DSC40B: Theoretical Foundations of Data Science II

Lecture 6: *Soring, and more on recurrences*

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Previously

- Binary search operation in an array
  - Require that the array is already sorted!
- Today: the sorting problem
  - Input: given an arbitrary array of numbers
  - Output: convert them into an array where all elements are either in non-decreasing or non-increasing order.
    - from now on, unless otherwise specified, in this class, we will assume a sorted array is in non-decreasing order.

Motivation

- There are many reasons why we want to solve the sorting problem
  - Given a list of tasks with different priority values, the CPU may want to process them in decreasing order of priority
  - Sorting can also make other problems easy
    - E.g, the search problem discussed last lecture,
    - or more generally, range search in multidimensional databases etc.

- But we will just focus on the simplest version
  - where the input is just a list of real numbers stored in an array.

(1) A simple sorting algorithm: Selection sort
(2) Correctness of algorithm: loop invariants

A simple idea

- Start with input array:
  - At each iteration, identify the smallest number in the remainder unsorted portion of the array
  - Put it at the end of the already-sorted portion
  - Iterate till the end

- Example:
  - Input array A = [12, 4, -1, 9, 10]

- How to implement this idea using an algorithm
  - *in-place* selection sort
    - meaning that it will only operate on the same array
  - separate "good" / "bad" part of the array by a barrier-id

- How to prove the correctness of the algorithm

- Time complexity

Algorithm selection_sort

```
def selection_sort(A):
    n = len(A)
    if n <= 1:
        return
    for barrier_id in range(n-1):
        # find index of min in A[start:]
        min_id = find_minimum(A, start=barrier_id)
        #swap
        A[barrier_id], A[min_id] = (
                A[min_id], A[barrier_id]
        )
```

Subroutine find_minimum

```
def find_minimum(A, start):
"""Finds index of minimum from [start, len(A)). Assumes non-empty."""
    n = len(A)
    min_value = A[start]
    min_id = start
    for i in range(start + 1, n):
        if A[i] < min_value:
            min_value = A[i]
            min_id = i
    return min_id
```

Note that instead of using this sub-routine, selection_sort can be written by using a nested loop.

Correctness

- How to convince us that this algorithm is correct?
  - Using loop invariants
    - Similar to the inductive idea mentioned earlier
  - A loop invariant is a statement that holds at the end of each iteration
    - to show that it holds for each iteration, we first show it holds for the base case
    - then we argue that if it holds at the end of (i- 1)-th iteration, which is the beginning of the i-th iteration, then it will also hold at the end of i-th iteration.
  - Using appropriate loop invariants, we can then argue the algorithm is correct after all iterations.

Algorithm selection_sort

```
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    n = len(A)
    if n <= 1:
        return
    for barrier_id in range(n-1):
        # find index of min in A[start:]
        min_id = find_minimum(A, start=barrier_id)
        #swap
        A[barrier_id], A[min_id] = (
                A[min_id], A[barrier_id]
        )
```

Loop invariants for selection_sort

- Loop invariant: after k iterations,
  - The first k numbers in A are sorted, and are smaller than all the remainder $n-k$ numbers.
    - k = barrier_id+1 in the code

- If this statement holds for any k, then after $k=n-1$ iterations, we will get a sorted array
  - as by the loop invariant, the first $n-1$ numbers are sorted, and the last one is the largest, meaning that all n numbers are sorted.

- Base case:
  - $k = 0:$ loop invariant holds trivially
- Inductive step:
  - if it holds for $k - 1$
  - then, we identify the smallest from the remainder $\mathrm{n} - k + 1$ numbers, which must be the k -th smallest of the original array
  - so after this k -th iteration, the loop invariant holds for k.
- Thus the algorithm is correct in the end
  - i.e., it returns sorted array after $n - 1$ iterations.

Time complexity

- Essentially nested for loops
  - $T(n) = cn + c(n-1) + c(n-2) + \ \ldots c \cdot 1$
    $= \Theta(n^2)$

