

DSC40B:
Theoretical Foundations of Data
Science II

Lecture 8: *Binary search tree*

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Set operations

- ▶ Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:

First approach: sort the array of keys

- ▶ Search
- ▶ Maximum
- ▶ Minimum
- ▶ Successor
- ▶ Predecessor

$\Theta(\lg n)$

$\Theta(1)$

$\Theta(1)$

$\Theta(1)$

$\Theta(1)$

- ▶ Insert
- ▶ Delete
- ▶ Extract-Max
- ▶ Increase-key

$\Theta(n)$

How to have a good data structure so we can support all these operations efficiently?

Today

- ▶ Binary search tree
 - ▶ support all the operations from previous slide
 - ▶ in time proportional to height of tree
- ▶ (Review): how to implement key operations, and time complexity
 - ▶ search, insert (and delete)
- ▶ Extension to **balanced** binary search tree
- ▶ *Select* query: **augmenting** data structure
 - ▶ median, order statistics

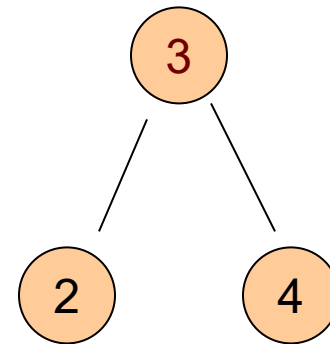


What is binary search tree?



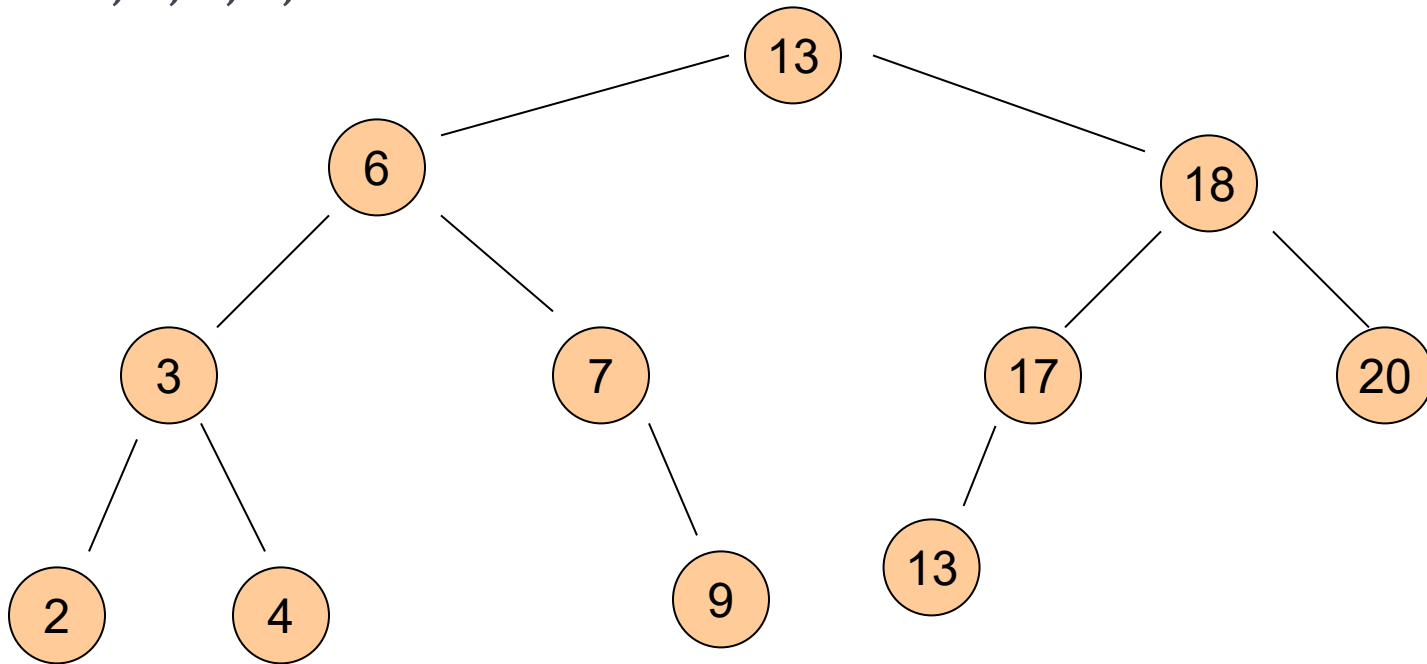
Binary tree

- ▶ A binary tree is a rooted tree
 - ▶ where each node has at most 2 children
- ▶ Represented by a linked data structure
- ▶ Each node contains at least fields:
 - ▶ *Key*
 - ▶ *Left*
 - ▶ *Right*
 - ▶ *Parent*



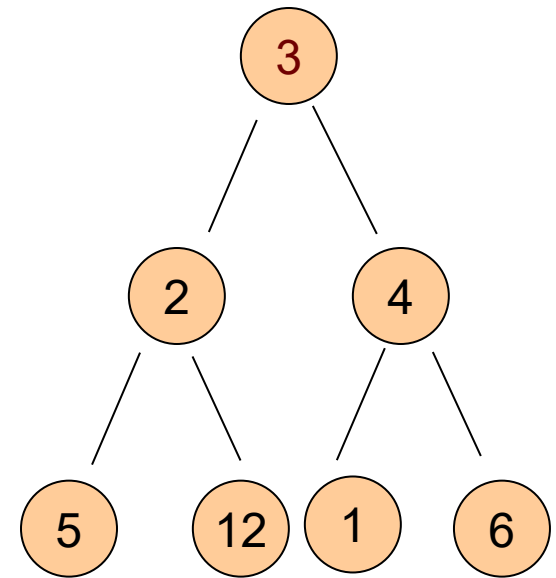
Example

- ▶ From root, following left pointers, we will visit
 - ▶ 13, 6, 3, 2, *Nil*



Binary tree

- ▶ A binary tree is a rooted tree where
 - ▶ each node has at most 2 children
- ▶ A node is the root of the tree
 - ▶ if its parent is Nil
- ▶ A node is a leaf
 - ▶ if both children are Nil
- ▶ Left sub-tree, right sub-tree



- ▶ A **complete binary tree** is a binary tree
 - ▶ where each node has two children other than leaves
 - ▶ and each level (except possibly last level) is filled and all nodes are as left as possible.
-



