DSC40B: Theoretical Foundations of Data Science II

Lecture 2: *Nested loops and asymptotic time complexity*

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More examples: Nest loops

The median problem

- Design an algorithm for the following problem
  - Input: given n numbers $X = \{x_1, x_2, \ldots, x_n\}$
  - Output: find h minimizing the total absolute loss:
    - $Loss(h) = \sum_{i=1}^{n} |x_i - h|$

- Recall from DSC40A
  - Solution: h^* is a median of numbers in X
    - i.e, the number whose order in X (if sorted) is $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$

- Question:
  - How to compute this median?

One algorithm

- Idea:
  - h^* has to be one of $\mathrm{X} = \{x_1, x_2, \ldots, x_n\}$
  - compute all $Loss(x_1), \ldots, Loss(x_n)$
  - return the one whose loss is smallest

One algorithm

Time/exec	#exec

```
def median(X):
        min_h = None
        min_value = float('inf’)
        for h in X:
            total_abs_loss = 0
            for x in X :
                total_abs_loss += abs(x - h)
            if total_abs_loss < min_value:
                min_value = total_abs_loss
                min_h = h
        return min_h
```

One algorithm

```
def median(X):
        min_h = None
        min_value = float('inf')
        for h in X:
                total_abs_loss = 0
                for x in X :
                        total_abs_loss += abs(x - h)
                if total_abs_loss < min_value:
                        min_value = total_abs_loss
                        min_h = h
        return min_h
```

$T(n) = \Theta(n^2)$

We will see how to do better later in class.

Caution!

- Not all nested loop takes $\Theta(n^2)$ time

```
def foo_1(n):
    for x in range(n):
        for y in range(n, n+10):
            print(x + y)
```

$$T_1(n) = \Theta(n)$$

```
def foo_2(n):
    for x in range(n):
        for y in range(n, 2n-10):
            print(x + y)
```

$$T_2(n) = \Theta(n^2)$$

Another example

- Alan is given n sticks. He needs to design an algorithm to compute the tallest pole he can make by stacking two sticks

```
def tallest_pole(heights):
1.  max_height = -float ('inf')
2.  n = len(heights)
3.  for i in range(n):
4.      for j in range(i+1, n):
5.          h = heights[i] + heights[j]
6.          if h > max_height
7.              max_height = h
8.  return max_height
```

On outer iter.#1, inner body runs _____ times

On outer iter.#2, inner body runs _____ times

On outer iter.#3, inner body runs _____ times

⋮

On outer iter.#n, inner body runs _____ times

- How many times does line $5-7$ (inner body) run?

Tallest_pole, cont.

- Total times line 5-7 (inner body) is executed:

$$\underbrace{(n-1)}_{\text{1st outer iter}} + \underbrace{(n-2)}_{\text{2nd outer iter}} + \ldots + \underbrace{(n-k)}_{\text{kth outer iter}} + \underbrace{(n-(n-1)}_{\text{(n-1)th outer iter}} + \underbrace{(n-n)}_{\text{nth outer iter}}$$

