest approximate solution and for this reason it is a good choice for real-time execution. Second, there are various implementations of approximate solutions for the problem. Some of them are fast but it is very possible to run into a singularity. Other methods have a smaller probability to run into a singularity, but they are slower. On the other hand, the analytical solution doesn't have any singularity. However, inverse kinematics have an analytical solution only if the chain has five or less DOF. If the chain has 6 DOF, it can have analytical solution to the problem of the inverse kinematics only if three sequential joints have intersecting axes.

Chapter 4

Related Work

The problem of forward and inverse kinematics is a familiar problem for all the teams that participate in the RoboCup SPL. The solution to the problem of forward kinematics is very easy and all the teams have implemented their own. There are not too many known solutions for the problem of inverse kinematics for NAO robot. On the other hand, Aldebaran provides an approximate solution for this problem but as we will see below we can't use this solution in our approach. Also, some teams have published their own analytical solutions.

4.1 Aldebaran Forward and Inverse Kinematics Solution

Aldebaran Forward Kinematics Solution

Aldebaran provides a forward kinematics mechanism but the problem is that it provides the solution only for the current state of the robot. So you can't find the position of the camera at the time a specific picture was taken given the joints values. Also, as we said before, we must find the solution for this problem if we want to solve inverse kinematics problem. However, Aldebaran provides us with the DH parameters for all the joints of the robot and that was very useful.

Aldebaran Inverse Kinematics Solution

Aldebaran has implemented in the API of the robot some functions that move an end effector to a given point in the $3-dimensional$ space. This functions are using the Jacobian approximate method to find the solution to the problem of inverse kinematics. The omni-directional walk that Aldebaran provide us with, uses this solution to execute the trajectories. Although this solution is accurate, it can easily fall into a singularity and if this happens, it will stuck in there. That is a very bad problem because then the whole motion of the robot gets stuck.

4.2 BHuman Inverse Kinematics Solution

B-Human is a RoboCup SPL team from the university of Bremen in Germany. Every year they publish the code release [5], the code that they used in the last RoboCup and a documentation for this code. In the code release they have an inverse kinematics solution for the legs of NAO but with an approximation. The solution provided, always makes the end effector parallel to the plane of the torso, that is defined by the z and x axis. So this approximation supplies us with a solution that moves the end effector to the target point but with different orientation from the target orientation.

4.3 QIAU Inverse Kinematics Solution

MRL SPL team is a team from the university of QIAU in Teheran. They have a paper [6] at `www.academia.edu` in which they present a solution that theoretically solves the problem of inverse kinematics for the legs. We have tried to implement this solution, but the results were not satisfactory.

Chapter 5

Kinematics for NAO: Our approach

As we mentioned before, there are already solutions for both problems but none of them is completely suitable for our needs. We want to be able to find a solution to the forward kinematics problem with any joint values for input and not only with the current values of the joint of NAO. Also, we need a real-time analytical solution for the problem of inverse kinematics without any approximations. Below we will describe our solutions for both of these problems.

5.1 Forward Kinematics for the NAO Robot

Aldebaran provides us with the DH parameters for each kinematic chain of the robot. The problem is that for some chains the given parameters are incorrect. More specifically, the parameters for the NAO arms are incorrect, so we found our own parameters for the arms and we used the parameters that Aldebaran gave us for the legs and the head.

NAO Zero Position

We must define the base frame of the NAO and the zero position before we continue to the solution. First, the base frame is the torso frame and in the figure 5.1 we can see the axis of this frame. In this figure we can also see the

Aldebaran's zero position, that is different than the zero position that we are using. Our zero position is with the arms down and not extended.

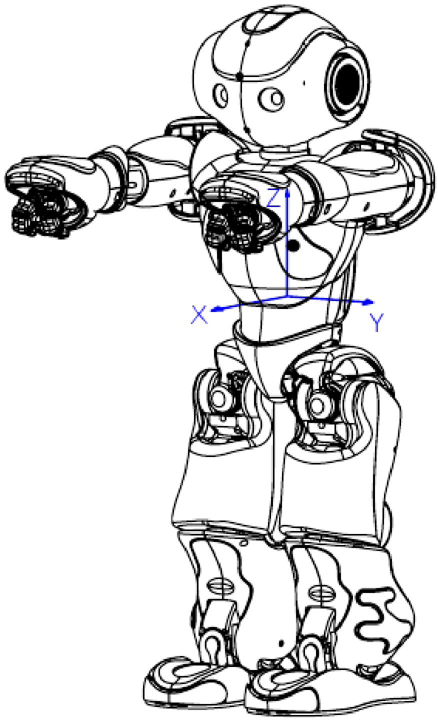


Figure 5.1: Torso Frame

Symbols

We are going to give a simple explanation to the symbols that we are using for our math calculations. First of all, all the matrices that we are using are affine transformation matrices. Second, we have three types of matrices: T, R, A. T is the transformation matrix, R_x, R_y, R_z are the basic rotations matrices and A is the translation matrix. The subscript of the symbol refers to the start frame and the superscript refers to the destination frame. The torso is the point that all the kinematic chains start and it is located on the center of the body of NAO. 'Base' will be the start frame and so will be the torso, and 'End' will be the end effector. The numbers will refer to one of the joints of the kinematic chain, as we see in the tables in Chapter 3. Also, we define the initialization of the translation matrix as: $A(d_x, d_y, d_z)$ and for the rotation matrices as: $R_x(\theta_x)$ or $R_y(\theta_y)$ or $R_z(\theta_z)$. We will present the DH parameters in tables and except from the DH parameters, in the tables we will have the translations from the base to the first joint and from the last joint to the end effector. Also, we will have some necessary rotations to adjust the frame of the last joint to the frame of the end effector.

Forward Kinematics equations

Forward kinematics for each chain of the NAO robot is an equation that transforms a point from the frame of the last joint to the base frame. In our case we will have an end effector that will be the point of interest. So, we can construct these equations with the use of transformation, rotation and translation matrices.

Extracting the position

As we will see, the result of forward kinematics is an affine transformation matrix with X submatrix to be a rotation matrix and Y a translation vector. We need to extract the p_x, p_y and p_z points and the a_x, a_y and a_z angles of the final position. So, we can extract p_x, p_y and p_z from the translation part of the matrix:

$$
\begin{aligned}
p_x &= T_{(1,4)} \\
p_y &= T_{(2,4)} \\
p_z &= T_{(3,4)}
\end{aligned}
$$

Now we must extract the a_x, a_y and a_z from the rotation table. The rotation of the final transformation table is a $R_z R_y R_z$ rotation table. In chapter '2' we present the analytical form of this table. Now it's easy to extract the angles:

$$
\begin{aligned}
a_x &= \arctan 2\left(T_{(3,2)}, T_{(3,3)}\right) \\
a_y &= \arctan 2\left(-T_{(3,1)}, \sqrt{T_{(3,2)}^2 + T_{(3,3)}^2}\right) \\
a_z &= \arctan 2\left(T_{(2,1)}, T_{(1,1)}\right)
\end{aligned}
$$

5.1.1 Forward Kinematics for the Head

The head of NAO is the simplest kinematic chain but, it has two possible end effectors, the top and the bottom camera position on the head. As we will see in our solution, it is easy to change the end effector by just changing the last translation matrix. Below, there is the table with all the DH parameters for this chain and the two possible end effectors.