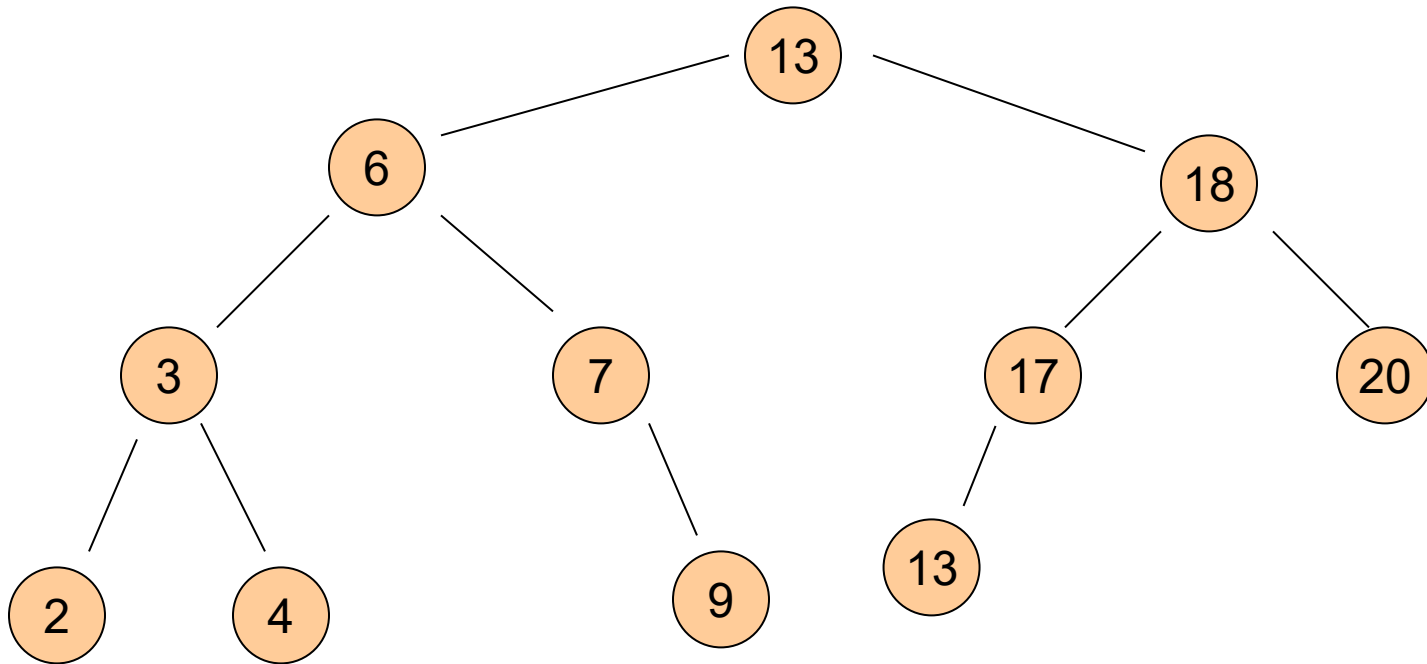
Binary search tree (BST)

- ▶ **Binary-search-tree property**
 - ▶ For any node $x \in T$,
 - ▶ $x.Key \geq y.Key$ if y is in the left subtree of x ; and
 - ▶ $x.Key \leq y.Key$ if y is in the right subtree of x
- ▶ A binary tree T is a **binary search tree (BST)** if
 - ▶ it satisfies the binary search tree property



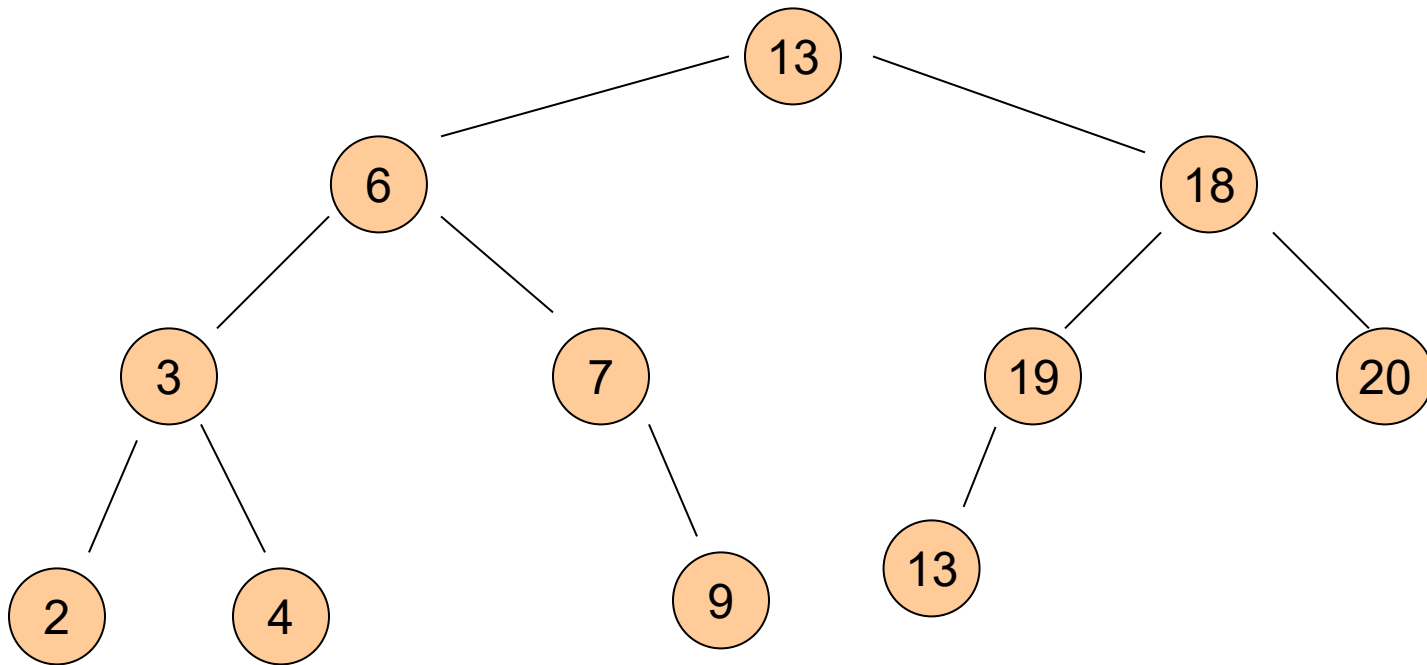
Example

► A valid BST



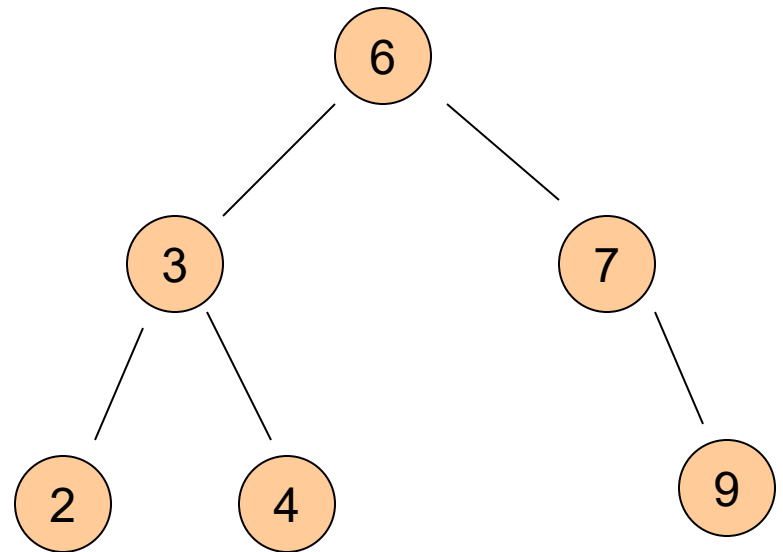
Example

► ?



Properties

- ▶ **Given the same set of elements**
 - ▶ there are many possible BSTs over them
- ▶ **Minimum?**
 - ▶ Does it have to be a leaf?



