DSC 40A

Lecture 13

Combinatorics

Example

- What is the probability of seeing exactly 2 heads in 3 flips of a fair coin?

- **Sample space**: ordered triples.

  (T, T, T), (T, T, H), (T, H, T), (H, T, T),
  (H, H, T), (H, T, H), (T, H, H), (H, H, H).

- **Event**: {(H, H, T), (H, T, H), (T, H, H)}

- All outcomes equally-likely, so:

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3}{8}$$

Example

- What is the probability of seeing exactly 40 heads in 100 flips of a fair coin?

- All outcomes equally-likely, so:

$$P(E) = \frac{|E|}{|\Omega|} = \frac{?}{?}$$

Today

How do we count the number of outcomes, besides enumerating them all?

- How many outcomes are possible if a die is rolled 100 times?
- How many different ways are there to shuffle 52 cards?
- How many ways are there to choose a jury of 12 people from a panel of 100?

The area of math concerned with counting is called **combinatorics**.

Sampling

- Many experiments involve choosing things from a set P called the **population**.
- Examples: drawing cards from a deck, selecting people for a survey, rolling a die.
- Two decisions to make:
  - With or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences, Permutations, and Combinations

- # of **sequences**: with replacement, order matters
- # of **permutations**: without replacement, order matters
- # of **combinations**: without replacement, order doesn't matter

Sequences

- A *k*-**sequence** is a **tuple**[1] obtained by selecting things from P **with replacement**.

- Example: draw a card, put it back, repeat four more times.

$$(A\heartsuit, 2\clubsuit, 6\spadesuit, A\heartsuit, 3\diamondsuit)$$

- Example: flip a coin 100 times.

$$(H, T, T, H, \dots, H, T, T, T)$$

[1] tuples are ordered!

Example: Flip a coin three times

- Possible outcomes:

- Two choices for first item.

- For each choice of first item, two choices for second.

- For each choice of first two items, two choices for third.

- In total: $2 \cdot 2 \cdot 2 = 2^3$

Counting Sequences

- How many sequences of length k are there?
  - Remember: P is the **population**.
- $|P|$ choices for first item.
- For each choice of first item, $|P|$ choices for second.
- ...
- For each choice of first $k - 1$ items, $|P|$ choices for kth.
- $\underbrace{|P| \cdot |P| \cdots |P|}_{k \text{ times}} = |P|^k$

Counting Sequences (Another View)

- A sequence of length k is the Cartesian product of P with itself, k times.

- there are $|\underbrace{P \times P \times \cdots \times P}_{k \text{ times}}| = \underbrace{|P| \cdot |P| \cdots |P|}_{k \text{ times}} = |P|^k$.

Example

- Draw a card, put it back, repeat four more times.
- $|P| =$
- $k =$
- Number of possible outcomes:

Discussion Question

How many possible outcomes are there if a coin is flipped 100 times?

A) 2^{100}
B) 100^2
C) $2 \cdot 100$
D) 4^{100}

Example

- Flip a coin 100 times.
- $|P| =$
- $k =$
- Number of possible outcomes:

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$
10	$2^{10} = 1024$

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$
10	$2^{10} = 1024$
15	$2^{15} = 32,758$

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$
10	$2^{10} = 1024$
15	$2^{15} = 32{,}758$
20	$2^{15} \approx 1$ million

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$
10	$2^{10} = 1024$
15	$2^{15} = 32{,}758$
20	$2^{15} \approx 1$ million
50	$2^{50} \approx$ # of grains of sand on Earth

Permutations

- A ***k*-permutation** is a **tuple** obtained by selecting k things from P **without replacement**.
- Example: draw a card, **don't** put it back, repeat four more times.

$$(A\heartsuit, 2\clubsuit, 6\spadesuit, 7\heartsuit, 3\diamondsuit)$$

- Example: rank all 6 colleges by preference.

(Warren, Sixth, Muir, Roosevelt, Marshall, Revelle)

- Example: rank top four movies from a list of 250.

(Fistful of Dollars, Parasite, Psycho, Hot Rod)

Example: Ranking top two cities

- How many ways are there to rank top two out of {LA, SD, SF, SJ}?

- Four choices for first city.

- For each choice of first city, three choices for second.

- $4 \cdot 3$ possible rankings.

