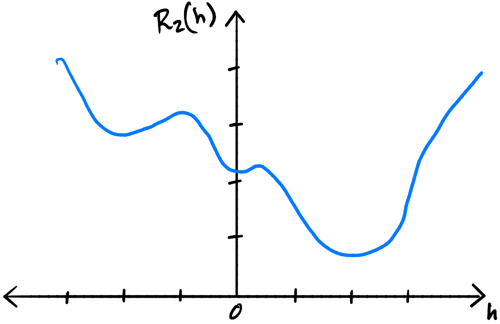


DSC 40A

Lecture 03

Learning via Optimization, pt III

Announcements

- First homework posted; due Friday at 5:00 pm.
- First discussion tonight! Will help with homework.
- My office hours: Tuesday, 5:30 – 6:30 pm and Thursday, 12 – 1 pm

Last Time

- To predict future salary:
  - Gather salaries $y_1, y_2, \dots, y_n$.
  - Find a prediction h^* which minimizes the **mean error**:

$$R(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

  - We saw that $R(h)$ is minimized by $\text{Median}(y_1, \dots, y_n)$.
- We turned learning into a math problem and solved it.

Two things we don't like

1. **Minimizing** the mean error wasn't so easy.

2. Actually **computing** the median isn't so easy, either.

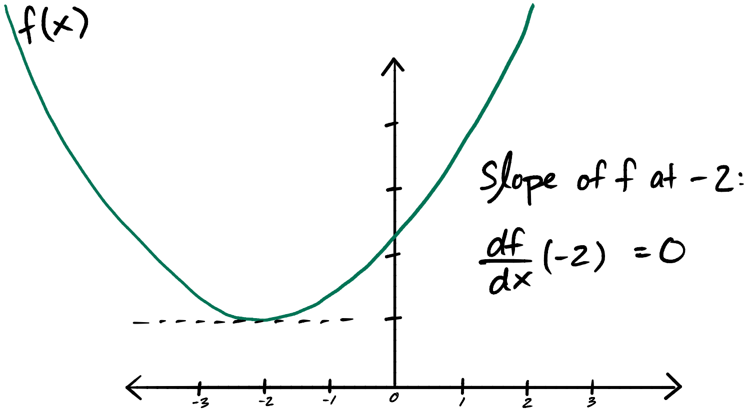
Today

Is there another way to measure the quality of a prediction that avoids these problems?

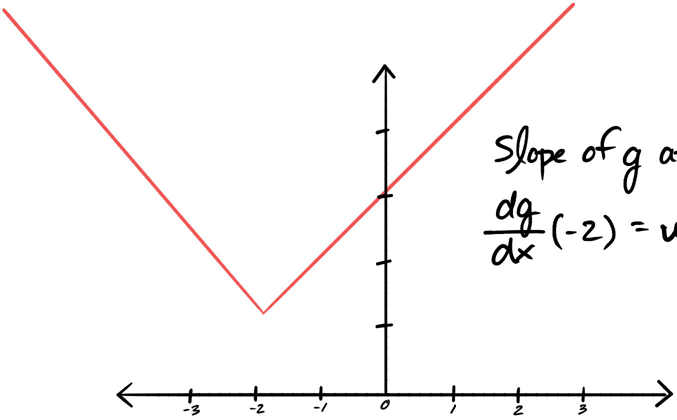
Minimizing via calculus

- Strategy: take derivative, set to zero, solve.
- Finding where the slope is zero.
- Only works if the function is **differentiable**.

Example: differentiable



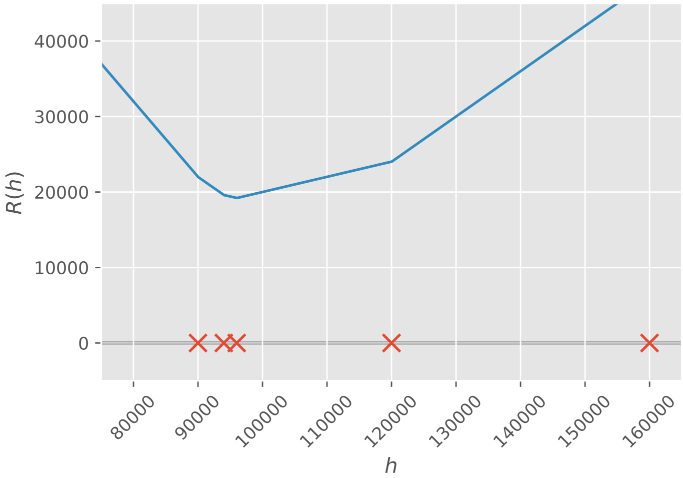
Example: not differentiable



Slope of g at -2:

