DSC40B: Theoretical Foundations of Data Science II

Lecture 12: *More on BFS: shortest path in (unweighted) graphs*

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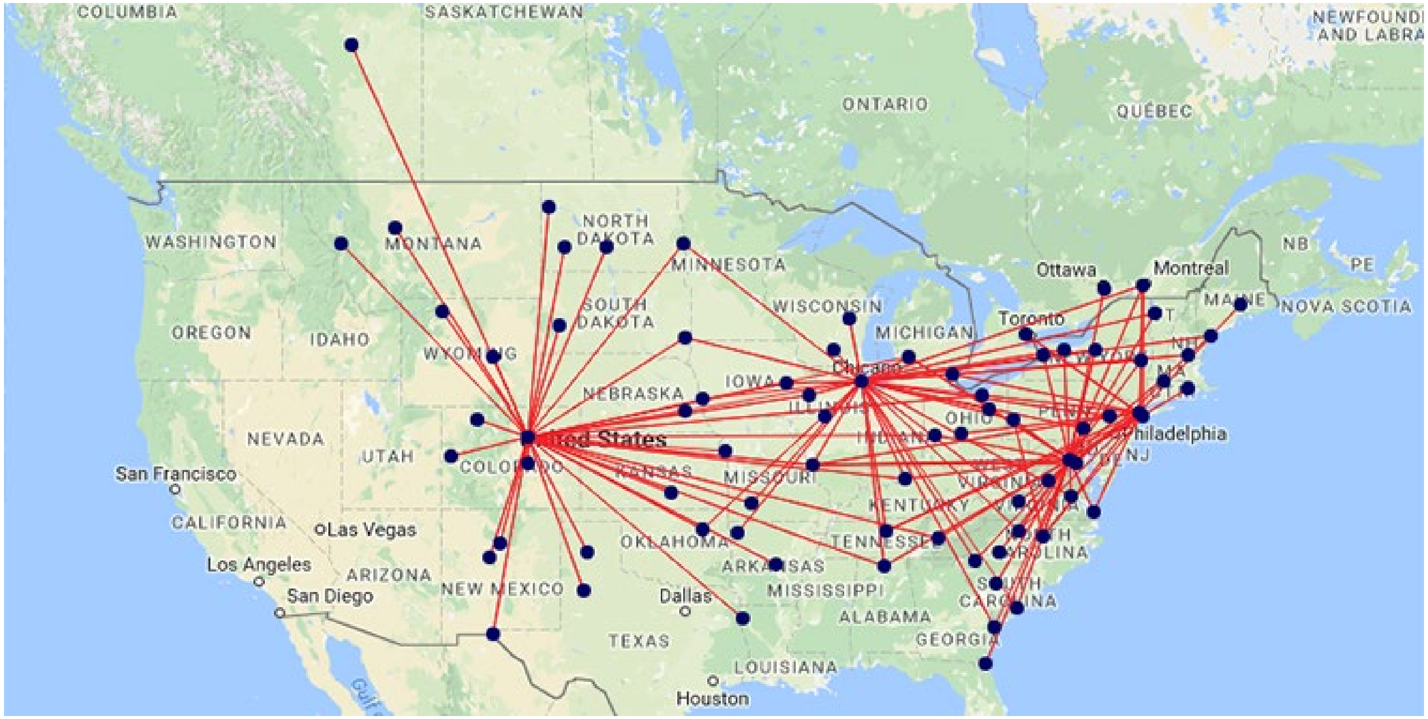
- Previously:
  - Introduced Breadth-first search (BFS) graph search algorithm
  - Can be used to check for connectivity etc

- Today:
  - Properties of BFS:
    - Computing shortest path from a source node
    - BFS tree

Shortest path in (unweighted) graphs



- How to fly to Denver from Columbus using fewest number of connections

- The length of a path is

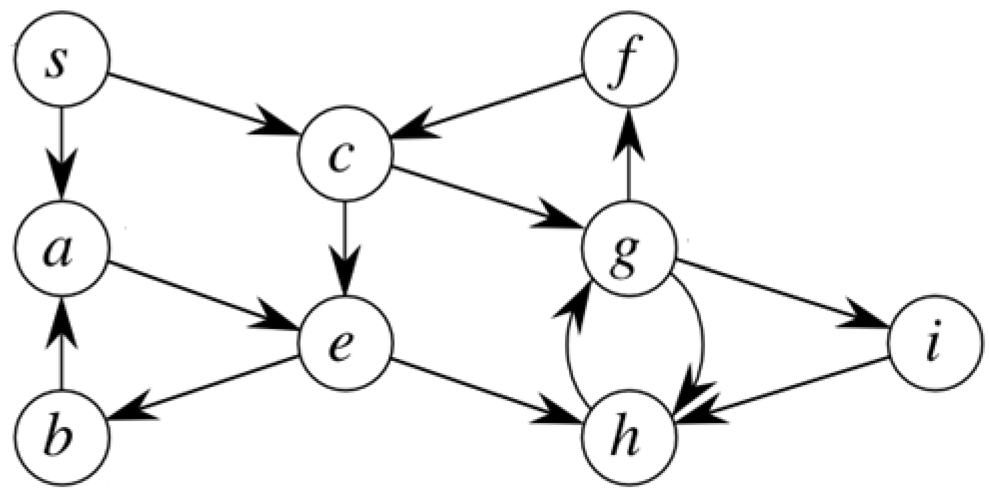
  (#of nodes in this path -1)

