DSC 40A

Lecture 13

Combinatorics

Example

- What is the probability of seeing exactly 2 heads in 3 flips of a fair coin?

- **Sample space**: ordered triples.

  (T, T, T), (T, T, H), (T, H, T), (H, T, T),
  (H, H, T), (H, T, H), (T, H, H), (H, H, H).

- **Event**: {(H, H, T), (H, T, H), (T, H, H)}

- All outcomes equally-likely, so:

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3}{8}$$

Example

- What is the probability of seeing exactly 40 heads in 100 flips of a fair coin?
- All outcomes equally-likely, so:

$$P(E) = \frac{|E|}{|\Omega|} = \frac{?}{?}$$

Today

How do we count the number of outcomes, besides enumerating them all?

- How many outcomes are possible if a die is rolled 100 times?
- How many different ways are there to shuffle 52 cards?
- How many ways are there to choose a jury of 12 people from a panel of 100?

The area of math concerned with counting is called **combinatorics**.

Sampling

- Many experiments involve choosing things from a set P called the **population**.
- Examples: drawing cards from a deck, selecting people for a survey, rolling a die.
- Two decisions to make:
  - With or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences, Permutations, and Combinations

- # of **sequences**: with replacement, order matters
- # of **permutations**: without replacement, order matters
- # of **combinations**: without replacement, order doesn't matter

Sequences

- A ***k*-sequence** is a **tuple**[1] obtained by selecting things from P **with replacement**.

- Example: draw a card, put it back, repeat four more times.

$$(A\heartsuit, 2\clubsuit, 6\spadesuit, A\heartsuit, 3\diamondsuit)$$

- Example: flip a coin 100 times.

$$(H, T, T, H, \ldots, H, T, T, T)$$

[1] tuples are ordered!

Example: Flip a coin three times

- Possible outcomes: $\{$
  $(H,H,H), (H,H,T), (H,T,H), (H,T,T),$
  $(T,H,H), (T,H,T), (T,T,H), (T,T,T)$
  $\}$
- Two choices for first item.
- For each choice of first item, two choices for second.
- For each choice of first two items, two choices for third.
- In total: $2 \cdot 2 \cdot 2 = 2^3$

Counting Sequences

- How many sequences of length k are there?
  - Remember: P is the **population**.
- $|P|$ choices for first item.
- For each choice of first item, $|P|$ choices for second.
- ...
- For each choice of first $k-1$ items, $|P|$ choices for kth.
- $\underbrace{|P| \cdot |P| \cdots |P|}_{k \text{ times}} = |P|^k$

Counting Sequences (Another View)

- A sequence of length k is the Cartesian product of P with itself, k times.
- there are $|\underbrace{P \times P \times \cdots \times P}_{k \text{ times}}| = \underbrace{|P| \cdot |P| \cdots |P|}_{k \text{ times}} = |P|^k$.

Example

- Draw a card, put it back, repeat four more times.
- $|P| = 52$
- $k = 4$
- Number of possible outcomes: $52^4 = 52 \cdot 52 \cdot 52 \cdot 52$

Discussion Question

How many possible outcomes are there if a coin is flipped 100 times?

A) 2^{100}
B) 100^2
C) $2 \cdot 100$
D) 4^{100}

Example

- Flip a coin 100 times.
- $|P| =$ 2
- $k =$ 100
- Number of possible outcomes: 2^{100}

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$

Exponential growth

- There are a lot of possible sequences.

n	# of Sequences of Length n
5	$2^5 = 32$
10	$2^{10} = 1024$

Exponential growth

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n	# of Sequences of Length n
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15	$2^{15} = 32{,}758$

Exponential growth

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n	# of Sequences of Length n
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Exponential growth

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5	$2^5 = 32$
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20	$2^{15} \approx 1$ million
50	$2^{50} \approx$ # of grains of sand on Earth

Permutations

- A ***k*-permutation** is a **tuple** obtained by selecting k things from P **without replacement**.

- Example: draw a card, **don't** put it back, repeat four more times.

$$(A\heartsuit, 2\clubsuit, 6\spadesuit, 7\heartsuit, 3\diamondsuit)$$

- Example: rank all 6 colleges by preference.

(Warren, Sixth, Muir, Roosevelt, Marshall, Revelle)

- Example: rank top four movies from a list of 250.

(Fistful of Dollars, Parasite, Psycho, Hot Rod)

Example: Ranking top two cities

- How many ways are there to rank top two out of {LA, SD, SF, SJ}? {
  (SD, SF), (SD, SJ), (SD, ZA),
  (SF, SD), (SF, SJ), (SF, ZA),
  (SJ, SD), (SJ, SF), (SJ, ZA),
  (ZA, SD), (ZA, SF), (ZA, SJ) }
- Four choices for first city.
- For each choice of first city, three choices for second.
- $4 \cdot 3$ possible rankings.

