DSC 40A - Discussion 02

January 21, 2020

1 Inequalities

Inequalities are a fundamental part of mathematical proofs. We will go over the basic properties to brush up on things.

- **Law of Trichotomy:** $\forall x, y \in \mathbb{R}$, either $x < y$, $x = y$ or $x > y$.
- **Transitive property:** if $x \leq y$ and $y \leq z$ then $\forall x, y, c \in \mathbb{R}$, $x \leq z$
- **Addition property:** if $x \leq y$, then $\forall x, y, c \in \mathbb{R}$, $x + c \leq y + c$
- **Multiplication property:** if $x \leq y$, then $\forall x, y \in \mathbb{R}$
  - $\forall c \geq 0 \in \mathbb{R}$, $cx \leq cy$
  - $\forall c \leq 0 \in \mathbb{R}$, $cx \geq cy$

Problem 1.

Which of the statements below are always true? If $a \leq b$ and $c \leq d$,

1. $a + c \leq b + d$
2. $a - c \leq b + d$
3. $a \leq bc$
4. $ac \leq bd$
5. $|ac| \leq |bd|$
6. $a^2 \leq b^2$
7. $min(a, c) \leq min(b, d)$
8. $min(a, c) \leq max(b, d)$
9. $min(a, max(b, d)) \leq min(c, max(b, d))$
10. $min(a, max(b, d)) \leq max(b, d)$

Solution:

1. **True.** We know $a \leq b$. First, we add d to both sides, $a + d \leq b + d$. We also know that $c \leq d$. Adding a to both sides gives us $a + c \leq a + d$. Then, we use transitivity to say that $a + c \leq b + d$.
2. **Not always true.** Pick $a = 3, b = 3, c = -2, d = -1$. Then, $a - c = 5 > b + d = 2$.
3. **Not always true.** Pick $a = 3, b = 3, c = -2, d = -1$. Then, $a = 3 > bc = -6$.
4. **Not always true.** Pick $a = 3, b = 10, c = -2, d = -1$. Then, $ac = -6 > bd = -10$.
5. **Not always true**. Pick $a = -100, b = 10, c = 1, d = 2$. Then, $|ac| = 100 > |bd| = 20$.
6. $a^2 \leq b^2$. **Not always true**. Pick $a = -2, b = 0$. Then, $a^2 = 4 > b^2 = 0$.
7. $min(a, c) \leq min(b, d)$. **True**. $min(a, c) \leq a$ and $min(a, c) \leq c$. Also, $a \leq b$ and $c \leq d$. By transitivity, $min(a, c) \leq b$ and $min(a, c) \leq d$. Since $min(b, d)$ is either b or d and $min(a, c)$ is smaller than or equal to both, $min(a, c) \leq min(b, d)$.
8. $min(a, c) \leq max(b, d)$. **True**. Using the same argument above, $min(a, c) \leq b$ and $min(a, c) \leq d$. Hence, $2 * min(a, c) \leq b + d \leq 2 * max(b, d)$. Then, $min(a, c) \leq max(b, d)$
9. $min(a, max(b, d)) \leq min(c, max(b, d))$. **Not always true**. Pick $a = 10, b = 20, c = -2, d = 100$.

Then, $min(10, max(20, 100)) = 10 > min(-2, max(20, 100)) = -2$.

10. $min(a, max(b, d)) \leq max(b, d)$. **True**. Let $max(b, d) = e$. Then the equation becomes $min(a, e) \leq e$. But we know that $min(a, e) \leq a$ and $min(a, e) \leq e$!

Challenge Problem.

Let $f(x, y)$ be a function from $\mathbb{R}^2 \rightarrow \mathbb{R}$. Show that

$$max_x min_y f(x, y) \leq min_y max_x f(x, y)$$

Solution: Let the maximizer in x dimension be x_{max} and the minimizer in y dimension be y_{min}. Then, we have to show:

$$max_x f(x, y_{min}) \leq min_y f(x_{max}, y)$$

First, we have to make the following observation.

$$f(x, y_{min}) \leq f(x, y) \leq f(x_{max}, y)$$

. By transitivity,

$$f(x, y_{min}) \leq f(x_{max}, y)$$

Take min_y of both sides.

$$min_y f(x, y_{min}) \leq min_y f(x_{max}, y)$$

The term on the left is unchanged as the y value is already fixed! So, we have

$$f(x, y_{min}) \leq min_y f(x_{max}, y)$$

Now, take max_x of both sides.

$$max_x f(x, y_{min}) \leq max_x min_y f(x_{max}, y)$$

Well, the term on the right is unchanged as the x value is already fixed! Hence,

$$max_x f(x, y_{min}) \leq min_y f(x_{max}, y)$$

If you're curious, you can look up the Minimax algorithm which relies on the principle above.

2 Convexity

In class, we saw how to minimize functions using gradient descent. This method will converge at a local minimum (provided that the step size is small enough). However, if the loss function is convex (and differentiable), it is guaranteed to find the global optimum! A function, $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex if and only if it satisfies the following inequality:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \qquad \forall a, b \in \mathbb{R}, t \in [0, 1]$$

What this means is that if we pick any two points on f and draw a line segment between them, all the points on the line segment should lie above f. If a function is not convex, it is nonconvex.

We can also prove if a function is convex with the **second derivative test**, but we will not touch upon it in today's discussion.

Problem 2.

(Sample problem with solution)

Prove that $f(x) = |x|$ is convex. Hint: Remember triangle inequality: $|a + b| \leq |a| + |b|$.

We want to show that $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$.

$$\begin{aligned}
f(ta + (1-t)b) &= |ta + (1-t)b| \\
&\leq |ta| + |(1-t)b| && \text{(triangle inequality)} \\
&= t|a| + (1-t)|b| && (t \in [0,1]\text{, can take it out}) \\
&= tf(a) + (1-t)f(b) && (\text{introduce } f)
\end{aligned}$$

Problem 3.

Let $h(x) : \mathbb{R} \to \mathbb{R} = max(f(x), g(x))$ where $f(x)$ and $g(x)$ are convex functions and $x \in \mathbb{R}$.

Prove that h convex.

Solution: We have to show that

$$h(ta + (1-t)b) \leq th(a) + (1-t)h(b)$$

Since $h(ta + (1-t)b) = max(f(ta + (1-t)b), g(ta + (1-t)b))$, it suffices to show that

$$f(ta + (1-t)b) \leq th(a) + (1-t)h(b)$$

AND

$$g(ta + (1-t)b) \leq th(a) + (1-t)h(b)$$

.

Let's start with the first inequality.

$$\begin{aligned}
f(ta + (1-t)b) &\leq tf(a) + (1-t)f(b) && \text{(f is convex)} \\
&\leq tmax(f(a), g(a)) + (1-t)max(f(b), g(b)) \\
&= th(a) + (1-t)h(b)
\end{aligned}$$

Similarly,

$$\begin{aligned}
g(ta + (1-t)b) &\leq tg(a) + (1-t)g(b) && \text{(g is convex)} \\
&\leq tmax(f(a), g(a)) + (1-t)max(f(b), g(b)) \\
&= th(a) + (1-t)h(b)
\end{aligned}$$

Hence, h is convex.