

DSC 40B

Theoretical Foundations II

Lecture 11 | Part 1

Adjacency Matrices (Recap)

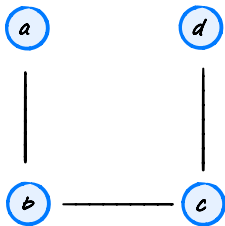
Representations

- ▶ How do we **store** a graph in a computer's memory?
- ▶ Three approaches:
 1. Adjacency matrices.
 2. Adjacency lists.
 3. "Dictionary of sets"

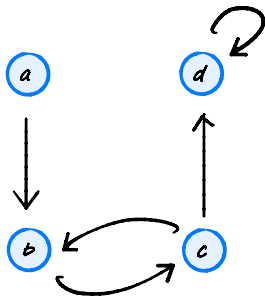
Adjacency Matrices

- ▶ Assume nodes are numbered $0, 1, \dots, |V| - 1$
- ▶ Allocate a $|V| \times |V|$ (Numpy) array
- ▶ Fill array as follows:
 - ▶ $\text{arr}[i, j] = 1$ if $(i, j) \in E$
 - ▶ $\text{arr}[i, j] = 0$ if $(i, j) \notin E$

Example



Example



Observations

- ▶ If G is undirected, matrix is symmetric.
- ▶ If G is directed, matrix may not be symmetric.

Time Complexity

operation	code	time
edge query	<code>adj[i,j] == 1</code>	$\Theta(1)$
<code>degree(i)</code>	<code>np.sum(adj[i,:])</code>	$\Theta(V)$

Space Requirements

- ▶ Uses $|V|^2$ bits, even if there are very few edges.
- ▶ But most real-world graphs are **sparse**.
 - ▶ They contain many fewer edges than possible.

Example: Facebook

- ▶ Facebook has 2 billion users.

$$(2 \times 10^9)^2 = 4 \times 10^{18} \text{ bits}$$

$$= 500 \text{ petabits}$$

$$\approx 6500 \text{ years of video at 1080p}$$

$$\approx 60 \text{ copies of the internet as it was in 2000}$$

Adjacency Matrices and Math

- ▶ Adjacency matrices are useful mathematically.
- ▶ Example: (i, j) entry of A^2 gives number of hops of length 2 between i and j .

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Adjacency Lists

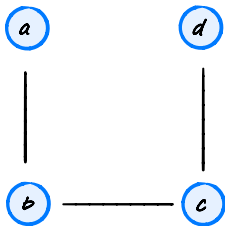
What's Wrong with Adjacency Matrices?

- ▶ Requires $\Theta(|V|^2)$ storage.
- ▶ Even if the graph has no edges.
- ▶ **Idea:** only store the edges that exist.

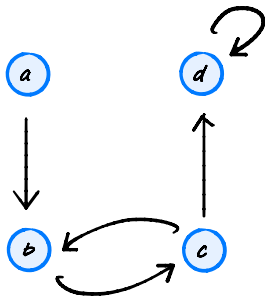
Adjacency Lists

- ▶ Create a list `adj` containing $|V|$ lists.
- ▶ `adj[i]` is list containing the neighbors of node i .

Example



Example



Observations

- ▶ If G is undirected, each edge appears twice.
- ▶ If G is directed, each edge appears once.

Time Complexity

operation	code	time
edge query	<code>j in adj[i]</code>	$\Theta(\text{degree}(i))$
<code>degree(i)</code>	<code>len(adj[i])</code>	$\Theta(1)$

Space Requirements

- ▶ Need $\Theta(|V|)$ space for outer list.
- ▶ Plus $\Theta(|E|)$ space for inner lists.
- ▶ In total: $\Theta(|V| + |E|)$ space.

Example: Facebook

- ▶ Facebook has 2 billion users, 400 billion friendships.
- ▶ If each edge requires 32 bits:

$$\begin{aligned} & (2 \text{ bits} \times 200 \times (2 \text{ billion})) \\ &= 64 \times 400 \times 10^9 \text{ bits} \\ &= 3.2 \text{ terabytes} \\ &= 0.04 \text{ years of HD video} \end{aligned}$$

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Dictionary of Sets

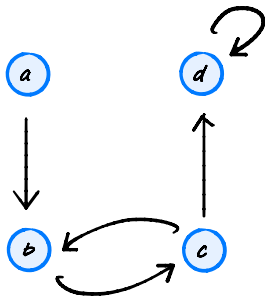
Tradeoffs

- ▶ Adjacency matrix: fast edge query, lots of space.
- ▶ Adjacency list: slower edge query, space efficient.
- ▶ Can we have the best of both?

Idea

- ▶ Use **hash tables**.
- ▶ Replace inner edge lists by **sets**.
- ▶ Replace outer list with **dict**.
 - ▶ Doesn't speed things up, but allows nodes to have arbitrary labels.