$$=$$

$$1 + 2 + 3 + \ldots + (n-3) + (n-2) + (n-1)$$

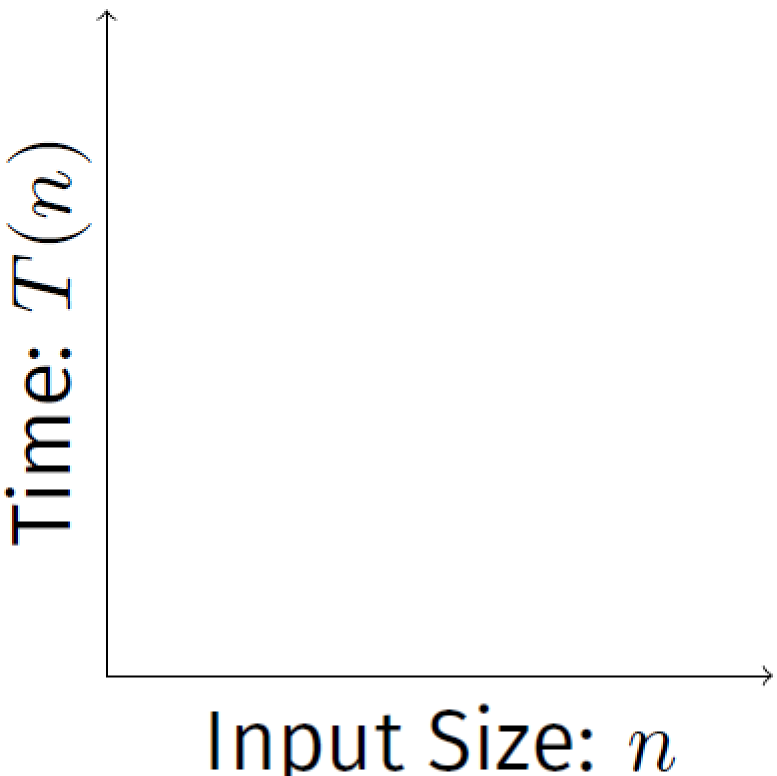
$$=$$

- Algorithm tallest_pole has $T(n) = \Theta(n^2)$
- Another way to see the time complexity:
  - Number of pairs of n objects is $\binom{n}{2} = \frac{\mathrm{n(n-1)}}{2} = \Theta(n^2)$

Exercise: Find a linear-time algorithm for this problem.

Linear vs quadratic growth

Scaling



- $T(n) = \Theta(n)$
  - means "$T(n)$ grows like n"
  - linear growth
- $T(n) = \Theta(n^2)$
  - means "$T(n)$ grows like n^2"
  - quadratic growth

- Suppose your algorithm runs 5 sec on 1000 points
- Linear growth
  - If the input has 100,000 points, then it takes 500 seconds (8.3 min)
- Quadratic growth
  - If the input has 100,000 points, then it will take 50,000 seconds (~14 hours)

Some common growth rates

- $\Theta(1)$: constant
- $\Theta(\log n)$: logarithmic
- $\Theta(n)$: linear
- $\Theta(n \log n)$: linearithmic
- $\Theta(n^2)$: quadratic
- $\Theta(n^3)$: cubic
- $\Theta(2^n)$: exponential

Asymptotic notations

- We have seen $\Theta(\cdot)$ allows us to ignore unnecessary details and focus on dominating terms of growth.
  - Simpler, easier to analyze, and focus on key growth rate
  - What exactly are we ignoring?

- Before we introduce this formally
  - First introduce big-O and big-Ω notations
  - Then big-Θ
  - Intuitively, big-O, big-Ω, and big-Θ "roughly" corresponds to $\leq$, $\geq$, and $=$, respectively (upto constants)

Big-O notation

Big-O notation

Definition

We write $f(n) = O\big(g(n)\big)$ if there are positive constants n_0 and c such that for all $n \geq n_0$:

$$f(h) \leq c \cdot g(n)$$

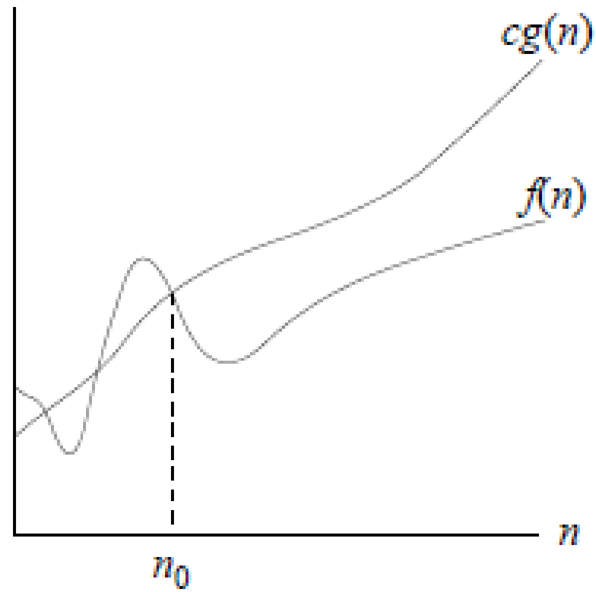
- More precisely, should be $f(n) \in O\big(g(n)\big)$
- Intuitively, $f(n) = O\big(g(n)\big)$ means that
  - $f(n)$ grows at most as fast as g(n) (up to multiplicative constant factor)
  - We also say $g(n)$ is an asymptotic upper bound for $f(n)$ in this case

