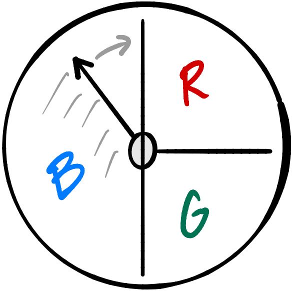



# DSC 40A

Lecture 12

Probability

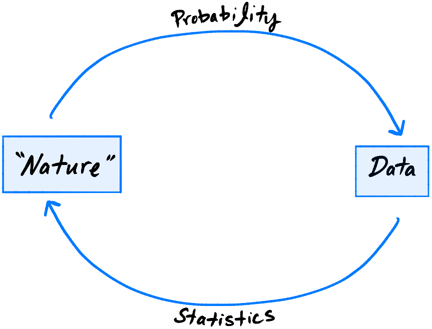
# Suggested Reading

Chapter 1.2 of Grinstead and Snell

# Why Probability

- We use data to make decisions.
- But the data could have been different.
- **Probability**: how different?

# Probability vs. Statistics



# The Language of Probability: Set Theory

- A set is a collection of distinct items.
  - Example: the six colleges. **Finite**.

$$\{\text{Marshall}, \text{Roosevelt}, \text{Warren}, \text{Muir}, \text{Revelle}, \text{Sixth}\}$$

  - Example: positive integers. **Discrete, infinite**.

$$\{1, 2, 3, 4, \ldots\}$$

  - Example: all real numbers. **Continuous, infinite**.

## Sets

- Sets are **unordered**.
- They do not contain **duplicates**.

## The Empty Set

- The **empty set** is the set with nothing in it.
- Written $\{\}$ or $\emptyset$.

# Elements

- The things in a set are called **elements**.

- Use $x \in A$ to denote that $x$ is an element of $A$:
  - $3 \in \{1, 2, 3, 4\}$
  - $1.7 \notin \{1, 2, 3, 4\}$

- The **size** of a set $A$, written $|A|$, is the number of elements it contains.
  - $|\{1, 2, 3\}| = 3$

# Subsets

- If every element of set $A$ is in set $B$, then $A$ is a **subset** of $B$.
- Written $A \subset B$ (or sometimes $A \subseteq B$).
- Examples:
  - $\{1, 4\} \subset \{1, 2, 3, 4\}$.
  - $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$.

$\{\} \subset \{1, 2, 3\}$

- If $A \subset B$ and $B \subset A$, then $A = B$.

## Discussion Question

Let $S = \{1, 2, 3, 4\}$. Which of these is true?

13 A) $\varnothing \not\subset S$ and $\varnothing \in S$.

16 B) $\varnothing \not\subset S$ and $\varnothing \notin S$.

14 C) $\varnothing \subset S$ and $\varnothing \in S$.

21 D) $\varnothing \subset S$ and $\varnothing \notin S$.

$\phi \notin \{1, 2, 3, 4\}$

$\phi \in \{\{1, 2, 3\}, \{1, 5\}, \{\}\}$

# Intersection

- The **intersection** of sets $A$ and $B$ is the set containing all elements that are in **both** $A$ and $B$.

- Written $A \cap B$.

- Examples:
  - $\{1, 2, 4\} \cap \{2, 3, 4\} = \{2, 4\}$

  - $\{1, 2\} \cap \{3, 4\} = \emptyset$

- If $A \cap B = \emptyset$, $A$ and $B$ are said to be **disjoint**.

# Union

- The **union** of sets $A$ and $B$ is the set containing all elements that are in **at least one** of $A$ or $B$.
- Written $A \cup B$.
- Examples:
  - $\underbrace{\{1, 2\}}_{A} \cup \underbrace{\{2, 3, 4\}}_{B} = \{1, 2, 3, 4\}$
  - $\{1\} \cup \{2\} \cup \{3\} \cup \emptyset = \{1, 2, 3\}$

$\{1, 2, 3\} \cup \emptyset = \{1, 2, 3\}$

$\{1, 2, 3\} \cup \{\emptyset\} = \{1, 2, 3, \emptyset\}$

$\{\{1, 2\}, \{3, 4\}\} \cup \{\{1, 3, 4\}\} = \{\{1, 2\}, \{3, 4\}, \{1, 3, 4\}\}$

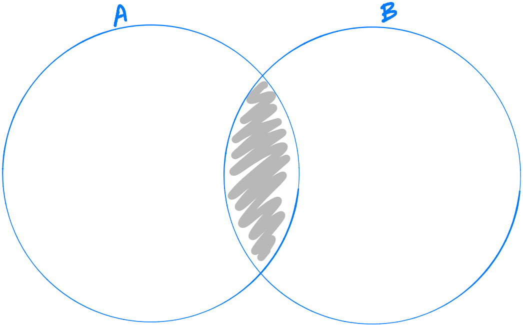
# Difference

- The **difference** $A - B$ is the set of all elements that are in $A$ and not in $B$.

