

DSC 40B

Theoretical Foundations II

Dynamic Sets

Queries: Easy to Hard

- ▶ We've been thinking about **queries**.
 - ▶ Given a collection of data, is x in the collection?
- ▶ Querying is a fundamental operation.
 - ▶ Useful in a data science sense.
 - ▶ But also frequently performed in algorithms.
- ▶ There are several situations to think about.

Situation #1: Static Set, One Query

- ▶ **Given:** a collection of n numbers (or strings, etc.).
- ▶ In future, you will be asked single query.
- ▶ Best approach: linear search, $\Theta(n)$ worst case.

Situation #2: Static Set, Many Queries

- ▶ **Given:** a collection of n numbers (or strings, etc.).
- ▶ In future, you will be asked **many** queries.
- ▶ Best approach: sort + binary search
 - ▶ $\Theta(n \log n)$ time preprocessing
 - ▶ $\Theta(\log n)$ worst case for subsequent queries

Situation #3: Dynamic Set, Many Queries

- ▶ **Given:** a collection of n numbers (or strings, etc.).
- ▶ In future, you will be asked **many** queries *and* to **insert** new elements.
- ▶ Best approach: ?

Binary Search?

- ▶ Can we still use binary search?
- ▶ **Problem:** To us binary search, we must maintain array in sorted order as we insert new elements.
- ▶ Inserting into array takes $\Theta(n)$ time in worst case.
 - ▶ Must “make room” for new element.

Today

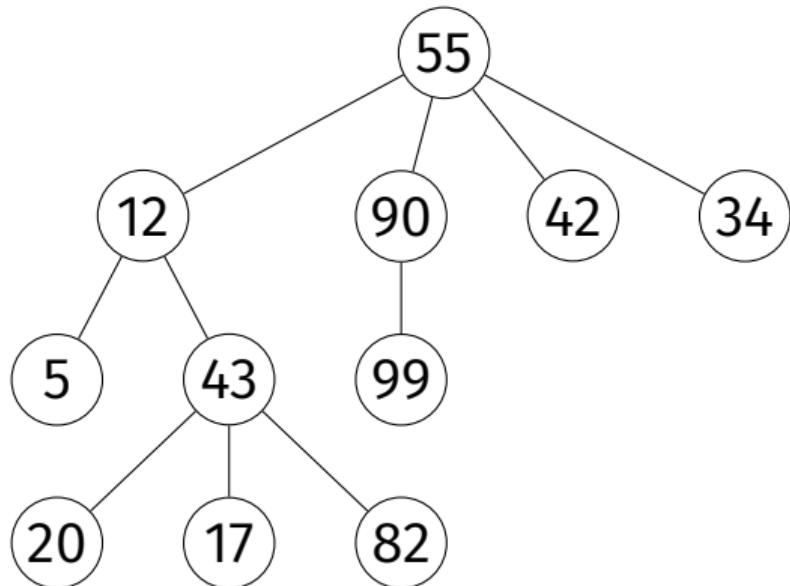
- ▶ Introduce (or review) **binary search trees**.
- ▶ BSTs support fast queries *and* insertions.
- ▶ Preserve sorted order of data after insertion.
- ▶ Can be modified to solve many problems efficiently.
 - ▶ Example: finding order statistics.

DSC 40B

Theoretical Foundations II

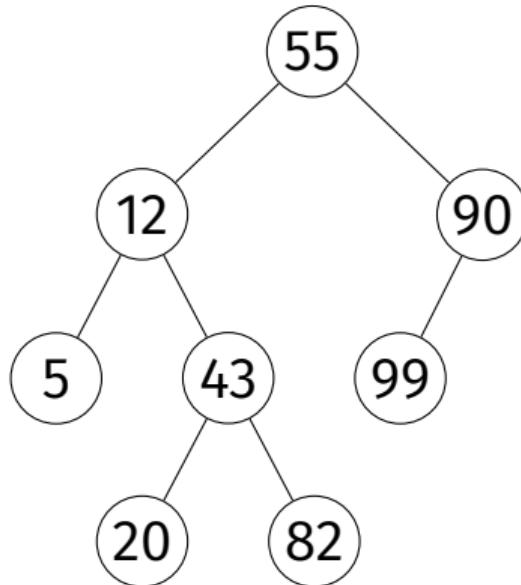
Binary Search Trees

Trees



Binary Trees

- ▶ Each node has *at most* two children (left and right).