- A shortest path from u to v
  - is a path from u to v using with smallest possible length.
  - Note that there may be multiple shortest paths from u to v, all of which has the same length.

- The shortest path distance from u to v
  - is the length of a shortest path from u to v
  - by convention, the distance is set to be $+\infty$ if there is no path from u to v

- Input:
  - A (directed or undirected) graph $G = (V, E)$ and a source node $s \in V$
- Output:
  - The shortest path distance from s to all other nodes in V

Example

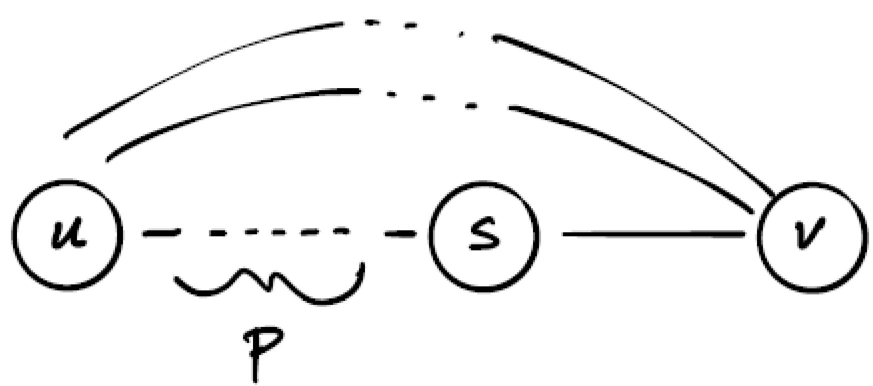


Property of shortest paths (i)

- Claim A:
  - Given any $u, v \in V$, if v is reachable from u, a shortest path from u to v has to be simple.
  - (Recall, a path is simple if no node in it is visited more than once)

Property of shortest paths (ii)

- Key structure of shortest paths:
  - Any sub-path of a shortest path must be a shortest path itself.
  - This implies that a shortest path of length k consists of a shortest path of length $(k-1)$ plus one edge.
    - e.g, given a shortest path $\pi = \langle v_0, v_1, \dots, v_{k-1}, v_k \rangle$ from v_0 to v_k, it consists of a shortest path $\langle v_0, v_1, \dots, v_{k-1} \rangle$ plus edge (v_{k-1}, v_k)



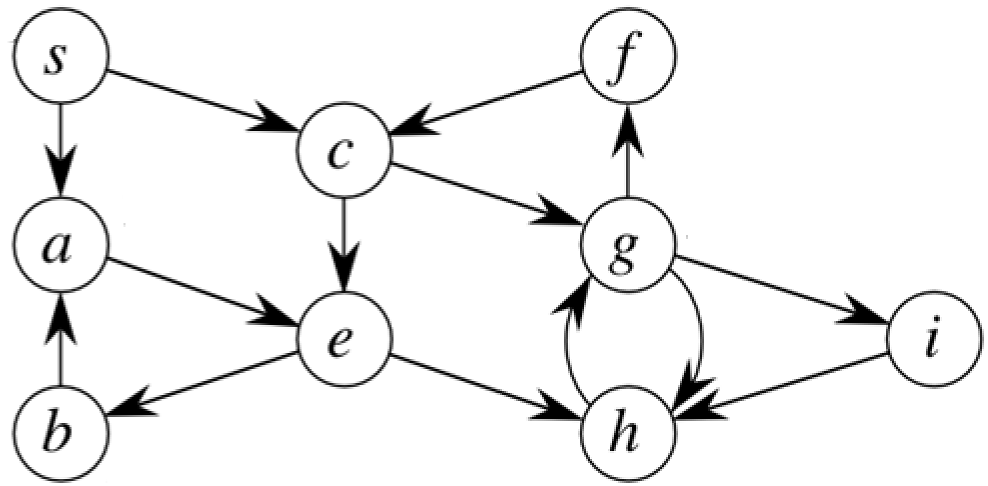
Shortest path and BFS

How do we find shortest paths?