Frame (Joint)	**a**	$\boldsymbol{\alpha}$	**d**	$\boldsymbol{\theta}$
Base	$A(0, 0, \text{NeckOffsetZ})$			
HeadYaw	0	0	0	θ_1
HeadPitch	0	$-\frac{\pi}{2}$	0	$\theta_2 - \frac{\pi}{2}$
Rotation	$R_x(\frac{\pi}{2})R_y(\frac{\pi}{2})$			
Top Camera	$A(\text{topCameraX}, 0, \text{topCameraZ})$			
Bottom Camera	$A(\text{bottomCameraX}, 0, \text{bottomCameraZ})$			

Table 5.1: DH parameters for Head chain

Now we can combine the tables and find the point of the end effector in the frame space of the torso:

$$T_{\text{Base}}^{\text{End}} = A_{\text{Base}}^{0} T_0^1 T_1^2 R_x(\tfrac{\pi}{2}) R_y(\tfrac{\pi}{2}) A_2^{\text{End}}$$

The DH transformation matrices are T_0^1 and T_1^2. The translation of the end effector can be replaced by one of the two possible translation matrices. Now we have the position of the end effector in the *three − dimensional* space of the torso.

5.1.2 Forward Kinematics for the Left Arm

The kinematic chain for the left arm consists of four joints. Below we can see the table with the DH parameters for all joints and with the necessary translations and rotations. Now we can easily calculate the final transformation table:

$$T_{\text{Base}}^{\text{End}} = A_{\text{Base}}^{0} T_0^1 T_1^2 T_2^3 T_3^4 R_z(\tfrac{\pi}{2}) A_4^{\text{End}}$$

Frame (Joint)	**a**	$\boldsymbol{\alpha}$	**d**	$\boldsymbol{\theta}$
Base	$A(0,$ ShoulderOffsetY+ElbowOffsetY, ShoulderOffsetZ$)$			
LShoulderPitch	0	$-\frac{\pi}{2}$	0	θ_1
LShoulderRoll	0	$\frac{\pi}{2}$	0	$\theta_2 - \frac{\pi}{2}$
LElbowYaw	0	$-\frac{\pi}{2}$	UpperArmLength	θ_3
LElbowRoll	0	$\frac{\pi}{2}$	0	θ_4
Rotation	$R_z(\frac{\pi}{2})$			
End effector	$A($HandOffsetX+LowerArmLength$, 0, 0)$			

Table 5.2: DH parameters for Left Arm chain

The DH transformation matrices are the T_0^1, T_1^2, T_2^3 and T_3^4. Now we have the transformation table for the position of the end effector of the left arm relatively to the frame of the torso.

5.1.3 Forward Kinematics for the Right Arm

The kinematic chain of the right arm is symmetric with the chain of the left arm relatively to the plain defined by x and z-axis. So, the differences with the left arm will be only in the distances along y-axis and in the joints that move the y-axis (angle roll joints). Also, in this chain we must add one extra rotation matrix after the final translation, because the z-axis is inverted.

Frame (Joint)	**a**	$\boldsymbol{\alpha}$	**d**	$\boldsymbol{\theta}$
Base	$A(0,$ -ShoulderOffsetY - ElbowOffsetY, ShoulderOffsetZ$)$			
RShoulderPitch	0	$-\frac{\pi}{2}$	0	θ_1
RShoulderRoll	0	$\frac{\pi}{2}$	0	$\theta_2 + \frac{\pi}{2}$
RElbowYaw	0	$-\frac{\pi}{2}$	- UpperArmLength	θ_3
RElbowRoll	0	$\frac{\pi}{2}$	0	θ_4
Rotation	$R_z(\frac{\pi}{2})$			
End effector	$A($-HandOffsetX-LowerArmLength$, 0, 0)$			
Rotation$'$	$R_z(-\pi)$			

Table 5.3: DH parameters for Right Arm chain

$$T_{\text{Base}}^{\text{End}} = A_{\text{Base}}^{0} T_0^1 T_1^2 T_2^3 T_3^4 R_z(\tfrac{\pi}{2}) A_4^{\text{End}} R_z(-\pi)$$

The DH transformation matrices are the T_0^1, T_1^2, T_2^3 and T_3^4. Now we have the transformation table for the position of the end effector of the right arm relatively to the frame of the torso.

5.1.4 Forward Kinematics for the Left Leg

The kinematic chain for the left leg has six joints and it is the biggest chain on the NAO. The DH parameters of these joints are slightly different because of the YawPitch joint.

Frame (Joint)	**a**	$\boldsymbol{\alpha}$	**d**	$\boldsymbol{\theta}$
Base	$A(0, \text{HipOffsetY}, \text{-HipOffsetZ})$			
LHipYawPitch	0	$-\frac{3\pi}{4}$	0	$\theta_1 - \frac{\pi}{2}$
LHipRoll	0	$-\frac{\pi}{2}$	0	$\theta_2 + \frac{\pi}{4}$
LHipPitch	0	$\frac{\pi}{2}$	0	θ_3
LKneePitch	-ThighLength	0	0	θ_4
LAnklePitch	-TibiaLength	0	0	θ_5
LAnkleRoll	0	$-\frac{\pi}{2}$	0	θ_6
Rotation	$R_z(\pi)R_y(-\frac{\pi}{2})$			
End effector	$A(0, 0, \text{-FootHeight})$			

Table 5.4: DH parameters for Left Leg chain

$$T_{\text{Base}}^{\text{End}} = A_{\text{Base}}^{0} T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 R_z(\pi) R_y(-\tfrac{\pi}{2}) A_6^{\text{End}}$$

The DH transformation matrices are the $T_0^1, T_1^2, T_2^3, T_3^4, T_4^5$ and T_5^6. Now we have the transformation table for the position of the end effector of the left leg relatively to the frame of the torso.

5.1.5 Forward Kinematics for the Right Leg

As for the hands, the kinematic chains for the legs are symmetric relatively to the plain defined by x and z-axis. So, the differences of the chain for the right leg will be only in the distances along y-axis and in the joints that move the y-axis.

Frame (Joint)	**a**	$\boldsymbol{\alpha}$	**d**	$\boldsymbol{\theta}$
Base	$A(0, \text{-HipOffsetY}, \text{-HipOffsetZ})$			
RHipYawPitch	0	$-\frac{\pi}{4}$	0	$\theta_1 - \frac{\pi}{2}$
RHipRoll	0	$-\frac{\pi}{2}$	0	$\theta_2 - \frac{\pi}{4}$
RHipPitch	0	$\frac{\pi}{2}$	0	θ_3
RKneePitch	$-ThighLength$	0	0	θ_4
RAnklePitch	$-TibiaLength$	0	0	θ_5
RAnkleRoll	0	$-\frac{\pi}{2}$	0	θ_6
Rotation	$R_z(\pi)R_y(-\frac{\pi}{2})$			
End effector	$A(0, 0, \text{-FootHeight})$			

Table 5.5: DH parameters for Right Leg chain

$$T_{\text{Base}}^{\text{End}} = A_{\text{Base}}^{0} T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 R_z(\pi) R_y(-\tfrac{\pi}{2}) A_6^{\text{End}}$$

The DH transformation matrices are the $T_0^1, T_1^2, T_2^3, T_3^4, T_4^5$ and T_5^6. Now we have the transformation table for the position of the end effector of the right leg relatively to the frame of the torso.