Big-O notation

Definition

We write $f(n) = O\big(g(n)\big)$ if there are positive constants n_0 and c such that for all $n \geq n_0$:

$$f(h) \leq c \cdot g(n)$$



Examples

- $n^3 - 3n^2 + 5n - 1 = O(n^3)$
- Proof:

- $5n^2 + 6\sqrt{n} + 8 = O(n^2)$?
- Proof:

More examples

- $\sqrt{6n^3 + 7n^2 + 3n} = O(n^2)$?
- $\sqrt{6n^3 + 7n^2 + 3n} = O(n^{1.5})$?
- $\frac{n}{\lg n} = O(n)$?
- $\frac{100\, n}{\lg n} = O(\frac{n}{\lg n})$?
- $n \lg n = O(n)$?

A useful result

Lemma [Upper Bound]

If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $f(n) = O\big(g(n)\big)$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c$$, where c is a positive constant.

Corollary [Upper Bound]

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$ then $f(n) = O\big(g(n)\big)$.

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty,$ then $f(n) = O\big(g(n)\big)$ does not hold.

More examples

- $n^{100} = O(n^2)$?
- $50 \lg n = O(n)$?
- $\log_2 n = O(\log_{10} n)$?
- $n^{100} = O(2^n)$?
- $2^n = O(3^n)$?

Big-Ω notation

Big-Ω notation

Definition

We write $f(n) = \Omega\big(g(n)\big)$ if there are positive constants n_0 and c such that for all $n \geq n_0$:

$$f(h) \geq c \cdot g(n)$$

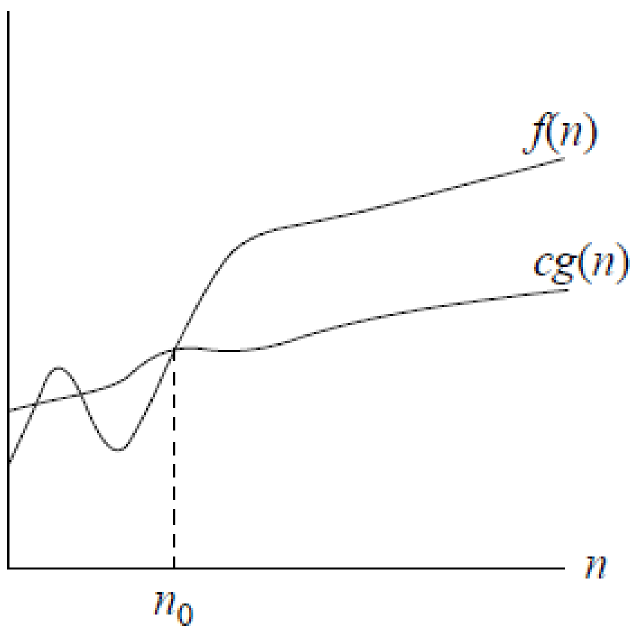
- More precisely, should be $f(n) \in \Omega\big(g(n)\big)$
- Intuitively, $f(n) = \Omega\big(g(n)\big)$ means that
  - $f(n)$ grows at least as fast as g(n) (up to multiplicative constant factor)
  - We also say $g(n)$ is an asymptotic lower bound for $f(n)$ in this case

Big-Ω notation

Definition

We write $f(n) = \Omega\big(g(n)\big)$ if there are positive constants n_0 and c such that for all $n \geq n_0$:

$$f(h) \geq c \cdot g(n)$$



Examples

- $n^3 - 3n^2 + 5n \ = \Omega(n^3)$?
- Proof:

- $n^2 = \Omega(100\, n^2)$?
- Proof:

A useful result

Lemma [Lower Bound]

If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $f(n) = \Omega\big(g(n)\big)$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq c$$, where c is a positive constant.

