

DSC 40B

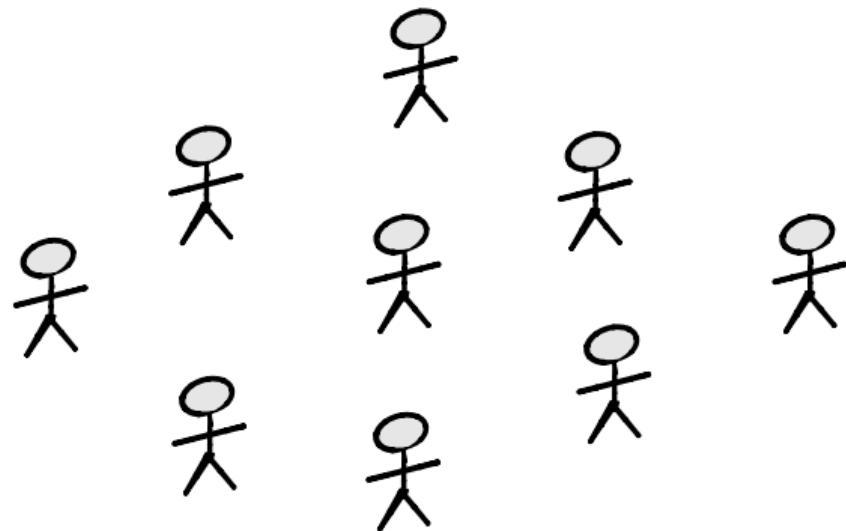
Theoretical Foundations II

Graphs

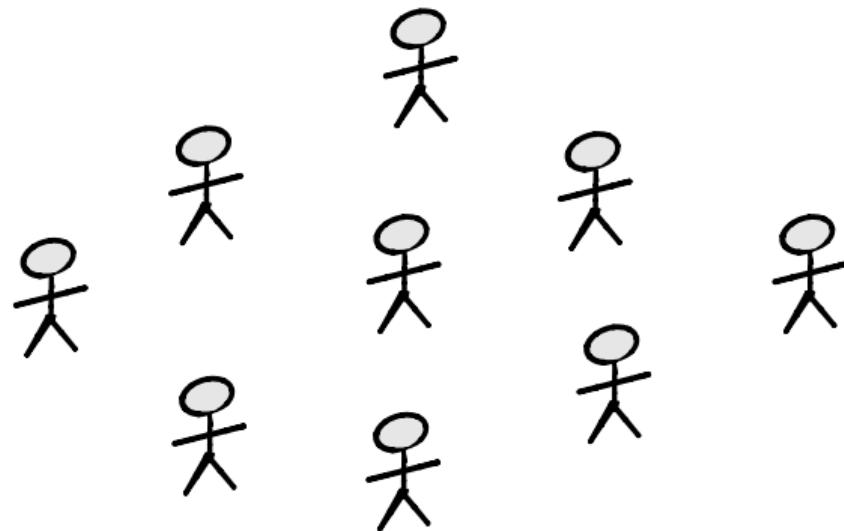
Data Types

- ▶ **Feature vectors**
 - ▶ We care about attributes of individuals.
- ▶ **Graphs**
 - ▶ We care about relationships between individuals.

Example: Facebook



Example: Twitter

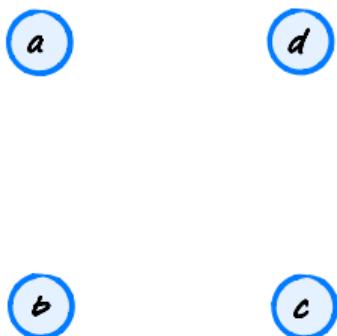


Definition

A **directed graph** (or **digraph**) G is a pair (V, E) where V is a finite set of **nodes** (or **vertices**) and E is a set of ordered pairs (the **edges**).

Example:

$$V = \{a, b, c, d\}$$
$$E = \{(a, c), (a, b), (d, b), (b, d), (b, b)\}$$



Directed Graphs (More Formally)

E is a subset of the **Cartesian product**, $V \times V$.