$\frac{dg}{dx}(-2) = \text{undefined}$

The mean error is not differentiable



The mean error is not differentiable

$$\frac{dR}{dh}(h) = \frac{d}{dh}\left[\frac{1}{n}\sum_{i=1}^{n}|y_i - h|\right]$$

$$= \frac{1}{n}\frac{d}{dh}\sum_{i=1}^{n}|y_i - h|$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{d}{dh}|y_i - h|$$

:(

not differentiable

$$\frac{d}{dx}[f+g] \overset{?}{=} \frac{df}{dx} + \frac{dg}{dx}$$

The core issue

- We can't compute $\frac{d}{dh}|y_i - h|$; it is **not differentiable**.
- Remember: $|y_i - h|$ measures how far h is from y_i.
- Is there something besides $|y_i - h|$ which:
  1. Measures how far h is from y_i; *and*
  2. is **differentiable**?

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  1. Measures how far h is from y_i; *and*
  2. is **differentiable**?

Discussion Question

Which of these would work?

a) $e^{|y_i - h|}$ 14

b) $|y_i - h|^2$ 43

c) $|y_i - h|^3$ 1

d) $\cos(y_i - h)$ 21

The Squared Error

- Let h be a prediction and y be the right answer. The **squared error** is:

$$|y - h|^2 = (y - h)^2$$

- Like error, measures how far h is from y.

- But unlike error, the squared error is **differentiable**:

$$\frac{d}{dh}(y - h)^2 = 2(y-h)\frac{d}{dh}(y-h)$$

$$= -2(y-h)$$

$$= 2(h-y)$$

The Mean Squared Error

- Suppose we predicted a future salary of $h_1 = 150{,}000$ *before* collecting data.

salary	error of h_1	squared error of h_1
90,000	60,000	$(60{,}000)^2$
94,000	56,000	$(56{,}000)^2$
96,000	54,000	$(54{,}000)^2$
120,000	30,000	$(30{,}000)^2$
160,000	10,000	$(10{,}000)^2$

total squared error: 1.0652×10^{10}
mean squared error: 2.13×10^{9}

- A good prediction is one with small **mean squared error**.

The Mean Squared Error

- Now suppose we had predicted $h_2 = 115{,}000$.

salary	error of h_2	squared error of h_2
90,000	25,000	$(25{,}000)^2$
94,000	21,000	$(21{,}000)^2$
96,000	19,000	$(19{,}000)^2$
120,000	5,000	$(5{,}000)^2$
160,000	45,000	$(45{,}000)^2$

total squared error: 3.47×10^9
mean squared error: 6.95×10^8

- A good prediction is one with small **mean squared error**.

The New Idea

- Make prediction by minimizing the **mean squared error**:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

- Strategy: Take derivative, set to zero, solve for minimizer.

The New Idea

- Make prediction by minimizing the **mean squared error**:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

- Strategy: Take derivative, set to zero, solve for minimizer.

Discussion Question

Which of these is dR_{sq}/dh?

a) $\frac{1}{n}\sum_{i=1}^{n}(y_i - h)$

b) 0

c) $\sum_{i=1}^{n} y_i$

d) $\frac{2}{n}\sum_{i=1}^{n}(h - y_i)$ 73

Solution

$$\frac{dR_{\text{sq}}}{dh} = \frac{d}{dh}\left[\frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{d}{dh}(y_i - h)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} 2(y_i - h)(-1)$$

$$= \frac{2}{n}\sum (y_i - h)(-1) = \frac{2}{n}\sum_{i=1}^{n}(h - y_i)$$

Set to zero and solve for minimizer

$$\frac{2}{n}\sum_{i=1}^{n}(h-y_i)=0 \implies \frac{2}{n}\sum h - \frac{2}{n}\sum y_i = 0$$

$$\implies \frac{1}{n}\underbrace{\sum_{i=1}^{n} h}_{= n \cdot h} = \frac{1}{n}\sum y_i$$

$$\implies \frac{1}{n}\cdot n \cdot h = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\implies h = \frac{1}{n}\sum_{i=1}^{n} y_i = \text{Mean}(y_1, \ldots, y_n)$$