Binary Search Tree

- ▶ A **binary search tree** (BST) is a binary tree that satisfies the following for any node x :
- ▶ if y is in x 's **left** subtree:

$$y.\text{key} \leq x.\text{key}$$

- ▶ if y is in x 's **right** subtree:

$$y.\text{key} \geq x.\text{key}$$

Assumption

- ▶ We'll assume that keys are unique (there are no duplicates).
- ▶ if y is in x 's **left** subtree:

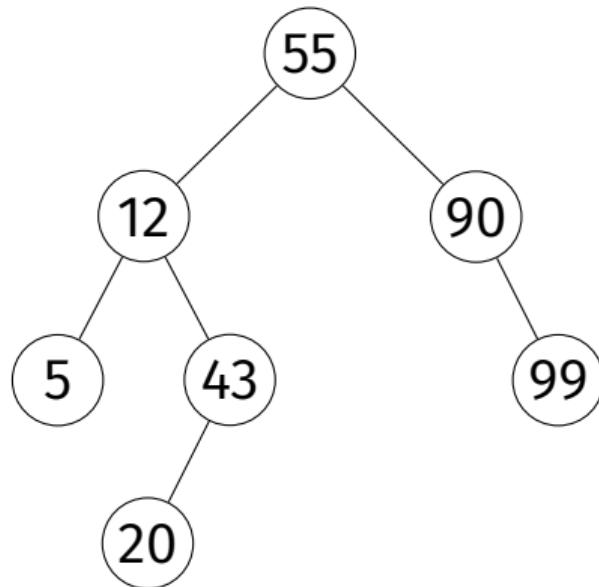
$$y.\text{key} < x.\text{key}$$

- ▶ if y is in x 's **right** subtree:

$$y.\text{key} > x.\text{key}$$

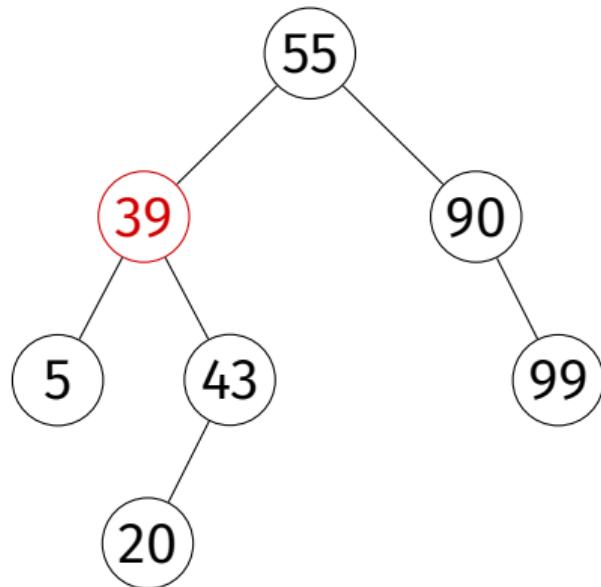
Example

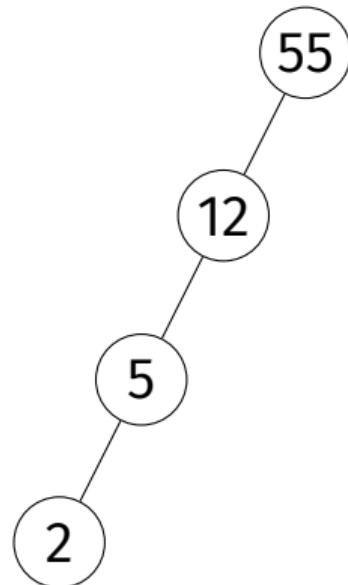
- ▶ This **is** a BST.



Example

- ▶ This is **not** a BST.





Exercise

Is this is a BST?