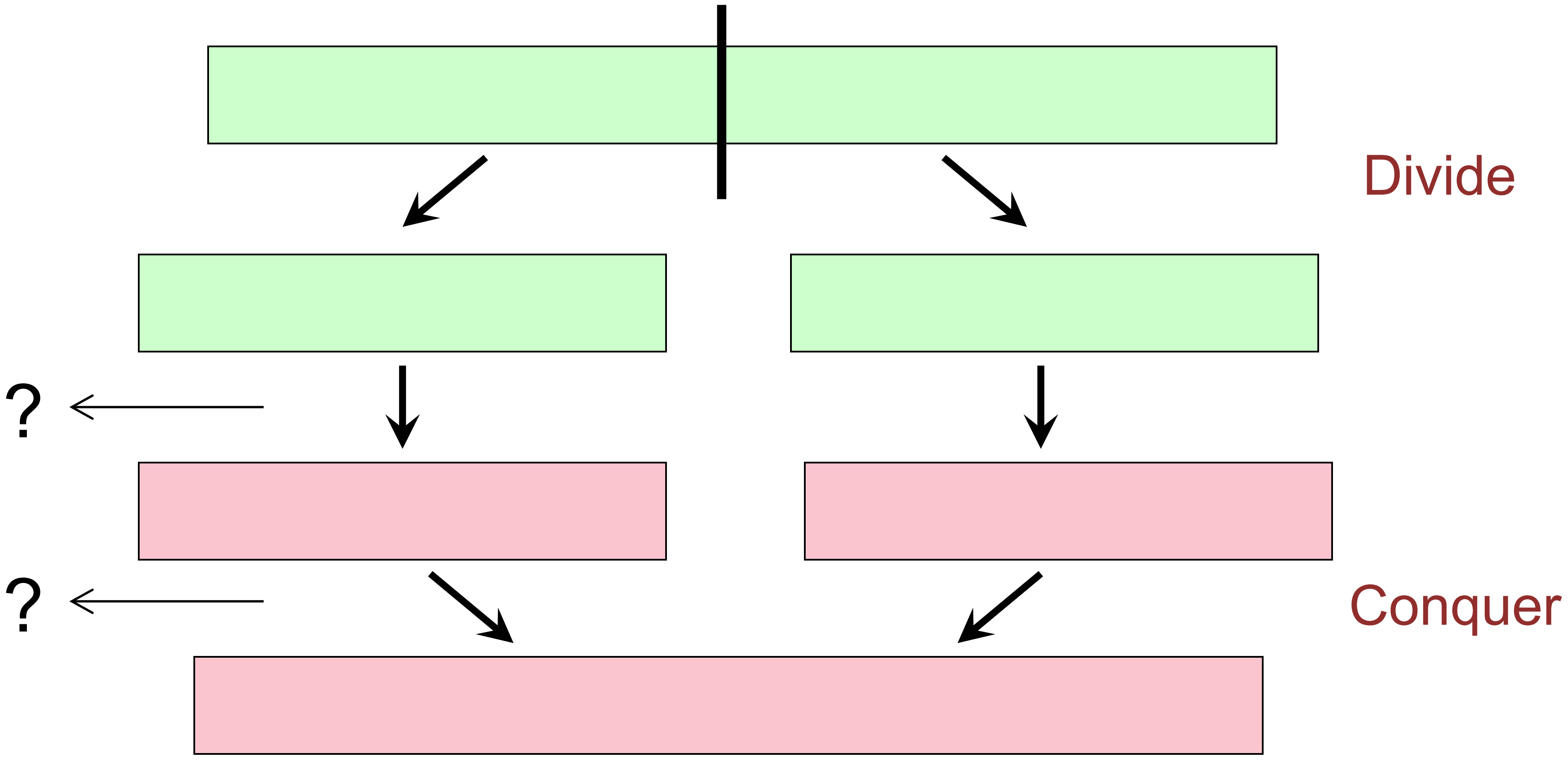
A more efficient sorting algorithm: Merge sort

MergeSort

- A faster sorting algorithm
  - has the optimal worst-case time complexity under the so-called comparison model.
- Use an idea called
  - divide-and-conquer to solve problems, which naturally leads to recursive algorithms.

Merge sort

- Use divide-and-conquer paradigm



Pseudo-code

```
MergeSort ( A, l, r )
    if (l ≥ r)  return;
    mid  = ⌊(l + r) / 2 ⌋;
    LeftA = MergeSort ( A, l, mid );
    RightA = MergeSort ( A, mid+1, r );
    B = Merge (A1,A2);
    return B;
```

Use recursive calls!

This is NOT in-place sorting!

- Input: an array A of length n
- Output: a new sorted array
- Call: MergeSort$(A, 1, n)$

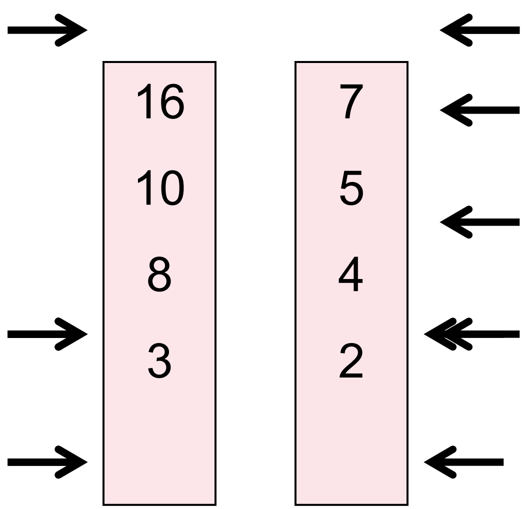
Correctness

- Recall for a recursive algorithm:
  - (1) Make sure algorithm works in the base case.
  - (2) Check that all recursive calls are on smaller problems, and that it terminates
  - (3) Assuming that the recursive calls work, does the whole algorithm work?

