DSC 40B

Theoretical Foundations II

Lecture -1 | Part 1

Complexity Theory

The quest for efficient algorithms is about finding clever ways to avoid taking exponential time. So far we have seen the most brilliant successes of this quest; now we meet the quest's most embarrassing and persistent failures.

- paraphrased from *Algorithms* by Dasgupta, Papadimitriou, Vazirani

Exponential to Polynomial

- Many problems have brute force solutions which take exponential time.
- Example: clustering to maximize separation
- The challenge of algorithm design: find a more “efficient” solution.

Polynomial Time

- If an algorithm's worst case time complexity is $O(n^k)$ for some k, we say that it runs in **polynomial time**.
  - Example: $\Theta(n \log n)$, since $n \log n = O(n^2)$.
- Any polynomial is much faster than exponential for big n.
  - But not necessarily for small n.
  - Example: n^{100} vs 1.0001^n.
- We therefore think of polynomial as "efficient".

Question

- Is every problem solvable in polynomial time?

Question

- Is every problem solvable in polynomial time?
- **No!** Problem: print all permutations of n numbers.

$n!$

Question

- Is every problem solvable in polynomial time?
- **No!** Problem: print all permutations of n numbers.
- **No!** Problem: given $n \times n$ checkerboard and current pieces, determine if red can force a win.

Ok, then...

- What problems can be solved in polynomial time?
- What problems can't?
- How can I tell if I have a hard problem?

Ok, then...

- What problems can be solved in polynomial time?
- What problems can't?
- How can I tell if I have a hard problem?
- Core questions in **computational complexity theory**.

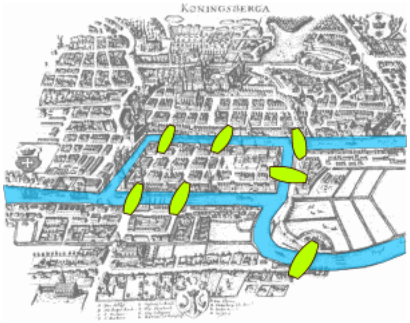
DSC 40B

Theoretical Foundations II

Lecture -1 | Part 2

Eulerian and Hamiltonian Cycles

Example: Bridges of Königsberg

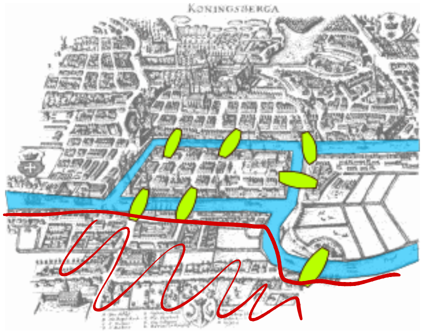


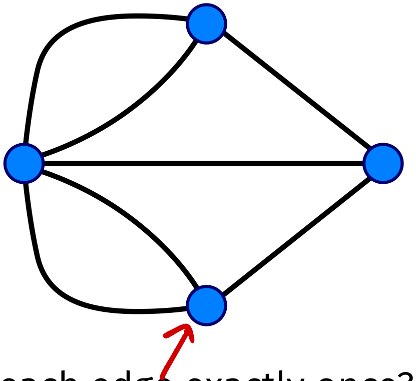
- **Problem**: Is it possible to start and end at same point while crossing each bridge exactly once?

Leonhard Euler

1707 - 1783

Eulerian Cycle

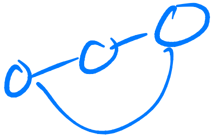




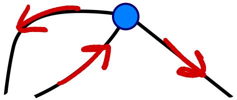
Is there a cycle which uses each edge exactly once?

Necessary conditions

- Graph must be connected.



- Each node must have even degree.
- Answer for Königsberg answer: it is **impossible**.



In General...

- These conditions are **necessary** and **sufficient**.
- A graph has a Eulerian cycle **if and only if**:
  - it is connected;
  - each node has even degree.

Exercise

Can we determine if a graph has an Eulerian cycle in time that is polynomial in the number of nodes?

Remember, an Eulerian cycle exists iff the graph is connected and each node has even degree.

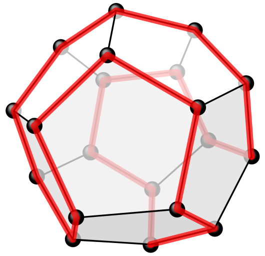
Answer

- We can check if it is connected in $\Theta(V + E)$ time.
- Compute every node's degree in $\Theta(V)$ time with adjacency list.
- Total: $\Theta(V + E) = O(V^2)$. **Yes!**

Gaming in the 19th Century

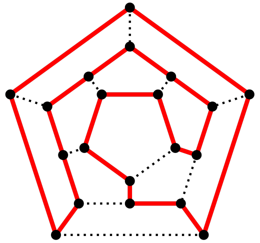
I have found that some young persons have been much amused by trying a new mathematical game which the Icosian furnishes [...]