Counting Permutations

- $|P|$ choices for first item.
- For each choice of first item, $|P| - 1$ choices for second.
- For each choice of first two items, $|P| - 2$ choices for second.
- ...
- For each choice of first $k - 1$ items, $|P| - (k - 1)$ choices for kth.
- $|P| \cdot (|P| - 1) \cdot (|P| - 2) \cdots (|P| - k + 1)$

Another Formula for Counting Permutations

- The number of k-permutations is $|P| \cdot (|P| - 1) \cdots (|P| - k + 1)$.

- Equivalently:
$$\frac{|P|!}{(|P| - k)!}$$

Special Case

- Suppose $k = |P|$.
- How many permutations are there of $|P|$ items?
- $|P|! = |P| \cdot (|P| - 1) \cdot (|P| - 2) \cdots 3 \cdot 2 \cdot 1$

Example

- Rank all 6 colleges by preference.
- $|P| =$
- $k =$
- Number of possible rankings:

Example

- Rank top four movies out of a list of 250.
- $|P| =$
- $k =$
- Number of possible rankings:

Combinations

- A ***k*-combination** is a **set** obtained by selecting k things from P without replacement.

- Example: draw a hand of five cards from a deck of 52.

$$\{A\heartsuit, 2\clubsuit, 6\spadesuit, 7\heartsuit, 3\diamondsuit\}$$

- Example: make a group of 5 people from a class of 100.

$$\{\text{Clint, Zelda, Alfred, Andy, Yvonne}\}$$

Counting Combinations

- How many ways are there to choose 2 unique things from $P = \{a, b, c\}$?
- Step 1) Enumerate all 3!/2! 2-**permutations**:

$$(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)$$

- Step 2) Group the k-permutations which have same elements.

$$\color{red}{(a, b)}, \color{green}{(a, c)}, \color{red}{(b, a)}, \color{blue}{(b, c)}, \color{green}{(c, a)}, \color{blue}{(c, b)}$$

- Step 3) Count number of groups: (3!/2!) / 2! = 3!/(2!·2!)

Example: Choose two cities

- How many ways are there of choosing two cities from {LA, SD, SF, SJ}?
- ?

Counting Combinations

- How many ways are there to choose k unique things from P?

- Step 1) Enumerate all |P|!/(|P| - k)! k-**permutations**:

- Step 2) Group the k-permutations which have same elements.

- Step 3) Count number of groups:

$$\#\text{ of } k\text{-combinations} = \frac{|P|!}{k!(|P| - k)!}$$

Counting Combinations

- The number of ways of choosing k items from n possibilities is often called **n choose k**, written:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Also called the **binomial coefficients**.

Example

- How many different hands of five cards are there?
- $|P| =$
- $k =$
- # of hands:

Example

- How many different ways are there to select a group of 5 people from 100?

- $|P| =$

- $k =$

- # of ways

Counting and Probability

- When outcomes are equiprobable, $P(E) = |E|/|\Omega|$
- To find $|E|$, we often need to count sequences, permutations, or combinations.
- Must decide if order matters.
- **Pro tip**: think about the sample space first!

Example: Groups from Warren

- A class of 100 contains 30 people from Warren college.
- What is the probability that a group of 5 randomly-selected people are all from Warren?
- Does order matter?
- What is the sample space? That is, what is an outcome?

Example: Groups from Warren

- An outcome is a **set** of five people.

- Sample space, Ω: all possible sets of five people.

$$|\Omega| =$$

- Event, E: all possible sets of five people from Warren.

$$|E| =$$

- $P(E) = |E|/|\Omega| =$

Example: 40 Heads

- What is the probability of seeing exactly 40 heads in 100 flips of a fair coin?

- Does order matter? No. But also yes.

- **Decide on your sample space!**

Example: 40 Heads

- An outcome is a sequence of 100 flips.
- Sample space, Ω: all possible sequences of 100 flips.

$$|\Omega| =$$

- Event, E: all sequences with exactly 40 heads.

$$|E| =$$

Example: 40 Heads

$$|E|$$
$$=$$
# of sequences of 100 flips with exactly 40 heads
$$=$$
# of ways of choosing here the 40 heads appear in 100 flips
$$=$$

$$P(E) = |E|/|\Omega| =$$