Corollary [Lower Bound]

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty,$ then $f(n) = \Omega\big(g(n)\big)$.

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$ then $f(n) = \Omega\big(g(n)\big)$ cannot hold.

More Examples

- $5n^2 + 6n + 8 = \Omega(n^3)$?
- $5n^2 + 6n + 8 = \Omega(n^2)$?
- $\sqrt{6n^3 - 7n^2 + 3n} = \Omega(n^{1.5})$?
- $2^n = \Omega(n^2)$?
- $3\lg n = \Omega(n)$?
- $3^n = \Omega(2^n)$?
- $\log_{10} n = \Omega(\log_2 n)$?

Big-Θ notation

Big-Θ notation

Definition

We write $f(n) = \Theta\big(g(n)\big)$ if there are positive constants n_0, c_1, and c_2 such that for all $n \geq n_0$:

$$c_1 \cdot g(n) \leq f(h) \leq c_2 \cdot g(n)$$

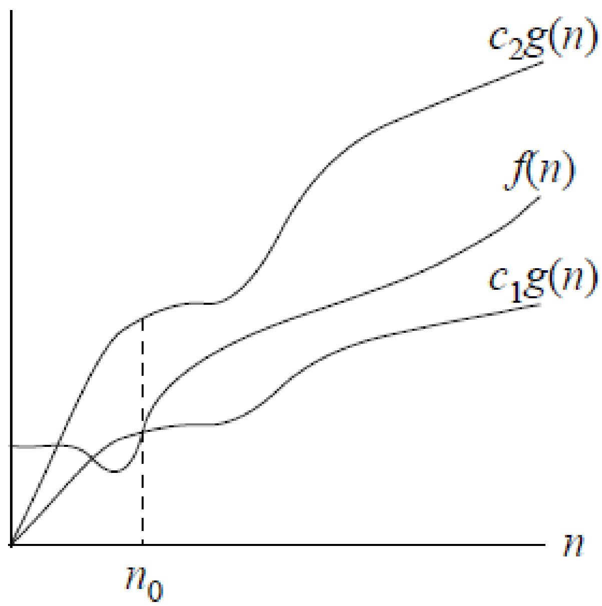
- More precisely, should be $f(n) \in \Theta\big(g(n)\big)$
- Intuitively, $f(n) = \Theta\big(g(n)\big)$ means that
  - $f(n)$ grows like g(n) (up to multiplicative constant factor)

Big-Θ notation

Definition

We write $f(n) = \Theta\big(g(n)\big)$ if there are positive constants n_0, c_1, and c_2 such that for all $n \geq n_0$:

$$c_1 \cdot g(n) \leq f(h) \leq c_2 \cdot g(n)$$



Examples

- $2n^3 - 3n^2 = \Theta(n^3)$
- Proof:

A useful result

Lemma [Big-Theta]

If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $f(n) = \Theta\big(g(n)\big)$ if and only if

$c_1 \leq \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c_2$, where c_1 and c_2 are positive constants.

Corollary [Big-Theta]

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some positive constant c, then $f(n) = \Theta\big(g(n)\big)$.

More Examples

- $5n^2 + 6n + 8 = \Theta(n^2)$?
- $\sqrt{6n^3 - 7n^2 + 3n} = \Theta(n^{1.5})$?
- $n^2 - \lg n = \Theta(n^2)$?
- $3^n = \Theta(2^n)$?
- $\log_{10} n = \Theta(\log_2 n)$?

More on nested loops analysis

A pseudo-code example

```
function func(n)
x ← 0;
i ← 7;
while (i ≤ n) do
    x ← x + i;
    i ← i + 3;
end
return (x);
```

#iteration	value of i	Cost of this iteration

A second example

```
function func(n)
x ← 0;
i ← 7;
while (i ≤ n) do
    x ← x + i;
    i ← i X 3;
end
return (x);
```

#iteration	value of i	Cost of this iteration

A third example

```
function func(n)
x ← 0;
for i ← 1 to n do
    j ← 1;
    while (j ≤ n) do
        x ← x + (i − j);
        j ← 2 ∗ j;
    end
end
return (x);
```

FIN