Teorema de Aproximación de Weierstrass

Enunciado Sea $f:[0,1] \longrightarrow \mathbb{R}$ continua. Entonces $\exists \{p_n\}$ sucesión de polinomios $p_n:[0,1] \longrightarrow \mathbb{R}$ que converge uniformemente a f en $[0,1]$.

Definición Llamamos polinomio de Berstein de grado n de f en $x \in [0,1]$.

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

Resultados previos

$$B_n(1;x) = 1; \quad B_n(x;x) = x; \quad B_n(x^2,x) = \frac{x(x(n-1)+1)}{n}$$

Demostración

Usaremos el binomio de Newton, es decir, $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. Como es un polinomio es derivable en $x \in \mathbb{R}$

$$n(x+y)^{n-1} = \sum_{k=0}^{n}\binom{n}{k} k x^{k-1} y^{n-k}$$

A su vez, vuelve a ser derivable en $x \in \mathbb{R}$

$$n(n-1)(x+y)^{n-2} = \sum_{k=0}^{n}\binom{n}{k}(k^2-k)x^{k-2}y^{n-k}$$

Tomamos ahora $y=(1-x)$ de donde obtenemos que:

$$1 = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = B_n(1;x)$$

$$n = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}(1-x)^{n-k} \overset{\cdot\frac{x}{n}}{\iff} x = \sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k} = B_n(x;x)$$

$$n(n-1) = \sum_{k=0}^{n}(k^2-k)\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n}k^2\binom{n}{k}x^{k-2}(1-x)^{n-k} - \sum_{k=0}^{n}k\binom{n}{k}x^{k-2}(1-x)^{n-k} \iff$$

$$\overset{\cdot\frac{x^2}{n}}{\iff} x^2(n-1) = \sum_{k=0}^{n}\frac{k^2}{n}\binom{n}{k}x^k(1-x)^{n-k} - \sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n}\frac{k^2}{n}\binom{n}{k}x^k(1-x)^{n-k} - x \iff$$

$$\Longleftrightarrow \quad x^2(n-1)+x=\sum_{k=0}^{n}\frac{k^2}{n}\binom{n}{k}x^k(1-x)^{n-k} \overset{\frac{1}{n}}{\Longleftrightarrow} \quad \frac{x^2(n-1)+x}{n}=B_n(x^2;x)=\frac{x(x(n-1)+1)}{n} \qquad \square$$

Demostración por Berstein

Sea $\{p_n\}$ una sucesión de polinomios de Berstein, es decir, $p_n := B_n(f;x)$ de grado n de f evaluados en $x\in[0,1]$

Como por hipótesis f está acotada (Teorema del Valor Intermedio) entonces $\exists M>0 \mid |f(x)|\leq M \ \forall x\in[0,1]$ En particular $|f(x)-f(y)|\leq 2M$.

Por el teorema de Heine tenemos que f es unif. continua en $x\in[0,1]$ por tanto

$$\forall \varepsilon>0,\ \frac{\varepsilon}{2}>0,\ \exists \delta>0 \mid x,y\in[0,1] \wedge |x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}<\varepsilon$$

$$\left|f(x)-B_n(f;x)\right|=\left|\sum_{k=0}^{n}\left[f(x)-f\left(\tfrac{k}{n}\right)\right]\binom{n}{k}x^k(1-x)^{n-k}\right|\leq \sum_{k=0}^{n}\left|f(x)-f\left(\tfrac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}=$$

$$=\sum_{\substack{|\frac{k}{n}-x|<\delta \\ 0\leq k\leq n}}\left|f(x)-f\left(\tfrac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}+\sum_{\substack{|\frac{k}{n}-x|\geq\delta \\ 0\leq k\leq n}}\left|f(x)-f\left(\tfrac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}\leq \frac{\varepsilon}{2}\sum_{\substack{|\frac{k}{n}-x|<\delta \\ 0\leq k\leq n}}\binom{n}{k}x^k(1-x)^{n-k}+2M\sum_{\substack{|\frac{k}{n}-x|\geq\delta \\ 0\leq k\leq n}}\binom{n}{k}x^k(1-x)^{n-k}$$

Lema

Sea $n\in\mathbb{N}$, $\delta>0$, $x\in[0,1]$, $k\in(\mathbb{N}\cap[1,n])\cup\{0\}$. Si $\left|\frac{k}{n}-x\right|\geq\delta \Rightarrow \displaystyle\sum_{\substack{|\frac{k}{n}-x|\geq\delta \\ 0\leq k\leq n}}\binom{n}{k}x^k(1-x)^{n-k}\leq\frac{1}{4\delta^2 n}$

Demostración

Como $\left|\frac{k}{n}-x\right|\geq\delta \Rightarrow \left(\frac{k}{n}-x\right)^2\geq\delta^2 \Rightarrow \frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2\geq 1$

$$\sum_{\substack{|\frac{k}{n}-x|\geq\delta \\ 0\leq k\leq n}}\binom{n}{k}x^k(1-x)^{n-k}\leq\frac{1}{\delta^2}\sum_{\substack{|\frac{k}{n}-x|\geq\delta \\ 0\leq k\leq n}}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}\overset{\frac{n^2}{n^2}}{\leq}\frac{1}{\delta^2 n^2}\sum_{k=0}^{n}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k}=$$

$$=\frac{1}{\delta^2 n^2}\sum_{k=0}^{n}(k^2-2knx+n^2x^2)\binom{n}{k}x^k(1-x)^{n-k}=\frac{1}{\delta^2 n^2}\left[\sum_{k=0}^{n}k^2\binom{n}{k}x^k(1-x)^{n-k}-2nx\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k}+n^2x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right]$$

$$\cdot\frac{n^2}{n^2}$$

$$=\frac{1}{\delta^2 n^2}\left[n^2B_n(x^2;x)-2xn^2B_n(x;x)+n^2x^2B_n(1;x)\right]=\frac{1}{\delta^2 n^2}\left[n\cancel{^2}\,\frac{x(x(n-1)+1)}{\cancel{n}}-2n^2x^2+n^2x^2\right]=$$

$$= \frac{1}{\delta^2 n^2}\left[n(x(n-1)x+1) - n^2x^2 \right] = \frac{1}{\delta^2 n^2}\left[n(n-1)x^2 + n - n^2x^2 \right] = \frac{1}{\delta^2 n^2}\left[\cancel{n^2x^2} - \cancel{n^2x^2} - x^2 n + nx \right] =$$

$$x(-xn+n)$$

$$= \frac{xn(1-x)}{\delta^2 n^2} = \frac{x(1-x)}{\delta^2 n} \leq \frac{1}{4\delta^2 n} \quad \text{pues} \quad x(1-x) \leq \frac{1}{4} \ \forall x \in [0,1]$$

□

$$\leq \frac{\varepsilon}{2} \sum_{\substack{|\frac{k}{n}-x|<\delta \\ 0\leq k\leq n}} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{\substack{|\frac{k}{n}-x|\geq\delta \\ 0\leq k\leq n}} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} + \frac{2M}{4\delta^2 n} \leq \varepsilon + \frac{M}{2\delta^2 n}$$

Como $\{\frac{1}{n}\} \to 0$ tenemos que $|f(x) - B_n(f;x)| \leq \varepsilon$ como se quería entonces $\{P_n(=B_n(f;x))\}$ converge uniformemente a f en $[0,1]$

□