Example



Time Complexity

operation	code	time
edge query	<code>j in adj[i]</code>	$\Theta(1)$ average
<code>degree(i)</code>	<code>len(adj[i])</code>	$\Theta(1)$ average

Space Requirements

- ▶ Requires only $\Theta(E)$.
- ▶ But there is overhead to using hash tables.

Dict-of-sets implementation

- ▶ Install with `pip install dsc40graph`
- ▶ Import with `import dsc40graph`
- ▶ Docs: <https://eldridgejm.github.io/dsc40graph/>
- ▶ Source code:
<https://github.com/eldridgejm/dsc40graph>
- ▶ Will be used in HW coding problems.

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Theoretical Foundations II

Lecture 11 | Part 4

Graph Search Strategies

How do we:

- ▶ determine if there is a path between two nodes?
- ▶ check if graph is connected?
- ▶ count connected components?

Search Strategies

- ▶ A **search strategy** is a procedure for exploring a graph.
- ▶ Different strategies are useful in different situations.

Node Statuses

At any point during a search, a node is in exactly one of three states:

- ▶ **visited**
- ▶ **pending** (discovered, but not yet visited)
- ▶ **undiscovered**

Rules

- ▶ At every step, next visited node chosen from among **pending** nodes.
- ▶ When a node is marked as **visited**, all of its neighbors have been marked as **pending**.

Choosing the next Node

How to choose among pending nodes?

- ▶ Idea 1: Visit **newest** pending (**depth-first search**).
- ▶ Idea 2: Visit **oldest** pending (**breadth-first search**).

Main Idea

DFS and BFS each discover different properties of the graph.

For example, we'll see that BFS is useful for finding shortest paths (DFS in general is not).

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Lecture 11 | Part 5

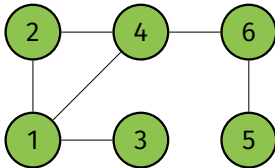
Breadth-First Search

Breadth-First Search

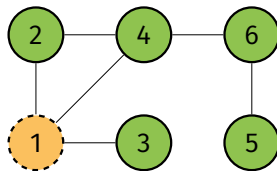
- ▶ At every step:
 1. Visit oldest pending node.
 2. Mark its undiscovered neighbors as pending.
- ▶ Convention: in this class, neighbors produced in sorted order.¹

¹In general, the order in which a node's neighbors produced is arbitrary.

Example



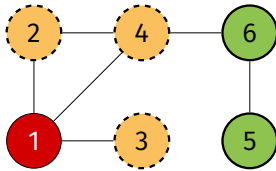
Example



pending = [1]

Before iterating.

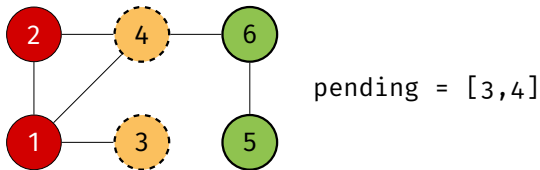
Example



pending = [2,3,4]

After 1st iteration.

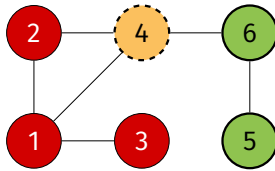
Example



After 2nd iteration.

Exercise: what will the picture look like after the next two iterations?

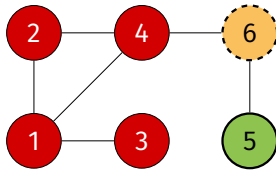
Example



pending = [4]

After 3rd iteration.

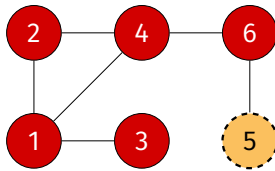
Example



pending = [6]

After 4th iteration.

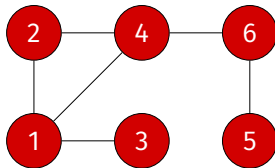
Example



pending = [5]

After 5th iteration.

Example



`pending = []`

After 6th iteration.

Implementation

- ▶ To store pending nodes, use a FIFO **queue**.
- ▶ While queue is not empty:
 - ▶ Pop a node, u .
 - ▶ Add undiscovered neighbors to queue.

Queues in Python

- ▶ Want $\Theta(1)$ time pops/appends on either side.
- ▶ `from collections import deque` (“deck”).
 - ▶ `.popleft()` and `.pop()`
 - ▶ `list` doesn't have right time complexity!
 - ▶ `import queue` isn't what you want!
- ▶ Keep track of node status attribute using dictionary.