- High level idea:
  - Starting from the source s
  - First find all nodes that are at distance 1 from s
  - Then use them to find those nodes at distance 2 from s
  - Then use them to find those nodes at distance 3 from s
  - ….
  - Till we find all reachable nodes

- Note:
  - to get a node at distance k from source s,
  - you have to first reach a node at distance $k-1$ from s, and extend it from that node via an edge.

Example



- It turns out that this idea is exactly what BFS is doing!
- Intuitively,
  - the first time that we discover a node turns out to encode the fastest way to reach it
    - that is also when we first change the status of a node from undiscovered to pending
  - the time this node is discovered relates to the distance to the source nodes
  - by visiting and exploring the oldest pending nodes (those with smallest distance to the source first), we find fastest way to reach a new node

Recall BFS

```
from collections import deque

def bfs(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

Property of BFS

- For any $k \geq 1$,
  - all nodes distance k from source are added to the queue before any node of distance k
  - Hence nodes are added to the queue in increasing order of their distances to the source

- Therefore, nodes are "processed" (popped from the queue) in order of distance from the source,
  - which further guarantees that the first time to find a undiscovered node, that must be the shortest path to reach this node from the source.

- Consider a node u that we just popped from the queue
- Suppose the distance from source s to u is k
- Our algorithm will then scan through all neighbors of u.
  - For a neighbor v of u, we have now found a new path π to reach v by the path from s to u, followed by the edge (u, v). The length is $k + 1$.
  - If this neighbor v is undiscovered
    - then the new path π must be shortest! Why?
    - hence the shortest path distance from s to u is $k + 1$
  - Otherwise, this neighbor v is already discovered (pending or visited)
    - then it means that we have already found a path to v before, whose length is at most $k + 1$.
    - so, the new path π we just found is not useful for shortest path – we already have a shortest path to v

Modified BFS – distance computation

```python
def bfs_shortest_paths(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}
    distance = {node: float('inf') for node in graph.nodes}

    status[source] = 'pending'
    distance[source] = 0

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                distance[v] = distance[u] + 1
                # append to right
                pending.append(v)
        status[u] = 'visited'

    return  distance
```

But we can do more!

- We can record information to help us recover shortest paths themselves later!

- Node u is set to be BFS-predecessor of v if v is discovered while visiting u.

- This means that u is the predecessor along a shortest path from the source s to v
  - In particular, the shortest path from s to u plus edge (u, v) is a shortest path from s to v !

- If all nodes remember their BFS-predecessors,
  - Then we have enough information to recover shortest paths from the source s to all reachable nodes!

Shortest-Path algorithm

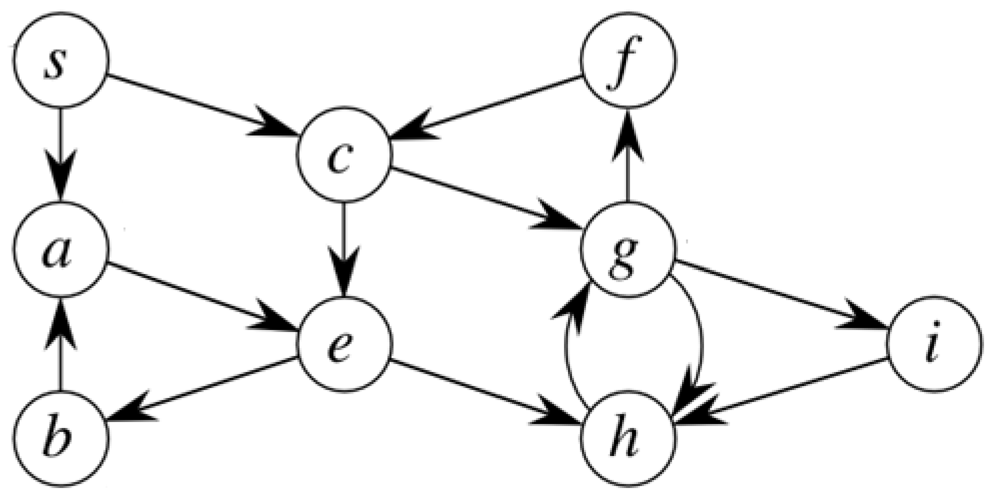
```
def bfs_shortest_paths(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}
    distance = {node: float('inf') for node in graph.nodes}
    predecessor = {node: None for node in graph.nodes}

    status[source] = 'pending'
    distance[source] = 0
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                distance[v] = distance[u] + 1
                predecessor[v] = u
                # append to right
                pending.append(v)
        status[u] = 'visited'

