DSC 190

DATA STRUCTURES & ALGORITHMS

Lecture 2 | Part 1

More about Memory

Is appending an array really so slow?

- `malloc()` vs. `realloc()`

Is appending an array really so slow?

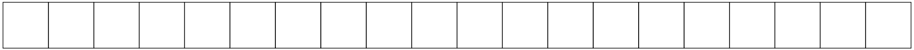
- If `realloc` doesn't copy: $\Theta(1)$.
- If `realloc` copies: $\Theta(n)$.
- Assume p is probability that `realloc` copies.
- **Expected time** is still[1] $\Theta(n)$.

---

[1] If p doesn't depend on n.

How is empty memory found?

- Basically: a linked list.



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DATA STRUCTURES & ALGORITHMS

Lecture 2 | Part 2

Dynamic Arrays

Motivation

- Can we have the best of both worlds?
- $\Theta(1)$ time access like an array.
- $\Theta(1)$ time append like a linked list.
- **Yes!** (sort of)

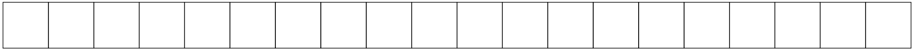
The Idea

- Allocate memory for an **underlying array**.
  - say, 512 elements
  - This is the **physical size**.
- To append element, insert into first unused slot.
  - Number of elements used is the **logical size**.
  - $\Theta(1)$ time.

The Idea

- We'll eventually run out of unused slots.
- Fix: allocate a new underlying array whose physical size is γ times as large.
  - γ is the **growth factor**.
  - Commonly, $\gamma = 2$; i.e., double its size.
  - Takes $\Theta(k)$ time, where k is current size.

Example



```
>>> arr = DynamicArray(initial_physical_size=4)
>>> arr.append(1)
>>> arr.append(2)
>>> arr.append(3)
>>> arr.append(4)
>>> arr.append(5)
```

(notebook)

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Data Structures & Algorithms

Lecture 2 | Part 3

Amortized Analysis

Analysis

- Appending takes $\Theta(1)$ time *usually*...
- ...but takes $\Theta(k)$ time when we run out of slots.
  - Where k is current size of sequence.

The Key

- Resizing is expensive, but rare.
  - If $\gamma = 2$, each new resize is twice as expensive, but happens half as often.
- Thus, the **cost per append** is small.
- **Amortize** the cost over all previous appends.

Amortized Time Complexity

- The **amortized** time for an append is:

$$T_{\text{amort}}(n) = \frac{\text{total time for } n \text{ appends}}{n}$$

- We'll see that $T_{\text{amort}}(n) = \Theta(1)$.

Amortized Analysis

total time for n appends

=

total time for **non-growing** appends

\+

total time for **growing** appends

Counting Growing Appends

- Want to calculate time taken by growing appends.
- First: how many appends caused a resize?
  - β: initial physical size
  - γ: growth factor

Counting Growing Appends

- Suppose initial physical size is $\beta = 512$, and $\gamma = 2$
- Resizes occur on append #:

$$512, 1024, 2048, 4096, \ldots$$

- In general, resizes occur on append #:

$$\beta\gamma^0, \beta\gamma^1, \beta\gamma^2, \beta\gamma^3, \ldots$$

Counting Growing Appends

- In a sequence of n appends, how many caused the physical size to grow?

- Simplification: Assume n is such that nth append caused a resize. Then, for some $x \in \{0, 1, 2, \ldots\}$:

$$n = \beta\gamma^x$$

- If $x = 0$ there was 1 resize; if $x = 1$ there were 2; etc.

