DSC 40A - Discussion 06 - Combinatorics and Conditional Probability

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Problem 1.

You want to plant an herb garden, so you go to a garden store that has 50 different herbs: 28 are culinary herbs, 12 are medicinal herbs, and 10 are aromatic herbs. You select 5 herbs for your herb garden by taking a random sample **without replacement** from the 50 available herbs.

a) If you consider the herbs you select as a sequence where the order in which you select each herb matters, how many sequences of 5 herbs are possible?

> **Solution:** There are $\frac{50!}{45!} = 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46$ possible sequences.

b) If you consider the herbs you select as a sequence where the order in which you select each herb matters, how many sequences of 5 herbs include 2 culinary herbs and 3 aromatic herbs?

> **Solution:**
>
> $$\binom{28}{2} \times \binom{10}{3} \times 5!$$
>
> There are $\binom{28}{2}$ ways of picking two culinary herbs and $\binom{10}{3}$ ways of picking three aromatic herbs, and so there are
>
> $$\binom{28}{2} \times \binom{10}{3}$$
>
> ways of picking 2 culinary herbs and three aromatic herbs. But the order matters. For each combination of 5 herbs, there are 5! different orderings. Hence:
>
> $$\binom{28}{2} \times \binom{10}{3} \times 5!.$$
>
> ---
>
> Here is another approach. We have to fill a sequence of 5 slots. To generate a sequence, we'll first pick the order in which the culinary herbs will appear. We'll then pick the order in which the aromatic herbs will appear. Then we will pick where the aromatic herbs will be placed in the order of five herbs.
>
> There are $28 \cdot 27$ ways in which to order 2 culinary herbs.
>
> There are $10 \cdot 9 \cdot 8$ ways in which to order 3 aromatic herbs.
>
> There are $\binom{5}{3} = 10$ ways of picking where to place the aromatic herbs in the final sequence.
>
> In total, there are:
>
> $$28 \cdot 27 \times 10 \cdot 9 \cdot 8 \times 10$$
>
> You can verify that this gives the same answer as above.

c) If you consider the herbs you select as a set where the order in which you select each herb does not matter, how many sets of 5 herbs are possible?

Solution: There are $\binom{50}{5}$ possible sets of 5 herbs.

d) If you consider the herbs you select as a set where the order in which you select each herb does not matter, how many sets of 5 herbs include 2 culinary herbs and 3 aromatic herbs?

Solution:

$$\binom{28}{2} \times \binom{10}{3}$$

There are $\binom{28}{2}$ ways to pick the culinary herbs and $\binom{10}{3}$ ways to pick the aromatic herbs, for a total of

$$\binom{28}{2} \times \binom{10}{3}$$

e) What is the probability that you choose 2 culinary herbs and 3 aromatic herbs for your garden?

Solution: There are $\binom{50}{5}$ sets of five herbs. From above, there are

$$\binom{28}{2} \times \binom{10}{3}$$

sets containing 2 culinary herbs and 3 aromatic herbs. Hence the probability is:

$$\frac{\binom{28}{2} \times \binom{10}{3}}{\binom{50}{5}}$$

Problem 2.

Suppose we throw a fair 6-sided die. If we get i after throwing the die, we throw coin C_i, which has $P(C_i = H) = \dfrac{i}{10}$ for $i \in \{1, 2, 3, 4, 5, 6\}$.

a) What is the probability that after repeating the above procedure for 3 times,

- we get all heads?

Solution: Let's represent toss 1 as T_1, toss 2 as T_2, toss 3 as T_3. We are asked to find $P(T_1 = H \cap T_2 = H \cap T_3 = H)$.

$$\begin{aligned} P(T_1 = H \cap T_2 = H \cap T_3 = H) &= P(T_1 = H)P(T_2 = H|T_1 = H)P(T_3 = H|T_2 = H \cap T_1 = H) \\ &= P(T_1 = H)P(T_2 = H)P(T_3 = H) \end{aligned}$$

Notice that the probability of getting a heads in any toss is equal. So, it is enough just to find one. Let's find $P(T_1 = H)$. We can represent the outcome of the die as D.

$$\begin{aligned} P(T_1 = H) &= P(D=1)P(T_1 = H|D=1) + P(D=2)P(T_1 = H|D=2) \\ &+ P(D=3)P(T_1 = H|D=3) + P(D=4)P(T_1 = H|D=4) \\ &+ P(D=5)P(T_1 = H|D=5) + P(D=6)P(T_1 = H|D=6) \\ &= (1/6)(1/10) + (1/6)(2/10) + (1/6)(3/10) + (1/6)(4/10) + (1/6)(5/10) + (1/6)(6/10) \\ &= (1/6)((1+2+3+4+5+6)/10) \\ &= (1/6)(21/10) \\ &= 7/20 \\ &= 35/100 \end{aligned}$$

Therefore, the answer is $(35/100)^3 = 0.042875$

- we get all tails?

