DSC40B: Theoretical Foundations of Data Science II

Lecture 9: *Hashing and Hash table*

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Recall: Set operations

- Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:
  - Search
  - Maximum
  - Minimum
  - Successor
  - Predecessor

  - Insert
  - Delete
  - Extract-Max
  - Increase-key

Using balanced BST, all these operations can be done in $O(\lg n)$ time

Dictionary operations

- Given a universe of elements U
  - these elements may not be numbers
- Need to store some keys
- Need to perform the following operations for keys
  - Insert
  - Search
  - Delete

We can use balanced BST for this purpose if keys can be compared.

But we can have even lighter weight data structure to handle these

Today

- Hashing in general

- Hash table
  - For dictionary operations

Hashing

Hashing

- An important idea used commonly in practice

- Many uses:
  - Fast queries on a large data set.
  - Verifying message integrity.
  - Identify if file has changed in version control.

Hash function

- Mathematically, a hash function is simply a function
  - $f\colon\ U \to X$ from one set to another

- To make it useful, in practice,
  - It often maps some potentially large or complex object to a much smaller and simpler "fingerprint" or "signature"
  - One also want to mapping to be "uniform" and cause few "collisions".
  - Note: a hash function needs to be deterministic!
    - Hashing the same object twice, we should get the same answer

Some examples

- A cryptographic hash function:
  - maps data of arbitrary size into an often fixed sized output of much smaller size
  - e.g, MD5: maps it to a 128-bit value
  - hard to "reverse engineer" input from hash.
  - two similar input could and should lead to very different hash values

```
> echo "My name is Justin" | md5
a741d8524a853cf83ca21eabf8cea190
> echo "My name is Justin!" | md5
f11eed2391bbd0a5a2355397c089fafd
```

```
> md5 slides.pdf
e3fd4370fda30ceb978390004e07b9df
```

A cryptographic hash function

- Why?
  - I release a piece of software.
  - I host it on Google Drive.
  - Someone (Google, US Gov., etc.) decides to insert extra code into software to spy on users.
  - You have no way of knowing.
- What do I do?
  - I release a piece of software and **publish the hash**
  - I host it on Google Drive.
  - Someone inserts extra code into software to spy on users.
  - You download the software and hash it. If hash is different, you know the file has been changed!

Some examples

- Want to place images into 100 bins.
  - How do we decide which bin an image goes into?
  - Hash function!
  - Takes in an image.
  - Outputs a number in {1, 2, …, 100}

Hash Table

Dictionary operations

- Given a universe of elements U
- Need to store some keys
- Need to perform the following operations for keys
  - Insert
  - Search
  - Delete
- In other words, the key operation is membership queries (search), but also allow dynamic updates (insert, delete).

Approach 0

- Use an array to organize all the keys
  - We could pre-sort the array.
    - Preprocessing time:
  - Search:
  - Insert / Delete:

Approach 1

- Organize all keys in a doubly-linked list
  - Pre-processing: None
  - Insert:
  - Delete
  - Search:
  - Space:

Approach 2

- Organize all keys in a balanced BST
  - Search:
  - Insert:
  - Delete
  - Space:

Approach 3

- Direct address tables
- Suppose we know that all the keys we ever care are from 0 to $N = 99{,}999$
  - e.g, we are querying for zipcodes
- Open an array A of size N
  - If a zipcode z is in, set $A[z] = 1$, and 0 otherwise
  - Given a query zipcode, say 223300, simply return $A[223300]$
  - Search:
  - Insert:
  - Delete:

What's the problem with this approach?

Not practical for if the size of universe U (which is N in this example) is huge!

Use Hash Table!

- U : universe
- $T[0 \ \ldots \ m-1]$: a hash table of size m
  - $m \ll |U|$
  - usually, we choose m to be around the size of data we will see