Example:

$$\{a, b, c\} \times \{1, 2\} =$$

Consequences

Because the edge set of a directed graph is allowed to be *any* subset of $V \times V$:

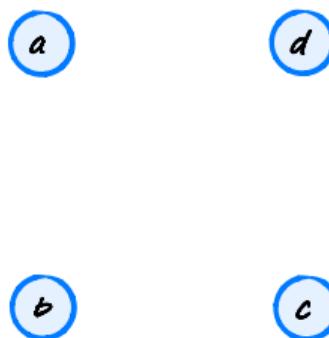
- ▶ the edges have directions.
 - ▶ e.g., (a, b) is “from a to b ”
- ▶ can have “opposite” edges.
 - ▶ e.g., (a, b) and (b, a) .
- ▶ can have “self-loops”
 - ▶ e.g., (a, a)

Definition

An **undirected graph** G is a pair (V, E) where V is a finite set of **nodes** (or **vertices**) and E is a set of unordered, distinct pairs (the **edges**).

Example:

$$V = \{a, b, c, d\}$$
$$E = \{\{a, c\}, \{a, b\}, \{d, b\}\}$$



Undirected Graphs (More Formally)

An edge in an undirected graph is a set $\{u, v\}$ where $u \neq v$. This has consequences:

- ▶ the edges have **no direction**.
 - ▶ e.g., $\{a, b\}$ is **not** “from” a “to” b .
- ▶ **cannot** have “opposite” edges.
 - ▶ e.g., $\{a, b\}$ and $\{b, a\}$ are the same.
- ▶ **cannot** have “self-loops”
 - ▶ e.g., $\{a, a\}$ is not a valid edge

Notational Note

Although edges in undirected graphs are sets, we typically write them as pairs: (u, v) instead of $\{u, v\}$.

Summary

- ▶ Edges have direction:
 - ▶ Directed: **yes**
 - ▶ Undirected: **no**
- ▶ Self-loops, (u, u) ?
 - ▶ Directed: **yes**
 - ▶ Undirected: **no**
- ▶ Opposite edges, (u, v) and (v, u) ?
 - ▶ Directed: **yes**
 - ▶ Undirected: **no** (they are the same edge)

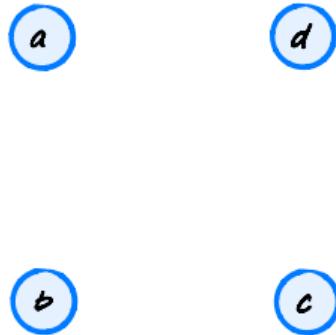
Note

Neither directed nor undirected graphs can have
duplicate edges¹

¹There are other definitions which allow duplicate edges.

Note

Graphs don't need to be "connected"²



²There are other definitions which allow duplicate edges.

Exercise

What is the greatest number edges possible in a **directed** graph?

Counting Edges

What is the greatest number edges possible in a **directed** graph?



Exercise

What is the greatest number edges possible in an **undirected** graph?

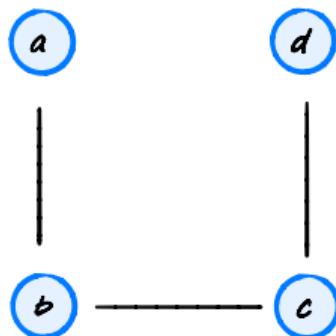
Counting Edges

What is the greatest number edges possible in an **undirected** graph?



Degree

The **degree** of a node in an undirected graph is the number of edges containing that node.



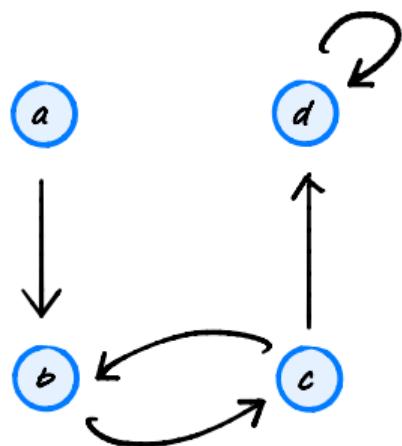
In-Degree/Out-Degree

The **in-degree** of a node in an directed graph is the number of edges **entering** that node.

The **out-degree** of a node in an directed graph is the number of edges **leaving** that node.