5.1.6 Forward Kinematics for merged chains

We have found the solution only for the chains that have as base frame the torso frame. In the real word, someone maybe wants to find the torso position relatively to one of the feet chains. If we get the final transformation table for, e.g., the left leg, we know that it is an affine transformation matrix and because of that, we can invert the table and then we will have the position of the torso relatively to the frame of the left leg.

$$T_{\text{Lleg}}^{\text{Torso}} = \left(T_{\text{Torso}}^{\text{Lleg}}\right)^{-1}$$

Also, it is possible to find the position of the head relatively to the left leg. The kinematic chains for the head and for the left leg are relative to the torso frame.

So, if we invert the chain of the left leg, we will have the torso relatively to the left leg frame. Then we can just multiply the transformation table that has the position of the head relatively to the torso frame and we will have the position of the head relatively to the left leg frame.

$$T_{\text{Lleg}}^{\text{Head}} = \left(T_{\text{Torso}}^{\text{Lleg}}\right)^{-1} T_{\text{Torso}}^{\text{Head}}$$

This property is very useful, because we can find any end effector relatively to any other end effector. For example, with this, we can find the height of the camera from the ground.

5.1.7 Calculation of Center Of Mass with Forward Kinematics

The calculation of the center of mass (CoM) is very important because NAO consists of a group of moving parts. Every moving part has a mass and with forward kinematics we can calculate the position of the CoM relatively to a given frame. Aldebaran provides us with the whole information that is needed to do the calculation. Aldebaran gives us the mass of the whole robot and the mass for every moving part of NAO. The separate masses are given according to every joint of the robot. Thus, every joint of the robot has a mass and the position of the CoM for this joint relatively to the joint frame.

The CoM is calculated relatively to the torso frame and the calculation order is simple. We construct smaller kinematic chains that stop to an earlier joint, and then we set as the end effector the position of CoM for the frame of the joint where we stop. Then we get the translation part and we multiply it by the mass of the certain part. At the end, we will have 21 chains plus the torso chain. Then we will add all the individual translation matrices that have been weighted by the mass of each joint and the result will be divided by the total mass. The result will be the position of the CoM relatively to the torso frame.

5.2 Inverse Kinematics for NAO

Forward kinematics can find the position of an end effector, relatively to the start frame, given the joints values. Now we have to solve the inverse problem: find the joints values given the desired end effector position relatively to the torso frame. The solutions for the inverse kinematic problem that will be present below is only for the kinematic chains that start from the torso.
The position of the end effector is the p_x, p_y and p_z points along with the a_x, a_y and a_z angles. As we mentioned before, the product of forward kinematics is an affine transformation matrix and it consists of a rotation matrix and a translation matrix. The rotation R is equal to $R_z R_y R_x$. Thus, we can construct the transformation table:

$$T = \begin{bmatrix} \cos a_y \cos a_z & -\cos a_x \sin a_z + \sin a_x \sin a_y \cos a_z & \sin a_x \sin a_z + \cos a_x \sin a_y \cos a_z & p_x \\ \cos a_y \sin a_z & \cos a_x \cos a_z + \sin a_x \sin a_y \sin a_z & -\sin a_x \cos a_z + \cos a_x \sin a_y \sin a_z & p_y \\ -\sin a_y & \sin a_x \cos a_y & \cos a_x \cos a_y & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As we mentioned before we can't solve the problem of inverse kinematics without the solution for forward kinematics. That's why the equations that we must solve to find the joints are the matrices from the forward kinematics but with the θ DH parameters as the unknown part. So, we can find the symbolic table that is the product of forward kinematics and equate it with the matrix above. Then, we will have twelve equations (we don't have sixteen because the last line of the matrix is always (0 0 0 1)) and n unknowns, where n is the number of joints in the chain. In fact, we will have $2n$ unknowns, because all the θ appear inside sin or cos.
As we will see below we, found some of the desired angles with arccos and arcsin. The problem is that arcsin returns a number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and arccos returns a number in $[0, \pi]$ but the possible range of the joints is in $[-\pi, \pi]$. Thus, we will have two possible solutions for every arcsin and arccos (because of the complimentary angles). Because of that, we must filter our results and keep only the correct angles. We archive that by doing a forward validation step when we find a complete set of angle values. Then we compare the result of the validation step with the reconstructed table and if they are equal, then we approve the set. Some

complimentary angles will be discarded a long time before the validation step due to the range restriction of each joint.

5.2.1 Inverse Kinematics for the Head

The head chain, as we said before, consists only of two joints. Thus, we can find both desired angles only by the a_z and a_y. On the other hand, someone maybe want to find the joints for the desired position only with the use of the target points p_x, p_y and p_z, so we implement an other solution that works only with the target points. Below we can see the symbolic result from forward kinematics:

$$T = \begin{bmatrix} -\cos\theta_1 \sin\theta_2 & -\sin\theta_1 & \cos\theta_1 \cos\theta_2 & l_2 \cos\theta_1 \cos\theta_2 - l_1 \cos\theta_1 \sin\theta_2 \\ -\sin\theta_1 \sin\theta_2 & \cos\theta_1 & \cos\theta_2 \sin\theta_1 & l_2 \cos\theta_2 \sin\theta_1 - l_1 \sin\theta_1 \sin\theta_2 \\ -\cos\theta_2 & 0 & -\sin\theta_2 & l_3 - l_2 \sin\theta_2 - l_1 \cos\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where: $l_1 = \text{cameraX}$, $l_2 = \text{cameraZ}$ and $l_3 = \text{NeckOffsetZ}$.
Because we only know the p_x, p_y and p_z and we can't reconstruct the rotation part of the matrix and we will use only the translation part that is reconstructible. Now we can see from the symbolic matrix that $T_{(3,4)} = l_3 - l_2 \sin\theta_2 - l_1 \cos\theta_2 = pz$ and we know from trigonometry that:

$$a \sin\theta + b \cos\theta = \sqrt{a^2 + b^2} \sin(\theta + \psi)$$

$$\psi = \arctan\left(\frac{b}{2}\right) + \begin{cases} 0 & \text{if } a \geq 0 \\ \pi & \text{if } a < 0 \end{cases}$$

So, we can now calculate θ_2:

$$\theta_2 = \arcsin\left(\frac{-p_z + l_3}{\sqrt{l_1^2 + l_2^2}}\right) + \arctan\left(\frac{l_1}{l_2}\right)$$

Now the final θ_2 is: $\theta_2 = \theta_2 - \frac{\pi}{2}$ because in the DH parameters we had added $\frac{\pi}{2}$ to the θ parameter for the second joint. From now on, we must substract $\frac{\pi}{2}$ from θ_2 if we want to use it to calculate a result from the equations.
Then we can easily extract θ_1 from $T_{(1,4)}$:

$$\theta_1 = \arccos\left(\frac{p_x}{l_1\left(\theta_2 + \frac{\pi}{2}\right) - l_2 \cos\left(\theta_2 + \frac{\pi}{2}\right)}\right)$$

Finally, we must filter our results because of arccos and arcsin. So, we will do a forward kinematics validation set and we will discard any set of values that doesn't move the end effector to the correct position.