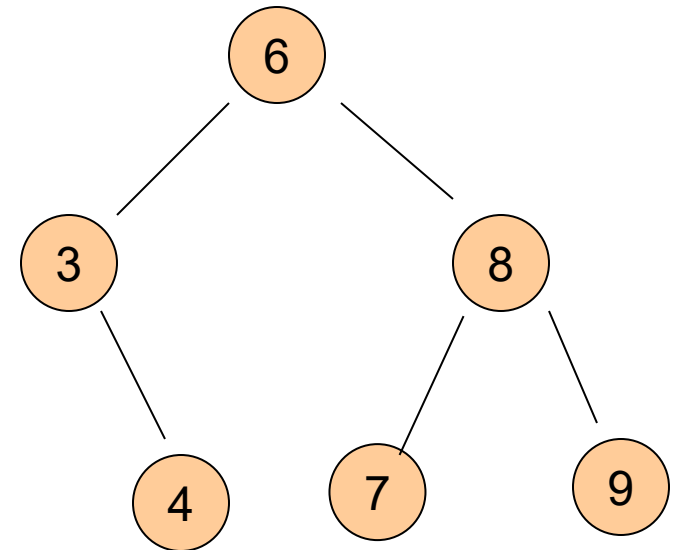
Properties

- ▶ Given the same set of elements
 - ▶ there are many possible BSTs over them

- ▶ Minimum?
 - ▶ Does it have to be a leaf?

- ▶ Maximum?

- ▶ Given n nodes,
 - ▶ Tallest possible BST tree has height $h = \underline{n}$
 - ▶ Shortest possible BST tree has height $h = \underline{\log_2 n} = \Theta(\lg n)$



Operations in BST



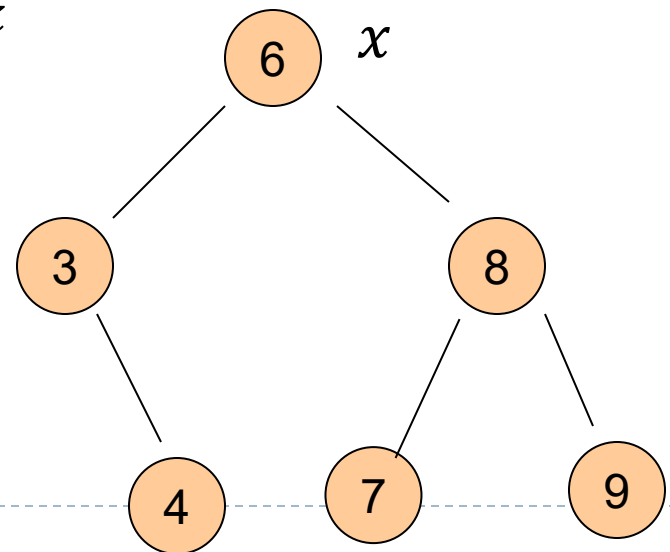
Search operation

- ▶ A BST T with n nodes can be viewed as a way to store n keys in a smart way, so that queries among these keys become easy.
- ▶ $\text{Tree-search}(x, k)$
 - ▶ Input: given a tree node x and a query key k
 - ▶ Output: search whether k is in the tree rooted at x
 - ▶ if it is in, return a node y s.t. $y.\text{key} = k$
 - ▶ otherwise, returns NIL

$\text{Tree-search}(x, 8)$

$\text{Tree-search}(x, 4)$

$\text{Tree-search}(x, 5)$



Tree-search algorithm, recursive version

Tree-search (x , k)

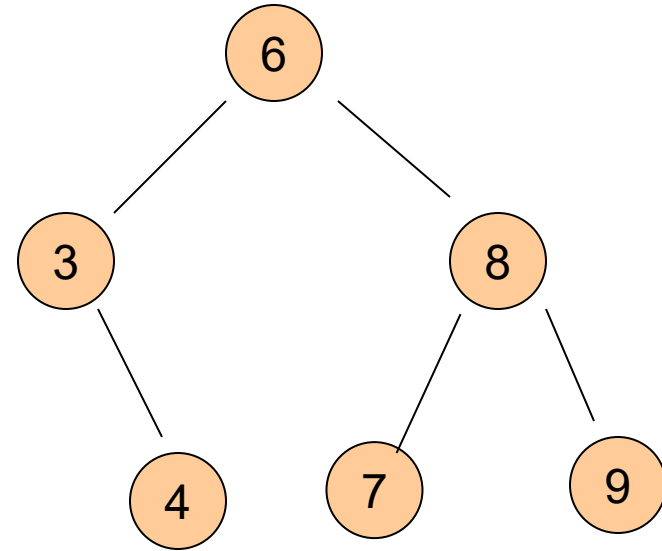
if $x = \text{Nil}$ or $k = x.\text{key}$

then return x

if $k < x.\text{key}$

then return Tree-search($x.\text{left}$, k)

else return Tree-search($x.\text{right}$, k)



- ▶ Given an input tree T and a key k
 - ▶ we will start by calling Tree-search($T.\text{root}$, k)



Tree-search algorithm, recursive version

Tree-search (x , k)

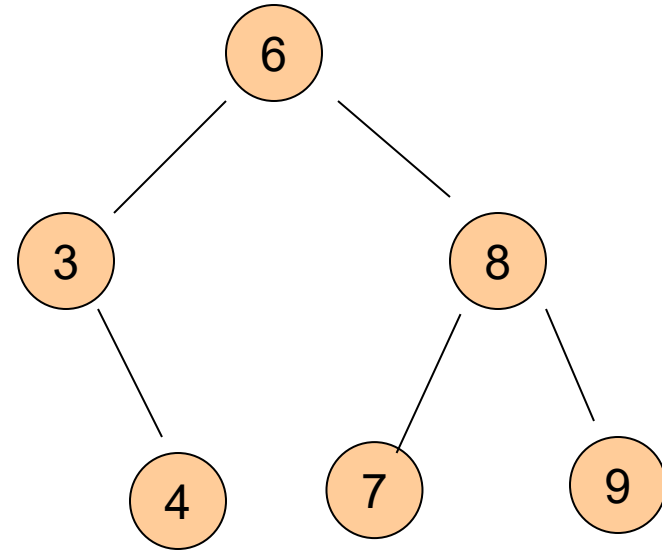
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► Time complexity analysis

- let $T(n)$ denote the worst case time complexity of procedure Tree-search() on any tree of n nodes



Tree-search algorithm, recursive version

Tree-search (x , k)

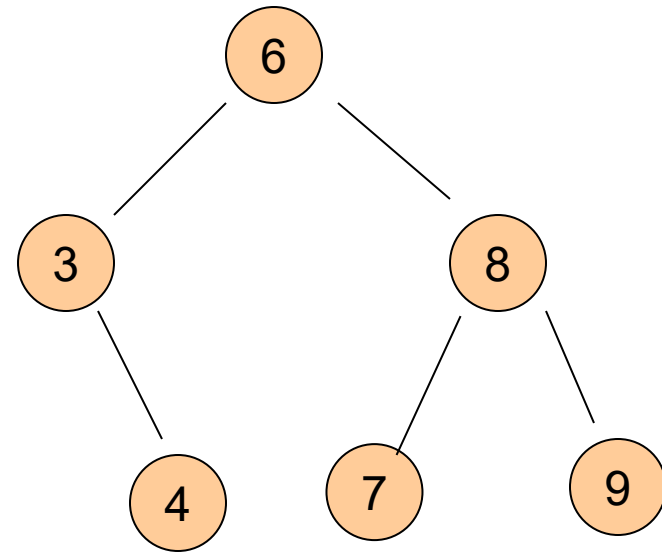
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then return x

if $k < x.\text{key}$

then return Tree-search($x.\text{left}$, k)

else return Tree-search($x.\text{right}$, k)



► Time complexity analysis

- other than recursive call, $\Theta(1)$ within each Tree-search call
- thus, $T(n)$ is proportional to the number of nodes x we will call Tree-search on
- $T(n) = \Theta(\text{tree-height}) = O(n)$