Solution: $P(T_1 = T) = 1 - P(T_1 = H) = 0.65$. Again, the probability does not change from toss to toss. Therefore, the answer is $(0.65)^3 = 0.274625$.

- we get exactly two heads?

Solution:

$$\begin{aligned} P(\#\ heads = 2) &= P(T_1 = T)P(T_2 = H)P(T_3 = H) \\ &+ P(T_1 = H)P(T_2 = T)P(T_3 = H) \\ &+ P(T_1 = H)P(T_2 = H)P(T_3 = T) \\ &= (0.65)(0.35)(0.35) + (0.35)(0.65)(0.35) + (0.35)(0.35)(0.65) \\ &= 3 * (0.65)(0.35)^2 \\ &= 0.238875 \end{aligned}$$

b) Suppose now that in the above procedure, whenever we throw a coin, if the outcome is tails, we replace the coin with a fair coin. What is the probability that after repeating it 2 times,

- we get all heads?

Solution: We are asked to find $P(T_1 = H \cap T_2 = H)$.

$$\begin{aligned} P(T_1 = H \cap T_2 = H) &= P(T_1 = H)P(T_2 = H|T_1 = H) \\ &= P(T_1 = H)P(T_2 = H) \end{aligned}$$

(Given that the first toss was a heads, we know that we will not replace any coins.)

$$\begin{aligned} &= (0.35)(0.35) \\ &= (0.35)^2 \\ &= 0.1225 \end{aligned}$$

- **challenge problem (beyond the scope of this course):** we get all tails?

Solution: We are asked to find $P(T_1 = T \cap T_2 = T)$.

$$P(T_1 = T \cap T_2 = T) = P(T_1 = T)P(T_2 = T|T_1 = T)$$

At this point, we know $P(T_1 = T) = 0.65$. However, we need to calculate $P(T_2 = T|T_1 = T)$. If we did not know the outcome of T_1, our prior belief of which coin we tossed would be equal for each coin with a probability of 1/6, since we pick it according to the outcome of a fair 6-sided die. However, notice that if we know that the outcome of T_1 is tails, this gives us information about which coin we might have used. As an example, C_1 has a 90% probability of tails. So, seeing a tails as the first outcome increases our belief that we tossed coin 1.

For this, we need to consider all the coins we could have replaced after the first toss. Let's represent the replaced coin with R (i.e $R = 1$ if we replaced the 1st coin, $R = 2$ if we replaced 2nd coin and so on). Then,

$$\begin{aligned}
P(R = i|T_1 = T) &= \frac{P(R = i \cap T_1 = T)}{P(T_1 = T)} \\
&= \frac{P(R = i)P(T_1 = T|R = i)}{P(T_1 = T)} \\
&= \frac{(1/6)(1 - (i/10))}{(0.65)} \\
&= \frac{(10 - i)}{60} * \frac{100}{65} \\
&= \frac{(10 - i)}{39}
\end{aligned}$$

This makes sense. According to what we found, the probability that we used and replaced
C_1 is $9/39 \approx 0.231$
C_2 is $8/39 \approx 0.205$
C_3 is $7/39 \approx 0.180$
C_4 is $6/39 \approx 0.154$
C_5 is $5/39 \approx 0.128$
C_6 is $4/39 \approx 0.103$
So, the prior distribution was $1/6 = 0.167$ for all coins, but now the distribution shifted in favor of coins with higher chances of getting a tails. Also, notice that the probabilities add up to $1.(4 + 5 + 6 + 7 + 8 + 9)/39 = 39/39 = 1)$

Now, we need to calculate $P(T_2 = T|T_1 = T) = \sum_{i=1}^{6} P(T_2 = T \cap R = i|T_1 = T)$. Before we do that, we need to calculate $P(T_2 = T|R = i)$. In other words, what's the probability of getting a tails on the second toss if we replaced coin i after the first toss?