- W.R. Hamilton, 1856



Hamiltonian Cycles

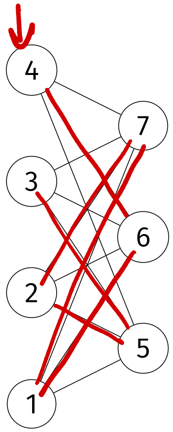
- A **Hamiltonian cycle** is a cycle which visits each *node* exactly once (except the starting node).
- Game: find a Hamiltonian cycle on the graph below:



Exercise

Can we determine whether a general graph has a Hamiltonian cycle in polynomial time?

Some cases are easy



In General

$n!$

- Could brute-force.
- How many possible cycles are there?

Hamiltonian Cycles are Difficult

- This is a **very difficult** problem.
- No polynomial algorithm is known for general graphs.
- In special cases, there may be a fast solution. But in general, worst case is hard.

Note

- Determining if a graph has a Hamiltonian cycle is **hard**.
- But if we're given a "hint" (i.e., $(v_1, v_2, \dots, v_n)$ is possibly a Hamiltonian cycle), we can check it very quickly!
- Hard to solve; but easy to verify "hints".

Similar Problems

- Eulerian: polynomial algorithm, **“easy”**.
- Hamiltonian: no polynomial algorithm known, **“hard”**.

Main Idea

Computer science is littered with pairs of similar problems where one easy and the other very hard.

DSC 40B

Theoretical Foundations II

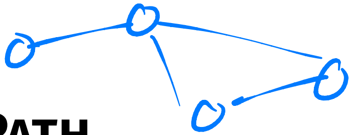
Lecture -1 | Part 3

Shortest and Longest Paths

Problem: SHORTPATH

- **Input:** Graph[1] G, source u, dest. v, number k.
- **Problem:** is there a path from u to v of length $\leq k$?
- **Solution:** BFS or Dijkstra/Bellman-Ford in polynomial time.
- **Easy!**

[1]Weighted with no negative cycles, or unweighted.



Problem: LongPath

- **Input:** Graph[2] G, source u, dest. v, number k.
- **Problem:** is there a **simple** path from u to v of length $\geq k$?
- Naïve solution: try all $V!$ path candidates.

[2]Weighted or unweighted.

Long Paths

- There is no known polynomial algorithm for this problem.
- It is a **hard problem**.
- But given a "hint" (a possible long path), we can verify it very quickly!

DSC 40B

Theoretical Foundations II

Lecture -1 | Part 4

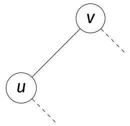
Reductions

Reductions

- Hamiltonian and LongPath are related.
- We can “convert” Hamiltonian into LongPath in polynomial time.
- We say that Hamiltonian **reduces** to LongPath.

Reduction

- Suppose we have an algorithm for LongPath.
- We can use it to solve Hamiltonian as follows:
  - Pick arbitrary node u.
  - For each neighbor v of u:
    - Create graph G' by copying G, deleting (u, v)
    - Use algorithm to check if a simple path of length $\geq |V| - 1$ from u to v exists in G'.
    - If yes, then there is a Hamiltonian cycle.



Reductions

- If Problem A reduces[3] to Problem B, it means "we can solve A by solving B".
- Best possible time for A ≤ best possible time for B + polynomial
- "A is no harder than B"

  $A \leq B$
- "B is at least as hard as A"

[3] We'll assume reduction takes polynomial time.

Relative Difficulty

- If Problem A reduces to Problem B, we say B is **at least as hard** as A.
- Example: HAMILTONIAN reduces to LONGPATH. LONGPATH is at least as hard as HAMILTONIAN.

DSC 40B

Theoretical Foundations II

Lecture -1 | Part 5

$\mathbf{P} \overset{?}{=} \mathbf{NP}$

Decision Problems

- All of today's problems are **decision problems**.
  - Output: yes or no.
  - Example: Does the graph have an Euler cycle?

P

- Some problems have polynomial time algorithms.
  - ShortPath, Euler
- The set of decision problems that can be solved in polynomial time is called **P**.
- Example: ShortPath and Euler are in P.