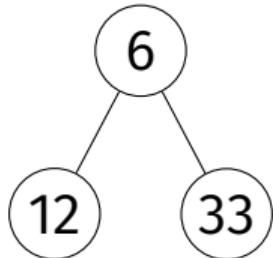
Height

- ▶ The **height** of a tree is the number of edges from the root to any leaf.
- ▶ Suppose a binary tree has n nodes.
- ▶ The **tallest** it can be is $\approx n$
- ▶ The **shortest** it can be is $\approx \log_2 n$

In Python

```
class Node:  
    def __init__(self, key, parent=None):  
        self.key = key  
        self.parent = parent  
        self.left = None  
        self.right = None  
  
class BinarySearchTree:  
    def __init__(self, root: Node):  
        self.root = root
```

In Python



```
root = Node(6)
n1 = Node(12, parent=root)
root.left = n1
n2 = Node(33, parent=root)
root.right = n2
tree = BinarySearchTree(root)
```

DSC 40B

Theoretical Foundations II

Queries and Insertions in BSTs

Operations on BSTs

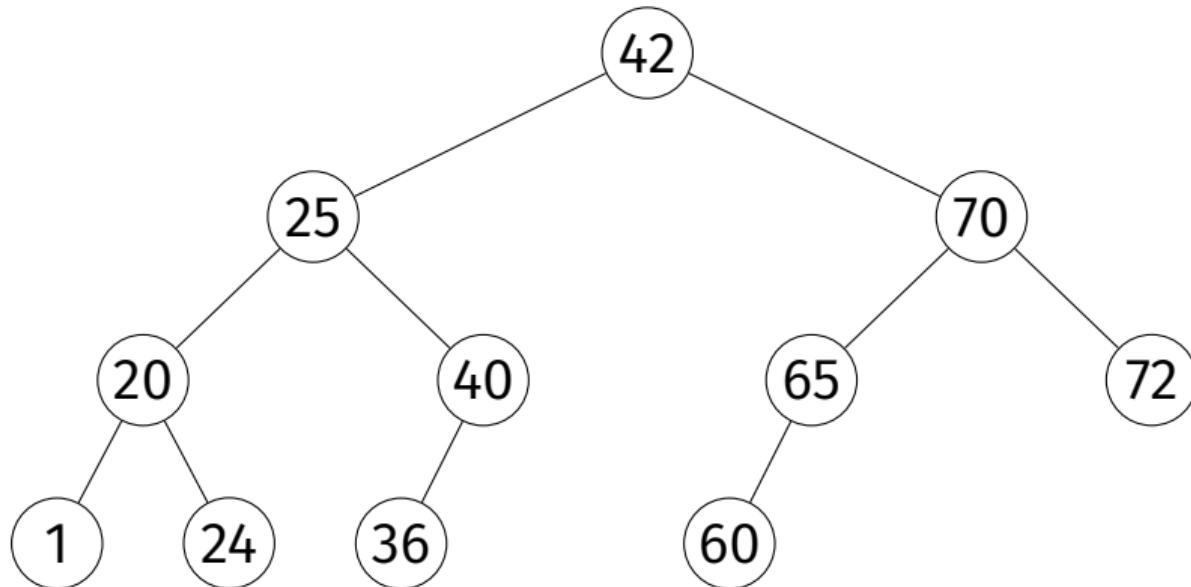
- ▶ We will want to:
 - ▶ **query** a key (is it in the tree?)
 - ▶ **insert** a new key

Queries

- ▶ **Given:** a BST and a target, t .
- ▶ **Return:** **True** or **False**, is the target in the collection?

Queries

- ▶ Is 36 in the tree? 65? 23?



Queries

- ▶ Start walking from root.
- ▶ If current node is:
 - ▶ equal to target, return **True**;
 - ▶ too large ($>$ target), follow left edge;
 - ▶ too small ($<$ target), follow right edge;
 - ▶ **None**, return **False**

Queries, in Python

```
def query(self, target):
    """As method of BinarySearchTree."""
    current_node = self.root
    while current_node is not None:
        if current_node.key == target:
            return current_node
        elif current_node.key < target:
            current_node = current_node.right
        else:
            current_node = current_node.left
    return None
```

Queries (Recursive)

```
def query_recursive(node, target):
    """As a 'free function'."""
    if node is None:
        return False

    if node.key == target:
        return node
    elif node.key < target:
        return query_recursive(node.right, target)
    else:
        return query_recursive(node.left, target)
```