Counting Growing Appends

- Solving for x:

$$x = \log_\gamma \frac{n}{\beta}$$

- Check: without assumption, $x = \lfloor \log_\gamma \frac{n}{\beta} \rfloor$

- Number of resizes is $\lfloor \log_\gamma \frac{n}{\beta} \rfloor + 1$

Counting Growing Appends

- Number of resizes is $\lfloor \log_\gamma \frac{n}{\beta} \rfloor + 1$
- Check with $\gamma = 2$, $\beta = 512$, $n = 400$
  - Correct # of resizes: 0
- Check with $\gamma = 2$, $\beta = 512$, $n = 1100$
  - Correct # of resizes: 2

Time of Growing Appends

- How much time was taken across all appends that caused resizes?
- Assumption: resizing an array with physical size k takes time $ck = \Theta(k)$.
  - c is a constant that depends on γ.

Time of Growing Appends

- Time for first resize: $c\beta$.
- Time for second resize: $c\gamma\beta$.
- Time for third resize: $c\gamma^2\beta$.
- Time for jth resize: $c\gamma^{j-1}\beta$.
- This is a **geometric progression**.

Time of Growing Appends

- Time for jth resize: $c\gamma^{j-1}\beta$.
- Suppose there are r resizes.
- Total time:

$$c\beta \sum_{j=1}^{r} \gamma^{j-1} = c\beta \sum_{j=0}^{r} \gamma^{j}$$

Recall: Geometric Sum

- From some class you've taken:

$$\sum_{p=0}^{N} x^p = \frac{1 - x^{N+1}}{1 - x}$$

- Example:

$$1 + 2 + 4 + 8 + 16 = \sum_{p=0}^{4} 2^p = \frac{1 - 2^5}{1 - 2} = 31$$

Time of Growing Appends

- Total time:

$$c\beta \sum_{j=0}^{r} \gamma^j = c\beta \frac{1 - \gamma^{r+1}}{1 - \gamma}$$

Time of Growing Appends

- Remember: in n appends there are $r = \lfloor \log_\gamma \frac{n}{\beta} \rfloor + 1$ resizes.

- Total time:

$$
\begin{aligned}
c\beta \frac{1 - \gamma^{r+1}}{1 - \gamma} &= c\beta \frac{1 - \gamma^{\lfloor \log_\gamma \frac{n}{\beta} \rfloor + 2}}{1 - \gamma} \\
&= \Theta(n)
\end{aligned}
$$

Amortized Analysis

total time for n appends

=

total time for **non-growing** appends

+

$\Theta(n)$ $\leftarrow$ total time for **growing** appends

Time of Non-Growing Appends

- In a sequence of n appends, how many are **non-growing**?

$$n - \left(\lfloor \log_\gamma \frac{n}{\beta} \rfloor + 1\right) = \Theta(n)$$

- Time for one such append: $\Theta(1)$.
- Total time: $\Theta(n) \times \Theta(1) = \Theta(n)$.

Amortized Analysis

total time for n appends

=

$\Theta(n)$ ← total time for **non-growing** appends

+

$\Theta(n)$ ← total time for **growing** appends

Amortized Time Complexity

- The **amortized** time for an append is:

$$
\begin{aligned}
T_{\text{amort}}(n) &= \frac{\text{total time for } n \text{ appends}}{n} \\
&= \frac{\Theta(n)}{n} \\
&= \Theta(1)
\end{aligned}
$$

Dynamic Array Time Complexities

- Retrieve *k*th element: $\Theta(1)$
- Append/pop element at end:
  - $\Theta(1)$ best case
  - $\Theta(n)$ worst case (where n = current size)
  - $\Theta(1)$ amortized
- Insert/remove in middle: $O(n)$
  - May or may not need resize, still $O(n)$!

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Data Structures & Algorithms

Lecture 2 | Part 4

Practicalities

Advantages

- Great cache performance (it’s an array).
- Fast access.
- Don’t need to know size in advance of allocation.

Downsides

- Wasted memory.
- Expensive deletion in middle.

Implementations

- Python: `list`
- C++: `std::vector`
- Java: `ArrayList`

Exercise

Why do we need `np.array`? Python's `list` is a dynamic array, isn't that better?

In defense of `np.array`

- Memory savings are one reason.
- Bigger reason: using Python's `list` to store numbers does not have good **cache** performance.