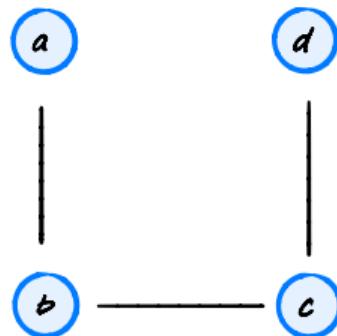
The **degree** of a node in a directed graph is the in-degree + out-degree.

Examples



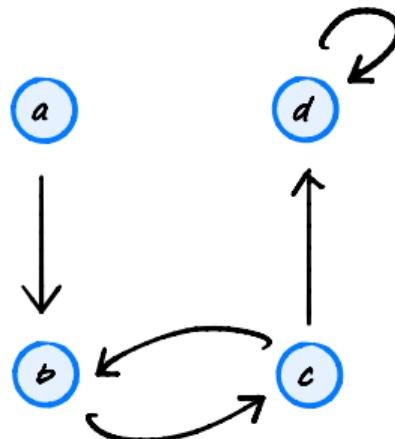
Neighbors

Definition: in an undirected graph, the set of **neighbors** of a node u is the set of all nodes which share an edge with u .



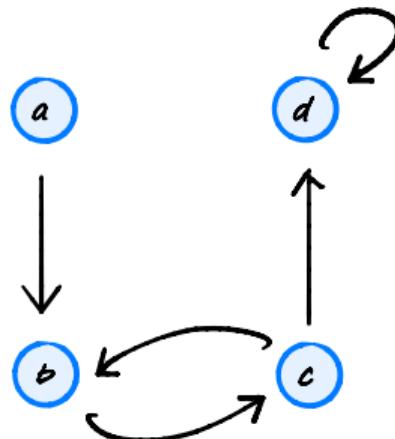
Predecessors

Definition: in an directed graph, the set of **predecessors** of a node u is the set of all nodes which are at the **start** of an edge **entering** u .



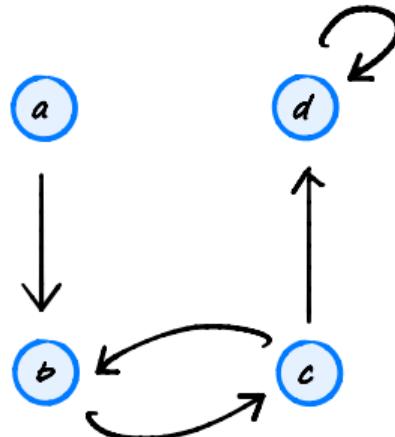
Successors

Definition: in an directed graph, the set of **successors** of a node u is the set of all nodes which are at the **end** of an edge **leaving** u .



A Convention

In a directed graph, the **neighbors** of u are the **successors** of u .



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Theoretical Foundations II

Paths

Example

- ▶ Consider a graph of direct flights.
- ▶ Each node is an airport.
- ▶ Each edge is a direct flight.
- ▶ Should the graph be directed or undirected?

Example



Example

- ▶ Can we get from San Diego to Columbus?
- ▶ Not with a single edge.
- ▶ But with a **path**.

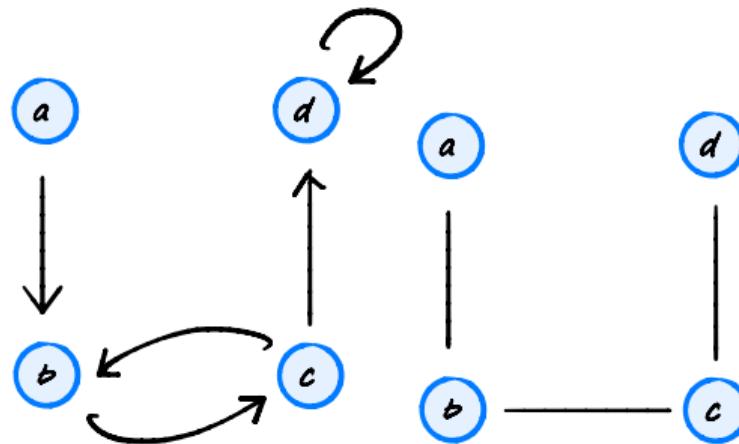
Definition

A **path** from u to u' in a (directed or undirected) graph $G = (V, E)$ is a sequence of one or more nodes $u = v_0, v_1, \dots, v_k = u'$ such that there is an edge between each consecutive pair of nodes in the sequence.

