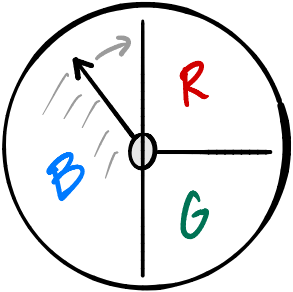


DSC 40A

Lecture 12

Probability

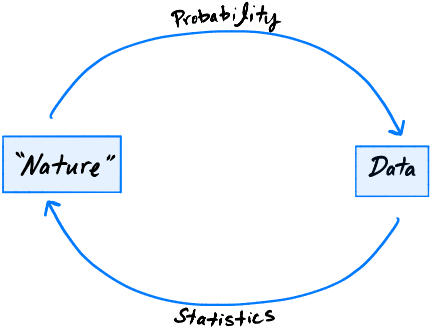
Suggested Reading

Chapter 1.2 of Grinstead and Snell

Why Probability

- We use data to make decisions.
- But the data could have been different.
- **Probability**: how different?

Probability vs. Statistics



The Language of Probability: Set Theory

- A set is a collection of distinct items.
  - Example: the six colleges. **Finite**.

$$\{\text{Marshall}, \text{Roosevelt}, \text{Warren}, \text{Muir}, \text{Revelle}, \text{Sixth}\}$$

  - Example: positive integers. **Discrete, infinite**.

$$\{1, 2, 3, 4, \ldots\}$$

  - Example: all real numbers. **Continuous, infinite**.

Sets

- Sets are **unordered**.
- They do not contain **duplicates**.

The Empty Set

- The **empty set** is the set with nothing in it.
- Written $\{\}$ or $\emptyset$.

Elements

- The things in a set are called **elements**.
- Use $x \in A$ to denote that x is an element of A:
  - $3 \in \{1, 2, 3, 4\}$
  - $1.7 \notin \{1, 2, 3, 4\}$
- The **size** of a set A, written $|A|$, is the number of elements it contains.
  - $|\{1, 2, 3\}| = 3$

Subsets

- If every element of set A is in set B, then A is a **subset** of B.
- Written $A \subset B$ (or sometimes $A \subseteq B$).
- Examples:
  - $\{1, 4\} \subset \{1, 2, 3, 4\}$.
  - $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$.
- If $A \subset B$ and $B \subset A$, then $A = B$.

Discussion Question

Let $S = \{1, 2, 3, 4\}$. Which of these is true?

A) $\varnothing \not\subset S$ and $\varnothing \in S$.
B) $\varnothing \not\subset S$ and $\varnothing \notin S$.
C) $\varnothing \subset S$ and $\varnothing \in S$.
D) $\varnothing \subset S$ and $\varnothing \notin S$.

Intersection

- The **intersection** of sets A and B is the set containing all elements that are in **both** A and B.

- Written $A \cap B$.

- Examples:
  - $\{1, 2, 4\} \cap \{2, 3, 4\} =$

  - $\{1, 2\} \cap \{3, 4\} =$

- If $A \cap B = \emptyset$, A and B are said to be **disjoint**.

Union

- The **union** of sets A and B is the set containing all elements that are in **at least one** of A or B.

- Written $A \cup B$.

- Examples:
  - $\{1, 2\} \cup \{2, 3, 4\} =$

  - $\{1\} \cup \{2\} \cup \{3\} \cup \varnothing =$

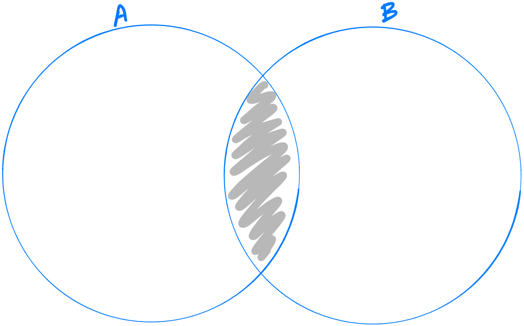
Difference

- The **difference** $A - B$ is the set of all elements that are in A and not in B.