Queries, Analyzed

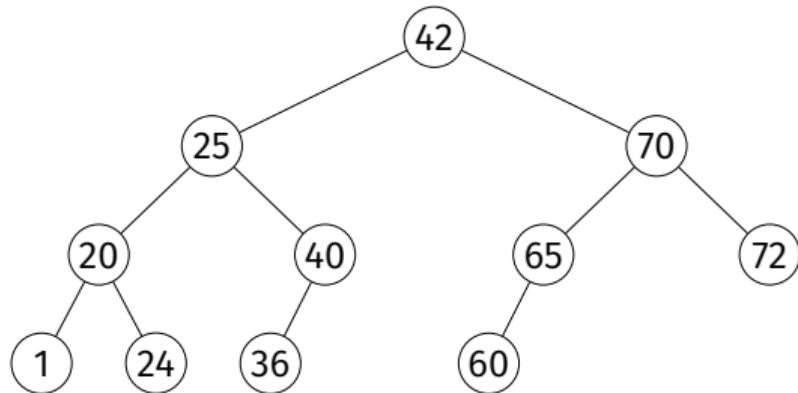
- ▶ Best case: $\Theta(1)$.
- ▶ Worst case: $\Theta(h)$, where h is **height** of tree.

Insertion

- ▶ **Given:** a BST and a new key, k .
- ▶ **Modify:** the BST, inserting k .
- ▶ Must **Maintain** the BST properties.

Insertion

- ▶ Insert 23 into the BST.



Insertion (The Idea)

- ▶ Traverse the tree as in query to find empty spot where new key should go, keeping track of last node seen.
- ▶ Create new node; make last node seen the parent, update parent's children.
- ▶ Be careful about inserting into empty tree!

```
def insert(self, new_key):
    # assume new_key is unique
    current_node = self.root
    parent = None

    # find place to insert the new node
    while current_node is not None:
        parent = current_node
        if current_node.key < new_key:
            current_node = current_node.right
        else: # current_node.key > new_key
            current_node = current_node.left

    # create the new node
    new_node = Node(key=new_key, parent=parent)

    # if parent is None, this is the root. Otherwise, update the
    # parent's left or right child as appropriate
    if parent is None:
        self.root = new_node
    elif parent.key < new_key:
        parent.right = new_node
    else:
        parent.left = new_node
```

Insertion, Analyzed

- ▶ Worst case: $\Theta(h)$, where h is **height** of tree.

Main Idea

Querying and insertion take $\Theta(h)$ time in the worst case, where h is the height of the tree.

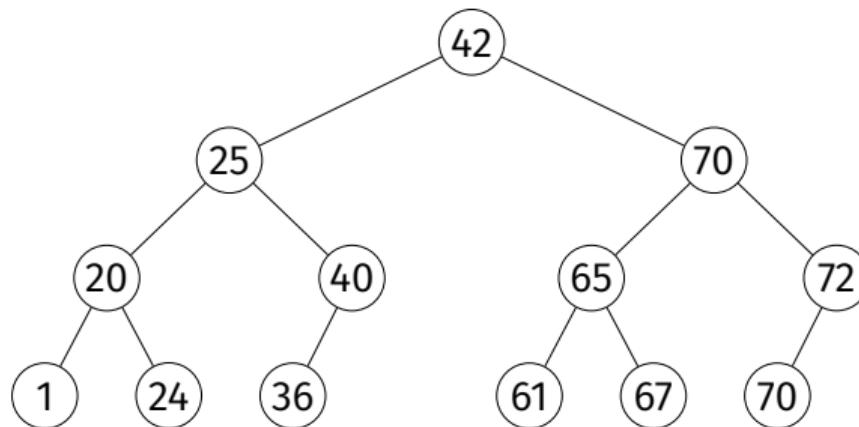
DSC 40B

Theoretical Foundations II

Balanced and Unbalanced BSTs

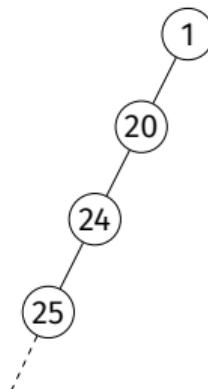
Binary Tree Height

- ▶ In case of very balanced tree, h grows **logarithmically** with n .
 - ▶ $h = \Theta(\log n)$
 - ▶ Query, insertion take worst case $\Theta(\log n)$ time.



Binary Tree Height

- ▶ In the case of very unbalanced tree, h grows **linearly** with n .
 - ▶ $h = \Theta(n)$
 - ▶ Query, insertion take worst case $\Theta(n)$ time.



Unbalanced Trees

- ▶ Occurs if we insert items in (close to) sorted or reverse sorted order.
- ▶ This is a **common** situation.