- Hash functions
  - $h: U \rightarrow \{0, 1, \ldots, m-1\}$

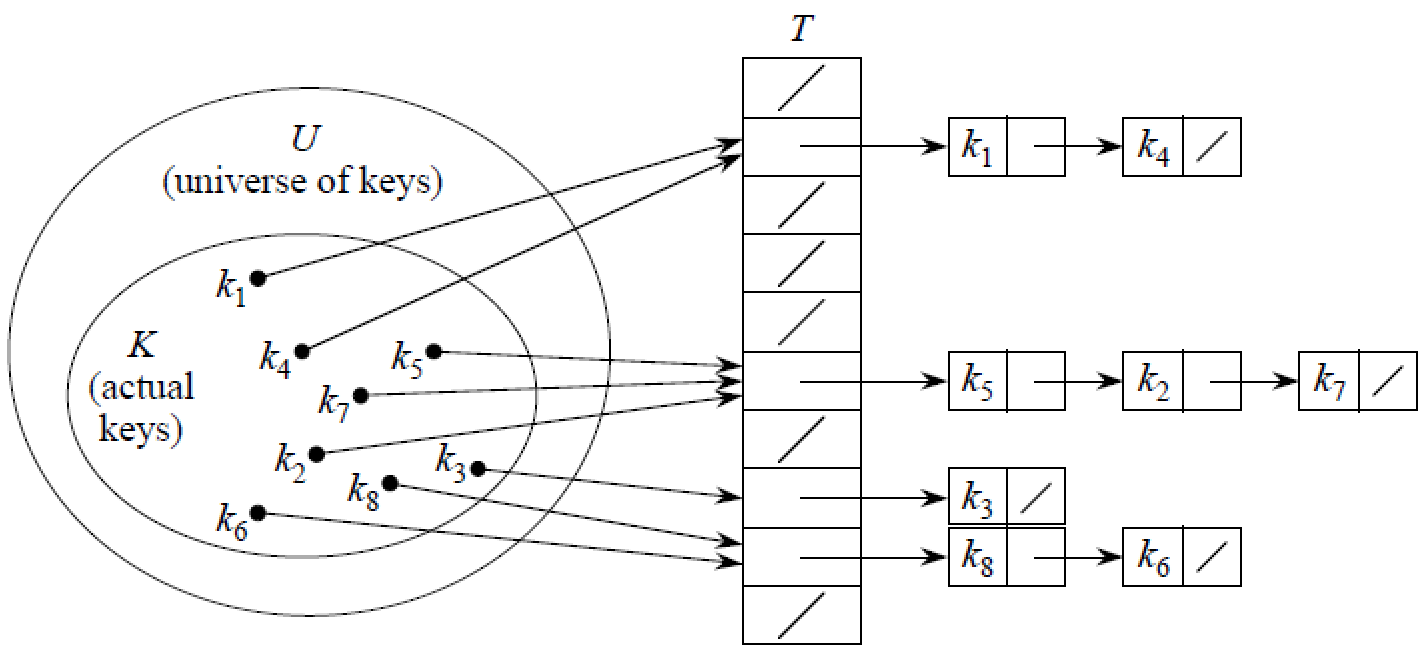
- $h(k)$ is called the hash value of key k.
  - Given a key k, we will store it in location $h(k)$ of hash table T,
    - i.e, store it at $T[h(k)]$

Collision

- Since the size of hash table is smaller than the universe:
  - Multiple keys may hash to the same slot.
  - A collision happens when $h(x) = h(y)$ for $x \neq y \in U$

- How to handle collisions?
  - Chaining
  - Open addressing
  - …

Collision Resolved by Chaining



- *T[j]*: a pointer to the head of the linked list of all stored elements that hash to *j*
- Nil otherwise

A simple example

- U: all positive integers $\mathbb{N}$
- Hash table $T[0, \ldots, 10]$
- Hash function: $h: \mathbb{N} \rightarrow [0, \ldots, 10]$
  - $h(x) = x \; mod \; 11$

Dictionary operations

- Chained-Hash-Insert (T, x)
  - Insert x at the head of list *T[h(key(x))]*

O(1)

- Chained-Hash-Search(T, k)
  - Search for an element with key k in list *T[h(k)]*

O(length(T[h(k)])

- Chained-Hash-Delete(T, x)
  - Delete x from the list *T[h(key(x))]*

O(length(T[h(key(x))])

Good Hash Function

- Performance of Hash table operations
  - depend on the number of elements hashed to each slot in hash table.
- Intuitively,
  - A good hash function should spread elements into the hash table uniformly
  - If there are n elements in the input data and m slots
    - then ideally there should be $\frac{n}{m}$ elements in each hash table slot.

Average case analysis

- n: # elements in the table
- m: size of table (# slots in the table)
- Load factor:
  - $\alpha = \frac{n}{m}$: average number of elements per linked list
  - Intuitively the optimal time needed
- Individual operation can be slow (*O(n)* time)
  - *Under certain assumption of the distribution of keys*, analyze expected performance.