So, the final inverse kinematics equations for the head are the following:

$$\theta_2 = \arcsin\left(\frac{-p_z + l_3}{\sqrt{l_1^2 + l_2^2}}\right) + \frac{\pi}{2}$$
$$\theta_1 = \arccos\left(\frac{p_x}{l_2 \cos\left(\theta_2 - \frac{\pi}{2}\right) - l_1 \sin\left(\theta_2 - \frac{\pi}{2}\right)}\right)$$

or only with angles

$$\theta_1 = a_z$$
$$\theta_2 = a_y$$

5.2.2 Inverse Kinematics for the Left Hand

The left arm chain is far more complicated than the head chain. First, we must construct the symbolic matrix:

$$T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = \sin\theta_4 (\sin\theta_1 \sin\theta_3 - \cos\theta_1 \cos\theta_2 \cos\theta_3) - \cos\theta_1 \cos\theta_4 \sin\theta_2$$

$$r_{12} = \cos\theta_4 (\sin\theta_1 \sin\theta_3 - \cos\theta_1 \cos\theta_2 \cos\theta_3) + \cos\theta_1 \sin\theta_2 \sin\theta_4$$

$$r_{13} = \cos\theta_3 \sin\theta_1 + \cos\theta_1 \cos\theta_2 \sin\theta_3$$

$$r_{14} = l_4 (\sin\theta_4 (\sin\theta_1 \sin\theta_3 - \cos\theta_1 \cos\theta_2 \cos\theta_3) - \cos\theta_1 \cos\theta_4 \sin\theta_2) - l_3 \cos\theta_1 \sin\theta_2$$

$$r_{21} = \cos\theta_2 \cos\theta_4 - \cos\theta_3 \sin\theta_2 \sin\theta_4$$

$$r_{22} = \cos\theta_2 \sin\theta_4 - \cos\theta_3 \cos\theta_4 \sin\theta_2$$

$$r_{23} = \sin\theta_2 \sin\theta_3$$

$$r_{24} = l_1 + l_3 \cos\theta_2 + l_4 (\cos\theta_2 \cos\theta_4 - \cos\theta_3 \sin\theta_2 \sin\theta_4)$$

$$r_{31} = \sin\theta_4 \left(\cos\theta_1 \sin\theta_3 + \cos\theta_2 \cos\theta_3 \sin\theta_1\right) + \cos\theta_4 \sin\theta_1 \sin\theta_2$$
$$r_{32} = \cos\theta_4 \left(\cos\theta_1 \sin\theta_3 + \cos\theta_2 \cos\theta_3 \sin\theta_1\right) - \sin\theta_1 \sin\theta_2 \sin\theta_4$$
$$r_{33} = \cos\theta_1 \cos\theta_3 - \cos\theta_2 \sin\theta_1 \sin\theta_3$$
$$r_{34} = l_2 + l_4 \left(\sin\theta_4 \left(\cos\theta_1 \sin\theta_3 + \cos\theta_2 \cos\theta_3 \sin\theta_1\right) + \cos\theta_4 \sin\theta_1 \sin\theta_2\right) + l_3 \sin\theta_1 \sin\theta_2$$

where: l_1 = ShoulderOffsetY+ElbowOffsetY, l_2 = ShoulderOffsetZ, l_3 = UpperArmLength and l_4 = HandOffsetX+LowerArmLength.
As we can see now, the problem of inverse kinematics for the arms is far more complicated than the problem for the head. It is very difficult to extract a joint value from all these equations, so, we can use trigonometry to find the value of one joint. More specifically this joint is the elbow roll joint. We can observe that the upper arm, the lower arm with the hand offset and the distance from the position of the shoulder pitch joint to the end effector is a triangle and we know the size of each side. The position of the shoulder pitch joint relative to the torso frame is known, and the position of the end effector is the target position. So, the distance can be calculated:

$$d = \sqrt{\left(s_x + p_x\right)^2 + \left(s_y + p_y\right)^2 + \left(s_z + p_z\right)^2}$$

where: $s_x = 0, s_y = l_1$ and $s_z = l_2$.
Now we can use the cosine law and find θ_4:

$$\theta_4 = \arccos\left(\frac{{l_3}^2 + {l_4}^2 - d^2}{2 l_3 l_4}\right)$$

Because θ_4 represents an interior angle and the elbow roll joint is being stretched in the zero-position, the resulting angle is computed by:

$$\theta_4 = \pi - \theta_4$$

The next angle that we can find is the θ_2, so we can look for equations from the table symbolic matrix, where we only have θ_2 and one more unknown θ. Using r_{22} we have:

$$T_{(2,2)} = -\cos\theta_2 - \cos\theta_3 \cos\theta_4 \sin\theta_2 \qquad \Leftrightarrow$$
$$\cos\theta_3 \sin\theta_2 = -\frac{\cos\theta_2 \sin\theta_4 + T_{(2,2)}}{\cos\theta_4}$$

We can do the division because θ_4 doesn't reach $\frac{\pi}{2}$ so, the cosine never becomes zero. Now we can go to r_{24}:

$$
\begin{aligned}
&T_{(2,4)} = p_y = l_1 + l_3 \cos\theta_2 + l_4 \left(\cos\theta_2 \cos\theta_4 - \cos\theta_3 \sin\theta_2 \sin\theta_4\right) && \Leftrightarrow \\
&p_y - l_1 = l_3 \cos\theta_2 + l_4 \cos\theta_2 \cos\theta_4 - l_4 \left(-\frac{\cos\theta_2 \sin\theta_4 + T_{(2,2)}}{\cos\theta_4}\right) \sin\theta_4 && \Leftrightarrow \\
&p_y - l_1 - \frac{l_4 \sin\theta_4 T_{(2,2)}}{\cos\theta_4} = \cos\theta_2 \left(l_3 + l_4 \cos\theta_4 + l_4 \frac{\sin\theta_4{}^2}{\cos\theta_4}\right) && \Leftrightarrow \\
&\theta_2 = \arccos\left(\frac{p_y - l_1 - \frac{l_4 \sin\theta_4 T_{(2,2)}}{\cos\theta_4}}{l_3 + l_4 \cos\theta_4 + l_4 \frac{\sin^2\theta_4}{\cos\theta_4}}\right) && \text{because } l_3 + l_4 \cos\theta_4 + l_4 \frac{\sin^2\theta_4}{\cos\theta_4} > 0
\end{aligned}
$$

Now the final θ_2 is: $\theta_2 = \theta_2 + \frac{\pi}{2}$ because in the DH parameters we had added $\frac{\pi}{2}$ to the θ parameter for the second joint of the arm. From now on, we must substract $\frac{\pi}{2}$ from θ_2 if we want to use it to calculate a result from the equations. Next we will calculate the θ_3 angle:

$$
\begin{aligned}
&T_{(2,3)} = \sin\left(\theta_2 - \frac{\pi}{2}\right) \sin\theta_3 && \Leftrightarrow \\
&\theta_3 = \arcsin\left(\frac{T_{(2,3)}}{\sin\left(\theta_2 - \frac{\pi}{2}\right)}\right) && \text{because } \theta_2 \neq \left|\frac{\pi}{2}\right|
\end{aligned}
$$