- Examples:
  - $\{1, 2, 3, 4\} - \{3, 5, 6\} =$

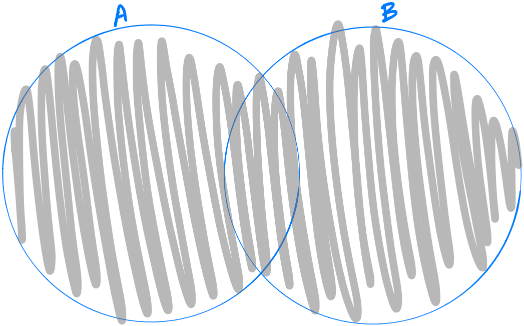
  - $\emptyset - \{1, 2, 3\} =$

Venn Diagrams



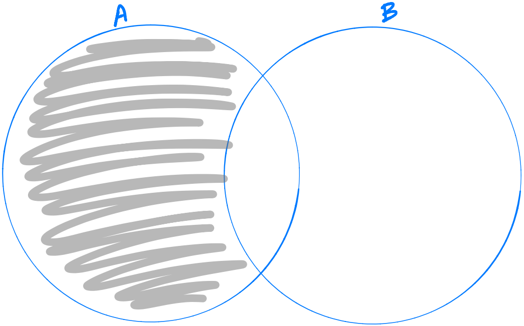
$A \cap B$

Venn Diagrams



$A \cup B$

Venn Diagrams



$A - B$

Tuples

- A **tuple** is an ordered sequence.
  - A 2-tuple is an **ordered pair**.

- Example: result of flipping coin four times.

$$(\text{Heads}, \text{Tails}, \text{Heads}, \text{Heads})$$

- Example: a point in three dimensions.

$$(3, -1, 2)$$

Tuples

- Tuples are **ordered**.
- Duplicates **are allowed**.

Products of Sets

- Options for dinner: {sushi, tacos}
- Options for dessert: {ice cream, milk tea, espresso}
- Set of all possibilities for dinner/dessert:

$$\begin{array}{lr} (\text{sushi}, & \text{ice cream}) \\ (\text{sushi}, & \text{milk tea}) \\ (\text{sushi}, & \text{espresso}) \\ (\text{tacos}, & \text{ice cream}) \\ (\text{tacos}, & \text{milk tea}) \\ (\text{tacos}, & \text{espresso}) \end{array}$$

Products of Sets

- The **Cartesian Product** of sets A and B, written $A \times B$, is the **set** of all ordered pairs (**2-tuples**) whose:
  - first element is in A
  - second element is in B
- Example: $\{1, 2\} \times \{a, b, c\}$.
- Example: $\{1, 2\} \times \{1, 2\}$.

Discussion Question

Which of these correctly gives the size of the Cartesian product of A and B?

A) $|A \times B| = |A| + |B|$
B) $|A \times B| = |A| \cdot |B|$
C) $|A \times B| = |A|^{|B|}$
D) $|A \times B| = |B|^{|A|}$

Experiments

- An **experiment** is something whose outcome appears to be random.
- Examples:
  - Rolling a die.
  - Flipping a coin, twice.
  - Asking someone what college they're in.
  - Looking for an open parking spot in Hopkins Parking Structure.

Outcomes

- An **outcome** is the result of an experiment.
- The **sample space**, Ω, is the set of all outcomes of an experiment.
  - Experiment: Rolling a die.
    Possible outcomes: {1, 2, 3, 4, 5, 6}
  - Experiment: Flipping a coin, twice.
    Possible outcomes:
    $$\{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$
  - Experiment: Looking for parking in Hopkins.
    Possible outcomes: {Spots, No Spots}

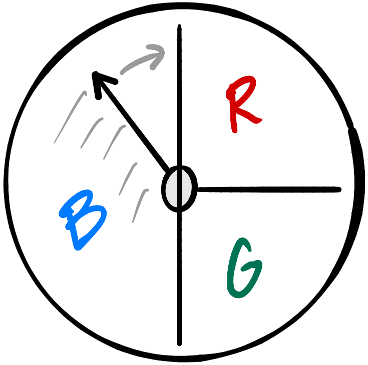
Discrete vs. Continuous Probability

- The sample space can be discrete or continuous.
- Discrete: rolling a die.
- Continuous: measuring temperature.
- We'll focus on **discrete** setting.

