

DSC 40B

Theoretical Foundations II

Lecture 6 | Part 1

Binary Search Recurrence

Binary Search

```
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

Binary Search

- ▶ What is the time complexity of `binary_search`?
- ▶ Best case: $\Theta(1)$.
- ▶ Worst case:

$$T(n) = \begin{cases} T(n/2) + \Theta(1), & n \geq 2 \\ \Theta(1), & n = 1 \end{cases}$$

Simplification

- ▶ When solving, we can replace $\Theta(f(n))$ with $f(n)$:

$$T(n) = \begin{cases} T(n/2) + 1, & n \geq 2 \\ 1, & n = 1 \end{cases}$$

- ▶ As long as we state final answer using Θ notation!

Another Simplification

- ▶ When solving, we can assume n is a power of 2.

Step 1: Unroll several times

$$T(n) = \begin{cases} T(n/2) + 1, & n \geq 2 \\ 1, & n = 1 \end{cases}$$

Step 2: Find general formula

$$\begin{aligned}T(n) &= T(n/2) + 1 \\&= T(n/4) + 2 \\&= T(n/8) + 3\end{aligned}$$

On step k :

Step 3: Find step # of base case

- ▶ On step k , $T(n) = T(n/2^k) + k$
- ▶ When do we see $T(1)$?

Step 4: Plug into general formula

- ▶ $T(n) = T(n/2^k) + k$
- ▶ Base case of $T(1)$ reached when $k = \log_2 n$.
- ▶ So:

Note

- ▶ Remember: $\log_b x = (\log_a x)/(\log_a b)$
- ▶ So we don't write $\Theta(\log_2 n)$
- ▶ Instead, just: $\Theta(\log n)$

Time Complexity of Binary Search

- ▶ Best case: $\Theta(1)$
- ▶ Worst case: $\Theta(\log n)$

Is binary search fast?

- ▶ Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- ▶ Goal: find a particular grain.
- ▶ Assume one basic operation takes 1 nanosecond.

Is binary search fast?

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- ▶ Goal: find a particular grain.
- ▶ Assume one basic operation takes 1 nanosecond.
- ▶ Linear search: 317 years.

Is binary search fast?

- ▶ Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- ▶ Goal: find a particular grain.
- ▶ Assume one basic operation takes 1 nanosecond.
- ▶ Linear search: 317 years.
- ▶ Binary search: ≈ 60 nanoseconds.

Exercise

Binary search seems so much faster than linear search. What's the caveat?

Caveat

- ▶ The array must be **sorted**.
- ▶ This takes $\Omega(n)$ time.

Why use binary search?

- ▶ If data is **not sorted**, sorting + binary search takes longer than linear search.
- ▶ But if doing **multiple queries**, looking for nearby elements, sort once and use binary search after.

Theoretical Lower Bounds

- ▶ A lower bound for searching a sorted list is $\Omega(\log n)$.
- ▶ This means that binary search has **optimal** worst case time complexity.

Databases

- ▶ Some database servers will **sort** by key, use binary search for queries.
- ▶ Often instead of sorting, **B-Tree indexes** are used.
- ▶ But sorting + binary search still used when space is limited.

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Theoretical Foundations II

Lecture 6 | Part 2

Selection Sort and Loop Invariants

Sorting

- ▶ Sorting is a very common operation.
- ▶ But why is it important?

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Sorting

- ▶ Sorting is a very common operation.
- ▶ But why is it important?
- ▶ A e s t h e t i c reasons?
- ▶ Sorting makes some problems easier to solve.

Today

- ▶ How do we sort?
- ▶ How fast can we sort?
- ▶ How do we use sorted structure to write faster algorithms?

Today

- ▶ **Also:** how to understand complex loops with **loop invariants**.

Selection Sort

- ▶ Repeatedly remove smallest element.
- ▶ Put it at beginning of new list.

Example: `arr = [5, 6, 3, 2, 1]`

In-place Selection Sort

- ▶ We don't need a separate list.
 - ▶ We can swap elements until sorted.
- ▶ Store “new” list at the beginning of input list.
- ▶ Separate the old and new with a **barrier**.

Example: `arr = [5, 6, 3, 2, 1]`