Finally, we must extract the value for θ_1:

$$
\begin{aligned}
&T_{(1,3)} = \cos\theta_3 \sin\theta_1 + \cos\theta_1 \cos\left(\theta_2 - \frac{\pi}{2}\right) \sin\theta_3 && \Leftrightarrow \\
&\sin\theta_1 = \frac{T_{(1,3)} - \cos\theta_1 \cos\left(\theta_2 - \frac{\pi}{2}\right) \sin\theta_3}{\cos\theta_3} && \text{when } \theta_3 \neq \left|\frac{\pi}{2}\right| \\
&\cos\theta_1 = \frac{T_{(1,3)}}{\cos\left(\theta_2 - \frac{\pi}{2}\right) \sin\theta_3} && \text{when} \quad \theta_3 = \left|\frac{\pi}{2}\right|
\end{aligned}
$$

So, now we have two posible solutions for θ_1 and we choose between them depending on the value of θ_3. So, when $\theta_3 = \left|\frac{\pi}{2}\right|$:

$$
\theta_1 = \arccos\left(\frac{T_{(1,3)}}{\cos\left(\theta_2 - \frac{\pi}{2}\right) \sin\theta_3}\right)
$$

Else if $\theta_2 \neq \left|\frac{\pi}{2}\right|$, we will find the θ_1 from r_{33} and we will replace $\sin\theta_1$ with the result above. So:

$$T_{(3,3)} = \cos\theta_1 \cos\theta_3 - \cos\left(\theta_2 - \frac{\pi}{2}\right)\sin\theta_1 \sin\theta_3 \qquad \Leftrightarrow$$

$$T_{(3,3)} = \cos\theta_1 \cos\theta_3 - \left(\frac{T_{(1,3)} - \cos\theta_1 \cos\left(\theta_2 - \frac{\pi}{2}\right)\sin\theta_3}{\cos\theta_3}\right)\sin\theta_3 \cos\left(\theta_2 - \frac{\pi}{2}\right) \qquad \Leftrightarrow$$

$$\theta_1 = \arccos\left(\frac{T_{(3,3)} - \frac{T_{(1,3)}\sin\theta_3\cos\left(\theta_2 - \frac{\pi}{2}\right)}{\cos\theta_3}}{\cos\theta_3 + \frac{\cos^2\left(\theta_2 - \frac{\pi}{2}\right)\sin^2\theta_3}{\cos\theta_3}}\right)$$

Now we have calculated all the joints for the left arm, but we will have a lot of invalid angle sets. We will find the correct set after the forward kinematics validation step. Below there are all the inverse kinematics joints equations for the left arm:

$$\theta_4 = \pi - \arccos\left(\frac{{l_3}^2 + {l_4}^2 - \left((s_x - p_x)^2 + (s_y - p_y)^2 + (s_z - p_z)^2\right)}{2l_3l_4}\right)$$

$$\theta_2 = \pm\arccos\left(\frac{p_y - l_1 - \left(\frac{l_4\sin\theta_4 T_{(2,2)}}{\cos\theta_4}\right)}{l_3 + l_4\cos\theta_4 + l_4\frac{\sin^2\theta_4}{\cos\theta_4}}\right) + \frac{\pi}{2}$$

$$\theta_3 = \arcsin\left(\frac{T_{(2,3)}}{\sin\left(\theta_2 - \frac{\pi}{2}\right)}\right)$$

$$\theta_3 = \pi - \arcsin\left(\frac{T_{(2,3)}}{\sin\left(\theta_2 - \frac{\pi}{2}\right)}\right)$$

$$\theta_1 = \pm\arccos\left(\frac{T_{(3,3)} - \frac{T_{(1,3)}\sin\theta_3\cos\left(\theta_2 - \frac{\pi}{2}\right)}{\cos\theta_3}}{\cos\theta_3 + \frac{\cos^2\left(\theta_2 - \frac{\pi}{2}\right)\sin^2\theta_3}{\cos\theta_3}}\right) \qquad \text{if } \theta_3 \neq \frac{\pi}{2}$$

$$\theta_1 = \pm\arccos\left(\frac{T_{(1,3)}}{\cos\left(\theta_2 - \frac{\pi}{2}\right)\sin\theta_3}\right) \qquad \text{if } \theta_3 = \frac{\pi}{2}$$

5.2.3 Inverse Kinematics for the Right Hand

As we said before, the right and left arm are symmetric, thus, the solution is almost the same. The only differences are in the distances over the y-axis and for the right hand the θ parameter does not subtract $\frac{\pi}{2}$ but it adds $\frac{\pi}{2}$. The kinematic chain for the right arm has an extra rotation matrix after the last translation:

$$T_{\text{Base}}^{\text{End}} = A_{\text{Base}}^{0} T_0^1 T_1^2 T_2^3 T_3^4 R_z(\tfrac{\pi}{2}) A_4^{\text{End}} R_z(-\pi)$$

We can remove this rotation from the reconstructed matrix , and then the chain will be the same as the chain for the left arm. So, first of all we reconstruct the final transformation matrix from the target point parameters and then we remove the rotation:

$$T_{\text{rhandUnrotated}} = T_{\text{rhand}}(R_z(-\pi))^{-1}$$

The only changes on the symbolic table are in the translation part:

$$r_{14} = l_4\left(\sin\theta_4\left(\sin\theta_1\sin\theta_3 - \cos\theta_1\cos\theta_2\cos\theta_3\right) - \cos\theta_1\cos\theta_4\sin\theta_2\right) + l_3\cos\theta_1\sin\theta_2$$
$$r_{24} = -l_1 - l_3\cos\theta_2 - l_4\left(\cos\theta_2\cos\theta_4 - \cos\theta_3\sin\theta_2\sin\theta_4\right)$$
$$r_{34} = l_2 - l_4\left(\sin\theta_4\left(\cos\theta_1\sin\theta_3 + \cos\theta_2\cos\theta_3\sin\theta_1\right) + \cos\theta_4\sin\theta_1\sin\theta_2\right) - l_3\sin\theta_1\sin\theta_2$$

As we can see, the changes doesn't have any impact in our solution for the left arm, e.g. no one denominator becomes zero with this changes. So,as in the previous section, we will get the functions below:

$$\theta_4 = \pi - \arccos\left(\frac{{l_3}^2 + {l_4}^2 - \left((s_x - p_x)^2 + (s_y - p_y)^2 + (s_z - p_z)^2\right)}{2l_3l_4}\right)$$
$$\theta_2 = \pm\arccos\left(\frac{-p_y - l_1 - \left(\frac{l_4\sin\theta_4 T_{(2,2)}}{\cos\theta_4}\right)}{l_3 + l_4\cos\theta_4 + l_4\frac{\sin^2\theta_4}{\cos\theta_4}}\right) - \frac{\pi}{2}$$
$$\theta_3 = \arcsin\left(\frac{T_{(2,3)}}{\sin\left(\theta_2 + \frac{\pi}{2}\right)}\right)$$
$$\theta_3 = \pi - \arcsin\left(\frac{T_{(2,3)}}{\sin\left(\theta_2 + \frac{\pi}{2}\right)}\right)$$

$$\theta_1 = \pm \arccos\left(\frac{T_{(3,3)} - \dfrac{T_{(1,3)} \sin\theta_3 \cos\left(\theta_2 + \frac{\pi}{2}\right)}{\cos\theta_3}}{\cos\theta_3 + \dfrac{\cos^2\left(\theta_2 + \frac{\pi}{2}\right)\sin^2\theta_3}{\cos\theta_3}}\right) \qquad \text{if } \theta_3 \neq \frac{\pi}{2}$$

$$\theta_1 = \pm \arccos\left(\frac{T_{(1,3)}}{\cos\left(\theta_2 + \frac{\pi}{2}\right)\sin\theta_3}\right) \qquad \text{if } \theta_3 = \frac{\pi}{2}$$

