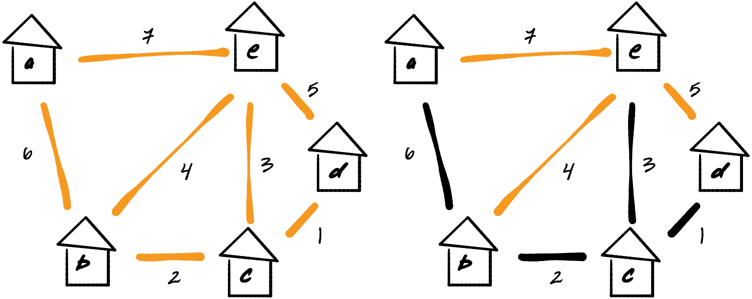
# DSC 40B

Theoretical Foundations II

## Minimum Spanning Trees

# Today's Problem




- Choose a set of dirt roads to pave so that:
  - can get between any two buildings only on paved roads;
  - total cost is minimized.

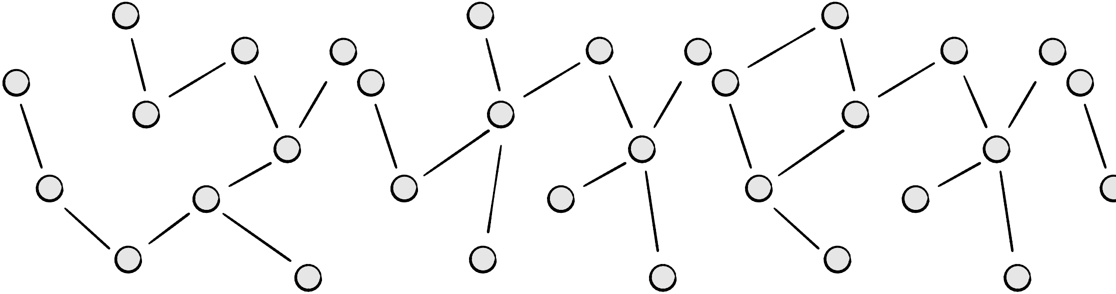
- Solution: compute a **minimum spanning tree**.

# Trees

An undirected graph $T = (V, E)$ is a **tree** if

- it is connected; and
- it is acyclic.

Example: a **tree**.Example: a **tree**.Example: **not** a tree.Example: **not** a tree.

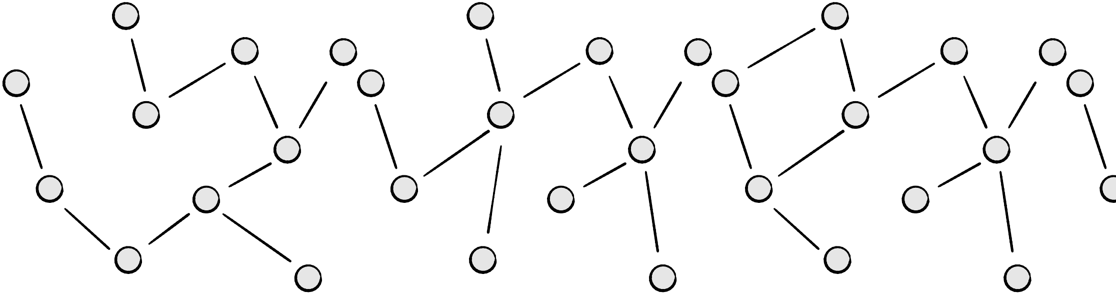


# Trees: Equivalent Definition

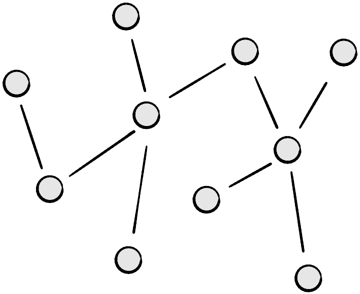
An undirected graph $T = (V, E)$ is a **tree** if

- it is connected; and
- $|E| = |V| - 1$.

Example: a **tree**.Example: a **tree**.Example: **not** a tree.Example: **not** a tree.