```
def selection_sort(arr):  
    """In-place selection sort."""  
    n = len(arr)  
    if n <= 1:  
        return  
    for barrier_ix in range(n-1):  
        # find index of min in arr[start:]  
        min_ix = find_minimum(arr, start=barrier_ix)  
        #swap  
        arr[barrier_ix], arr[min_ix] = (  
            arr[min_ix], arr[barrier_ix]  
        )
```

```
def find_minimum(arr, start):  
    """Finds index of minimum. Assumes non-empty."""  
    n = len(arr)  
    min_value = arr[start]  
    min_ix = start  
    for i in range(start + 1, n):  
        if arr[i] < min_value:  
            min_value = arr[i]  
            min_ix = i  
    return min_ix
```

Loop Invariants

- ▶ How we understand an iterative algorithm?
- ▶ A **loop invariant** is a statement that is true after every iteration.
 - ▶ And before the loop begins!

Loop Invariant(s)

After the α th iteration of selection sort, each of the first α elements is \leq each of the remaining elements.

Example: `arr = [5, 6, 3, 2, 1]`

Loop Invariant(s)

After the α th iteration, the first α elements are sorted.

Example: `arr = [5, 6, 3, 2, 1]`

Loop Invariants

- ▶ Plug the total number of iterations into the loop invariant to learn about the result.
 - ▶ `selection_sort` makes $n - 1$ iterations:
 - ▶ After the $(n - 1)$ th iteration, the first $(n - 1)$ elements are sorted.
 - ▶ After the $(n - 1)$ th iteration, each of the first $(n - 1)$ elements is \leq each of the remaining elements.

Time Complexity

```
def selection_sort(arr):  
    """In-place selection sort."""  
    n = len(arr)  
    if n <= 1:  
        return  
    for barrier_ix in range(n-1):  
        # find index of min in arr[barrier_ix:]  
        min_value = arr[barrier_ix]  
        min_ix = barrier_ix  
        for i in range(barrier_ix + 1, n):  
            if arr[i] < min_value:  
                min_value = arr[i]  
                min_ix = i  
  
        #swap  
        arr[barrier_ix], arr[min_ix] = (  
            arr[min_ix], arr[barrier_ix]  
        )
```

Time Complexity

- ▶ Selection sort takes $\Theta(n^2)$ time.

Exercise

Modify `selection_sort` so that it computes a **median** of the input array. What is the time complexity?

```
def selection_sort(arr):  
    """In-place selection sort."""  
    n = len(arr)  
    if n <= 1:  
        return  
    for barrier_ix in range(n-1):  
        # find index of min in arr[start:]  
        min_ix = find_minimum(arr, start=barrier_ix)  
        #swap  
        arr[barrier_ix], arr[min_ix] = (  
            arr[min_ix], arr[barrier_ix]  
        )
```

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Theoretical Foundations II

Lecture 6 | Part 3

Mergesort

Can we sort faster?

- ▶ The tight theoretical lower bound for **comparison** sorting is $\Theta(n \log n)$.
- ▶ Selection sort is quadratic.
- ▶ How do we sort in $\Theta(n \log n)$ time?

Mergesort

- ▶ Mergesort is a fast sorting algorithm.
- ▶ Has **best possible** (worst-case) time complexity: $\Theta(n \log n)$.
- ▶ Implements **divide/conquer/recombine** strategy.

The Idea

- ▶ **Divide:** split the array into halves
 - ▶ $[6, 1, 9, 2, 4, 3] \rightarrow [6, 1, 9], [2, 4, 3]$
- ▶ **Conquer:** sort each half, recursively
 - ▶ $[6, 1, 9] \rightarrow [1, 6, 9]$ and $[2, 4, 3] \rightarrow [2, 3, 4]$
- ▶ **Combine:** merge sorted halves together
 - ▶ $[1, 6, 9], [2, 3, 4] \rightarrow [1, 2, 3, 4, 6, 9]$

Aside: splitting arrays

- ▶ Splitting an array in half by **slicing**:

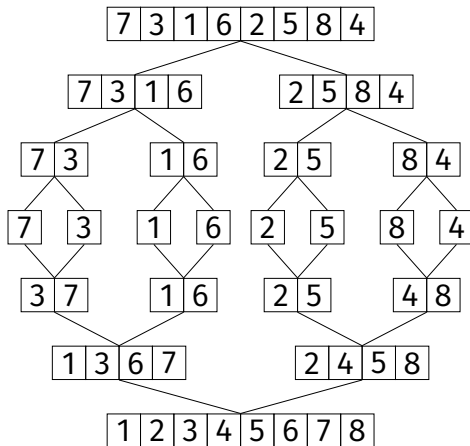
```
»> arr = [9, 1, 4, 2, 5]
»> middle = math.floor(len(arr) / 2)
»> arr[:middle]
[9, 1]
»> arr[middle:]
[4, 2, 5]
```

- ▶ **Warning!** Creates a copy!

Mergesort