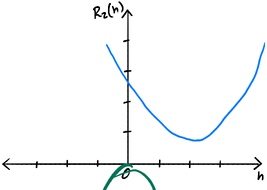
The mean minimizes the mean squared error

- That is:

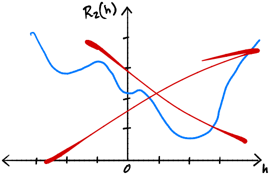
$$\arg\min_{h \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2 = \text{Mean}(y_1, \dots, y_n)$$

Discussion Question

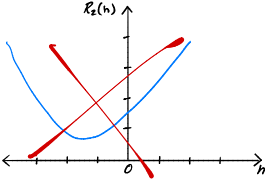
Suppose $y_1, \ldots, y_n$ are salaries. Which plot could be $R_{sq}(h)$?



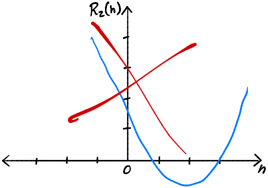
(a) 61



(b) 10



(c) 10



(d)

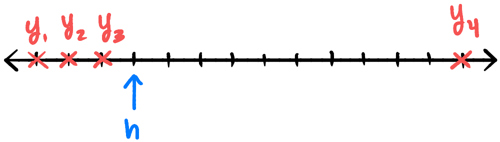
The mean is easy to compute

```
def mean(numbers):
    total = 0
    for number in numbers:
        total = total + number
    return total / len(numbers)
```

- Time complexity: $\Theta(n)$
- Median by sorting: $\Theta(n \log n)$
- But there's a $\Theta(n)$ way to find median: `quickselect`.
- DSC 40B.

Outliers

- The mean is quite **sensitive** to outliers.

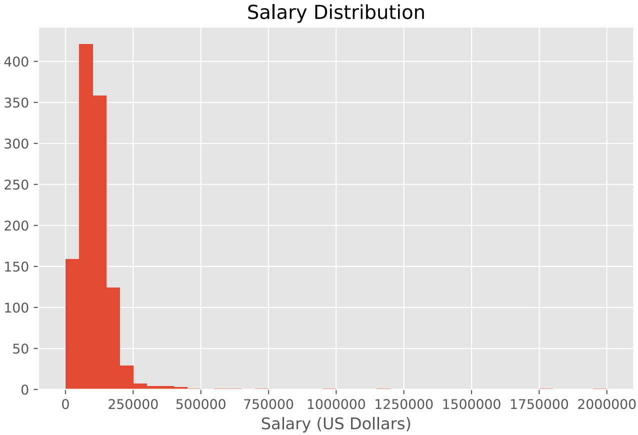


- $|y_4 - h|$ is 10 times as big as $|y_3 - h|$.
- But $(y_4 - h)^2$ is 100 times as big as $(y_3 - h)^2$.
- Squared error can be dominated by outliers.

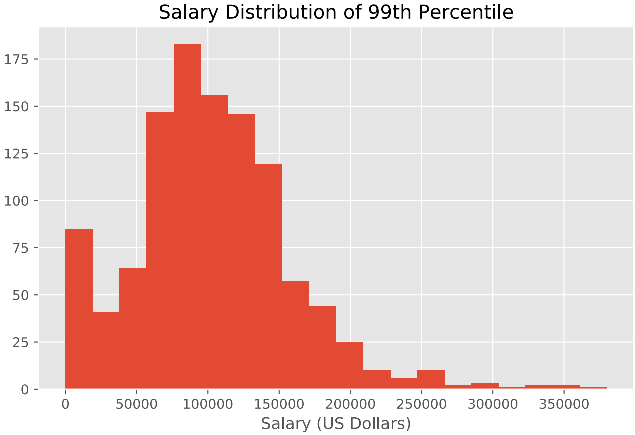
Example: Data Science Salaries

- Data set of 1121 self-reported data science salaries in the United States from the 2018 StackOverflow survey.
- Median = $100,000
- Mean = $111,032
- Max = $2,000,000
- Min = $52
- 95th Percentile: $200,000