5.2.4 Inverse Kinematics for the Left Leg

The kinematic chain for the legs has six joints, so, it will be much more difficult to find a solution. The symbolic matrix for this chain was too huge, therefore we want to make it smaller. First, to make the matrix smaller, we will remove the known rotations and translations from the kinematic chain:

$$T_{\text{lleg}} = A^0_{\text{Base}} T^1_0 T^2_1 T^3_2 T^4_3 T^5_4 T^6_5 R_z(\pi) R_y(-\tfrac{\pi}{2}) A^{\text{End}}_6$$

$$T_{\text{lleg}'} = T_{\text{lleg}} \left(A^{\text{End}}_6\right)^{-1}$$

$$T_{\text{lleg}''} = \left(A^0_{\text{Base}}\right)^{-1} T_{\text{lleg}'}$$

Now we have the chain from the base frame of the first joint to the frame of the last joint. The first joint, HipYawPitch, is rotated by $\frac{-3\pi}{4}$ in the x-axis. We will add a rotation a the start of the chain to rotate the x-axis by $\frac{\pi}{4}$. Then the first joint will be a yaw joint and then we will have three intersected joints and the problem will be solvable:

$$T_{\text{llegRotated}} = R_x(\tfrac{\pi}{4}) T_{\text{lleg}''}$$

Now the end effector is the last joint (ankle roll) and the base is the first joint (rotated HipYawPitch). The first four joints are responsible for the position and orientation of the end effector and the other two joints are only responsible for the orientation. It would be easier if we have only three joints responsible for the position of the end effector, thus, we will invert the transformation matrix. Now only the ankle roll, ankle pitch and knee pitch are responsible for the position:

$$T_{\text{llegInv}} = \left(T_{\text{llegRotated}}\right)^{-1}$$

The resulted symbolic matrix is pretty large and we will use only the translation part, so:

$$\begin{aligned}
r_{14} &= l_2 \sin\theta_5 - l_1 \sin(\theta_4 + \theta_5) \\
r_{24} &= (l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \sin\theta_6 \\
r_{34} &= (l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \cos\theta_6
\end{aligned}$$

where $l_1 =$ ThightLength and $l_2 =$ TibiaLength.
We can now find θ_4 with the same way that we find θ_4 for the arms. So, we have a triangle with the following sides: ThighLength, TibiaLength and the distance from the base to the end effector:

$$d = \sqrt{(s_x - p_x)^2 + (s_y - p_y)^2 + (s_z - p_z)^2}$$

where $s_x = 0$, $s_y =$HipOffsetY, $s_z =$HipOffsetZ, $p_x = T_{\text{llegInv}(1,4)}$, $p_y = T_{\text{llegInv}(2,4)}$ and $p_z = T_{\text{llegInv}(3,4)}$. Now from the law of cosines we have:

$$\theta_4 = \arccos\left(\frac{{l_1}^2 + {l_2}^2 - d^2}{2 l_1 l_2}\right)$$

Because θ_4 represents an interior angle and the knee roll joint is being stretched in the zero-position, the resulting angle is computed by:

$$\theta_4 = \pi - \theta_4$$

Next, we will extract the θ_6 angle. We can extract this angle from the translation matrix using r_{24} and r_{34}:

$$\begin{aligned}
\frac{r_{24}}{r_{34}} &= \frac{p_y}{p_z} && \Leftrightarrow \\
\frac{(l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \sin\theta_5}{(l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \cos\theta_5} &= \frac{p_y}{p_z} && \Leftrightarrow \\
\theta_6 &= \arctan\left(\frac{p_y}{p_z}\right) \quad \text{if} \, (l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \neq 0
\end{aligned}$$

Because we don't know θ_5, we can't find when this equation is zero. So we will have division by zero, but in reality we will not have this problem, because we are using the atan2 function in our program and the result will be just undefined.

So, the program will continue to run but the validation step will reject all the solutions. In figure 5.2 we can see the locus of the equation, wich gives the undefined points. In Subsection 5.2.6 we present the locus along with the values of the ankle pitch (θ_5) and knee pitch (θ_4) for a series of moves of the robot and we discuss what we do when we are in this locus.

Now we can reconstruct and remove the rotation at the end and the transformation from ankle pitch to ankle roll from the chain to make it simpler:

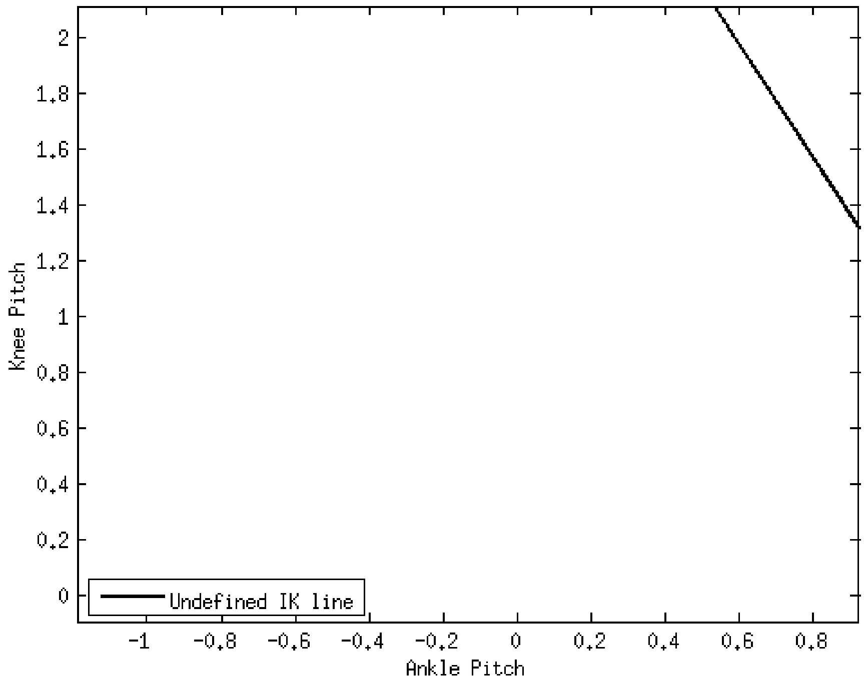


Figure 5.2: Undefined Locus For Legs

$$T_{\text{llegRotated}'} = T_{\text{llegRotated}} \left(T_5^6 R_z\left(\pi\right) R_y(-\tfrac{\pi}{2})\right)^{-1}$$

$$T_{\text{llegInv}'} = \left(T_{\text{llegRotated}'}\right)^{-1}$$

Now we have $p_x = T_{\text{llegInv}'(1,4)}$, $p_y = T_{\text{llegInv}'(2,4)}$ and $p_z = T_{\text{llegInv}'(3,4)}$. From the new symbolic transformation matrix we have:

$$r_{14} = l_2 \cos\theta_5 + l_1\left(\cos\theta_5\cos\theta_4 - \sin\theta_5\sin\theta_4\right)$$

$$
\begin{aligned}
r_{24} &= -l_2 \sin\theta_5 - l_1 \left(\sin\theta_5 \cos\theta_4 + \cos\theta_5 \sin\theta_4\right) \\
r_{34} &= 0
\end{aligned}
$$