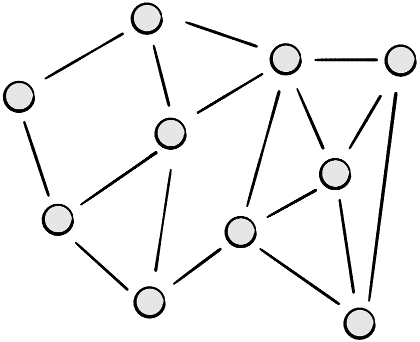
# Tree Properties



- There is a unique path between any two nodes in a tree.
- Adding a new edge to a tree creates a cycle (no longer a tree).
- Removing an edge from a tree disconnects it (no longer a tree).
- Out of all graphs with $n$ nodes, trees with $n$ nodes have the least edges: $n - 1$.

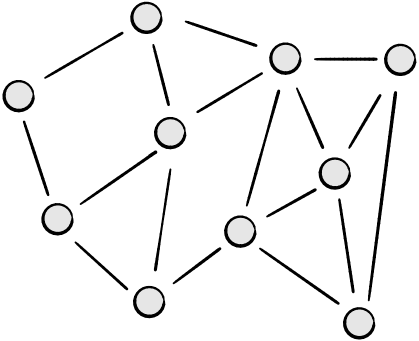
# Spanning Trees

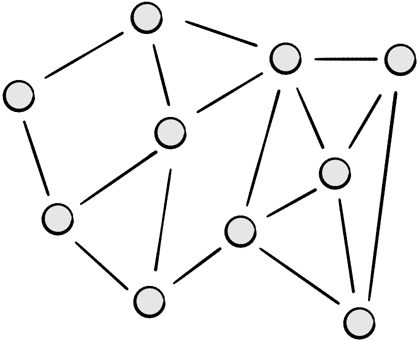
Let $G = (V, E)$ be a **connected** graph. A **spanning tree** of $G$ is a tree $T = (V, E_T)$ with the same nodes as $G$, and a subset of $G$'s edges.



# Many Spanning Trees

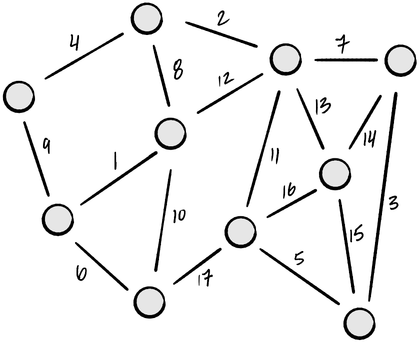
The same graph can have many spanning trees.





# Spanning Tree Cost

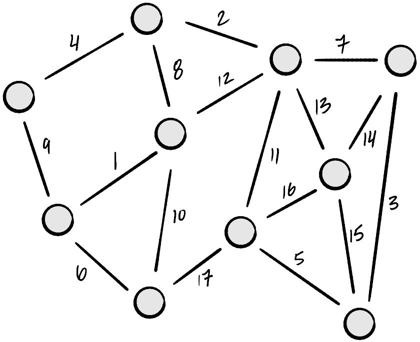
If $G = (V, E, \omega)$ is a weighted undirected graph, the **cost** (or **weight**) of a spanning tree is the total weight of the edges in the spanning tree.




Cost:

# Spanning Tree Cost

Different spanning trees of the same graph can have different costs.




Cost:

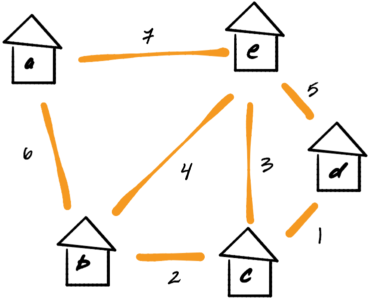
# Minimum Spanning Tree

- The **minimum spanning tree** problem is as follows:
  - GIVEN: A weighted, undirected graph $G = (V, E, \omega)$.
  - COMPUTE: a spanning tree of $G$ with minimum cost (i.e., minimum total edge weight).
- For a given graph, the MST may not be unique.

**Exercise**

Suppose the edges of a graph $G = (V, e, \omega)$ all have the same weight. How can we compute an MST of the graph?

# Today's Problem




- Choose a set of dirt roads to pave so that:
  - can get between any two buildings only on paved roads;
  - total cost is minimized.

- Solution: compute a **minimum spanning tree**.