Example: Data Science Salaries



Example: Data Science Salaries



Example: Income Inequality

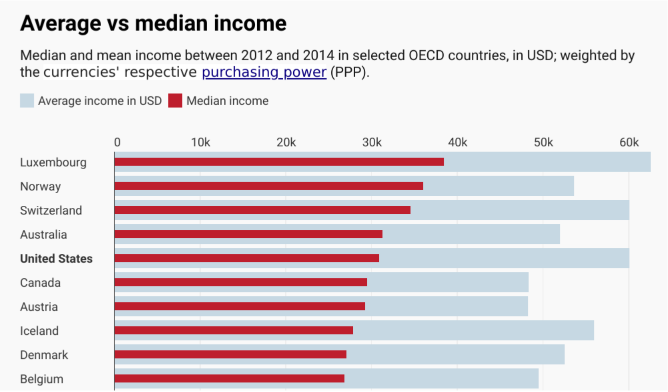
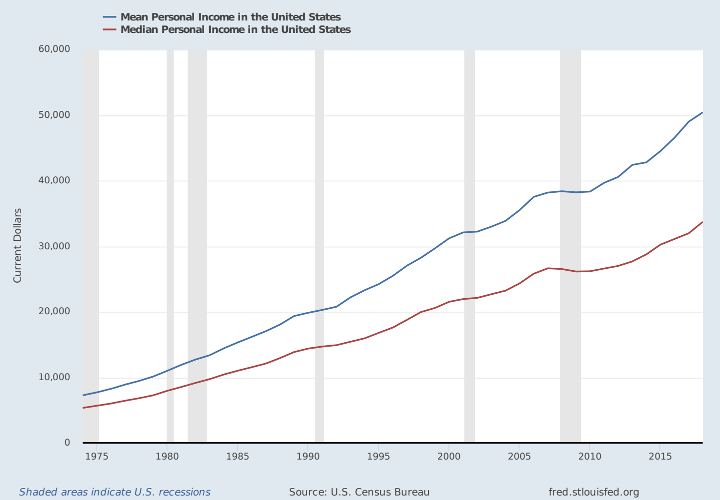


Chart: Lisa Charlotte Rost, Datawrapper

Example: Income Inequality



A General Framework

- We started with the **mean error**:

$$R(h) = \frac{1}{n} \sum_{i=1} |h - y_i|$$

- Today, we introduced the **mean squared error**:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1} (h - y_i)^2$$

- They have the same form: average difference between h and data.

A General Framework

- Definition: A **loss function** $L(h, y)$ takes in a prediction h and a right answer, y, and outputs a number measuring how far h is from y (bigger = further).
- The **absolute loss**:

$$L_{\text{abs}}(h, y) = |y - h|$$

- The **square loss**:

$$L_{\text{sq}}(h, y) = (y - h)^2$$

A General Framework

- Suppose that $y_1, \dots, y_n$ are some data points, h is a prediction, and L is a loss function. The **empirical risk** is the average loss:

$$R_L(h) = \frac{1}{n} \sum_{i=1}^{n} L(h, y_i)$$

- The goal of learning: find h that minimizes R_L. This is called **empirical risk minimization (ERM)**.

Designing a learning algorithm using ERM

1. Pick a loss function.

2. Pick a way to minimize the average loss (empirical risk)

- **Key Idea**: The choice of loss function determines the properties of the result and the difficulty of finding it.

Loss	Minimizer	Outliers	Differentiable	Algorithm
L_{abs}	median	**insensitive**	**no**	**not simple**
L_{sq}	mean	**sensitive**	**yes**	**simple, fast**

Status Update

- We introduced the **mean squared error** because it is differentiable.
- The minimizer of the mean squared error is the **mean**.
- The mean error and the mean squared error fit into a general framework of **empirical risk minimization**.

Next Time

- We’ll design our own loss function.
- We’ll develop a general way of solving minimization problems: **gradient descent**.