```
def mergesort(arr):  
    """Sort array in-place."""  
    if len(arr) > 1:  
        middle = math.floor(len(arr) / 2)  
        left = arr[:middle]  
        right = arr[middle:]  
        mergesort(left)  
        mergesort(right)  
        merge(left, right, arr)
```

The Idea



Understanding Mergesort

1. What is the base case?
2. Are the recursive problems smaller?
3. Assuming the recursive calls work, does the whole algorithm work?

1. Base Case: $n = 1$

- ▶ Arrays of size one are trivially sorted.
- ▶ Returns immediately. **Correct!**

2. Smaller Problems?

- ▶ Are `arr[:middle]` and `arr[middle:]` always smaller than `arr`?
- ▶ Try it for `len(arr) == 2`.

3. Does it Work?

- ▶ Assume mergesort works on arrays of size $< n$.
- ▶ Does it work on arrays of size n ?

Mergesort

```
def mergesort(arr):  
    """Sort array in-place."""  
    if len(arr) > 1:  
        middle = math.floor(len(arr) / 2)  
        left = arr[:middle]  
        right = arr[middle:]  
        mergesort(left)  
        mergesort(right)  
        merge(left, right, arr)
```

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Lecture 6 | Part 4

Merge

Merging

- ▶ We have sorted each half.
- ▶ Now we need to **merge** together.

Merging

- ▶ We have sorted each half.
- ▶ Now we need to **merge** together.
- ▶ **Note:** this is an example of a problem that is made easier by sorting.

merge

3

1

merge

3

2

1

merge

3

6

1

2

merge

5

6

1

2

3

merge

7

6

1

2

3

5

merge

7

1 2 3 5 6

merge

8

1

2

3

5

6

7

merge

1 2 3 5 6 8

merge

```
def merge(left, right, out):  
    """Merge sorted arrays, store in out."""  
    left.append(float('inf'))  
    right.append(float('inf'))  
    left_ix = 0  
    right_ix = 0  
  
    for ix in range(len(out)):  
        if left[left_ix] < right[right_ix]:  
            out[ix] = left[left_ix]  
            left_ix += 1  
        else:  
            out[ix] = right[right_ix]  
            right_ix += 1
```

Loop Invariant

- ▶ Assume `left` and `right` are sorted.
- ▶ **Loop invariant:** After α th iteration,
`out[: α] == sorted(left + right)[: α]`

Key of mergesort

- ▶ merge is where the **actual sorting** happens.
- ▶ Example: `merge([3], [1], ...)` results in `[1,3]`

Time Complexity of merge

```
def merge(left, right, out):  
    """Merge sorted arrays, store in out."""  
    left.append(float('inf'))  
    right.append(float('inf'))  
    left_ix = 0  
    right_ix = 0  
  
    for ix in range(len(out)):  
        if left[left_ix] < right[right_ix]:  
            out[ix] = left[left_ix]  
            left_ix += 1  
        else:  
            out[ix] = right[right_ix]  
            right_ix += 1
```

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Lecture 6 | Part 5

Time Complexity of Mergesort

Time Complexity