Path Length

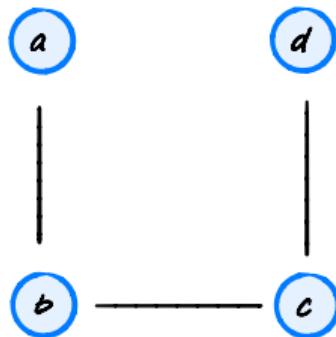
Definition: The **length** of a path is the number of nodes in the sequence, minus one. Paths of length zero are possible!

Examples



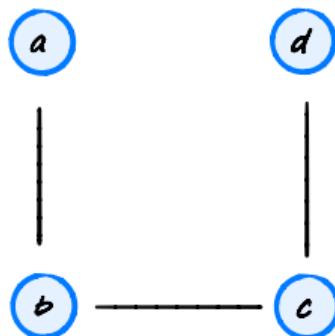
Note

Paths **can** go through the same node more than once!



Simple Paths

Definition: A **simple path** is a path in which every node is unique.



Reachability

Definition: node v is **reachable** from node u if there is a path from u to v .

Reachability and Directedness

- ▶ If G is undirected, reachability is symmetric.
 - ▶ If u reachable from v , then v reachable from u .
- ▶ If G is directed, reachability is **not** symmetric.
 - ▶ If u reachable from v , then v may/may not be reachable from u .

Important Trivia

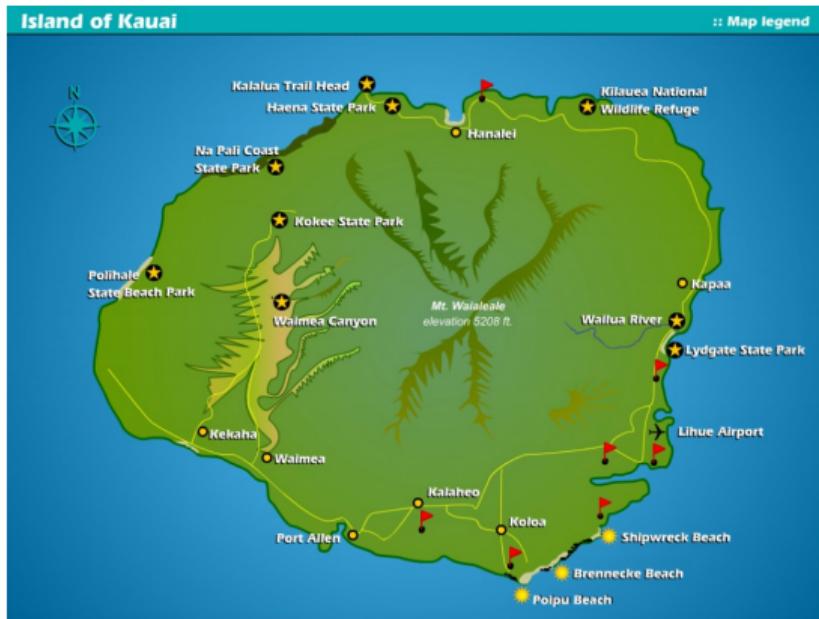
- ▶ In any graph, any node is **reachable** from **itself**.