- (1) Base case:
  - Portion of array to be inspected is of size at most 1
  - Obviously already sorted!
- (2) Work on smaller subproblems? Terminate?
  - Yes
- (3) If recursive calls return correct output, does the entire algorithm works ?
  - Yes, as long as Merge (B, C) is correct.

Conquer: Merge(B, C)

- Input: Given two sorted arrays B and C
- Output: Merge into a single sorted array



2
2 3 4 5 7 8 10 16

Pseudo-code

```
Merge ( B, C )
    n_b = len(B); n_c = len(C);  n_o = n_b + n_c;
    init (outA, n_o);      //initialize outA to be an array of size n
    id_b = 0;  id_c = 0;
    for (i = 0; i < n_o; i++) {
        if (B[id_b] ≤ C[id_c]) or (id_b ≥ n_b )
                outA[i] = C[id_c];
                id_c++;
        else
                outA[i] = B[id_b];
                id_b++;
    }
    return outA;
```

Time complexity analysis

- First: worst case time complexity for Merge(B, C)
  - Let $n_b = len(B); n_c = len(C)$
  - Then the time $T_{merge(B,C)} = \Theta(n_b + n_c)$

Pseudo-code

```
MergeSort ( A, l, r )
    if (l ≥ r)  return;
    mid  = ⌊(l + r) / 2 ⌋;
    LeftA = MergeSort ( A, l, mid );
    RightA = MergeSort ( A, mid+1, r );
    B = Merge (A1,A2);
    return B;
```

- $T(n)$:
  - the worst case time complexity of MergeSort performed on a subarray of size n
- $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + cn = 2T\left(\frac{n}{2}\right) + cn$

Solving Recurrence relations

- $T(n) = 2T\left(\frac{n}{2}\right) + cn$

Solving Recurrence

- One way is via the following strategy:
  - 1. "Unroll" several times to find a pattern.
  - 2. Write general formula for kth unroll.
  - 3. Solve for # of unrolls needed to reach base case.
  - 4. Plug this number into general formula.

Solving Recurrence relations

- $T(n) = 2T\left(\frac{n}{2}\right) + cn$

$$= 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn = 4T\left(\frac{n}{4}\right) + 2cn$$

$$= 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn = 8T\left(\frac{n}{8}\right) + 3cn$$

$\ldots$ $$= 2^k T\left(\frac{n}{2^k}\right) + kcn$$

Terminates when $\frac{n}{2^k} = 1 \Rightarrow 2^k = n \Rightarrow k = \log_2 n$

Thus: $T(n) = 2^k T\left(\frac{n}{2^k}\right) + kcn = n\,T(1) + cn\log_2 n$

$$= \Theta(n \lg n)$$

Sorting problem

- The sorting problem can be solved in $\Theta(n \lg n)$ worst-case time.
- This is the optimal asymptotic time complexity
  - if we assume the so-called comparison model.
  - So under the comparison model, we can have an asymptotically faster algorithm than the merge sort.
- This algorithm is not in-place.
  - in practice, quicksort tends to be rather popular

More on solving recurrences

Another MergeSort

MergeSort (A, l, r)

if $(l \geq r)$ return;

$m_1 = l + (r - l) / 3$;

$m_2 = l + 2(r - l) / 3$;

$A1$ = MergeSort (A, l, m_1);

$A2$ = MergeSort ($A, m_1 + 1, m_2$);

$A3$ = MergeSort ($A, m_2 + 1, r$);

Merge ($A1, A2, A3$);

- Recurrence relation for MergeSort(A,1,n)
  - $T(n) = 3T(\frac{n}{3}) + cn$

Solving recurrence

- $T(n) = 3T(\frac{n}{3}) + cn$

Another example

- $T(n) = T(\frac{n}{2}) + cn$

The Movie problem revisited

Recall

- The Movie problem
  - Input: Given a list of length of movies available, stored in array $movies$, and a flight duration D
  - Output: Return two movies whose total length = D; None otherwise.

- Previously,
  - we gave an algorithm with worst-case time complexity $\Theta(n^2)$
- Can we do better?
  - Yes, if we first sort the input array of movie times.
- Example:
  - Flight time: 170
  - Movie times (sorted): $60, 80, 90, 110, 130$

Code

```
def optimize_entertainment(times, target):
    n = len(times)
    MergeSort(times, 0, n-1)
    shortest = 0
    longest = n - 1
    for i in range(n - 1):
        total_time = times[shortest] + times[longest]
        if total_time == target:
            return (shortest, longest)
        elif total_time < target:
            shortest += 1
        else: # total_time > target
            longest -= 1
    return None
```

Worst-case time complexity:

$$T(n) = \Theta(n \lg n) + \Theta(n)$$
$$= \Theta(n \lg n)$$

Take-home messages

- Sorting can be done in $\Theta(n \lg n)$ time
- More examples on solving recurrences
- Using sorted structures can sometimes help solve other problems more efficiently
  - e.g, binary search, and the movie problems.

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