# MSTs in Data Science?

- Do we need to find MSTs in data science?
- Actually, yes! (Next lecture)

# DSC 40B

Theoretical Foundations II

## Prim's Algorithm

# Building MSTs

- How do we build a MST efficiently?

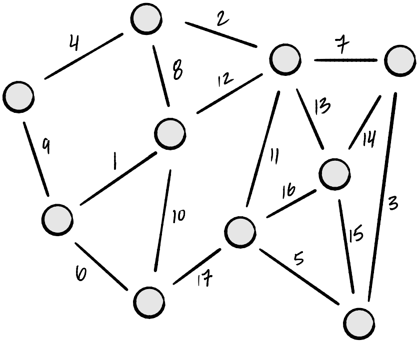
- We'll adopt a **greedy** approach.
  - Build a tree edge-by-edge.
  - At every step, doing what looks best at the moment.

- This strategy isn't guaranteed to work in all of life's situations, but it works for building MSTs.

# Two Greedy Approaches

- We'll look at two greedy algorithms:
  - Today: Prim's Algorithm
  - Next time: Kruskal's Algorithm
- Differ in the order in which edges are added to tree.
- Also differ in time complexity.

# Prim's Algorithm, Informally




- Start by picking any node to add to "tree", *T*.
- While *T* is not a tree, greedily add **lightest** edge from a node in *T* to a node not in *T*.
  - "lightest" = edge of smallest weight
- **Is this guaranteed to work?** Yes, as we'll see.

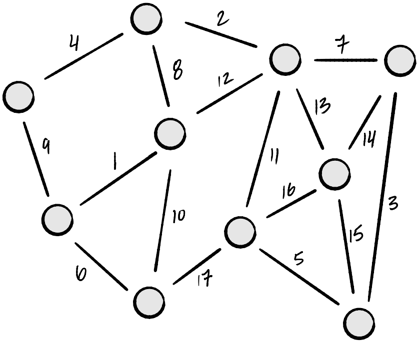
# Prim's Algorithm, Equivalently

- For each node $u$, store:
  - estimated cost of adding node to tree;
  - estimated "predecessor" $v$ in the tree.

- At each step,
  - Find node with smallest cost.
  - Add to tree $T$ by including edge with estimated "predecessor".
  - Update cost of neighbors.

- Same as adding lightest edge from $T$ to outside $T$ at every step!

# Prim’s Algorithm, Equivalently




- While $T$ is not a tree:
  - find the node $u \notin T$ with smallest cost
  - add the edge between $u$ and its estimated “predecessor” to $T$
  - update estimated cost/pred. of $u$’s neighbors which aren’t already in tree.

# Recall: Priority Queues

- How do we efficiently find node with smallest cost?
- Priority Queues:
  - `PriorityQueue(priorities)`: creates priority queue from dictionary whose values are priorities.
  - `.extract_min()`: removes and returns key with smallest value.
  - `.decrease_priority(key, value)`: changes key's value.
- We'll use a priority queue to hold nodes not yet added to tree.

```python
def prim(graph, weight):
    tree = UndirectedGraph()

    estimated_predecessor = {node: None for node in graph.nodes}
    initial_costs = {node: float('inf') for node in graph.nodes}
    priority_queue = PriorityQueue(costs)

    while priority_queue:
        u = priority_queue.extract_min()
        tree.add_edge(estimated_predecessor[u], u)
        for v in graph.neighbors(u):
            if weight(u, v) < cost[v] and v not in tree.nodes:
                priority_queue.decrease_priority(v, weight(u, v))
                estimated_predecessor[v] = u
    return tree
```

# Prim and Dijkstra

- This is a lot like Dijkstra's Algorithm for s.p.d.!
- Both: at each step, extract node with smallest cost, update its edges. (Prim: only those edges to nodes not in tree).
- Dijkstra update of $(u, v)$:

```
cost[v] = min(cost[v], cost[u] + weight(u, v))
```

- Prim update of $(u, v)$:

```
cost[v] = min(cost[v], weight(u, v))
```

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Theoretical Foundations II

## Time Complexity

# Time Complexity

- A priority queue can be implemented using a **heap**.

- If a **binary min-heap** is used:
  - `PriorityQueue(est)` takes $\Theta(V)$ time.
  - `.extract_min()` takes $O(\log V)$ time.
  - `.decrease_priority()` takes $O(\log V)$ time.

