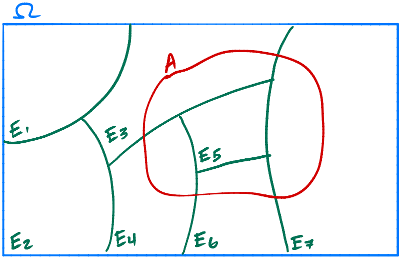


DSC 40A

Lecture 14

Conditional Probability

Getting to Campus

- 100 people were surveyed.
- How did you get to campus today? Walk, bike, or drive?
- Were you late or on-time?
- Results:

(Walk, Late)	6%
(Walk, Not Late)	24%
(Bike, Late)	3%
(Bike, Not Late)	7%
(Drive, Late)	36%
(Drive, Not Late)	24%
	100%

Example

- What is the probability that a randomly-selected person was late?

(Walk, Late)	6%
(Walk, Not Late)	24%
(Bike, Late)	3%
(Bike, Not Late)	7%
(Drive, Late)	36%
(Drive, Not Late)	24%

$6\% + 3\% + 36\% = 45\%$

$P(\text{Late}) = 45\%$

Example

- What is the probability that a randomly-selected person drove?

(Walk, Late)	6%
(Walk, Not Late)	24%
(Bike, Late)	3%
(Bike, Not Late)	7%
(Drive, Late)	36%
(Drive, Not Late)	24%

$P(\text{Drive}) = 36\% + 24\% = 60\%$

Getting to Campus

$P(\text{Walk} \cap \text{Late}) = P(\text{Walk}) \cdot P(\text{Late} \mid \text{Walk})$

- Suppose we no longer have this table:

(Walk, Late)	6%	= 30% × 20%
(Walk, Not Late)	24%	= 30% × 80%
(Bike, Late)	3%	= 10% × 30%
(Bike, Not Late)	7%	
(Drive, Late)	36%	= 60% × 60%
(Drive, Not Late)	24%	

- Instead, we are told only that:
  - $P(\text{Walk})$ 30% of people walk; 20% $P(\text{Late} \mid \text{Walk})$ of them are late. 80% are not late $P(\text{Not Late} \mid \text{Walk})$
  - 10% of people bike; 30% of them are late.
  - 60% of people drive; 60% of them are late.

- Can we recover the table?

Conditional Probabilities

- Of those who walked, 20% were late.
- We say the **conditional probability** of being late **given** walking is 20%.
- Written: $P(\text{Late} \mid \text{Walk}) = 0.20$
- We saw:

$$P(\text{Walk} \cap \text{Late}) = P(\text{Walk}) \cdot P(\text{Late} \mid \text{Walk})$$

- So:

$$P(\text{Late} \mid \text{Walk}) = \frac{P(\text{Walk} \cap \text{Late})}{P(\text{Walk})}$$

Conditional Probability

- Let A and B be events, with $P(B) > 0$.

- The **conditional probability** of A **given** B, written $P(A \mid B)$, is defined by:
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Useful: $P(A \cap B) = P(A \mid B) \cdot P(B)$

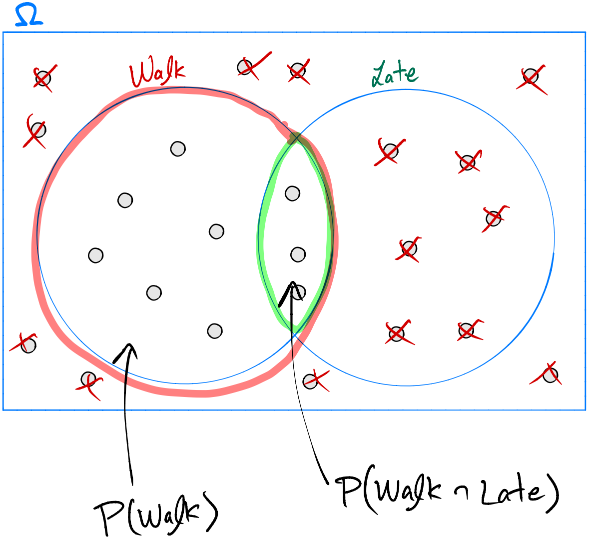
Example

- Suppose someone **tells you** that they walked. What is the probability that they were late?

- That is, what is $P(\text{Late} \mid \text{Walk})$?

(Walk, Late)	6%
(Walk, Not Late)	24%
~~(Bike, Late)~~	~~3%~~
~~(Bike, Not Late)~~	~~7%~~
~~(Drive, Late)~~	~~36%~~
~~(Drive, Not Late)~~	~~24%~~

Venn Diagram: Late given walk



$$\frac{P(Walk \cap Late)}{P(Walk)}$$

Conditional Probability

- Use the definition:

$$P(\text{Late} \mid \text{Walk}) = \frac{P(\text{Late} \cap \text{Walk})}{P(\text{Walk})} = \frac{6\%}{6\% + 24\%}$$

$$= \frac{6\%}{30\%}$$

$$= 1/5 = 20\%$$

(Walk, Late)	6%
(Walk, Not Late)	24%
(Bike, Late)	3%
(Bike, Not Late)	7%
(Drive, Late)	36%
(Drive, Not Late)	24%

Discussion Question

The probability of driving is 30%. The probability of being late, given that they drove, is 50%. What is the probability that a randomly-selected person drove **and** was late?