We only need the translation part to extract θ_5, so:

$$
\begin{aligned}
&\cos\theta_5 \left(l_2 + l_1 \cos\theta_4\right) = p_x + l_2 \sin\theta_5 \sin\theta_4 && \Leftrightarrow \\
&\cos\theta_5 = \frac{p_x + l_1 \sin\theta_5 \sin\theta_4}{l_2 + l_1 \cos\theta_4} && \text{if } l_2 + l_1 \cos\theta_4 \neq 0
\end{aligned}
$$

The $l_2 + l_1 \cos\theta_4$ is zero if and only if $\cos\theta_4 = 1.029$. So it is always greater than zero, because $-1 \leq \cos \leq 1$:

$$
\begin{aligned}
&\sin\theta_5 \left(-l_2 - l_1 \cos\theta_4\right) - l_2 \cos\theta_5 \sin\theta_4 = p_y && \Leftrightarrow \\
&\sin\theta_5 \left(-l_2 - l_1 \cos\theta_4\right) - l_2 \frac{p_x + l_1 \sin\theta_5 \sin\theta_4}{l_2 + l_1 \cos\theta_4} \sin\theta_4 \ = p_y && \Leftrightarrow \\
&- \sin\theta_5 \left(l_2 + l_1 \cos\theta_4\right) - \frac{l_2 p_x \sin\theta_4}{l_2 + l_1 \cos\theta_4} - \frac{- {l_2}^2 \sin\theta_5 \sin^2\theta_4}{l_2 + l_1 \cos\theta_4} = p_y && \Leftrightarrow \\
&- \sin\theta_5 \left(l_2 + l_1 \cos\theta_4\right)^2 - {l_2}^2 \sin\theta_5 \sin^2\theta_4 = p_y \left(l_2 + l_1 \cos\theta_4\right) + l_2 p_x \sin\theta_4 && \Leftrightarrow \\
&\theta_5 = \arcsin\left(-\frac{p_y \left(l_2 + l_1 \cos\theta_4\right) + l_1 p_x \sin\theta_4}{{l_1}^2 \sin^2\theta_4 + \left(l_2 + l_1 \cos\theta_4\right)}\right)
\end{aligned}
$$

We can do the division because ${l_1}^2 \sin^2\theta_4 + \left(l_2 + l_1 \cos\theta_4\right)^2$ is obviusly greater than zero. Now we can remove the two transformations of θ_4 and θ_5 from the symbolic matrix and then we will have a transformation matrix that is a rotation matrix:

$$
T_{\text{llegRotated}''} = T_{\text{llegRotated}'} \left(T_3^4 T_4^5\right)^{-1}
$$

$$
T_{\text{llegInv}''} = \left(T_{\text{llegRotated}''}\right)^{-1}
$$

And the rotation part is:

$$
\begin{aligned}
r_{11} &= \cos\theta_1 \cos\theta_2 \cos\theta_4 - \sin\theta_1 \sin\theta_3 \\
r_{12} &= -\cos\theta_3 \sin\theta_1 - \cos\theta_1 \cos\theta_2 \sin\theta_3
\end{aligned}
$$

$$
\begin{aligned}
r_{13} &= \cos\theta_1 \sin\theta_2 \\
r_{21} &= -\cos\theta_3 \sin\theta_2 \\
r_{22} &= \sin\theta_2 \sin\theta_3 \\
r_{23} &= \cos\theta_2 \\
r_{31} &= -\cos\theta_2 \cos\theta_3 \sin\theta_1 - \cos\theta_1 \sin\theta_3 \\
r_{32} &= -\cos\theta_1 \cos\theta_3 + \cos\theta_2 \sin\theta_1 \sin\theta_3 \\
r_{33} &= -\sin\theta_1 \sin\theta_2
\end{aligned}
$$

Finally we can extract, from the transformation matrix, the remaining three angles:

$$
\begin{aligned}
\theta_2 &= \arccos T_{\text{llegInv}''(2,3)} \\
\theta_2 &= \theta_2 - \frac{\pi}{4} \\
\theta_3 &= \arcsin\left(\frac{T_{\text{llegInv}''(2,2)}}{\sin\left(\theta_2 + \frac{\pi}{4}\right)}\right) \\
\theta_1 &= \arccos\left(\frac{T_{\text{llegInv}''(3,3)}}{\sin\left(\theta_2 + \frac{\pi}{4}\right)}\right) \\
\theta_1 &= \theta_1 + \frac{\pi}{2}
\end{aligned}
$$

The equations above don't have any problem with division by zero because of the restriction of NAO. The HipRoll joint (θ_2) doesn't reach $-\frac{\pi}{4}$ or $\frac{3\pi}{4}$ so the denominator never becomes zero. Finally, we will do a forward validation step for all possible set of angles.
Below there are all the equations of inverse kinematics for the left leg:

$$
\begin{aligned}
T_{\text{llegInv}} &= \left(R_x(\tfrac{\pi}{4})\left(\left(A^0_{\text{Base}}\right)^{-1} T_{\text{lleg}} \left(A_6^{\text{End}}\right)^{-1}\right)\right)^{-1} \\
\theta_4 &= \pi - \arccos\left(\frac{{l_1}^2 + {l_2}^2 - \sqrt{(s_x - p_x)^2 + (s_y - p_y)^2 + (s_z - p_z)^2}^2}{2 l_1 l_2}\right) \\
\theta_6 &= \arctan\left(\frac{p_y}{p_z}\right) \qquad \text{if}\,(l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \neq 0 \\
T_{\text{llegInv}'} &= T_{\text{llegInv}} \left(T_5^6 R_z(\pi) R_y(-\tfrac{\pi}{2})\right)^{-1}
\end{aligned}
$$

$$\theta_5 = \arcsin\left(-\frac{p_y\left(l_2 + l_1\cos\theta_4\right) + l_1 p_x \sin\theta_4}{{l_1}^2\sin^2\theta_4 + \left(l_2 + l_1\cos\theta_4\right)}\right)$$
$$\theta_5 = \pi - \arcsin\left(-\frac{p_y\left(l_2 + l_1\cos\theta_4\right) + l_1 p_x \sin\theta_4}{{l_1}^2\sin^2\theta_4 + \left(l_2 + l_1\cos\theta_4\right)}\right)$$
$$T_{\text{llegInv}''} = T_{\text{llegInv}'}\left(T_3^4 T_4^5\right)^{-1}$$
$$\theta_2 = \pm\arccos T_{\text{llegInv}''(2,3)} - \frac{\pi}{4}$$
$$\theta_3 = \arcsin\left(\frac{T_{\text{llegInv}''(2,2)}}{\sin\left(\theta_2 + \frac{\pi}{4}\right)}\right)$$
$$\theta_3 = \pi - \arcsin\left(\frac{T_{\text{llegInv}''(2,2)}}{\sin\left(\theta_2 + \frac{\pi}{4}\right)}\right)$$
$$\theta_1 = \pm\arccos\left(\frac{T_{\text{llegInv}''(3,3)}}{\sin\left(\theta_2 + \frac{\pi}{4}\right)}\right) + \frac{\pi}{2}$$