Tree-search: iterative version

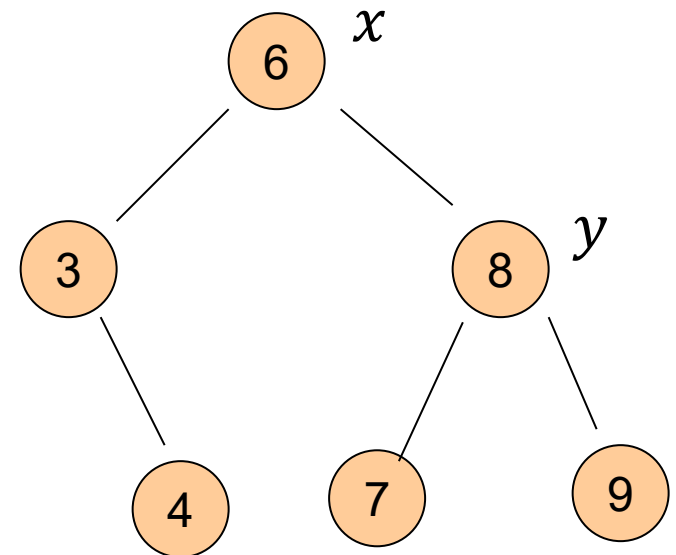
```
    procedure IterativeTreeSearch( $x, K$ )  
1  while ( $x = \text{NIL}$ ) and ( $K \neq x.\text{key}$ ) do  
2      |   if ( $K \leq x.\text{key}$ ) then  
3          |    $x \leftarrow x.\text{left};$   
4      |   else  
5          |    $x \leftarrow x.\text{right};$   
6      |   end  
7  end  
8  return ( $x$ );
```



Minimum / Maximum

▶ Tree-minimum(x)

- ▶ Input: a node x of a BST T
- ▶ Output: return the node containing minimum key in the subtree rooted at x



Minimum / Maximum

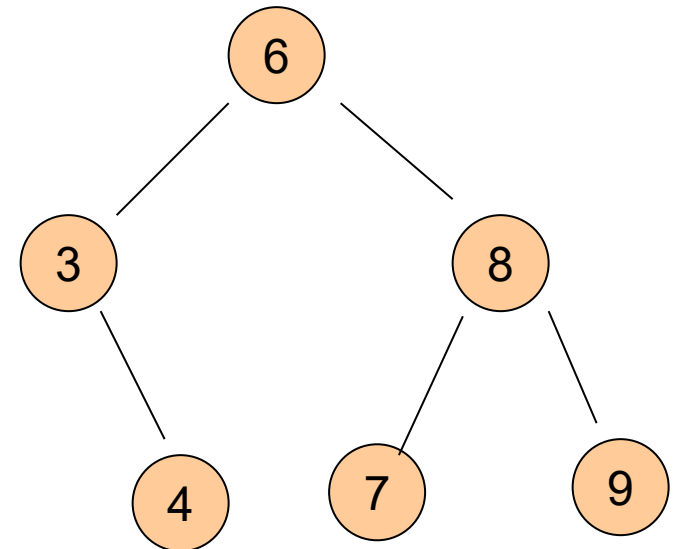
▶ Tree-minimum(x)

- ▶ Input: a node x of a BST T
- ▶ Output: return the node containing a minimum key in the subtree rooted at x

```
Tree-minimum( $x$ )  
    while (  $x.\text{left} \neq \text{Nil}$  )  
        do  $x = x.\text{left};$   
    return  $x$ ;
```

▶ Time complexity

- ▶ $T(n) = \Theta(h)$ where h is height of input tree



Minimum / Maximum

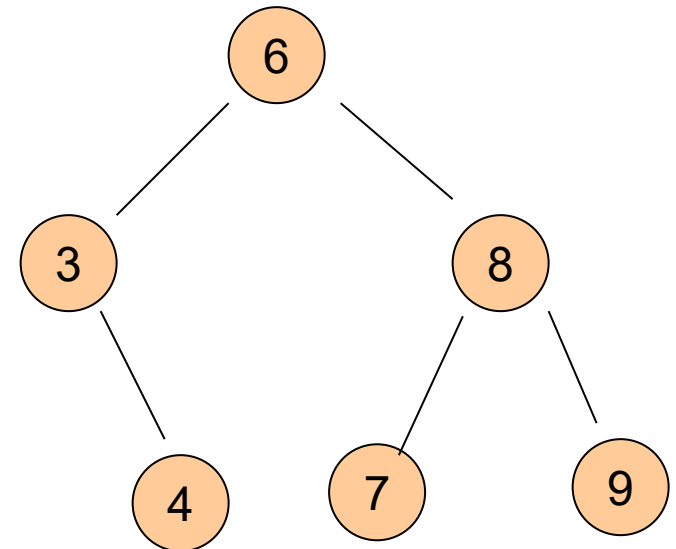
▶ Tree-maximum(x)

- ▶ Input: a node x of a BST T
- ▶ Output: return the node containing a maximum key in the subtree rooted at x

```
Tree-maximum( $x$ )  
    while (  $x.right \neq \text{Nil}$  )  
        do  $x = x.right$ ;  
    return  $x$ ;
```

▶ Time complexity

- ▶ $T(n) = \Theta(h)$ where h is height of input tree



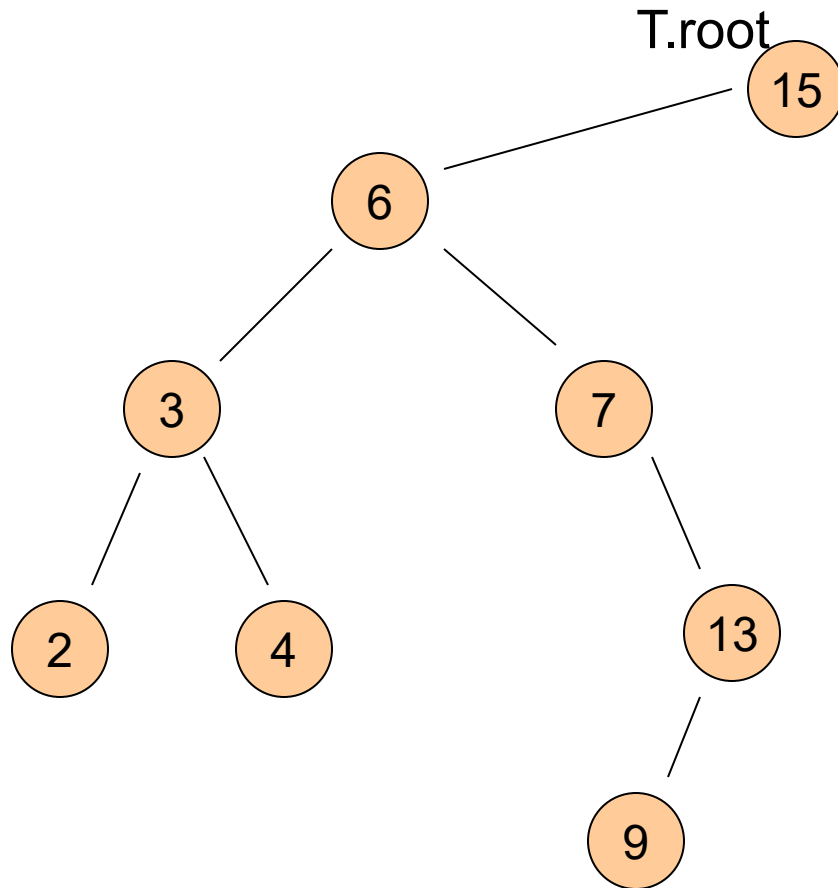
Tree-insert

- ▶ **Tree-insert(x, k)**

- ▶ Input: a BST tree node x and a key k
- ▶ Output: insert k to the tree rooted at x such that the resulting tree is still a binary search tree



Examples



Tree-Insert(T.root, 8)

Tree-Insert(T.root, 6.5)

Use tree-search !



