# DSC 140B

## Representation Learning

Lecture 17 | Part 1

## Least Squares Classifiers

# Movie Ratings

- Five of your friends rate a movie from 0-10:
  - $x_1$: 9
  - $x_2$: 3
  - $x_3$: 7
  - $x_4$: 2
  - $x_5$: 8
- **Task**: Will you like the movie? (yes / no)

# Classification

- Linear prediction functions can be used in classification, too.

$$H(\vec{x}) = w_0 + w_1 x_1 + w_2 x_2 + \ldots + w_d x_d$$

- Same ERM paradigm also useful.

# A Classifier from a Regressor

- Binary classification can be thought of as regression where the targets are 1 and -1
  - (or 0 and 1, or ...)

- $H(\vec{x})$ outputs a real number. Use the **sign** function to turn it into –1, 1:

$$\text{sign}(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \\ 0 & \text{otherwise} \end{cases}$$

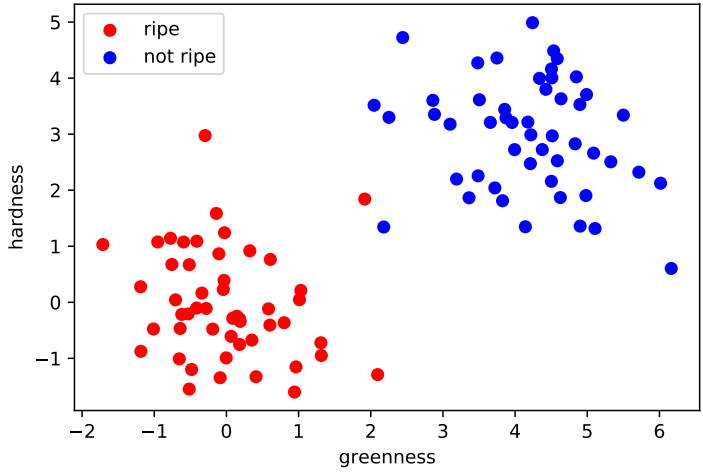
- Final prediction: $\text{sign}(H(\vec{x}))$

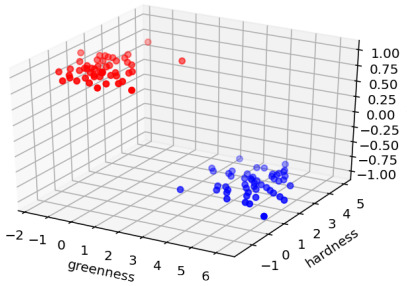
# Example: Mango Ripeness

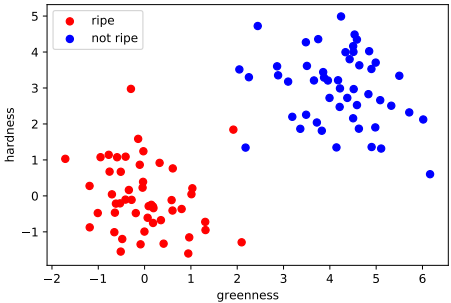
- Predict whether a mango is ripe given greenness and hardness.

- Idea: gather a set of labeled **training data**.
  - Inputs along with correct output (i.e., "the answer").

| Greenness | Hardness | Ripe |
|---|---|---|
| 0.7 | 0.9 | 1 |
| 0.2 | 0.5 | -1 |
| 0.3 | 0.1 | -1 |
| ⋮ | ⋮ | ⋮ |


ripe
not ripe
hardness
greenness
5
4
3
2
1
0
−1
−2
−1
0
1
2
3
4
5
6


greenness
hardness
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00


ripe
not ripe
hardness
greenness

# Decision Boundary

- The **decision boundary** is the place where the output of $H(x)$ switches from "yes" to "no".
  - If $H > 0 \mapsto$ "yes" and $H < 0 \mapsto$ "no", the decision boundary is where $H = 0$.