A) 20%
B) 30%
C) 6%
D) 15%

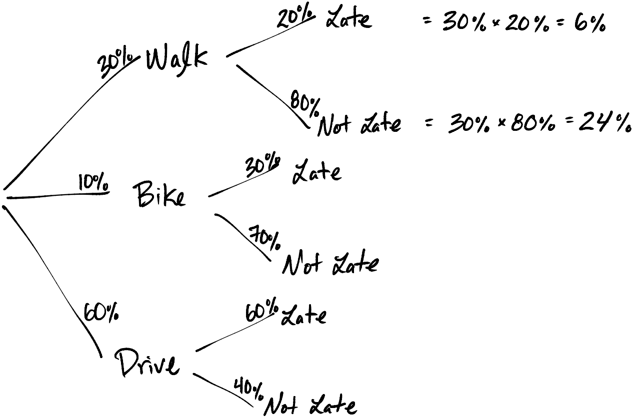
$$P(\text{Drive}) = .3$$

$$P(\text{Late} \mid \text{Drive}) = .5$$

$$P(\text{Late} \cap \text{Drive}) = P(\text{Late} \mid \text{Drive})\,P(\text{Drive})$$
$$= .5 \times .3$$
$$= .15$$

Tree Diagrams

- In what ways can a person arrive on campus?
  - P(Walk) = 30%; P(Late | Walk) = 20%.
  - P(Bike) = 10%; P(Late | Bike) = 30%.
  - P(Drive) = 60%; P(Late | Drive) = 60%.



Law of Total Probability

- What is $P(\text{Late})$?

(Walk, Late)	6%
(Walk, Not Late)	24%
(Bike, Late)	3%
(Bike, Not Late)	7%
(Drive, Late)	36%
(Drive, Not Late)	24%

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

Law of Total Probability

- What is $P(\text{Late})$?
  - $P(\text{Walk}) = 30\%$; $P(\text{Late} \mid \text{Walk}) = 20\%$.
  - $P(\text{Bike}) = 10\%$; $P(\text{Late} \mid \text{Bike}) = 30\%$.
  - $P(\text{Drive}) = 60\%$; $P(\text{Late} \mid \text{Drive}) = 60\%$.

- Remember: $P(A \cap B) = P(A \mid B) \cdot P(B)$

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

$$= P(\text{Late} \mid \text{Walk})P(\text{Walk}) + P(\text{Late} \mid \text{Bike})P(\text{Bike})$$

$$+ P(\text{Late} \mid \text{Drive})P(\text{Drive})$$

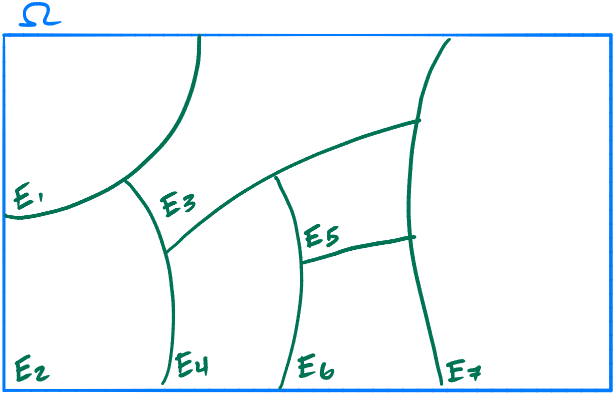
$$= (0.2)(0.3) + (0.3)(0.1) + (0.6)(0.6)$$

$$= 6\% + 3\% + 36\% = 45\%$$

Partitions

- Suppose events $E_1, \ldots, E_k$ are events such that, whatever the outcome, **exactly one** of the events is satisfied.
- That is:
  - No two events can happen simultaneously; they are mutually disjoint.
  - One of the events must happen. $P(E_1) + \ldots + P(E_k) = 1$.
- We say that $E_1, \ldots, E_k$ **partition** the outcome space.

Partitions



Example: Partitions

- Examples of events which partition the outcome space:
  - In getting to campus, the events Walk, Bike, Drive.
  - In getting to campus, the events Late, On-Time.
  - In rolling a die, the events Even, Odd.
  - In rolling a die, the events ≤ 3, > 3.
  - In drawing a card, the events Spades, Clubs, Diamonds, Hearts.

Law of Total Probability

- Let A be an event, let $E_1, \ldots, E_k$ be events partitioning Ω.