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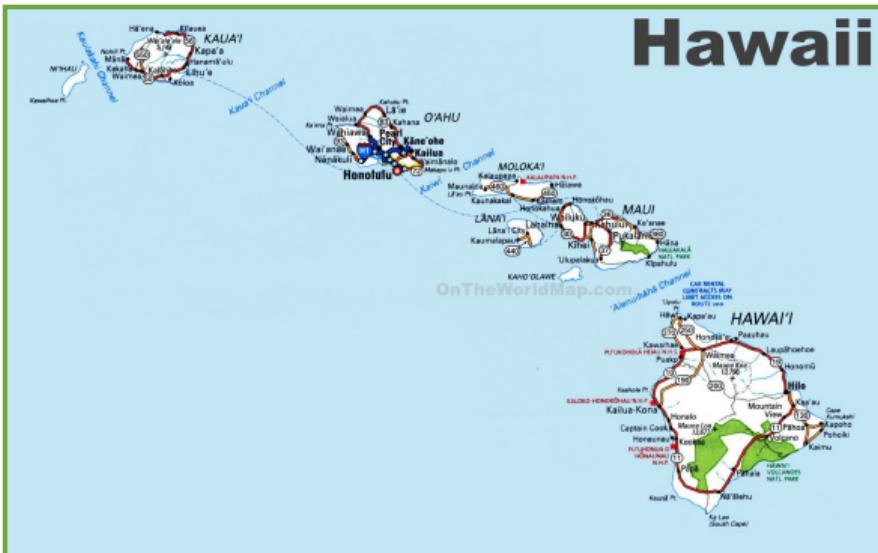
Theoretical Foundations II

Connected Components

Example



Example



Connectedness

A graph is **connected** if every node u is reachable from every other node v . Otherwise, it is **disconnected**.

Equivalent: there is a path between every pair of nodes.

Connected Components

A **connected component** is a maximally-connected set of nodes.

I.e., if $G = (V, E)$ is an undirected graph, a connected component is a set $C \subset V$ such that

- ▶ any pair $u, u' \in C$ are reachable from one another; and
- ▶ if $u \in C$ and $z \notin C$ then u and z are not reachable from one another.

Exercise

What are the connected components?

$$V = \{0, 1, 2, 3, 4, 5, 6\}$$

$$E = \{(0, 2), (1, 5), (3, 1), (2, 4), (0, 4), (5, 3)\}$$

Example

What are the connected components?

$$V = \{0, 1, 2, 3, 4, 5, 6\}$$

$$E = \{(0, 2), (1, 5), (3, 1), (2, 4), (0, 4), (5, 3)\}$$

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Theoretical Foundations II

Adjacency Matrices

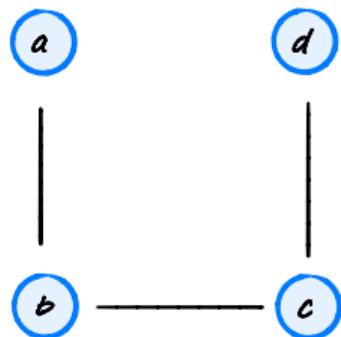
Representations

- ▶ How do we **store** a graph in a computer's memory?
- ▶ Three approaches:
 1. Adjacency matrices.
 2. Adjacency lists.
 3. "Dictionary of sets"

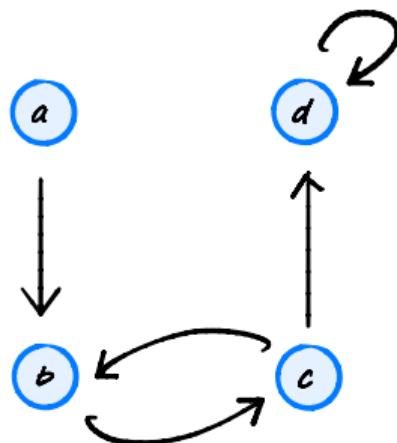
Adjacency Matrices

- ▶ Assume nodes are numbered $0, 1, \dots, |V| - 1$
- ▶ Allocate a $|V| \times |V|$ (Numpy) array
- ▶ Fill array as follows:
 - ▶ $\text{arr}[i, j] = 1$ if $(i, j) \in E$
 - ▶ $\text{arr}[i, j] = 0$ if $(i, j) \notin E$

Example



Example



Observations

- ▶ If G is undirected, matrix is symmetric.
- ▶ If G is directed, matrix may not be symmetric.

Time Complexity

operation	code	time
edge query	<code>adj[i, j] == 1</code>	$\Theta(1)$
$\text{degree}(i)$	<code>np.sum(adj[i, :])</code>	$\Theta(V)$

Space Requirements

- ▶ Uses $|V|^2$ bits, even if there are very few edges.
- ▶ But most real-world graphs are **sparse**.
 - ▶ They contain many fewer edges than possible.