- If $H$ is a linear predictor and[1]
  - $\vec{x} \in R^1$, then the decision boundary is just a number.
  - $\vec{x} \in \mathbb{R}^2$, the boundary is a straight line.
  - $\vec{x} \in \mathbb{R}^d$, the boundary is a $d - 1$ dimensional (hyper) plane.

[1] when plotted in the original feature coordinate space!

# Empirical Risk Minimization

- Step 1: choose a **hypothesis class**
  - Let's assume we've chosen linear predictors
- Step 2: choose a **loss function**
- Step 3: minimize **expected loss (empirical risk)**

## Exercise

Can we use the square loss for classification?

$$(H(\vec{x}^{(i)}) - y_i)^2$$

# Least Squares and Outliers

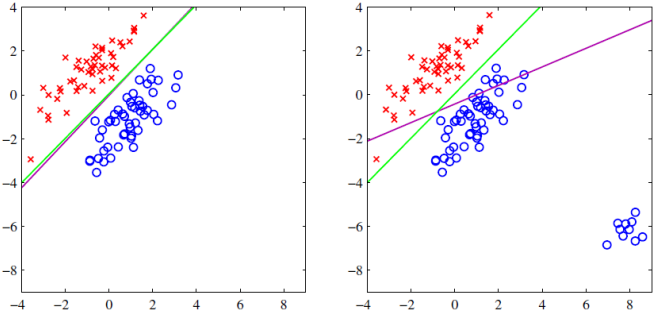



**Figure 4.4** The left plot shows data from two classes, denoted by red crosses and blue circles, together with the decision boundary found by least squares (magenta curve) and also by the logistic regression model (green curve), which is discussed later in Section 4.3.2. The right-hand plot shows the corresponding results obtained when extra data points are added at the bottom left of the diagram, showing that least squares is highly sensitive to outliers, unlike logistic regression.



[2]Bishop, *Pattern Recognition and Machine Learning*

# Square Loss for Classification

- We **can** use the square loss for classification
  - The “least squares classifier”
- However, the square loss penalizes being “too correct”
- **Example**: suppose the correct label is 1. What is the square loss of predicting 10? -9?

# Loss Functions

- There are many different loss functions for classification.
- Each leads to a different classifier:
  - Logistic Regression
  - Support Vector Machine
  - Perceptron
  - etc.
- But that's for another class... (DSC 140A)

# DSC 140B

## Representation Learning

Lecture 17 | Part 2

**Linear Limitations**

# Linear Predictors

- Last time, we saw linear prediction functions:

$$H(\vec{x}; \vec{w}) = w_0 + w_1 x_1 + \ldots + w_d x_d$$
$$= \text{Aug}(\vec{x}) \cdot \vec{w}$$

# Linear Decision Functions

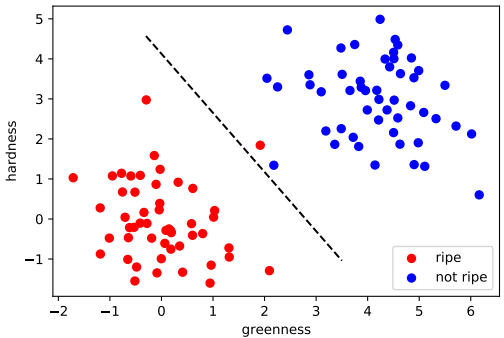
- A linear prediction function $H$ outputs a number.
- What if classes are +1 and -1?
- Can be turned into a **decision function** by taking:

$$\text{sign}(H(\vec{x}))$$

- **Decision boundary** is where $H = 0$
  - Where the sign switches from positive to negative.

# Decision Boundaries

- A linear decision function's decision boundary is linear.
  - A line, plane, hyperplane, etc.



# An Example: Parking Predictor

- **Task**: Predict (yes / no): Is there parking available at UCSD right now?
- What training data to collect? What features?

# Useful Features

- Time of day?
- Day's high temperature?
- ...