Tree-insert

Tree-insert(T, k)

$y = \text{Nil}; x = T.\text{root}$

$z.\text{key} = k; z.\text{left} = \text{Nil}; z.\text{right} = \text{Nil}$

while ($x \neq \text{Nil}$) do

$y = x$

 if ($z.\text{key} < x.\text{key}$)

 then $x = x.\text{left}$

 else $x = x.\text{right}$

$z.\text{parent} = y$

if ($y = \text{Nil}$) then $T.\text{root} = z$

else if ($z.\text{key} < y.\text{key}$)

 then $y.\text{left} = z$

 else $y.\text{right} = z$

- z is the new node to be inserted
- Locate potential parent y of z .

- Set up z as appropriate child of y

Tree-insert

Tree-insert(T, k)

$y = \text{Nil}; x = T.\text{root}$

$z.\text{key} = k; z.\text{left} = \text{Nil}; z.\text{right} = \text{Nil}$

while ($x \neq \text{Nil}$) do

$y = x$

 if ($z.\text{key} < x.\text{key}$)

 then $x = x.\text{left}$

 else $x = x.\text{right}$

$z.\text{parent} = y$

if ($y = \text{Nil}$) then $T.\text{root} = z$

else if ($z.\text{key} < y.\text{key}$)

 then $y.\text{left} = z$

 else $y.\text{right} = z$

► Time complexity

► $T(n) = \Theta(h)$, where h is height of input tree

Summary

- ▶ Suppose n input keys are already stored in a BST of height h

Time complexity

▶ Search	$\Theta(h)$
▶ Maximum	$\Theta(h)$
▶ Minimum	$\Theta(h)$
▶ Successor	$\Theta(h)$
▶ Predecessor	$\Theta(h)$
▶ Insert	$\Theta(h)$
▶ Delete	$\Theta(h)$
▶ Extract-Max	$\Theta(h)$
▶ Increase-key	$\Theta(h)$

- However, performance depending on height!
- Height $h = O(n)$ and $h = \Omega(\lg n)$

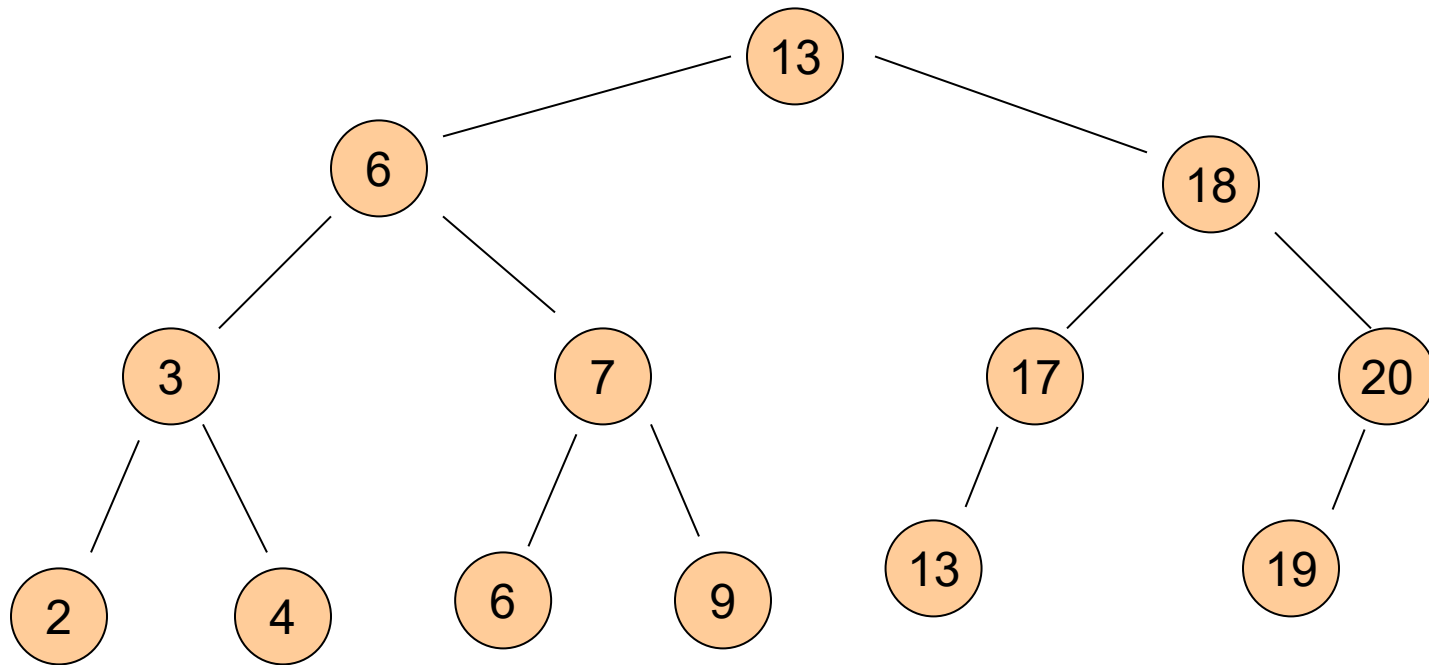
- To have good performance, we want to keep the tree height low!