NP

- The set of decision problems with “hints” that can be verified in polynomial time is called **NP**.
- All of today’s problems are in NP.
    - All problems in P are also in NP.
- Example: SHORTPATH, EULER, HAMILTONIAN, LONGPATH are all in NP.

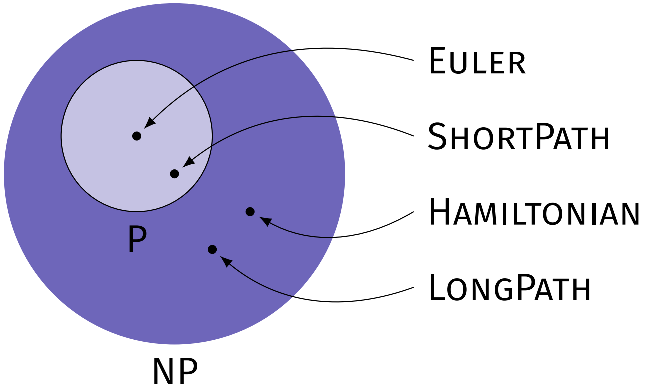
$\mathbf{P} \subset \mathbf{NP}$

- P is a subset of NP.
- It *seems* like some problems in NP aren't in P.
  - Example: HAMILTONIAN, LONGPATH.
  - We don't know polynomial time algorithms for these problems.
- But that doesn't such an algorithm is impossible!

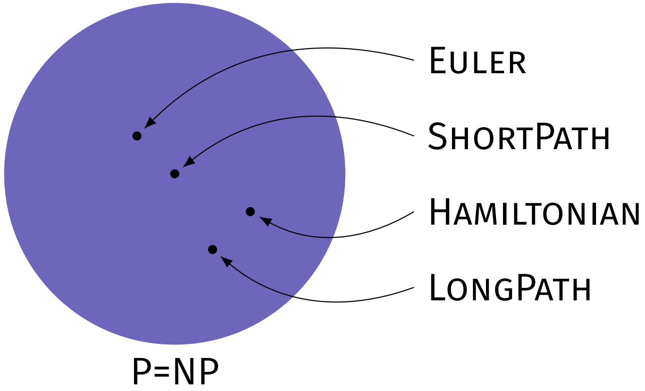
P = NP?

- Are there problems in NP that aren’t in P?
  - That is, is P ≠ NP?
- Or is any problem in NP also in P?
  - That is, is P = NP?

P ≠ NP



P = NP



P = NP?

- Is P = NP?

[4]If you solve it, you'll be rich and famous.

P = NP?

- Is P = NP?
- **No one knows!**
- Biggest open problem in Math/CS.[4]
- Most think P $\neq$ NP.

[4] If you solve it, you'll be rich and famous.

What if P = NP?

- Possibly Earth-shattering.
    - Almost all cryptography instantly becomes obsolete;
    - Logistical problems solved exactly, quickly;
    - *Mathematicians* become obsolete.

- But maybe not...
    - Proof could be non-constructive.
    - Or, constructive but really inefficient. E.g., $\Theta(n^{10000})$

DSC 40B

Theoretical Foundations II

Lecture -1 | Part 6

NP-Completeness

Problem: 3-SAT

- Suppose `x_1`, …, `x_n` are boolean variables (`True`,`False`)

- A **3-clause** is a combination made by `or`-ing and possibly negating three variables:
  - `x_1 or x_5 or (not x_7)`
  - `(not x_1) or (not x_2) or (not x_4)`

Problem: 3-SAT

- **Given:** *m* clauses over *n* boolean variables.
- **Problem:** Is there an assignment of `x_1, …, x_n` which makes all clauses true simultaneously?
- No polynomial time algorithm is known.
- But it is easy to verify a solution, given a hint.
    - 3-SAT is in NP.

Cook’s Theorem

Every problem in NP is polynomial-time reducible to 3-SAT.

- …including Hamiltonian, long path, etc.
- 3-SAT is at least as hard as every problem in NP.
- “hardest problem in NP”

Cook's Theorem (Corollary)

- If 3-SAT is solvable in polynomial time, then all problems in NP are solvable in polynomial time.
  - ...including Hamiltonian, long path, etc.

NP-Completeness

- We say that a problem is **NP-complete** if:
  - it is in NP;
  - every problem in NP is reducible to it.

- HAMILTONIAN, LONGPATH, 3-SAT are all NP-complete.

- NP-complete problems are the “hardest” in NP.

Equivalence

- In some sense, NP-complete problems are equivalent to one another.
- E.g., a fast algorithm for HAMILTONIAN gives a fast algorithm for 3-SAT, LONGPATH, and all problems in NP.