## Exercise

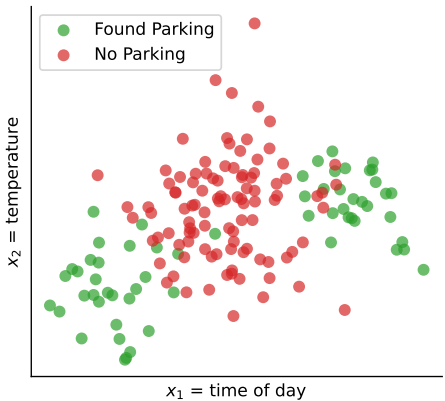
Imagine a scatter plot of the training data with the two features:

- $x_1$ = time of day
- $x_2$ = temperature

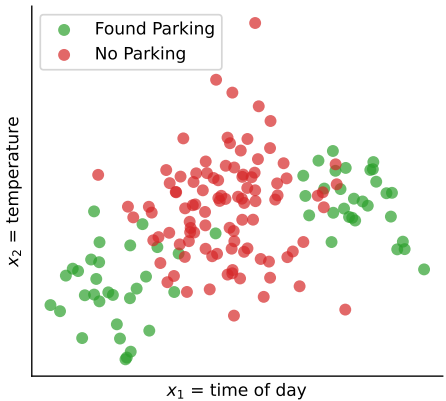
“yes” examples are green, “no” are red.

What does it look like?

# Parking Data



# Uh oh




- A linear decision function won't work.
- What do we do?

# Today's Question

- How do we learn non-linear patterns using linear prediction functions?

# DSC 140B

## Representation Learning

Lecture 17 | Part 3

**Feature Maps**

# Representations

- We **represented** the data with two features: time and temperature

- In this **representation**, the trend is **nonlinear**.
  - There is no good linear decision function
  - Learning is “difficult”.

# Idea

- **Idea**: We'll make a new **representation** by creating **new features** from the **old features**.
- The "right" representation makes the problem easy again.
- What new features should we create?

# New Feature Representation

- Linear prediction functions[3] work well when relationship is linear
  - When $x$ is small we should predict -1
  - When $x$ is large we should predict +1

- But parking's relationship with time is not linear:
  - When time is small we should predict +1
  - When time is medium we should predict -1
  - When time is large we should predict +1

[3] Remember: they are weighted votes.

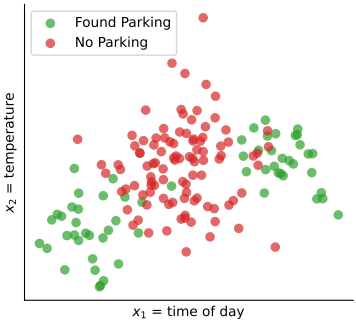
## Exercise

How can we "transform" the time of day $x_1$ to create a new feature $x_1'$ satisfying:

- When $x_1'$ is small, we should predict -1
- When $x_1'$ is large, we should predict +1

What about the temperature, $x_2$?

# Idea




- Transform “time” to “absolute time until/since Noon”
- Transform “temp.” to “absolute difference between temp. and 72˚”

# Basis Functions

- We will transform:
  - the time, $x_1$, to $|x_1 - \text{Noon}|$
  - the temperature, $x_2$, to $|x_2 - 72^\circ|$

- Formally, we've designed non-linear **basis functions**:

$$\varphi_1(x_1, x_2) = |x_1 - \text{Noon}|$$
$$\varphi_2(x_1, x_2) = |x_2 - 72^\circ|$$

- In general a basis function $\varphi$ maps $\mathbb{R}^d \to \mathbb{R}$

# Feature Mapping

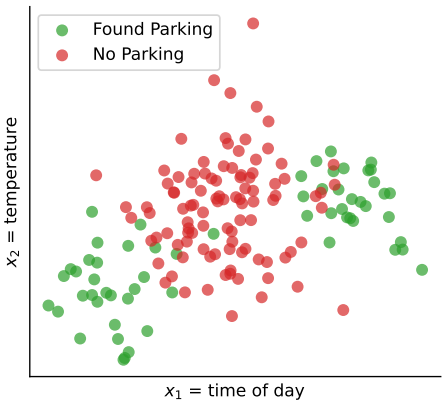
- Define $\vec{\varphi}(\vec{x}) = (\varphi_1(\vec{x}), \varphi_2(\vec{x}))^T$. $\vec{\varphi}$ is a **feature map**
  - Input: vector in "old" representation
  - Output: vector in "new" representation