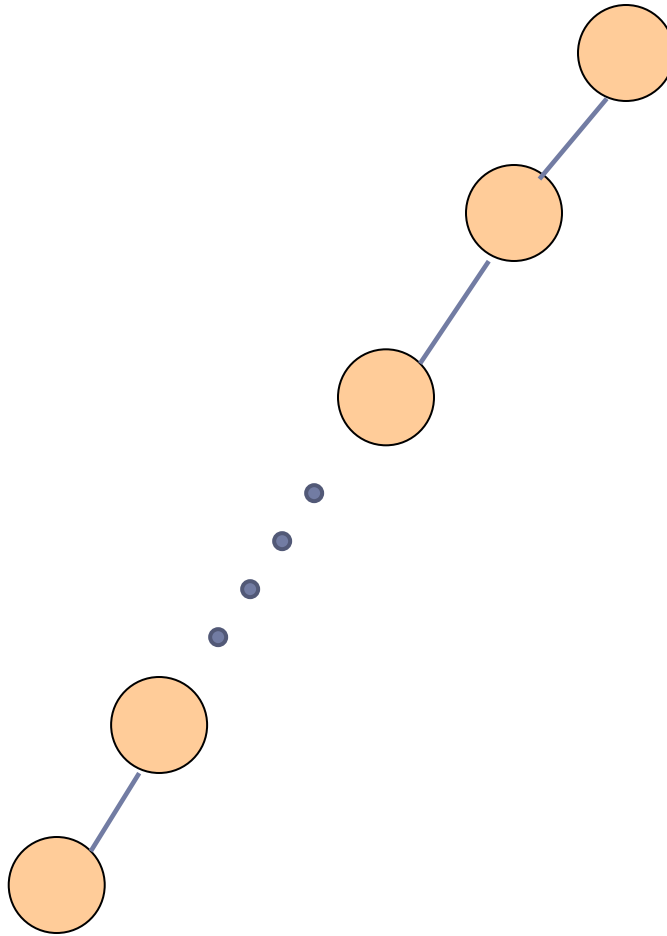
Balanced binary search tree



Good tree



Bad Tree



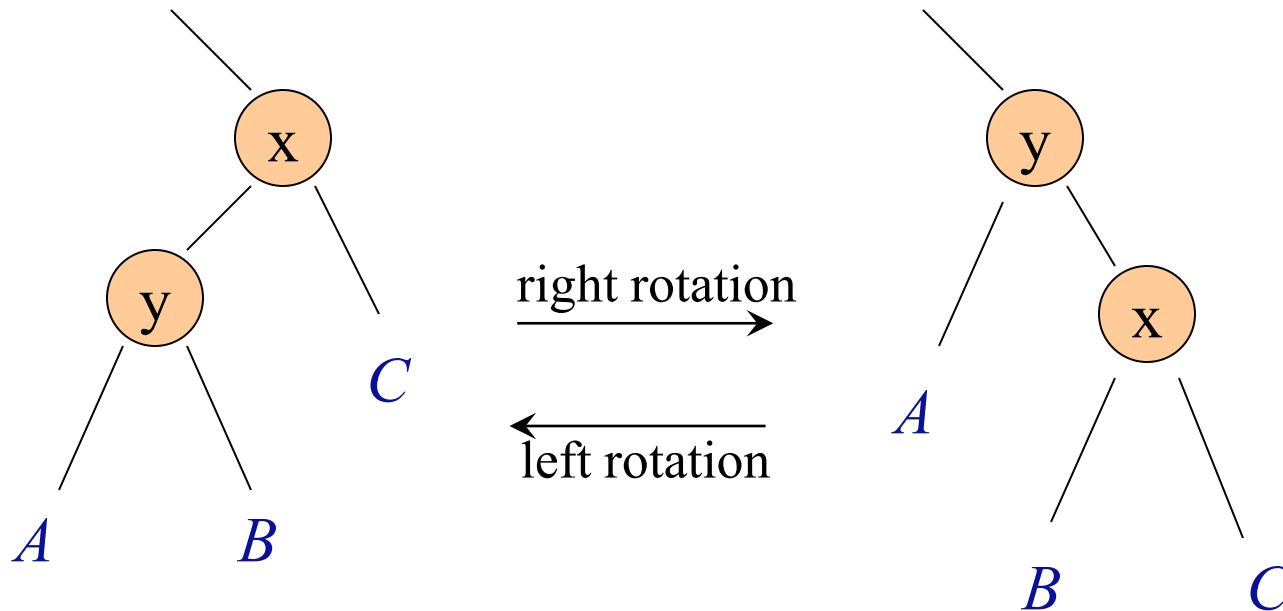
Balanced binary search tree

- ▶ It turns out that there are ways to add extra conditions to binary search trees, so that their height is $\Theta(\lg n)$
 - ▶ E.g, red-black tree, AVL tree, etc
- ▶ Once such a tree is created,
 - ▶ it can support search, minimum, maximum etc in $\Theta(h) = \Theta(\lg n)$ time using the same algorithms described before
 - ▶ the extra work comes at handling dynamic operations: insertion, deletion, and so on. Re-balancing is needed
 - ▶ however, for standard balanced BSTs, all these operations can be handled in $\Theta(\lg n)$ time.



Rotation operation

- ▶ Left rotation or Right rotation to keep tree height low



With balanced BST

- ▶ Suppose n input keys are already stored in a balanced BST

Time complexity	
-----------------	--

▶ Search	$\Theta(\lg n)$
▶ Maximum	$\Theta(\lg n)$
▶ Minimum	$\Theta(\lg n)$
▶ Successor	$\Theta(\lg n)$
▶ Predecessor	$\Theta(\lg n)$
▶ Insert	$\Theta(\lg n)$
▶ Delete	$\Theta(\lg n)$
▶ Extract-Max	$\Theta(\lg n)$
▶ Increase-key	$\Theta(\lg n)$

- Height of tree will be $\Theta(\lg n)$, where n is number of nodes in the tree



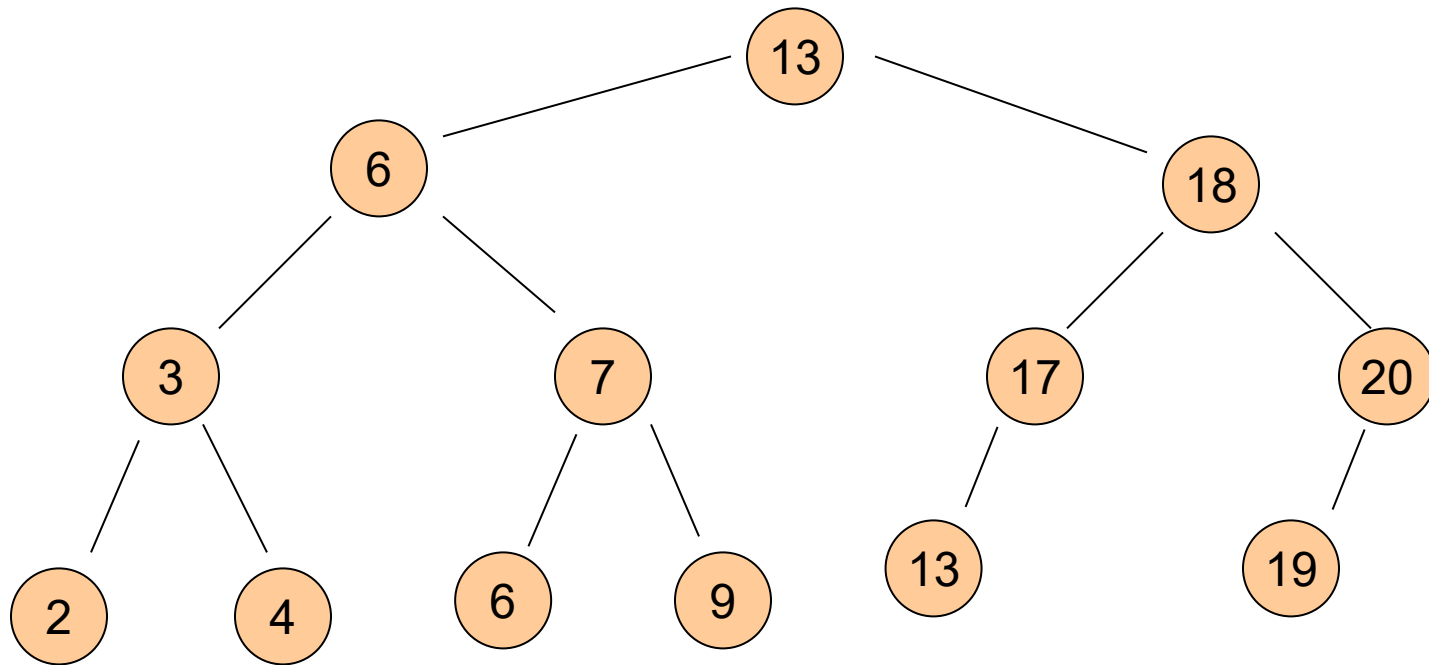
Select queries
augmenting data structure



-
- ▶ What if we also want to perform Select operation
 - ▶ BST-Select (x, k):
 - ▶ Given a list of records whose keys are stored in a tree rooted at x , return the node whose key has rank k .
 - ▶ We can do linear search to find it. But can we do better?