Example

- ▶ Insert 1, 2, 3, 4, 5, 6, 7, 8 (in that order).

Time Complexities

query	$\Theta(h)$
insertion	$\Theta(h)$

Where h is height, and $h = \Omega(\log n)$ and $h = O(n)$.

Time Complexities (Balanced)

query	$O(\log n)$
insertion	$O(\log n)$

Where h is height, and $h = \Omega(\log n)$ and $h = O(n)$.

Worst Case Time Complexities (Unbalanced)

query	$\Theta(n)$
insertion	$\Theta(n)$

- ▶ The worst case is **bad**.
 - ▶ Worse than using a sorted array!
- ▶ The worst case is **not rare**.

Main Idea

The operations take linear time in the worst case **unless** we can somehow ensure that the tree is **balanced**.

Self-Balancing Trees

- ▶ There are variants of BSTs that are **self-balancing**.
 - ▶ Red-Black Trees, AVL Trees, etc.
- ▶ Quite complicated to implement correctly.
- ▶ But their height is guaranteed to be $\Theta(\log n)$.
- ▶ So insertion, query take $\Theta(\log n)$ in worst case.

Warning!

If asked for the time complexity of a BST operation, be careful! A common mistake is to say that insertion/query are $\Theta(\log n)$ without being told that the tree is balanced.

Main Idea

In general, insertion/query take $\Theta(h)$ time in worst case. If tree is balanced, $h = \Theta(\log n)$, so they take $\Theta(\log n)$ time. If tree is badly unbalanced, $h = O(n)$, and they can take $O(n)$ time.

DSC 40B

Theoretical Foundations II

Augmenting BSTs

Modifying BSTs

- ▶ Perhaps more than most other data structures, BSTs must be modified (**augmented**) to solve unique problems.

Order Statistics

- ▶ Given n numbers, the **k th order statistic** is the k th smallest number in the collection.

Example

[99, 42, -77, -12, 101]

- ▶ 1st order statistic:
- ▶ 2nd order statistic:
- ▶ 4th order statistic:

Dynamic Set, Many Order Statistics

- ▶ Quickselect finds any order statistic in linear expected time.
- ▶ This is efficient for a static set.
- ▶ Inefficient if set is dynamic.

Goal

- ▶ Create a **dynamic** set data structure that supports fast computation of **any** order statistic.

BST Solution

- ▶ For each node, keep attribute `.size`, containing # of nodes in subtree rooted at current node
- ▶ Property:¹
 $x.size = x.left.size + x.right.size + 1$

¹If a left or right child doesn't exist, consider its size zero.

Computing Sizes

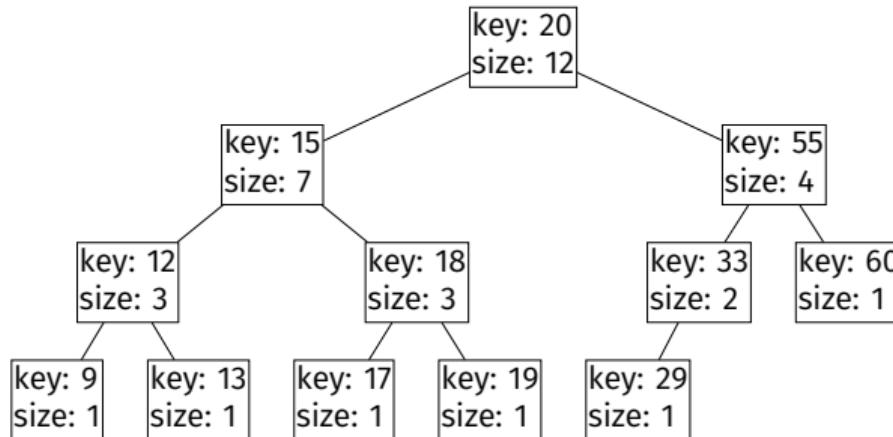
```
def add_sizes_to_tree(node):
    if node is None:
        return 0
    left_size = add_sizes_to_tree(node.left)
    right_size = add_sizes_to_tree(node.right)
    node.size = left_size + right_size + 1
    return node.size
```

Note

- ▶ Also need to maintain size upon inserting a node.

Computing Order Statistics

- ▶ 8th? 2nd? 12th



Augmenting Data Structures

- ▶ This is just one example, but many more.
- ▶ Understanding how BSTs work is key to augmenting them.