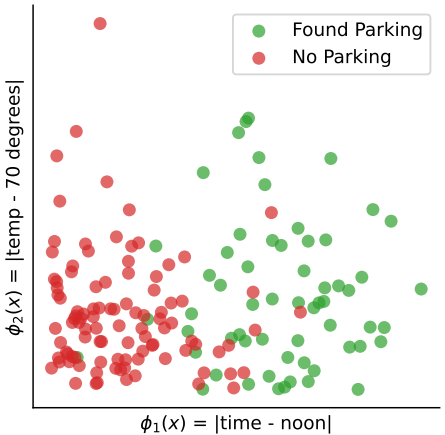
- Example:

$$\vec{\varphi}((10\text{a.m.}, 75^\circ)^T) = (2 \text{ hours}, 3^\circ)^T$$

- $\vec{\varphi}$ maps raw data to a **feature space**.

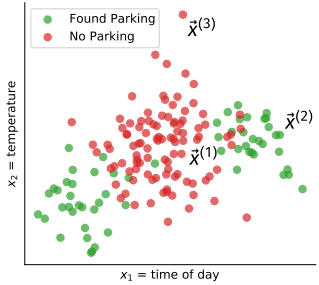
# Feature Space, Visualized

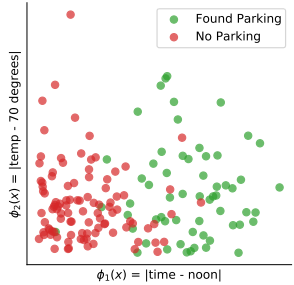


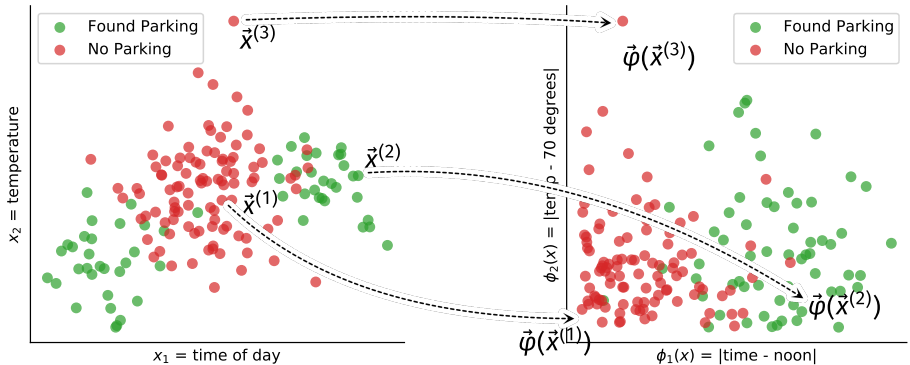
# Exercise

Where does $\vec{\varphi}$ map $\vec{x}^{(1)}$, $\vec{x}^{(2)}$, and $\vec{x}^{(3)}$?



# Solution


Found Parking
No Parking
$\vec{x}^{(3)}$
$\vec{x}^{(2)}$
$\vec{x}^{(1)}$
$x_2$ = temperature
$x_1$ = time of day
$\phi_2(x)$ = |temp - 70 degrees|
$\vec{\varphi}(\vec{x}^{(3)})$
$\vec{\varphi}(\vec{x}^{(2)})$
$\vec{\varphi}(\vec{x}^{(1)})$
Found Parking
No Parking
$\phi_1(x)$ = |time - noon|

# After the Mapping

- The basis functions $\varphi_1, \varphi_2$ give us our “new” features.
- This gives us a new **representation**.
- In this representation, learning (classification) is easier.