BST



-
- ▶ What if we also want to perform Select operation
 - ▶ BST-Select (x, k):
 - ▶ Given a list of records whose keys are stored in a tree rooted at x , return the node whose key has rank k .
 - ▶ We can do linear search to find it. But can we do better?
 - ▶ Goal:
 - ▶ **Augment** the binary search tree data structure so as to support Select (x, k) efficiently
-



In particular,

- ▶ BST-Select (x, k)
- ▶ Goal:
 - ▶ Augment the binary search tree data structure so as to support BST-Select (x, k) efficiently
- ▶ Ordinary binary search tree T
 - ▶ $O(h)$ time for BST-Select($T.root, k$) where h is height of tree T
- ▶ Using balanced search tree)
 - ▶ $O(\lg n)$ time for BST-Select($T.root, k$)

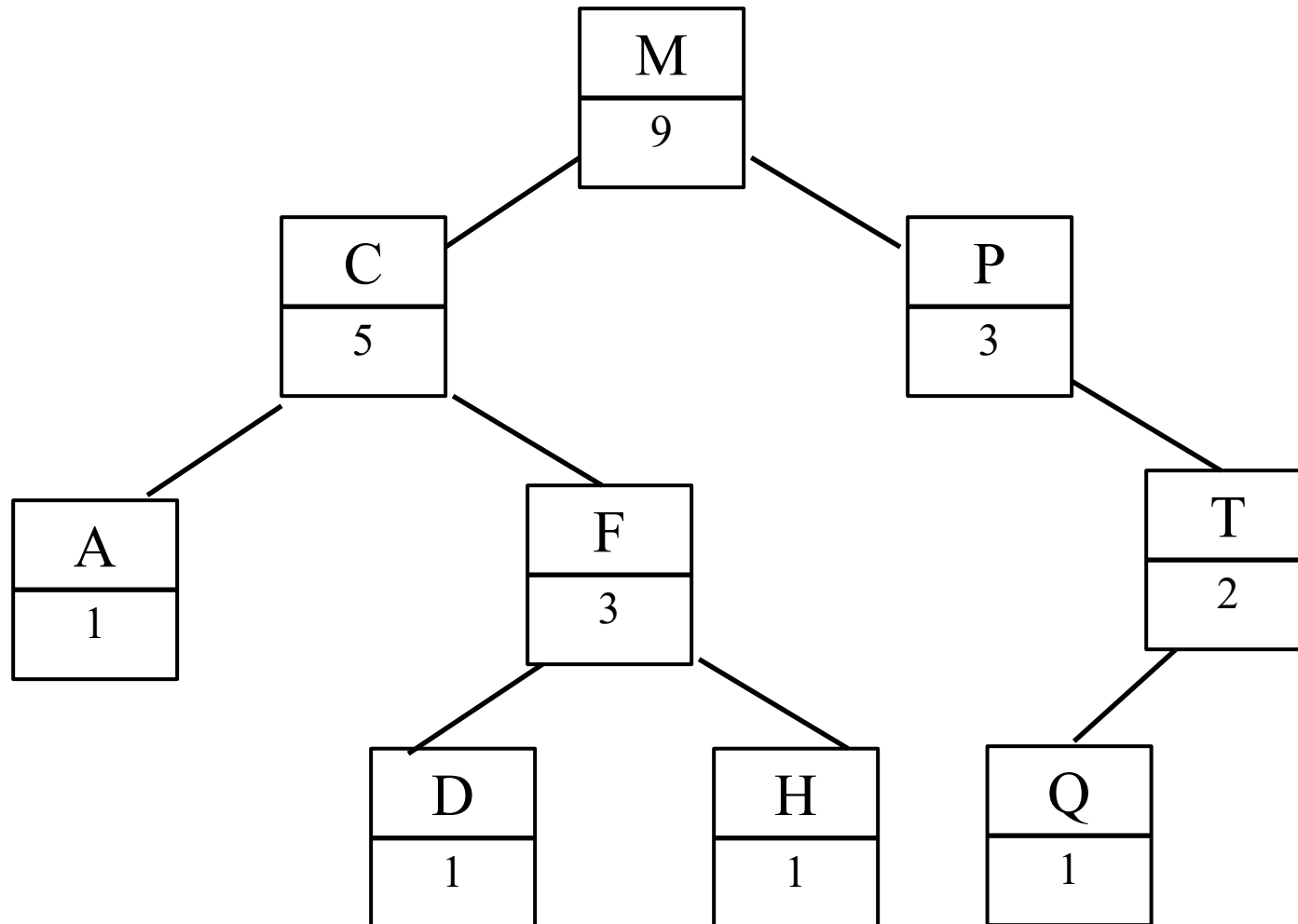


How do we augment a BST T ?

- ▶ At each node x of the tree T
 - ▶ store $x.size = \#$ nodes in the subtree rooted at x
 - ▶ Include x itself
 - ▶ If a node (leaf) is NIL, its size is 0.
- ▶ Space of an augmented tree:
 - ▶ $\Theta(n)$