Example: Facebook

- ▶ Facebook has 2 billion users.

$$(2 \times 10^9)^2 = 4 \times 10^{18} \text{ bits}$$

= 500 petabits

≈ 6500 years of video at 1080p

≈ 60 copies of the internet as it was in 2000

Adjacency Matrices and Math

- ▶ Adjacency matrices are useful mathematically.
- ▶ Example: (i, j) entry of A^2 gives number of hops of length 2 between i and j .

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Theoretical Foundations II

Adjacency Lists

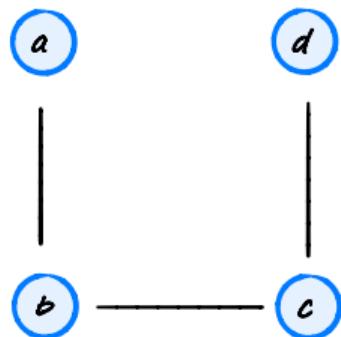
What's Wrong with Adjacency Matrices?

- ▶ Requires $\Theta(|V|^2)$ storage.
- ▶ Even if the graph has no edges.
- ▶ **Idea:** only store the edges that exist.

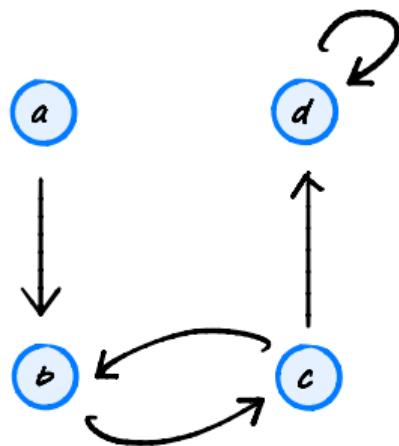
Adjacency Lists

- ▶ Create a list adj containing $|V|$ lists.
- ▶ $\text{adj}[i]$ is list containing the neighbors of node i .

Example



Example



Observations

- ▶ If G is undirected, each edge appears twice.
- ▶ If G is directed, each edge appears once.

Time Complexity

operation	code	time
edge query	<code>j in adj[i]</code>	$\Theta(\text{degree}(i))$
$\text{degree}(i)$	<code>len(adj[i])</code>	$\Theta(1)$

Space Requirements

- ▶ Need $\Theta(|V|)$ space for outer list.
- ▶ Plus $\Theta(|E|)$ space for inner lists.
- ▶ In total: $\Theta(|V| + |E|)$ space.

Example: Facebook

- ▶ Facebook has 2 billion users, 400 billion friendships.
- ▶ If each edge requires 32 bits:

$$\begin{aligned}& (2 \text{ bits} \times 200 \times (2 \text{ billion})) \\&= 64 \times 400 \times 10^9 \text{ bits} \\&= 3.2 \text{ terabytes} \\&= 0.04 \text{ years of HD video}\end{aligned}$$

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Theoretical Foundations II

Dictionary of Sets

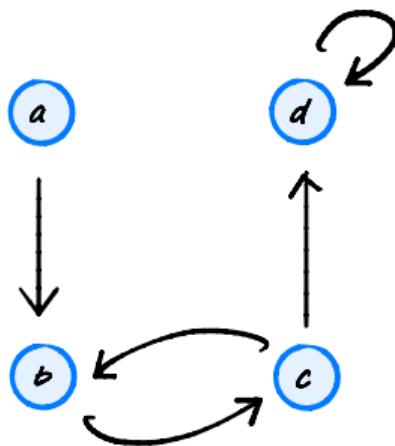
Tradeoffs

- ▶ Adjacency matrix: fast edge query, lots of space.
- ▶ Adjacency list: slower edge query, space efficient.
- ▶ Can we have the best of both?

Idea

- ▶ Use **hash tables**.
- ▶ Replace inner edge lists by **sets**.
- ▶ Replace outer list with **dict**.
 - ▶ Doesn't speed things up, but allows nodes to have arbitrary labels.

Example



Time Complexity

operation	code	time
edge query	<code>j in adj[i]</code>	$\Theta(1)$ average
degree(i)	<code>len(adj[i])</code>	$\Theta(1)$ average

Space Requirements

- ▶ Requires only $\Theta(E)$.
- ▶ But there is overhead to using hash tables.

Dict-of-sets implementation

On datahub:

```
import dsc4ograph
```

Or download from the course page.