# Training

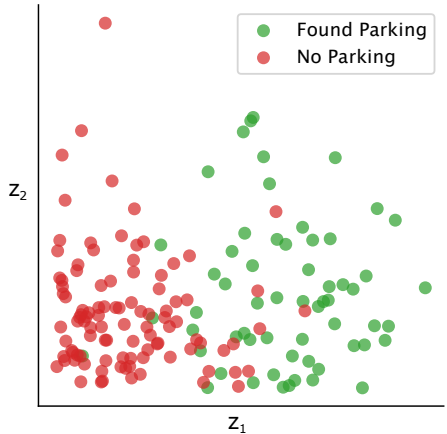
- Map each training example $\vec{x}^{(i)}$ to feature space, creating new training data:

$$\vec{z}^{(1)} = \vec{\varphi}(\vec{x}^{(1)}), \quad \vec{z}^{(2)} = \vec{\varphi}(\vec{x}^{(2)}), \quad \ldots, \quad \vec{z}^{(n)} = \vec{\varphi}(\vec{x}^{(n)})$$

- Fit linear prediction function $H$ in usual way:

$$H_f(\vec{z}) = w_0 + w_1 z_1 + w_2 z_2 + \ldots + w_d z_d$$

# Training Data in Feature Space



# Prediction

- If we have $\vec{z}$ in feature space, prediction is:

$$H_f(\vec{z}) = w_0 + w_1 z_1 + w_2 z_2 + \ldots + w_d z_d$$

# Prediction

- But if we have $\vec{x}$ from original space, we must "convert" $\vec{x}$ to feature space first:

$$
\begin{aligned}
H(\vec{x}) &= H_f(\vec{\varphi}(\vec{x})) \\
&= H_f(\,(\varphi_1(\vec{x}),\, \varphi_2(\vec{x}),\, \ldots,\, \varphi_d(\vec{x}))^T\,) \\
&= w_0 + w_1\varphi_1(\vec{x}) + w_2\varphi_2(\vec{x}) + \ldots + w_d\varphi_d(\vec{x})
\end{aligned}
$$

# Overview: Feature Mapping

- A basis function can involve any/all of the original features:

$$\varphi_3(\vec{x}) = x_1 \cdot x_2$$

- We can make more basis functions than original features:

$$\vec{\varphi}(\vec{x}) = (\varphi_1(\vec{x}), \varphi_2(\vec{x}), \varphi_3(\vec{x}))^T$$

# Overview: Feature Mapping

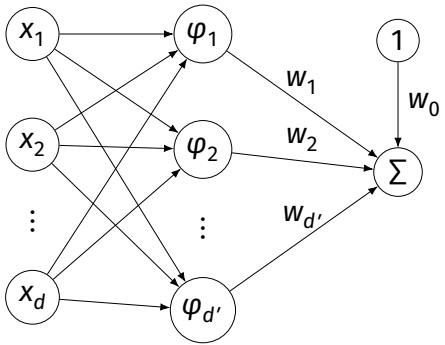
1. Start with data in original space, $\mathbb{R}^d$.
2. Choose some basis functions, $\varphi_1, \varphi_2, \ldots, \varphi_{d'}$
3. Map each data point to **feature space** $\mathbb{R}^{d'}$:

$$\vec{x} \mapsto (\varphi_1(\vec{x}), \varphi_2(\vec{x}), \ldots, \varphi_{d'}(\vec{x}))^t$$

4. Fit linear prediction function in new space:

$$H(\vec{x}) = w_0 + w_1\varphi_1(\vec{x}) + w_2\varphi_2(\vec{x})$$

$$H(\vec{x}) = w_0 + w_1\varphi_1(\vec{x}) + w_2\varphi_2(\vec{x})$$



# Today's Question

- Q: How do we learn non-linear patterns using linear prediction functions?
- A: Use non-linear basis functions to map to a feature space.

# DSC 140B

## Representation Learning

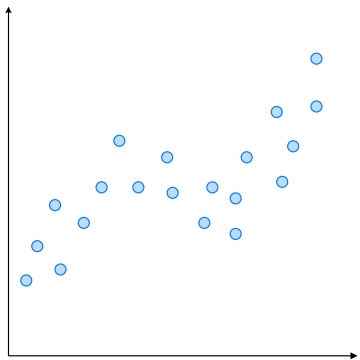
Lecture 17 | Part 4

## Basis Functions and Regression

# By the way...

- You've (probably) seen basis functions used before.
- Linear regression for non-linear patterns in DSC 40A.