An example

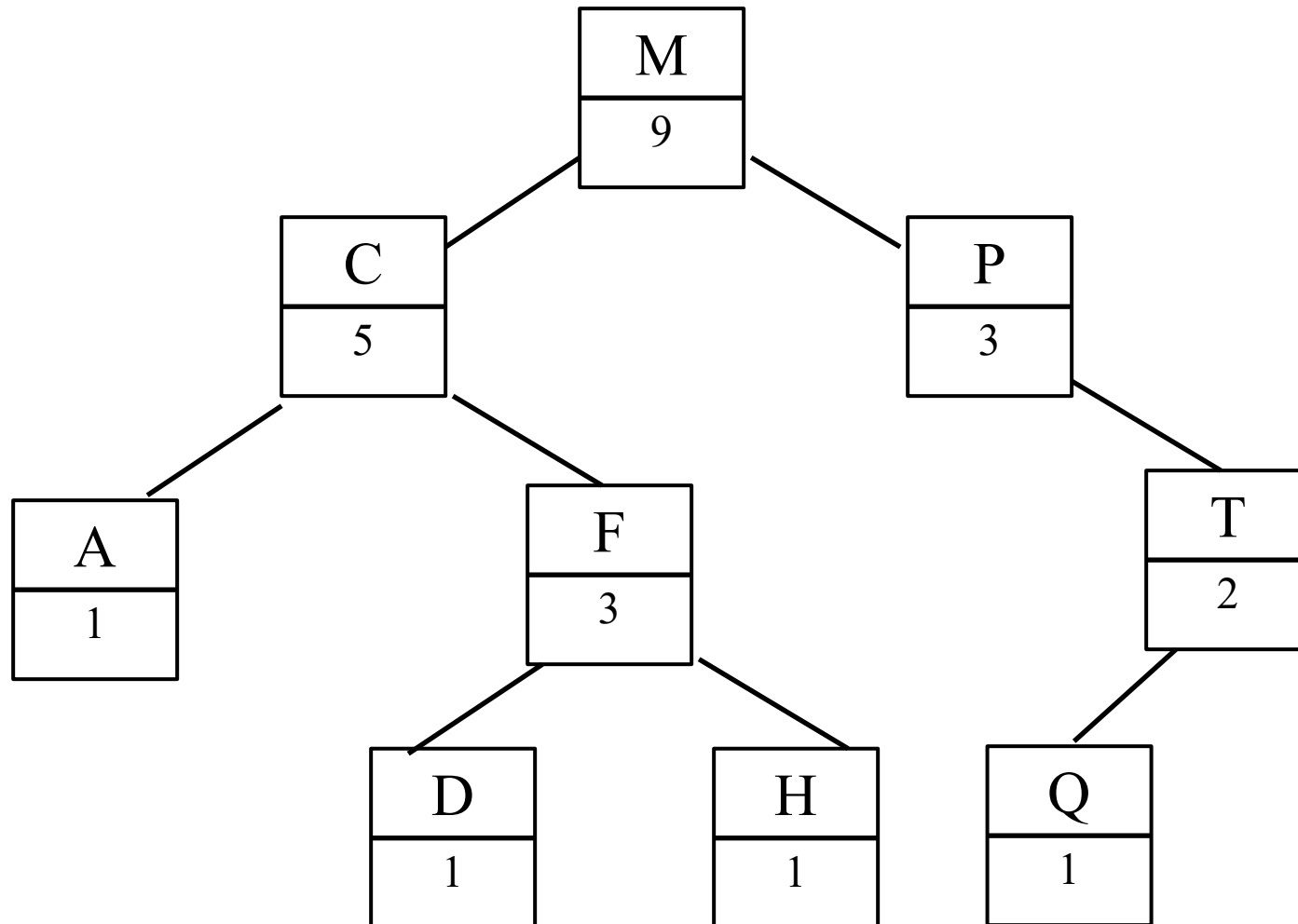


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- ▶ At each node x of the tree T
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 - ▶ Include x itself
 - ▶ If a node (leaf) is NIL, its size is 0.
- ▶ Space of an augmented tree:
 - ▶ $\Theta(n)$
- ▶ Basic property:
 - ▶ $x.size = x.left.size + x.right.size + 1$



How to set up size information ?



How to setup size information?

► procedure *AugmentSize*(*treenode* x)

 If ($x \neq NIL$) then

$Lsize = \text{AugmentSize}(x.left);$

$Rsize = \text{AugmentSize}(x.right);$

$x.size = Lsize + Rsize + 1;$

 Return($x.size$);

end

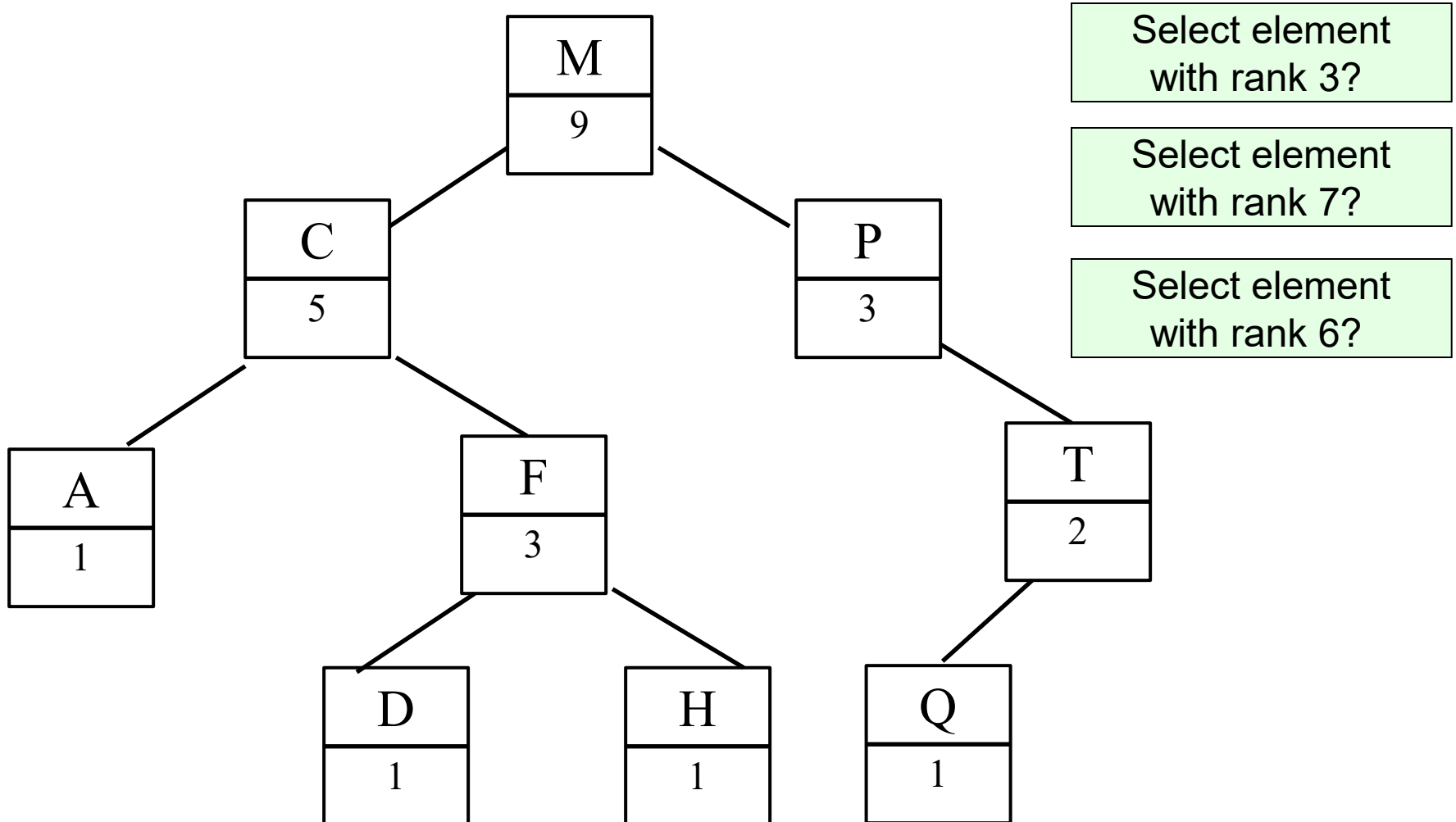
Return (0);

Postorder traversal
of the tree !

Time complexity for Augmentsize:
 $\Theta(\text{size of tree})$



How to perform select with aug-BST?



-
- ▶ Let T be an augmented binary search tree
 - ▶ $BST\text{-}Select(x, k)$:
 - ▶ Return the k -th smallest element in the subtree rooted at x
 - ▶ $BST\text{-}Select(T.root, k)$ returns the k -th smallest elements in the entire tree.
 - ▶ Using ideas just described, $BST\text{-}Select(x, k)$ can be implemented to have $\Theta(\text{height of tree})$ time complexity
 - ▶ which is $\Theta(\lg n)$ for a balanced binary tree.
 - ▶ See homework.
-



Are we done?

- ▶ Need to maintain the augmented information under dynamic changes of the tree!
 - ▶ i.e, under insertions / deletions
 - ▶ in this case, just adjusting this size count as we update nodes, or under rotations, and it does not increase asymptotic time complexity of these operations
- ▶ Remark:
 - ▶ Select() in an sorted array can be done in $\Theta(1)$ time.
 - ▶ However, an array does not support dynamic operations (insert/delete) efficiently. That's augmented BST is a better data structure in this case.



Summary

- ▶ Simple example of augmenting data structures
- ▶ In general, the augmented information can be quite complicated
 - ▶ Can be a separate data structure!
- ▶ Need to consider how to maintain such information under dynamic changes



FIN