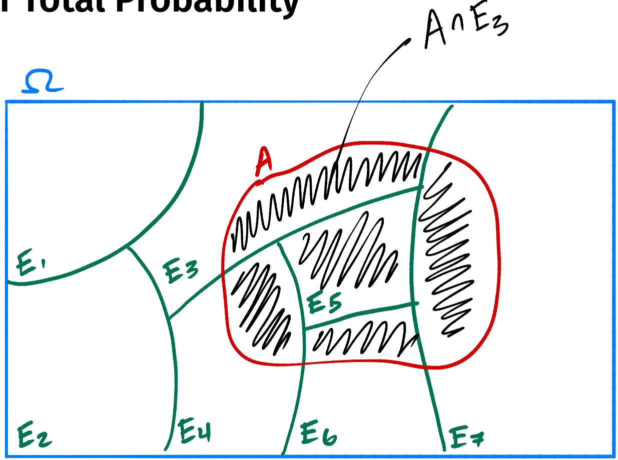
- Then:

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$$
$$= \sum_{i=1}^{k} P(A \cap E_i)$$

- And since $P(A \cap E) = P(A \mid E) \cdot P(E)$:

$$P(A) = P(A \mid E_1) \cdot P(E_1) + \ldots + P(A \mid E_k) \cdot P(E_k)$$
$$= \sum_{i=1}^{k} P(A \mid E_i) \cdot P(E_i)$$

Law of Total Probability



Bayes' Theorem

- Someone tells you that they were late. What is the probability that they drove to campus?
- We know: $P(\text{Late}) = 45\%$; $P(\text{Late} \mid \text{Drive}) = 60\%$.
- We want: $P(\text{Drive} \mid \text{Late})$.
- Using the definition:

$$= \frac{P(\text{Late} \cap \text{Drive})}{P(\text{Drive})} \Rightarrow P(\text{Late} \cap \text{Drive}) = P(\text{Late} \mid \text{Drive}) \times P(\text{Drive})$$

$$P(\text{Drive} \mid \text{Late}) = \frac{P(\text{Drive} \cap \text{Late})}{P(\text{Late})}$$

$$= \frac{P(\text{Late} \mid \text{Drive})P(\text{Drive})}{P(\text{Late})}$$

Bayes' Theorem

- Let A and B be events (with $P(A) > 0$ and $P(B) > 0$).
- Then:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Example

- A certain disease occurs in only 1% of the population.
- A test for the disease is 95% accurate.
- You've tested positive for the disease; what is the probability that you actually have it?

$P(\text{Sick}) = .01$

$P(+|\text{Sick}) = .95$

$P(-|\text{Sick}) = .05$

$P(+|\text{Healthy}) = .05$

$P(-|\text{Healthy}) = .95$

$$P(\text{Sick} \mid +) = \frac{P(+|\text{Sick})P(\text{Sick})}{P(+)}$$

$$= \frac{(.95)(.01)}{P(+)}$$

$$= \frac{(.95)(.01)}{(.05)(.99) + (.95)(.01)} = \frac{.0095}{.059} = .161$$

$$\approx 16\%$$

$$
\begin{aligned}
P(+) &= P(+ \cap \text{Healthy}) + P(+ \cap \text{Sick}) \\
&= P(+ \mid \text{Healthy})\,P(\text{Healthy}) + P(+ \mid \text{Sick})\,P(\text{Sick}) \\
&= (.05)(.99) + (.95)(.01)
\end{aligned}
$$

Bayes' Theorem: Alternate Form

- Let A be an event.
- Let $E_1, \ldots, E_k$ be events partitioning Ω.
- Then, using the law of total probability:

$$
\begin{aligned}
P(E_1 \mid A) &= \frac{P(A \mid E_1) \cdot P(E_1)}{P(A)} \\
&= \frac{P(A \mid E_1) \cdot P(E_1)}{P(A \cap E_1) + \ldots + P(A \cap E_k)} \\
&= \frac{P(A \mid E_1) \cdot P(E_1)}{P(A \mid E_1) \cdot P(E_1) + \ldots + P(A \mid E_k) \cdot P(E_k)}
\end{aligned}
$$

Example

In a collection of 65 coins, one has two heads (the rest are fair). You select a coin at random and flip it six times, seeing Heads each time. What is the probability that the coin you selected is Unfair?

$$P(\text{Unfair} \mid 6\text{ Heads}) = \frac{\overbrace{P(6\text{ Heads} \mid \text{Unfair})}^{=1}\ \overbrace{P(\text{Unfair})}^{=1/65}}{P(6\text{ Heads})} = \frac{1/65}{2/65} = 1/2$$

$$P(6\text{ Heads}) = P(6\text{ Heads} \mid \text{Unfair})\,P(\text{Unfair}) + P(6\text{ Heads} \mid \text{Fair})\,P(\text{Fair})$$

$$= (1)(1/65) + \left(\frac{1}{2}\right)^6 \frac{64}{65}$$

$$= \frac{1}{65} + \frac{1}{64} \cdot \frac{64}{65} = \frac{1}{65} + \frac{1}{65} = \frac{2}{65}$$

Example

A deck of five cards is numbered: 2, 4, 6, 8, 10. Three cards are drawn, one at a time with replacement; the sum of their values is 12. What is the probability that 2 was drawn twice?

$$P(2 \text{ twice} \mid 12) = \frac{P(12 \mid 2 \text{ twice})\, P(2 \text{ twice})}{P(12)} = 30\%$$