Who cares?

- Complexity theory is a fascinating piece of science.
- But it’s practically useful, too, for recognizing hard problems when you stumble upon them.

DSC 40B
Theoretical Foundations II

Lecture -1 | Part 7

Hard Optimization Problems

Hard Optimization problems

- NP-completeness refers to **decision problems**.
- What about optimization problems?
- We can typically state a similar decision problem.
- If that decision problem is hard, then optimization is at least as hard.

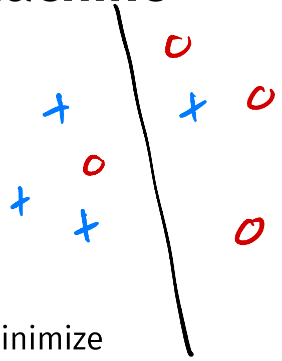
Problem: bin packing

- Optimization problem:
  - **Given:** bin size B, n objects of size $\alpha_1, \dots, \alpha_n$..
  - **Problem:** find minimum number of bins k that can contain all n objects.
- Decision problem version:
  - **Given:** bin size B, n objects of size $\alpha_1, \dots, \alpha_n$, integer k.
  - **Problem:** is it possible to pack all n objects into k bins?
- Decision problem is NP-complete, reduces to optimization problem.

Example: traveling salesperson

- Optimization problem:
    - **Given:** set of n cities, distances between each.
    - **Problem:** find shortest Hamiltonian cycle.
- Decision problem:
    - **Given:** set of n cities, distance between each, length ℓ.
    - **Problem:** is there a Hamiltonian cycle of length $\leq \ell$?
- Decision problem is NP-complete, reduces to optimization problem.

NP-complete problems in machine learning



- Many machine learning problems are NP-complete.
- Examples:
  - Finding a linear decision boundary to minimize misclassifications in non-separable regime.
  - Minimizing k-means objective.

So now what?

- Just because a problem is NP-Hard, doesn’t mean you should give up.
- Usually, an approximation algorithm is fast, “good enough”.
- Some problems are even hard to *approximate*.

Summary

- Not every problem can be solved efficiently.
- Computer scientists are able to categorize these problems.

DSC 40B

Theoretical Foundations II

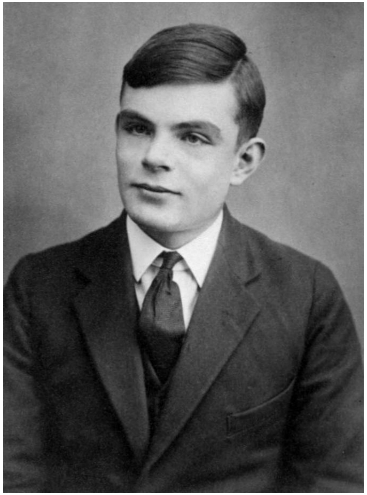
Lecture -1 | Part 8

The Halting Problem

Really hard problems

- Some decision problems are harder than others.
- That is, it takes more time to solve them.
- Given enough time, all decision problems can be solved, right?

Alan Turing



1912–1954

Turing's Halting Problem

- **Given:** a function `f` and an input `x`.
- **Problem:** does `f(x)` halt, or run forever?
- Algorithm must work for all functions/inputs!

```
foo(x):
    while True:
        pass
```

Turing's Argument

- Turing says: no such algorithm can exist.
- Suppose there is a function `halts(f, x)`:
  - Returns `True` if `f(x)` halts.
  - Returns `False` if `f(x)` loops forever.

Turing's Argument

$f(f)$

```
def evil_function(f):
    if halts(f, f):
        # loop forever
    else: # it runs forever
        return
```

- Consider `evil_function(evil_function)`.
  - Does it halt or not?

Turing's Argument

```python
def evil_function(f):
    if halts(f, f):
        # loop forever
    else: # it runs forever
        return
```

- Consider `evil_function(evil_function)`.
  - Does it halt or not?

- Assuming that `halt` works leads to logical impossibility!
  - So a working `halt` cannot exist.

Undecidability

- The halting problem is **undecidable**.

- Fact of the universe: there can be no algorithm for solving it which works on all functions/inputs.

- All of these problems are undecidable:
  - Does the program terminate?
  - Does this line of code ever run?
  - Does this function compute what its specification says?
  - Many others...

Reality

- **Physics:** can’t go faster than the speed of light.
- **Computer science:**
    - There’s a speed limit for certain problems, too.
    - And some problems can’t even be solved!

The End