```
def mergesort(arr):  
    """Sort array in-place."""  
    if len(arr) > 1:  
        middle = math.floor(len(arr) / 2)  
        left = arr[:middle]  
        right = arr[middle:]  
        mergesort(left)  
        mergesort(right)  
        merge(left, right, arr)
```

Aside: Copying

- ▶ What is `arr[:middle]` doing “under the hood”?
- ▶ What is the time complexity?

The Recurrence

```
def mergesort(arr):  
    """Sort array in-place."""  
    if len(arr) > 1:  
        middle = math.floor(len(arr) / 2)  
        left = arr[:middle]  
        right = arr[middle:]  
        mergesort(left)  
        mergesort(right)  
        merge(left, right, arr)
```

Solving the Recurrence

$$T(n) = 2T(n/2) + \Theta(n)$$

Optimality

- ▶ **Theorem:** Any (comparison) sorting algorithm's worst-case time complexity must be $\Omega(n \log n)$.
- ▶ Mergesort is **optimal**!

Be Careful!

- ▶ It is possible for a sorting algorithm to have a **best case** time complexity smaller than $n \log n$.
 - ▶ Insertion sort, for example.
- ▶ Mergesort has best case time complexity of $\Theta(n \log n)$.
- ▶ Mergesort is **sub-optimal** in this sense!

Be Careful!

- ▶ The $\Theta(n \log n)$ lower-bound is for **comparison sorting**.
- ▶ It is possible to sort in worst-case $\Theta(n)$ time without comparing.¹

¹Bucket sort, radix sort, etc.

What if?

- ▶ **Divide:** split the array into halves
- ▶ **Conquer:** sort each half **using selection sort**
- ▶ **Combine:** merge sorted halves together

mergeselectionsort

```
def mergeselectionsort(arr):  
    """Sort array in-place."""  
    if len(arr) > 1:  
        middle = math.floor(len(arr) / 2)  
        left = arr[:middle]  
        right = arr[middle:]  
        selection_sort(left)  
        selection_sort(right)  
        merge(left, right, arr)
```

Exercise

What is the time complexity of this algorithm?

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Theoretical Foundations II

Lecture 6 | Part 6

Using Sorted Structure

Sorted structure is useful

- ▶ Some problems become **much easier** if input is sorted.
 - ▶ For example, median, minimum, maximum.
- ▶ Sorting is useful as a **preprocessing** step.

Recall: The Movie Problem

- ▶ You're on a flight that will last D minutes.
- ▶ You want to pick two movies to watch.
- ▶ You want the total time of the two movies to be **as close as possible** to D .

The Movie Problem

- ▶ Brute force algorithm: $\Theta(n^2)$
- ▶ We can do better, if movie times are **sorted**.

Example

- ▶ Flight duration $D = 155$
- ▶ Movie times: 60, 80, 90, 120, 130

	60	80	90	120	130
60					
80					
90					
120					
130					

Best pair:

The Algorithm

- ▶ Keep index of shortest and longest remaining.
- ▶ On every iteration, pair the shortest and longest.
- ▶ If this pair is too long, remove longest movie; otherwise remove shortest.
 - ▶ If times are **sorted**, finding new longest/shortest movie takes $\Theta(1)$ time!

60, 80, 90, 120, 130

The Algorithm

```
def optimize_entertainment(times, target):  
    """assume times is sorted."""  
    shortest = 0  
    longest = len(times) - 1  
  
    best_pair = (shortest, longest)  
    best_objective = None  
  
    for i in range(len(times) - 1):  
        total_time = times[shortest] + times[longest]  
  
        if abs(total_time - target) < best_objective:  
            best_objective = abs(total_time - target)  
            best_pair = (shortest, longest)  
  
        if total_time == target:  
            return (shortest, longest)  
        elif total_time < target:  
            shortest += 1  
        else: # total_time > target  
            longest -= 1  
  
    return best_pair
```

Main Idea

Sorted structure allows you to rule out possibilities without explicitly checking them. But, it requires you to spend the time sorting first.

Tip: when designing an algorithm, think about sorting the input first.