# Example



# Fitting Non-Linear Patterns

- Fit function of the form

$$H(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4$$

- Linear function of $\vec{w}$, non-linear function of $x$.

# The Trick

- Treat $x$, $x^2$, $x^3$, $x^4$ as **new** features.
- Create design matrix:

$$X = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 \end{pmatrix}$$

- Solve $X^T X\vec{w} = X^T\vec{w}$ for $\vec{w}$, as usual.
- Works for more than just polynomials.

# Another View

- We have changed the representation of a point:

$$x \mapsto (x, x^2, x^3, x^4)$$

- Basis functions:

$$\varphi_1(x) = x \quad \varphi_2(x) = x^2 \quad \varphi_3(x) = x^3 \quad \varphi_4(x) = x^4$$

# DSC 140B

## Representation Learning

Lecture 17 | Part 5

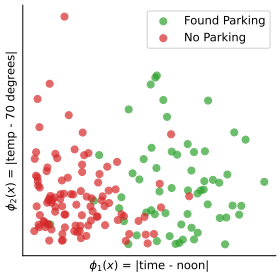
## A Tale of Two Spaces

# A Tale of Two Spaces

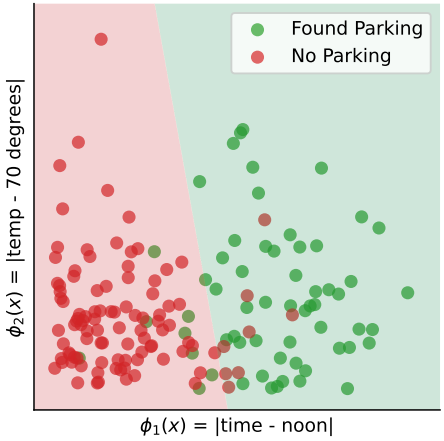
- The **original space**: where the raw data lies.
- The **feature space**: where the data lies after feature mapping $\vec{\varphi}$
- Remember: we fit a linear prediction function in the **feature space**.

## Exercise

- In **feature space**, what does the decision boundary look like?
- What does the prediction function surface look like?

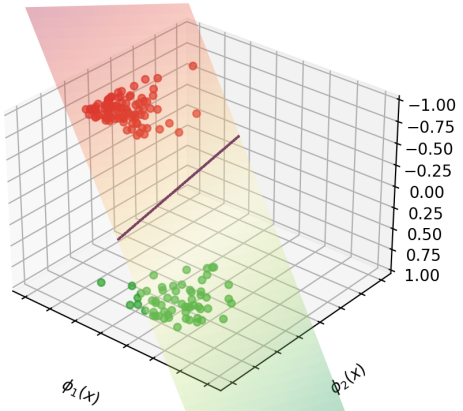


# Decision Boundary in Feature Space[4]



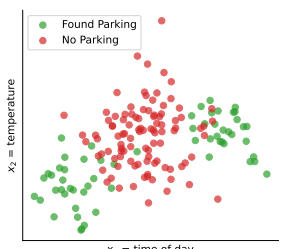

[4]Fit by minimizing square loss

# Prediction Surface in Feature Space

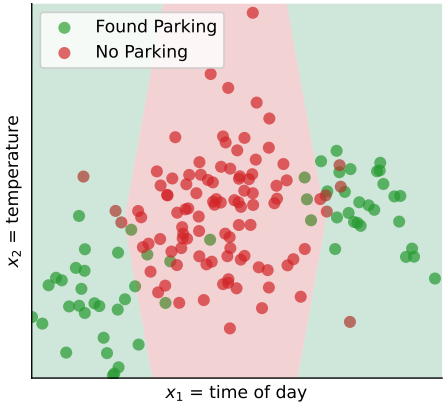


## Exercise

- In the **original space**, what does the decision boundary look like?
- What does the prediction function surface look like?