- Examples:
  - $\{1, 2, \cancel{3}, 4\} - \{3, 5, 6\} =$ $\{1, 2, 4\}$

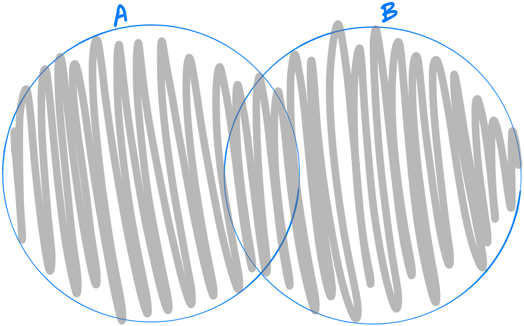
  - $\varnothing - \{1, 2, 3\} =$ $\{\ \}$

# Venn Diagrams



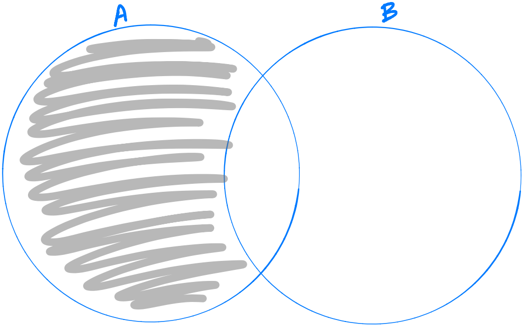

$A \cap B$

# Venn Diagrams




$A \cup B$

# Venn Diagrams




$A - B$

# Tuples

- A **tuple** is an ordered sequence.
  - A 2-tuple is an **ordered pair**.

- Example: result of flipping coin four times.

$$(\text{Heads}, \text{Tails}, \text{Heads}, \text{Heads})$$

- Example: a point in three dimensions.

$$(3, -1, 2)$$

# Tuples

- Tuples are **ordered**.
- Duplicates **are allowed**.

# Products of Sets

- Options for dinner: {sushi, tacos}
- Options for dessert: {ice cream, milk tea, espresso}
- Set of all possibilities for dinner/dessert:

$$\begin{array}{lr}
(\text{sushi}, & \text{ice cream}) \\
(\text{sushi}, & \text{milk tea}) \\
(\text{sushi}, & \text{espresso}) \\
(\text{tacos}, & \text{ice cream}) \\
(\text{tacos}, & \text{milk tea}) \\
(\text{tacos}, & \text{espresso})
\end{array}$$

# Products of Sets

- The **Cartesian Product** of sets $A$ and $B$, written $A \times B$, is the **set** of all ordered pairs (**2-tuples**) whose:
  - first element is in $A$
  - second element is in $B$
- Example: $\underset{A}{\{1,2\}} \times \underset{B}{\{a,b,c\}}$. $= \{$
  $(1,a), (1,b), (1,c),$
  $(2,a), (2,b), (2,c)$
  $\}$

  $|\{1,2\} \times \{a,b,c\}| = 6$
- Example: $\underset{A}{\{1,2\}} \times \underset{A}{\{1,2\}}$. $= \{(1,1), (1,2), (2,1), (2,2)\}$

## Discussion Question

Which of these correctly gives the size of the Cartesian product of $A$ and $B$?

A) $|A \times B| = |A| + |B|$
B) $|A \times B| = |A| \cdot |B|$
C) $|A \times B| = |A|^{|B|}$
D) $|A \times B| = |B|^{|A|}$

# Experiments

- An **experiment** is something whose outcome appears to be random.
- Examples:
  - Rolling a die.
  - Flipping a coin, twice.
  - Asking someone what college they're in.
  - Looking for an open parking spot in Hopkins Parking Structure.

# Outcomes

- An **outcome** is the result of an experiment.
- The **sample space**, $\Omega$, is the set of all outcomes of an experiment.
  - Experiment: Rolling a die.
    Possible outcomes: $\{1, 2, 3, 4, 5, 6\} = \Omega$
  - Experiment: Flipping a coin, twice.
    Possible outcomes:
    $$\{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$
  - Experiment: Looking for parking in Hopkins.
    Possible outcomes: {Spots, No Spots}

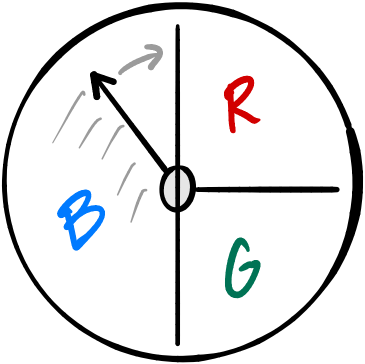
# Discrete vs. Continuous Probability

- The sample space can be discrete or continuous.
- Discrete: rolling a die.
- Continuous: measuring temperature.
- We’ll focus on **discrete** setting.

# Probability

- The **probability** of an outcome is the proportion of times it happens if the experiment is repeated an infinite number of times.
- Example: probability of seeing Heads is 1/2.
- Example: probability of rolling a 3 is 1/6.
- Outcomes need not be equally-probable!

# Example




- Outcomes: {R, G, B}
- Probability of B: 1/2. Probability of R and G: 1/4, each.