The coin we replaced (i) / Prob. of getting tails	1	2	3	4	5	6
$P(C_1 = T\|R = i)$	**5/10**	9/10	9/10	9/10	9/10	9/10
$P(C_2 = T\|R = i)$	8/10	**5/10**	8/10	8/10	8/10	8/10
$P(C_3 = T\|R = i)$	7/10	7/10	**5/10**	7/10	7/10	7/10
$P(C_4 = T\|R = i)$	6/10	6/10	6/10	**5/10**	6/10	6/10
$P(C_5 = T\|R = i)$	5/10	5/10	5/10	5/10	**5/10**	5/10
$P(C_6 = T\|R = i)$	4/10	4/10	4/10	4/10	4/10	**5/10**
$P(T_2 = T\|R = i)$	35/60	36/60	37/60	38/60	39/60	40/60

We know from question 1 that the probability of getting a heads or a tails is just the average of the probabilities of each coin, since we toss the coins after throwing a fair 6-sided die. That's how we got the final row of our table.

Let us proceed:

$$\begin{aligned}
\sum_{i=1}^{6} P(T_2 = T \cap R = i | T_1 = T) &= \sum_{i=1}^{6} P(R = i | T_1 = T) P(T_2 = T | R = i \cap T_1 = T) \\
&= \sum_{i=1}^{6} P(R = i | T_1 = T) P(T_2 = T | R = i) \\
&= \sum_{i=1}^{6} \frac{10 - i}{39} * \frac{34 + i}{60} \\
&= \frac{1}{39 * 60} \sum_{i=1}^{6} 340 - 24i - i^2 \\
&= \frac{1}{39 * 60} \left(\sum_{i=1}^{6} 340 - 24 \sum_{i=1}^{6} i - \sum_{i=1}^{6} i^2 \right) \\
&= \frac{1}{39 * 60} \left(6 * 340 - 24 * \frac{6 * 7}{2} - \frac{6 * 7 * 13}{6} \right) \\
&= \frac{1}{39 * 60} \left(2040 - 504 - 91 \right) \\
&= \frac{1445}{39 * 60} \\
&= \frac{1445}{39 * 60} \\
\sum_{i=1}^{6} P(T_2 = T \cap R = i | T_1 = T) &\approx 0.61752
\end{aligned}$$

Hence, $\sum_{i=1}^{6} P(T_2 = T \cap R = i | T_1 = T) = P(T_2 = T | T_1 = T) \approx 0.61752$.
Bonus: Does this probability make sense? Why? (Hint: it is less than 0.65)

Finally, the answer is $P(T_1 = T) P(T_2 = T | T_1 = T) = (0.65)(0.61752) \approx 0.40139$

Problem 3.

There are two boxes. Box 1 contains three red and five white balls and box 2 contains two red and five white balls. A box is chosen at random P(box = 1) = P(box = 2) = 0.5 and a ball chosen at random from this box turns out to be red. What is the probability that the red ball came from box 1?

Solution: We are asked to find $P(box = 1|ball = red)$.

$$
\begin{aligned}
P(box = 1|ball = red) &= \frac{P(box = 1 \cap ball = red)}{P(ball = red)} \\
&= \frac{P(box = 1)P(ball = red|box = 1)}{\sum_{b \in \{1,2\}} P(ball = red \cap box = b)} \\
&= \frac{P(box = 1)P(ball = red|box = 1)}{\sum_{b \in \{1,2\}} P(box = b)P(ball = red|box = b)} \\
&= \frac{P(box = 1)P(ball = red|box = 1)}{P(box = 1)P(ball = red|box = 1) + P(box = 2)P(ball = red|box = 2)} \\
&= \frac{(1/2)(3/8)}{(1/2)(3/8) + (1/2)(2/7)} \\
&= \frac{3/16}{3/16 + 1/7} \\
&= \frac{3/16}{(21 + 16)/(16 * 7)} \\
&= \frac{3}{16} * \frac{16 * 7}{37} \\
&= \frac{21}{37}
\end{aligned}
$$

Problem 4.

Challenge follow-up: Two balls are placed in a box as follows: A fair coin is tossed and a white ball is placed in the box if a head occurs, otherwise a red ball is placed in the box. The coin is tossed again and a red ball is placed in the box if a tail occurs, otherwise a white ball is placed in the box. Balls are drawn from the box three times in succession (always with replacing the drawn ball back in the box). It is found that on all three occasions a red ball is drawn. What is the probability that both balls in the box are red?

Solution: The answer is 4/5.