DSC 40A - Discussion 02

January 21, 2020

1 Inequalities

Inequalities are a fundamental part of mathematical proofs. We will go over the basic properties to brush up on things.

- **Law of Trichotomy:** $\forall x, y \in \mathbb{R}$, either $x < y$, $x = y$ or $x > y$.
- **Transitive property:** if $x \leq y$ and $y \leq z$ then $\forall x, y, c \in \mathbb{R}$, $x \leq z$
- **Addition property:** if $x \leq y$, then $\forall x, y, c \in \mathbb{R}$, $x + c \leq y + c$
- **Multiplication property:** if $x \leq y$, then $\forall x, y \in \mathbb{R}$
  - $\forall c \geq 0 \in \mathbb{R}$, $cx \leq cy$
  - $\forall c \leq 0 \in \mathbb{R}$, $cx \geq cy$

Problem 1.

Which of the statements below are always true? If $a \leq b$ and $c \leq d$,

1. $a + c \leq b + d$
2. $a - c \leq b + d$
3. $a \leq bc$
4. $ac \leq bd$
5. $|ac| \leq |bd|$
6. $a^2 \leq b^2$
7. $min(a, c) \leq min(b, d)$
8. $min(a, c) \leq max(b, d)$
9. $min(a, max(b, d)) \leq min(c, max(b, d))$
10. $min(a, max(b, d)) \leq max(b, d)$

Challenge Problem.

Let $f(x, y)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$. Show that

$$max_x min_y f(x, y) \leq min_y max_x f(x, y)$$

2 Convexity

In class, we saw how to minimize functions using gradient descent. This method will converge at a local minimum (provided that the step size is small enough). However, if the loss function is convex (and differentiable), it is guaranteed to find the global optimum! A function, $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if it satisfies the following inequality:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \qquad \forall a, b \in \mathbb{R}, t \in [0, 1]$$

What this means is that if we pick any two points on f and draw a line segment between them, all the points on the line segment should lie above f. If a function is not convex, it is nonconvex.

We can also prove if a function is convex with the **second derivative test**, but we will not touch upon it in today's discussion.

Problem 2.

(Sample problem with solution)

Prove that $f(x) = |x|$ is convex. Hint: Remember triangle inequality: $|a + b| \leq |a| + |b|$.

We want to show that $f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b)$.

$$\begin{aligned} f(ta + (1-t)b) &= |ta + (1-t)b| \\ &\leq |ta| + |(1-t)b| && \text{(triangle inequality)} \\ &= t|a| + (1-t)|b| && (t \in [0,1], \text{ can take it out}) \\ &= tf(a) + (1-t)f(b) && (\text{introduce } f) \end{aligned}$$

Problem 3.

Let $h(x) : \mathbb{R} \to \mathbb{R} = max(f(x), g(x))$ where $f(x)$ and $g(x)$ are convex functions and $x \in \mathbb{R}$.

Prove that h convex.