Simple uniform hashing assumption

- Simple uniform hashing assumption:
  - any given element is equally likely to hash into any of the m slots in T
- Let n_j be length of list $T[j]$
  - $n = n_0 + n_1 + \cdots + n_{m-1}$
  - Under simple uniform hashing assumption:
    - expected value $E[n_j] = \alpha = \frac{n}{m}$

Why?

Why

- Let $\{k_1, k_2, \dots, k_n\}$ be the set of keys
- Goal: Estimate $E(n_j)$
- Let $X_i = 1$ if $h(k_i) = j$

  0 otherwise
- Note: $n_j = \sum_{i=1}^{n} X_i$!
- $E[X_i] = \Pr[h(k_i) = j] = \frac{1}{m}$
- Hence $E[n_j] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{m} = \frac{n}{m}$

Expected running time

- Under Simple uniform hashing assumption

- Search:
  - Expected time: $\mathrm{ET(n)} = \Theta(1 + \frac{n}{m})$
    - (worst case time $T(n) = \Theta(n)$)
- Insert:
  - $\mathrm{T(n)} = \Theta(1)$
- Delete:
  - Expected time: $\mathrm{ET(n)} = \Theta(1 + \frac{n}{m})$
    - (worst case time $T(n) = \Theta(n)$)

Key message:
if the load factor $\alpha = \frac{n}{m} = \Theta(1)$
then $\Theta(1)$ expected time for these operations!

Hashing in Python

- dict and set implement hash tables

- Querying is done using in:

```
>>> # make a set
>>> L = {3, 6, -2, 1, 7, 12}
>>> 1 in L # Theta(1)
False
>>> 7 in L # Theta(1)
True
```

Some examples of using hash tables

Recall the movie problem

- The Movie problem
  - Input: Given a list of length of movies available, stored in array $movies$, and a flight duration D
  - Output: Return two movies whose total length = D; None otherwise.

Recall

- The naïve algorithm solves it in $\Theta(n^2)$ time
- But earlier, we mentioned we can do better by
  - First sorting the array of movie lengths
  - Then check for each movie *x*, whether there exits one whose length is *D – length(x)*
  - Worst case time complexity $T(n) = \Theta(n \lg n)$

New approach via Hashing

- Use Hash table, and frame this as a membership query problem

```
def optimize_entertainment_hash(times, D):
    hash_table = dict()
    for i, time in enumerate(times):
        hash_table[time] = i

    for i, time in enumerate(times):
        target = D - time
        if target in hash_table:
            return i, hash_table[target]
    return None
```

Expected time?
$\Theta(n)$

Another example

- Anagrams
  - Two strings w_1 and w_2 are anagrams if the letters of w_1 can be permuted to make w_2.
    - E.g, "eat" and "tea" , or "listen" and "silent"

- The *Anagrams problem*
  - Given a collection of n strings, determine if any two of them are anagrams.

Using Hash table

- **Observation:**
  - two strings are anagrams if their sorted lists are equal
    - sorted(w_1) == sorted(w_2)

```
def any_anagrams(words):
    seen = set()
    for word in words:
        w = sorted(word)
        if w in seen
            return True
        else:
            seen.add(w)
    return False
```

Expected time?
$\Theta(n)$

Hashing Downsides

- Only support dictionary queries
  - i.e, membership queries + insert / delete
  - Example 1: cannot be used to query for the two movies whose total time is closest to *D*
  - Example 2: cannot be used for performing a range query in a list of numbers
    - Say report the number of numbers fall within range [a, b]

Hashing Downsides

- Only support dictionary queries
  - i.e, membership queries + insert / delete

- No locality: similar items map to very different bins
  - Necessarily so for the performance of hashing in most cases!
  - But, in practice, we often query similar objects continuously
    - May result in many cache misses, slow

Summary

- Hashing
  - Very useful idea
  - Generates a small signature for a potentially large complex object

- Hash Table
  - Very practical data structure for dictionary operations
    - Very efficient for these operations, can be constant expected time if the load factor $\alpha = \frac{n}{m} = \Theta(1)$
  - Especially when the number of keys necessary is much smaller than the size of universe where input objects could come from!
  - In practice, need to choose hash functions properly
    - There exist intelligent hashing schemes

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