5.2.5 Inverse Kinematics for the Right Leg

As we mentioned before, the chains of the legs are symmetric, so we will have a similar solution for the problem as we have with the arms. The only changes are in the rotation matrix that we use to rotate the HipYawPitch joint. Now we must rotate by $-\frac{\pi}{4}$.

$$T_{\text{rleg}} = A^0_{\text{Base}} T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 R_z(\pi) R_y(-\tfrac{\pi}{2}) A_6^{\text{End}}$$
$$T_{\text{rleg}'} = T_{\text{rleg}}\left(A_6^{\text{End}}\right)^{-1}$$
$$T_{\text{rleg}''} = \left(A^0_{\text{Base}}\right)^{-1} T_{\text{rleg}'}$$
$$T_{\text{rlegRotated}} = R_x(-\tfrac{\pi}{4}) T_{\text{rleg}''}$$
$$T_{\text{rlegInv}} = \left(T_{\text{rlegRotated}}\right)^{-1}$$

After that the symbolic matrix for the right leg is exactly the same as the symbolic matrix for the left leg. From this point of view, if we follow all the steps as we did with the left leg, we will conclude to the same equations. So, the final equations are:

$$T_{\text{rlegInv}} = \left(R_x(\tfrac{\pi}{4})\left(\left(A^0_{\text{Base}}\right)^{-1} T_{\text{rleg}}\left(A_6^{\text{End}}\right)^{-1}\right)\right)^{-1}$$

$$\theta_4 = \pi - \arccos\left(\frac{{l_1}^2 + {l_2}^2 - \sqrt{(s_x - p_x)^2 + (s_y - p_y)^2 + (s_z - p_z)^2}^2}{2 l_1 l_2}\right)$$

$$\theta_6 = \arctan\left(\frac{p_y}{p_z}\right) \qquad \text{if}\,(l_2 \cos\theta_5 + l_1 \cos(\theta_4 + \theta_5)) \neq 0$$

$$T_{\text{rlegInv}'} = T_{\text{rlegInv}} \left(T_5^6 R_z(\pi) R_y(-\tfrac{\pi}{2})\right)^{-1}$$

$$\theta_5 = \arcsin\left(-\frac{p_y (l_2 + l_1 \cos\theta_4) + l_1 p_x \sin\theta_4}{{l_1}^2 \sin^2\theta_4 + (l_2 + l_1 \cos\theta_4)}\right)$$

$$\theta_5 = \pi - \arcsin\left(-\frac{p_y (l_2 + l_1 \cos\theta_4) + l_1 p_x \sin\theta_4}{{l_1}^2 \sin^2\theta_4 + (l_2 + l_1 \cos\theta_4)}\right)$$

$$T_{\text{rlegInv}''} = T_{\text{rlegInv}'} \left(T_3^4 T_4^5\right)^{-1}$$

$$\theta_2 = \pm \arccos T_{\text{rlegInv}''(2,3)} + \frac{\pi}{4}$$

$$\theta_3 = \arcsin\left(\frac{T_{\text{rlegInv}''(2,2)}}{\sin\left(\theta_2 - \frac{\pi}{4}\right)}\right)$$

$$\theta_3 = \pi - \arcsin\left(\frac{T_{\text{rlegInv}''(2,2)}}{\sin\left(\theta_2 - \frac{\pi}{4}\right)}\right)$$

$$\theta_1 = \pm \arccos\left(\frac{T_{\text{rlegInv}''(3,3)}}{\sin\left(\theta_2 - \frac{\pi}{4}\right)}\right) + \frac{\pi}{2}$$

5.2.6 Undefined Target Points For Legs

As we mentioned before, we have some target points for which we can't find an inverse kinematics solution. These target points are presented in the figure below, alongside with all the values of the angles that are responsible for this singularity. As we can see, we let the robot to do a lot of movements in the field, and none of those movements was close to the points of the singularity. In real world, it is very rare for someone to give target points in that area. Also, when someone is executing a trajectory, they "feed" inverse kinematics with a lot of target points per second (usually they give target points with frequency 50 to 100 HZ), so if one of those target point is in the area of singularity, we will find a solution for the next target point. Because we are working in high frequency rate, the singularity will be unnoticed and the movement will continue normally.

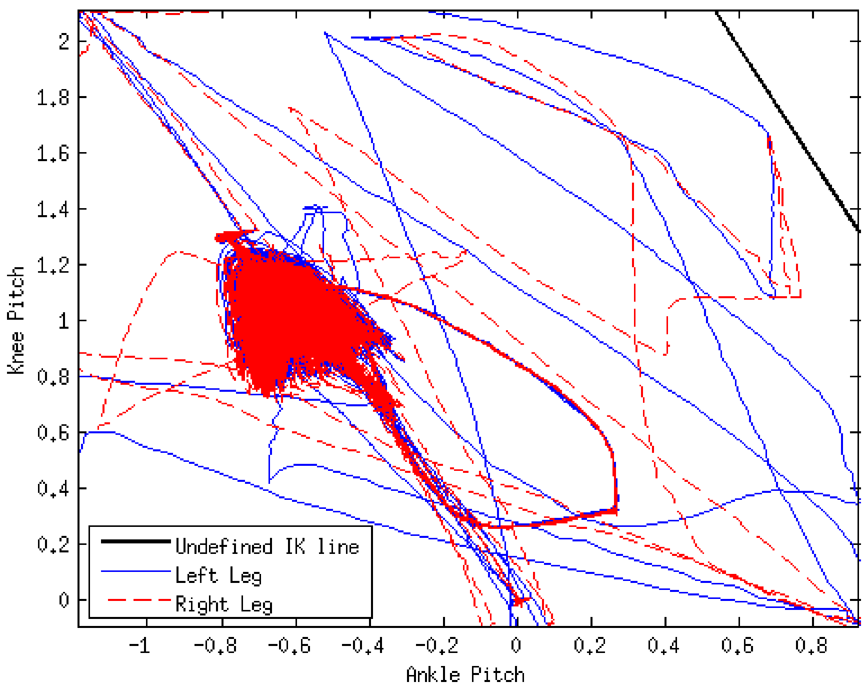


Figure 5.3: Undefined Target Points For Legs

Chapter 6

Implementation

6.1 KMat: Kouretes Math Library

6.2 Nao Kinematics in c++

Chapter 7

Results

7.1 Demo Program

Chapter 8

Conclusion

8.1 Future Work

References

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