# Probability Distribution Function

- A **probability distribution function** $m(\omega)$ assigns a probability to every outcome $\omega \in \Omega$.

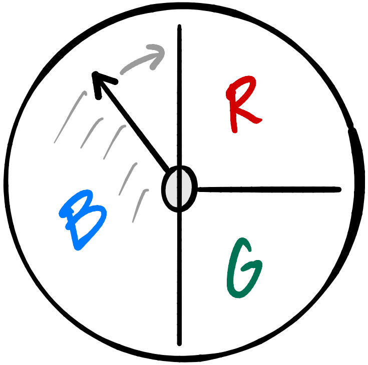
- Requirement #1: probabilities are $\geq 0$.

$$m(\omega) \geq 0$$

- Requirement #2: probabilities sum to 1.

$$\sum_{\omega \in \Omega} m(\omega) = 1$$

# Example




- $m(\mathbf{B}) = 1/2, \quad m(\mathbf{R}) = 1/4, \quad m(\mathbf{G}) = 1/4.$

# Events

- An **event** is a set of outcomes.
- An event "happens" if the result of the experiment is contained in the event.
- Example:
  - Experiment: rolling a die.
  - Sample space: $\{1, 2, 3, 4, 5, 6\}. = \Omega$
  - Event: $\{2, 4, 6\}$ (i.e., rolling an even number).

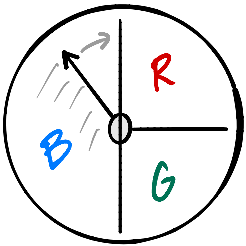
# Probability of an Event

- The **probability** of an event $E$, written $P(E)$ is the sum of the probabilities of the elements of $E$:

$$P(E) = \sum_{\omega \in E} m(\omega)$$

# Example

- What is the probability of spinning either a **G** or a **B**?




- $E = \{B, G\}$
- $P(E) = \sum_{\omega \in E} m(\omega) = [m(B) + m(G)] = [1/2 + 1/4] = 3/4$

# Equally-Probable Outcomes

- If all of the $\boxed{\text{outcomes are equally-probable,}}$ then

$$P(E) = \frac{|E|}{|\Omega|}$$

- Proof:

$$P(E) = \sum_{\omega \in E} m(\omega) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

# Example

- what is the probability of rolling an even number?
  - $E = \{2, 4, 6\}$
  - $|E| = 3$
  - $|\Omega| = 6$
  - $P(E) = |E| / |\Omega| = 3/6 = \frac{1}{2}$

$$P(E) = \sum_{\omega \in E} m(\omega) = [m(2) + m(4) + m(6)] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

# Combining Events

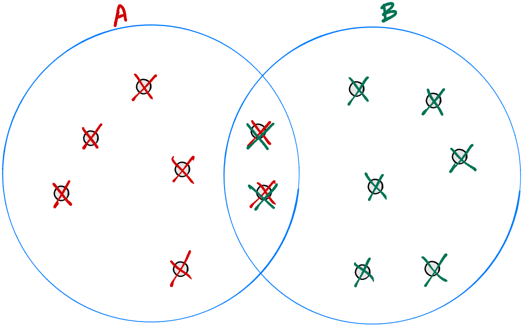
- The event that "$A$ **or** $B$" happens = $A \cup B$.
- The event that "$A$ **and** $B$" happens = $A \cap B$.
- The event that "$A$ **but not** $B$" happens = $A - B$.
- The event that "$A$ **doesn't**" happen = $\Omega - A$.

# Example

rolling a # that is even and $\leq 3$?

- What is the probability of rolling an even number ≤ 3?
  - $A =$ roll even # $= \{2,4,6\}$
  - $B =$ roll $\# \leq 3$ $= \{1,2,3\}$
  - $|A \cap B| =$ $|\{2,4,6\} \cap \{1,2,3\}| = |\{2\}| = 1$
  - $P(A \cap B) =$ $\frac{|E|}{|\Omega|} = \frac{|A \cap B|}{|\Omega|} = \frac{1}{6}$

# Probability of a Union




- $P(A \cup B) = P(A) + P(B)$?
- $P(A \cup B) = \sum_{\omega \in A \cup B} m(\omega) \cancel{\overset{?}{=}} \overbrace{\sum_{\omega \in A} m(\omega)}^{P(A)} + \overbrace{\sum_{\omega \in B} m(\omega)}^{P(B)}$

$$\sum_{\omega \in A \cup B} m(\omega) = \sum_{\omega \in A} m(\omega) + \sum_{\omega \in B} m(\omega) - \sum_{\omega \in A \cap B} m(\omega)$$

$$= P(A) + P(B) - P(A \cap B)$$

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$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Example: $A$ = roll a 1, 2, 3
$B$ = roll a 2, 3, 4

$$P(A \cup B) = P(\{1,2,3,4\}) = \frac{4}{6} = \frac{2}{3}$$

Example: $A$ = roll a 1, 2, 3
$B$ = roll a 4, 5, 6

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 1/2 + 1/2 - 0$$
$$= 1$$