Demostrar que

$$M_{X_1,\dots,X_k}(t_1,\dots,t_k) = \left(p_1e^{t_1}+\dots+p_ke^{t_k}+\left(1-\sum_{i=1}^{k}p_i\right)\right)^n,\quad t_1,\dots,t_k\in\mathbb{R}$$

-Demostración-

Notaremos por $X=(X_1,\dots,X_k)$ al vector aleatorio de dimensión $k\in\mathbb{N}$.

Por definición

$$M_X(t_1,\dots,t_k) = \mathcal{E}\left[e^{t_1X_1+\dots+t_kX_k}\right] = \sum_{\substack{X=0\\ \sum_{i=1}^{k}x_i\le n}} e^{t_1x_1+\dots+t_kx_k}\frac{n!}{\prod_{i=1}^{k}x_i!\left(n-\sum_{i=1}^{k}x_i\right)!}\prod_{i=1}^{k}p_i^{x_i}\left(1-\sum_{i=1}^{k}p_i\right)^{n-\sum_{i=1}^{k}x_i} =$$

$$= \sum_{\substack{X=0\\ \sum_{i=1}^{k}x_i\le n}} \frac{n!}{\prod_{i=1}^{k}x_i!\left(n-\sum_{i=1}^{k}x_i\right)!}\prod_{i=1}^{k}\left(e^{t_i}p_i\right)^{x_i}\left(1-\sum_{i=1}^{k}p_i\right)^{n-\sum_{i=1}^{k}x_i}$$

Haciendo uso de la fórmula de Leibniz para la potencia de un polinomio:

$$(q_1+\dots+q_m)^n = \sum_{\substack{k_1,\dots,k_m=0\\ \sum_{i=1}^{m}k_i=n}}\frac{n!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}q_i^{k_i} = \sum_{\substack{k_1,\dots,k_{m-1}=0\\ \sum_{i=1}^{m-1}k_i\le n}}\frac{n!}{\prod_{i=1}^{m-1}q_i!\left(n-\sum_{i=1}^{m-1}q_i\right)!}\prod_{i=1}^{m-1}q_i^{k_i}\,q_m^{\,n-\sum_{i=1}^{m-1}q_i}$$

obtenemos directamente que

$$M_X(t_1,\dots,t_k) = \left(p_1e^{t_1}+\dots+p_ke^{t_k}+\left(1-\sum_{i=1}^{k}p_i\right)\right)^n \qquad \square$$

Demostrar que, si $X=(X_1,\dots,X_k)\sim M_k(m;p_1,\dots,p_k)$ entonces

$$(X_1,\dots,X_h)/(X_{h+1}=x_{h+1},\dots,X_k=x_k)\sim M_h\left(m-\sum_{i=h+1}^{k}x_i;\ \frac{p_1}{1-\sum_{i=h+1}^{k}p_i},\dots,\frac{p_h}{1-\sum_{i=h+1}^{k}p_i}\right)$$

-Demostración-

Notaremos por $X^{(1)}=(X_1,\dots,X_h)$ y $X^{(2)}=(X_{h+1},\dots,X_k)$, buscaremos calcular la función masa de probabilidad condicionada a un valor $x^{(2)}=(x_{h+1},\dots,x_k)$ de $X^{(2)}$ para ver que se rige por la multinomial que buscamos.

Por definición

$$P\left[X^{(1)}=x^{(1)}/X^{(2)}=x^{(2)}\right] = \frac{P\left[X^{(1)}=x^{(1)},X^{(2)}=x^{(2)}\right]}{P\left[X^{(2)}=x^{(2)}\right]}$$

donde $x^{(1)}=(x_1,\dots,x_u)$.

Entonces:

$$P[X^{(1)}=x^{(1)}/X^{(2)}=x^{(2)}]=\frac{\dfrac{n!}{\prod_{i=1}^{u} x_i!\,\prod_{i=u+1}^{k} x_i!\,\left(n-\sum_{i=u+1}^{k} x_i-\sum_{i=1}^{u} x_i\right)!}\prod_{i=1}^{u} p_i^{x_i}\prod_{i=u+1}^{k} p_i^{x_i}\left(1-\sum_{i=u+1}^{k} p_i-\sum_{i=1}^{u} p_i\right)^{n-\sum_{i=1}^{u} x_i-\sum_{i=u+1}^{k} x_i}}{\dfrac{n!}{\prod_{i=u+1}^{k} x_i!\,\left(n-\sum_{i=u+1}^{k} x_i\right)!}\prod_{i=u+1}^{k} p_i^{x_i}\left(1-\prod_{i=u+1}^{k} p_i\right)^{n-\sum_{i=u+1}^{k} x_i}}$$

donde simplificando tenemos

$$\frac{\left(n-\sum_{i=u+1}^{k} x_i\right)!}{\prod_{i=1}^{u} x_i!\,\left(n-\sum_{i=u+1}^{k} x_i-\sum_{i=1}^{u} x_i\right)!}\prod_{i=1}^{u} p_i^{x_i}\,\frac{\left(1-\sum_{i=u+1}^{k} p_i-\sum_{i=1}^{u} p_i\right)^{n-\sum_{i=1}^{u} x_i-\sum_{i=u+1}^{k} x_i}}{\left(1-\sum_{i=u+1}^{k} p_i\right)^{n-\sum_{i=u+1}^{k} x_i}}$$

Donde si sumamos y restamos $\sum_{i=1}^{u} x_i$ en el exponente del denominador de la parte derecha:

$$\frac{\left(1-\sum_{i=u+1}^{k} p_i-\sum_{i=1}^{u} p_i\right)^{n-\sum_{i=1}^{u} x_i-\sum_{i=u+1}^{k} x_i}}{\left(1-\sum_{i=u+1}^{k} p_i\right)^{\sum_{i=1}^{u} x_i}\left(1-\sum_{i=u+1}^{k} p_i\right)^{n-\sum_{u+1}^{k} x_i-\sum_{i=1}^{u} x_i}}=\frac{1}{\left(1-\sum_{i=u+1}^{k} p_i\right)^{\sum_{i=1}^{u} x_i}}\left(1-\frac{\sum_{i=1}^{u} p_i}{1-\sum_{i=u+1}^{k} p_i}\right)^{n-\sum_{i=1}^{u} x_i-\sum_{i=u+1}^{k} x_i}$$

Sustituyendo en la expresión inicial y desarrollando los cálculos obtenemos

$$P[X^{(1)}=x^{(1)}/X^{(2)}=x^{(2)}]=\frac{\left(n-\sum_{i=u+1}^{k} x_i\right)!}{\prod_{i=1}^{u} x_i!\,\left(n-\sum_{i=u+1}^{k} x_i-\sum_{i=1}^{u} x_i\right)!}\prod_{i=1}^{u}\frac{p_i^{x_i}}{\left(1-\sum_{i=u+1}^{k} p_i\right)^{x_i}}\left(1-\sum_{i=1}^{u}\frac{p_i}{1-\sum_{i=u+1}^{k} p_i}\right)^{n-\sum_{i=1}^{u} x_i-\sum_{i=u+1}^{k} x_i}$$

Donde comparando con la función masa de probabilidad de la distribución multinomial obtenemos el resultado buscado $\square$