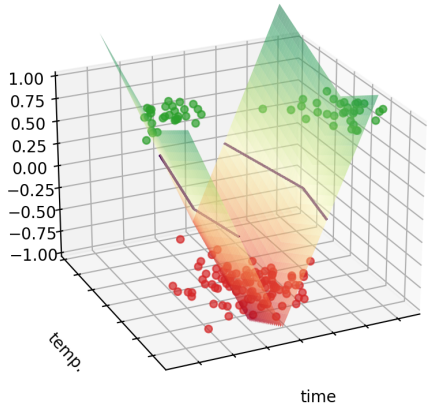


# Decision Boundary in Original Space[5]




[5]Fit by minimizing square loss

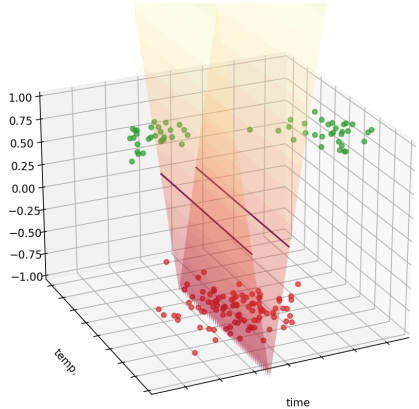
# Prediction Surface in Original Space



# Insight

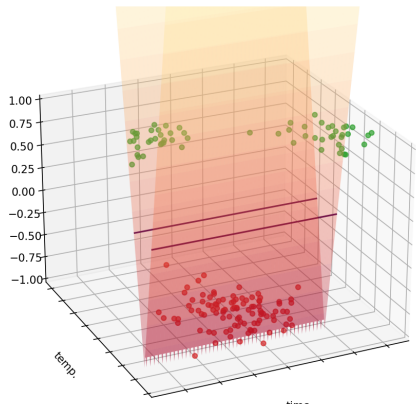
- $H$ is a sum of basis functions, $\varphi_1$ and $\varphi_2$.
  - $H(\vec{x}) = w_0 + w_1\varphi_1(\vec{x}) + w_2\varphi_2(\vec{x})$
- The prediction surface is a sum of other surfaces.
- Each basis function is a “building block”.

# Visualizing the Basis Function $\varphi_1$



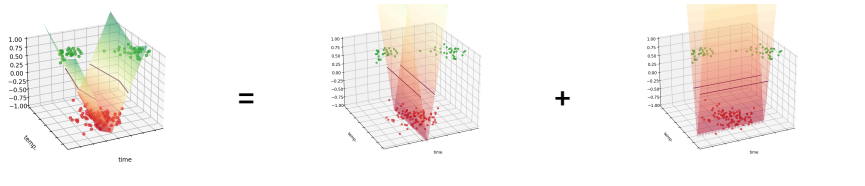

- $w_0 + w_1 |x_1 - \text{noon}|$

# Visualizing the Basis Function $\varphi_2$

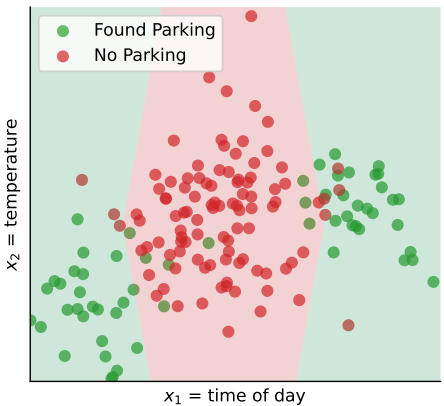



- $w_0 + w_2 |x_2 - 72^\circ|$

# Visualizing the Prediction Surface



# View: Function Approximation




- Find a function that is ≈ 1 near green points and ≈ −1 near red points.

# What's Wrong?

- We've discovered how to learn non-linear patterns using linear prediction functions.
  - Use non-linear basis functions to map to a feature space.

- Something should bug you, though...