# Time Complexity

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                estimated_predecessor[v] = u
    return tree
```

# Time Complexity

- Using a **binary heap**...
- Overall: $\Theta(V \log V + E \log V)$.
- Since graph is assumed connected, $E = \Omega(V)$.
- So this simplifies to $\Theta(E \log V)$.

# Fibonacci Heaps

- A priority queue can be implemented using a **heap**.
- If a **Fibonacci min-heap** is used:
  - `PriorityQueue(est)` takes $\Theta(V)$ time.
  - `.extract_min()` takes $\Theta(\log V)$ time[1].
  - `.decrease_priority()` takes $O(1)$ time.

[1] Amortized

# Time Complexity

```
def prim(graph, weight):
    tree = UndirectedGraph()

    estimated_predecessor = {node: None for node in graph.nodes}
    initial_costs = {node: float('inf') for node in graph.nodes}
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                priority_queue.decrease_priority(v, weight(u, v))
                estimated_predecessor[v] = u
    return tree
```

# Time Complexity

- Using a **Fibonacci heap**...
- Overall: $\Theta(V \log V + E)$.

# Fibonacci vs. Binary Heaps

- Using Fibonacci heaps improves time complexity when $E \ll V$.

- E.g., if $E = \Theta(V^2)$:
  - Prim's with Fibonacci: $\Theta(E) = \Theta(V^2)$
  - Prim's with binary: $\Theta(E \log E) = \Theta(V^2 \log V)$.

- But Fibonacci heaps are **hard** to implement; have large constants.

- Binary heaps used more in practice despite complexity.

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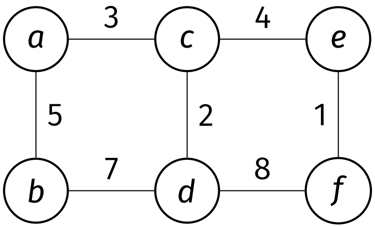
Theoretical Foundations II

# Correctness of Prim's Algorithm

# Being Greedy

- At every step, we add the lightest edge.
- Is this “safe”?
- Yes! This is guaranteed to find an MST.

# Promising Subtrees



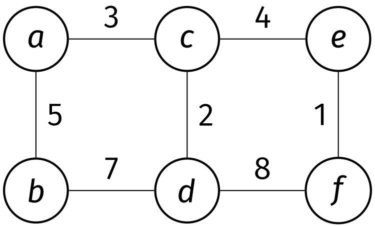

- Let $G = (V, E, \omega)$ be a weighted graph.
- A subgraph $T' = (V', E')$ is **promising** if it is "part" of some MST.
  - That is, it is an "MST in progress"
  - Not necessarily a tree!
- That is, there exists an MST $T = (V, E_{\text{mst}})$ such that $E' \subset E_{\text{mst}}$.
- Hint: a "promising subtree" where $V' = V$ is an MST!

## Main Idea

Prim's starts with a promising subtree $T$. At each step, adds lightest edge from a node within $T$ to a node outside of $T$.

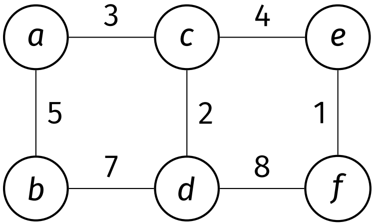
We'll show each new edge results in a larger promising subtree. Eventually the promising subtree becomes a full MST.

# Claim




- Let $G = (V, E, \omega)$ be a weighted graph.
- Suppose $T' = (V', E')$ is a promising subtree for an MST of $G$.
- Let $e = (u, v)$ be a **lightest edge** from a node in $T'$ to a node outside of $T'$. (Prim).
- Then adding $(u, v)$ to $T'$ results in another **promising subtree**.

# Proof




- Suppose $T_{\text{mst}}$ is an MST that includes $T'$.
- If $T_{\text{mst}}$ includes $e$, we're done: $T' + e$ is promising.
- If it doesn't include $e$, it must have an edge $f$ that connects $T'$ to rest of the graph.
- Swap $f$ with $e$ in $T_{\text{mst}}$. The result is a tree, and it must be a MST since $\omega(e) \leq \omega(f)$.
- So there is an MST that contains $T' + e$.