Exercise

```
from collections import deque

def bfs(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}
    status[source] = 'pending'
    pending = deque([source])
    # while there are still pending nodes
    while pending:
        # EXERCISE: fill this in...
```

BFS

```
from collections import deque

def bfs(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

Note

- ▶ What does this code actually *return*?

Note

- ▶ What does this code actually *return*?
- ▶ Nothing, yet. It is a *foundation*.

Note

- ▶ BFS works just as well for directed graphs.

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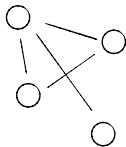
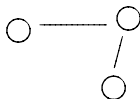
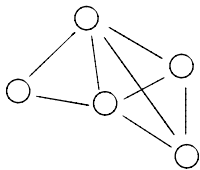
Theoretical Foundations II

Lecture 11 | Part 6

Analysis of BFS

Exercise

What will bfs do when run on a disconnected graph?



Claim

- ▶ bfs with source u will visit all nodes reachable from u (and only those nodes).
- ▶ Useful!
 - ▶ Is there a path between u and v ?
 - ▶ Is graph connected?

Exploring with BFS

- ▶ BFS will visit all nodes reachable from source.
- ▶ If **disconnected**, BFS will not visit all nodes.
- ▶ We can do so with a **full BFS**.
 - ▶ Idea: “re-start” BFS on undiscovered node.
 - ▶ Must pass statuses between calls.

Making Full BFS

Modify bfs to accept statuses:

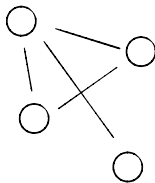
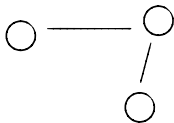
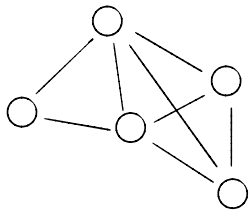
```
def bfs(graph, source, status=None):  
    """Start a BFS at `source`."""  
    if status is None:  
        status = {node: 'undiscovered' for node in graph.nodes}  
    # ...
```

Making Full BFS

Call bfs multiple times:

```
def full_bfs(graph):  
    status = {node: 'undiscovered' for node in graph.nodes}  
    for node in graph.nodes:  
        if status[node] == 'undiscovered':  
            bfs(graph, node, status)
```


Example



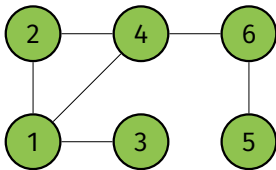
Observation

- If there are k connected components, `bfs` in line 5 is called exactly k times.

```
1 def full_bfs(graph):  
2     status = {node: 'undiscovered' for node in graph.nodes}  
3     for node in graph.nodes:  
4         if status[node] == 'undiscovered':  
5             bfs(graph, node, status)
```

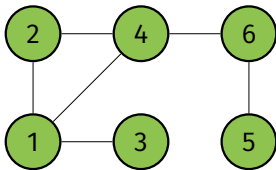
Exercise

How many times is each node added to the queue in a BFS of the graph below?



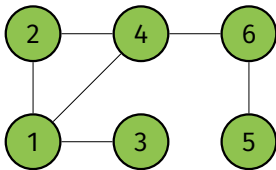
Exercise

How many times is each edge “explored” in a BFS of the graph below?



Exercise

How many times is each edge “explored” in a BFS of the *directed* graph below?



Key Properties of full_bfs

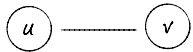
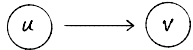
- ▶ Each node added to queue **exactly once**.
- ▶ Each edge is explored **exactly**:
 - ▶ **once** if graph is **directed**.
 - ▶ **twice** if graph is **undirected**.

Time Complexity of full_bfs

- ▶ Analyzing full_bfs is easier than analyzing bfs.
 - ▶ full_bfs visits all nodes, no matter the graph.
- ▶ Result will be **upper bound** on time complexity of bfs.
- ▶ We'll use an **aggregate analysis**.

BFS

```
def bfs(graph, source, status=None):  
    """Start a BFS at `source`."""  
    if status is None:  
        status = {node: 'undiscovered' for node in graph.nodes}  
  
    status[source] = 'pending'  
    pending = deque([source])  
  
    # while there are still pending nodes  
    while pending:  
        u = pending.popleft()  
        for v in graph.neighbors(u):  
            # explore edge (u,v)  
            if status[v] == 'undiscovered':  
                status[v] = 'pending'  
                # append to right  
                pending.append(v)  
        status[u] = 'visited'
```



Time Complexity

```
def full_bfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered':
            bfs(graph, node, status)

def bfs(graph, source, status=None):
    """Start a BFS at `source`."""
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

Time Complexity of Full BFS

- ▶ $\Theta(V + E)$
- ▶ If $|V| > |E|$: $\Theta(V)$
- ▶ If $|V| < |E|$: $\Theta(E)$
- ▶ Namely, if graph is **complete**: $\Theta(V^2)$.
- ▶ Namely, if graph is **very sparse**: $\Theta(V)$.

Notational Note

- ▶ We'll often write $\Theta(V + E)$ instead of $\Theta(|V| + |E|)$.
- ▶ You can use whichever.

Next Time

- ▶ Finding **shortest paths** using BFS.