Probability

- The **probability** of an outcome is the proportion of times it happens if the experiment is repeated an infinite number of times.
- Example: probability of seeing Heads is $1/2$.
- Example: probability of rolling a 3 is $1/6$.
- Outcomes need not be equally-probable!

Example



- Outcomes: {R, G, B}
- Probability of B: 1/2. Probability of R and G: 1/4, each.

Probability Distribution Function

- A **probability distribution function** $m(\omega)$ assigns a probability to every outcome $\omega \in \Omega$.

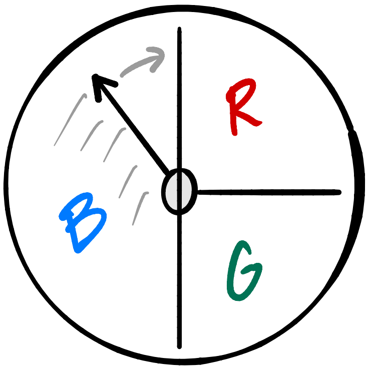
- Requirement #1: probabilities are ≥ 0.

$$m(\omega) \geq 0$$

- Requirement #2: probabilities sum to 1.

$$\sum_{\omega \in \Omega} m(\omega) = 1$$

Example



- $m(\mathbf{B}) = 1/2, \quad m(\mathbf{R}) = 1/4, \quad m(\mathbf{G}) = 1/4.$

Events

- An **event** is a set of outcomes.
- An event "happens" if the result of the experiment is contained in the event.
- Example:
  - Experiment: rolling a die.
  - Sample space: $\{1, 2, 3, 4, 5, 6\}$.
  - Event: $\{2, 4, 6\}$ (i.e., rolling an even number).

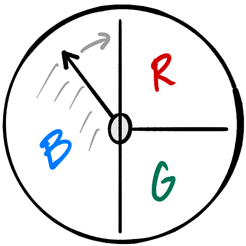
Probability of an Event

- The **probability** of an event E, written $P(E)$ is the sum of the probabilities of the elements of E:

$$P(E) = \sum_{\omega \in E} m(\omega)$$

Example

- What is the probability of spinning either a **G** or a **B**?



- E =

- $P(E)$ =

Equally-Probable Outcomes

- If all of the outcomes are equally-probable, then

$$P(E) = \frac{|E|}{|\Omega|}$$

- Proof:

$$P(E) = \sum_{\omega \in E} m(\omega) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Example

- what is the probability of rolling an even number?
  - $E =$
  - $|E| =$
  - $|\Omega| =$
  - $P(E) =$

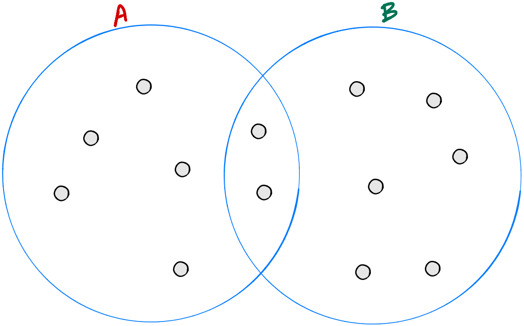
Combining Events

- The event that "A **or** B" happens = $A \cup B$.
- The event that "A **and** B" happens = $A \cap B$.
- The event that "A **but not** B" happens = $A - B$.
- The event that "A **doesn't**" happen = $\Omega - A$.

Example

- What is the probability of rolling an even number ≤ 3?
  - $A =$
  - $B =$
  - $|A \cap B| =$
  - $P(A \cap B) =$

Probability of a Union



- $P(A \cup B) = P(A) + P(B)$?
- $P(A \cup B) =$