Counting Permutations

- $|P|$ choices for first item.
- For each choice of first item, $|P| - 1$ choices for second.
- For each choice of first two items, $|P| - 2$ choices for second.
- ...
- For each choice of first $k - 1$ items, $|P| - (k - 1)$ choices for kth.
- $|P| \cdot (|P| - 1) \cdot (|P| - 2) \cdots (|P| - k + 1)$

Another Formula for Counting Permutations

- The number of k-permutations is $|P| \cdot (|P|-1) \cdots (|P|-k+1)$.
- Equivalently:

$$\frac{|P|!}{(|P|-k)!}$$

$$\frac{|P|!}{(|P|-k)!} = \frac{|P| \cdot (|P|-1) \cdots (|P|-k+1)\cancel{(|P|-k)!}}{\cancel{(|P|-k)!}}$$

Special Case

- Suppose $k = |P|$.
- How many permutations are there of $|P|$ items?
- $|P|! = |P| \cdot (|P| - 1) \cdot (|P| - 2) \cdots 3 \cdot 2 \cdot 1$

Example

- Rank all 6 colleges by preference.
- $|P| =$ 6
- $k =$ 6
- Number of possible rankings: $\frac{6!}{(6-6)!} = \frac{6!}{0!}$

$0! = 1 \Rightarrow \frac{6!}{0!} = 6!$

Example

- Rank top four movies out of a list of 250.
- $|P| = 250$
- $k = 4$
- Number of possible rankings: $\frac{250!}{(250-4)!} = \frac{250!}{246!}$

Combinations

- A ***k*-combination** is a **set** obtained by selecting k things from P without replacement.

- Example: draw a hand of five cards from a deck of 52.

$$\{A\heartsuit, 2\clubsuit, 6\spadesuit, 7\heartsuit, 3\diamondsuit\}$$

- Example: make a group of 5 people from a class of 100.

$$\{\text{Clint, Zelda, Alfred, Andy, Yvonne}\}$$

Counting Combinations

- How many ways are there to choose 2 unique things from $P = \{a, b, c\}$?
- Step 1) Enumerate all 3!/2! 2-**permutations**:

$$(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)$$

- Step 2) Group the k-permutations which have same elements.

$$\color{red}{(a, b)}, \color{green}{(a, c)}, \color{red}{(b, a)}, \color{blue}{(b, c)}, \color{green}{(c, a)}, \color{blue}{(c, b)}$$

- Step 3) Count number of groups: (3!/2!) / 2! = 3!/(2!·2!)

Example: Choose two cities

- How many ways are there of choosing two cities from {LA, SD, SF, SJ}? {

  {SD, SF}, {SD, SJ}, {SD, LA},

  {LA, SJ}, {LA, SF}, {SJ, SF}

  }

- ?

Counting Combinations

- How many ways are there to choose k unique things from P?
- Step 1) Enumerate all |P|!/(|P| - k)! k-**permutations**:
- Step 2) Group the k-permutations which have same elements.
- Step 3) Count number of groups:

$$\#\text{ of } k\text{-combinations} = \frac{|P|!}{k!(|P| - k)!}$$

Counting Combinations

- The number of ways of choosing k items from n possibilities is often called **n choose k**, written:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Also called the **binomial coefficients**.

Example

- How many different hands of five cards are there?
- $|P| = 52$
- $k = 5$
- # of hands: $\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!\,47!}$

Example

- How many different ways are there to select a group of 5 people from 100?
- $|P| = 100$
- $k = 5$
- # of ways $\binom{100}{5} = \frac{100!}{5!\,95!}$

Counting and Probability

- When outcomes are equiprobable, $P(E) = |E|/|\Omega|$
- To find $|E|$, we often need to count sequences, permutations, or combinations.
- Must decide if order matters.
- **Pro tip**: think about the sample space first!

Example: Groups from Warren

- A class of 100 contains 30 people from Warren college.
- What is the probability that a group of 5 randomly-selected people are all from Warren?
- Does order matter?
- What is the sample space? That is, what is an outcome?

Example: Groups from Warren

- An outcome is a **set** of five people.

- Sample space, Ω: all possible sets of five people.

$$|\Omega| = \binom{100}{5}$$

- Event, E: all possible sets of five people from Warren.

$$|E| = \binom{30}{5}$$

- $P(E) = |E|/|\Omega| = \dfrac{\binom{30}{5}}{\binom{100}{5}}$

Example: 40 Heads

- What is the probability of seeing exactly 40 heads in 100 flips of a fair coin?
- Does order matter? No. But also yes.
- **Decide on your sample space!**

Example: 40 Heads

- An outcome is a sequence of 100 flips.

- Sample space, Ω: all possible sequences of 100 flips.

$$|\Omega| = 2^{100}$$

- Event, E: all sequences with exactly 40 heads.

$$|E| = ?$$

Example: 40 Heads

$$|E|$$
$$=$$
$$\text{\# of sequences of 100 flips with exactly 40 heads}$$
$$=$$
$$\text{\# of ways of choosing here the 40 heads appear in 100 flips}$$
$$=$$
$$\binom{100}{40}$$

$$P(E) = |E|/|\Omega| = \frac{\binom{100}{40}}{2^{100}}$$