    return predecessor, distance
```

Example



Time complexity

- Note that this has the same asymptotic time complexity as BFS algorithm
  - $O(|V| + |E|)$

BFS Trees
and
recovering shortest paths

Results of BFS_shortest_path

- Every node reachable from the source has a single BFS-predecessor
  - except for the source s itself

- Connecting each node to its BFS-predecessor gives a rooted tree, where the source is the root
  - in particular, the parent of each node in the tree is its BFS-predecessor

- This tree is called the BFS-tree associated to source s

Trees

- A (free) tree is a connected graph $T = (V, E)$ where $|E| = |V| - 1$.

- For any two nodes in a tree, there is only one simple path connecting them.

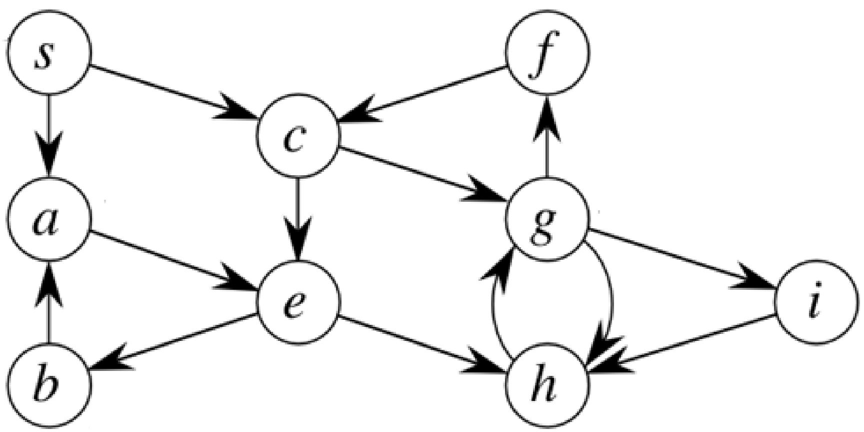
- A rooted tree has a root, and each node other than the root has a parent.
  - The parent v is the predecessor of v along the unique simple path from the root to v

- A collection of trees is called a forest.

BFS-tree

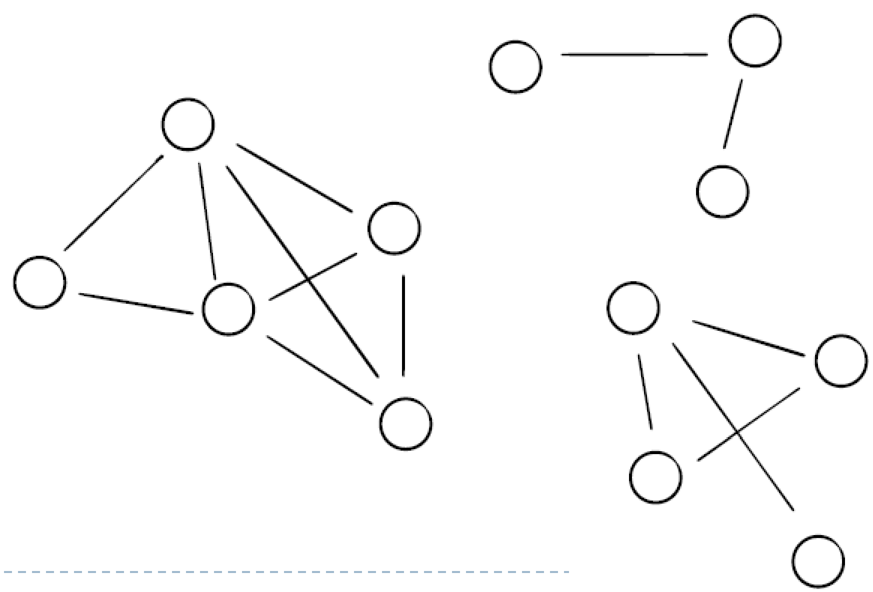
- BFS-tree associated to source s can be used to recover
  - both the shortest path and shortest path distance from s to each reachable node.

- Example:



- Claim: Given a graph $G = (V, E)$, let T be its BFS-tree from source s. Then for the unique path from s to u in T is a shortest path in G, and its length is the shortest path distance from s to u in G.

- In general, if we run the full-BFS algorithm (with augmentation of computing also predecessors and distances), then we will obtain a collection of BFS-trees, called a BFS-forest.
- Example:



Summary

- BFS algorithm:
  - It explores nodes in order of their first discovery time
  - In particular, it will explore them in order of their shortest path distance to the source
  - It will propagate a wavefront, first visit all nodes distance 1 to source, then distance 2, then distance 3, and so on
  - So it explores as broad as possible before moving deeper (meaning further away from the source)
  - Thus the name: "breadth-first" search.
- It can be used to compute the shortest path distance to a source node
  - Time complexity is $O(|V| + |E|)$

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