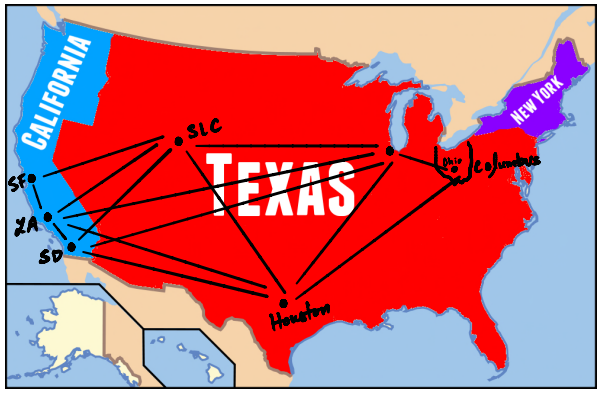
DSC 40B

Theoretical Foundations II

Shortest Paths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Recall

- The **length** of a path is

  (# of nodes) − 1

Definitions

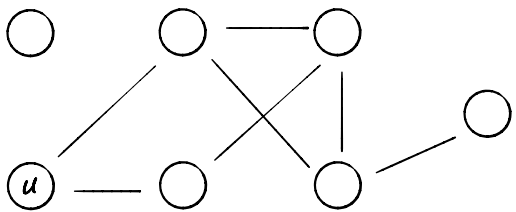
- A **shortest path** between u and v is a path between u and v with smallest possible length.
  - There may be several, or none at all.

- The **shortest path distance** is the length of a shortest path.
  - Convention: ∞ if no path exists.
  - "the distance between u and v" means spd.

Today: Shortest Paths

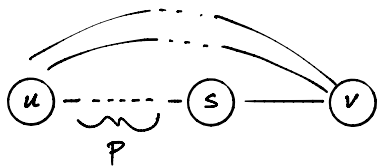
- **Given**: directed/undirected graph G, source u
- **Goal**: find shortest path from u to every other node

Example



Key Property

- A shortest path of length k is composed of:
  - A **shortest path** of length $k - 1$.
  - Plus one edge.



Algorithm Idea

- Find all nodes distance 1 from source.
- Use these to find all nodes distance 2 from source.
- Use these to find all nodes distance 3 from source.
- ...

It turns out...

...this is exactly what BFS does.

DSC 40B

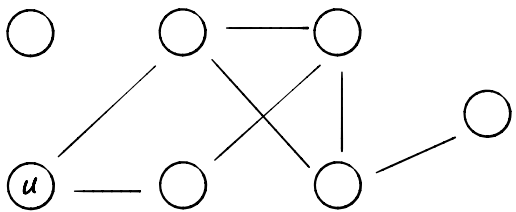
Theoretical Foundations II

BFS for Shortest Paths

Key Property of BFS

- For any $k \geq 1$ you choose:
- All nodes distance $k - 1$ from source are added to the queue before any node of distance k.
- Therefore, nodes are “processed” (popped from queue) in order of distance from source.

Example



Discovering Shortest Paths

- We “discover” shortest paths when we pop a node from queue and look at its neighbors.
- But the neighbor’s status matters!

Consider This

- We pop a node *s*.
- It has a neighbor *v* whose status is **undiscovered**.
- We've discovered a **shortest path** to *v* through *s*!

Consider This

- We pop a node *s*.
- It has a neighbor *v* whose status is **pending** or **visited**.
- We already have a shortest path to *v*.

Modifying BFS

- Use BFS “framework”.
- Return dictionary of **search predecessors**.
  - If v is discovered while visiting u, we say that u is the **BFS predecessor** of v.
  - This encodes the shortest paths.
- Also return dictionary of shortest path distances.

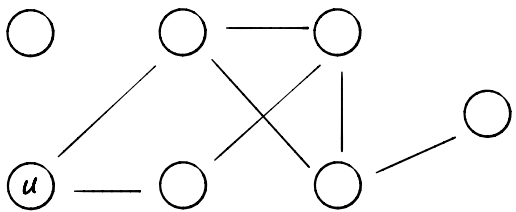
```python
def bfs_shortest_paths(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}
    distance = {node: float('inf') for node in graph.nodes}
    predecessor = {node: None for node in graph.nodes}

    status[source] = 'pending'
    distance[source] = 0
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                distance[v] = distance[u] + 1
                predecessor[v] = u
                # append to right
                pending.append(v)
        status[u] = 'visited'

    return predecessor, distance
```

Example



DSC 40B

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BFS Trees

Result of BFS

- Each node reachable from source has a single BFS predecessor.
  - Except for the source itself.
- The result is a **tree** (or forest).

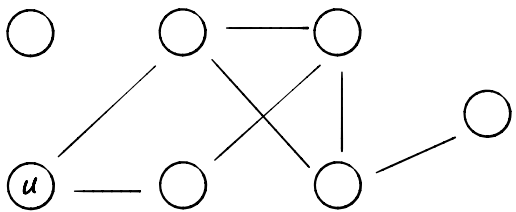
Trees

- A (free) **tree** is an undirected graph $T = (V, E)$ such that T is connected and $|E| = |V| - 1$.
- A **forest** is graph in which each connected component is a tree.

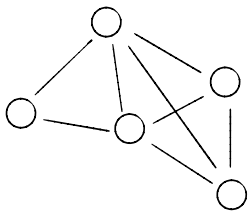
BFS Trees (Forests)

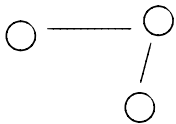
- If the input is connected, BFS produces a **tree**.
- If the input is not connected, BFS produces a **forest**.

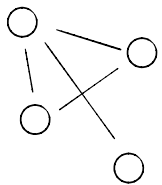
Example



Example







BFS Trees

- BFS trees and forests encode shortest path distances.