DSC 40B - Midterm 01

February 9, 2023

Name: SOLUTIONS

PID:

Exam Version: A

By signing below, you agree that you will behave honestly and fairly during and after this exam. You should not discuss any part of this exam with anyone who has not yet taken it.

Signature:

Name of student to your **left**:

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(Write "N/A" if a wall/aisle is to your left/right.)

Instructions:

- Your exam version should be different from those to your left/right.
- Write your solutions to the following problems in the spaces provided.
- No calculators are permitted, but a page of notes is.
- Write your name or PID at the top of each sheet in the space provided.

(Please do not open your exam until instructed to do so.)

Problem 1.

What is the time complexity of the following function? State your answer as a function of n using asymptotic notation in the simplest form possible. E.g., "$\Theta(n)$".

```
def foo(n):
    for i in range(n**3):
        for j in range(n):
            print(i + j)
        for j in range(n**2):
            print(i + j)
```

$\Theta(n^5)$

Problem 2.

What is the time complexity of the following function? State your answer as a function of n using asymptotic notation in the simplest form possible. E.g., "$\Theta(n)$".

```
def foo(n):
    for i in range(n):
        for j in range(i):
            for k in range(n):
                print(i + j + k)
```

$\Theta(n^3)$

Problem 3.

What is the time complexity of the following function? State your answer as a function of n using asymptotic notation in the simplest form possible. E.g., "$\Theta(n)$".

```
def foo(n):
    for i in range(200, n):
        for j in range(i, 2*i + n**2):
            print(i + j)
```

$\Theta(n^3)$

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Problem 4.

What is the time complexity of the following function? State your answer as a function of n using asymptotic notation in the simplest form possible. E.g., "$\Theta(n)$".

```
import math

def foo(arr):
    """`arr` is an array with n elements."""
    n = len(arr)
    ix = 1
    s = 0

    while ix < n:
        s = s + arr[ix]
        ix = ix * 5 + 2

    return s
```

$\Theta(\log n)$

Problem 5.

The code below takes in an array of n numbers and checks whether there is a pair of numbers in the array which, when added together, equal the maximum element of the array.

What is the **best case** time complexity of this code as a function of n? State your answer using asymptotic notation.

```
def exists_pair_summing_to_max(arr):
    n = len(arr)
    maximum = max(arr)
    for i in range(n):
        for j in range(i + 1, n):
            if arr[i] + arr[j] == maximum:
                return True
    return False
```

$\Theta(n)$

Problem 6.

What is the **worst case** time complexity of the function in the last problem? State your answer using asymptotic notation.

$\Theta(n^2)$

Problem 7.

Consider again the problem of determining whether there exists a pair of numbers in an array which, when added together, equal the maximum number in the array. Additionally, **assume that the array is sorted**.

True or False: $\Theta(n)$ is a **tight** theoretical lower bound for this problem.

● True

○ False

Solution: Any algorithm must take $\Omega(n)$ time in the worst case, since in the worst case all elements of the array must be read.

This is tight because there is an algorithm that will compute the answer in worst-case $\Theta(n)$ time. Namely, this is an instance of the "movie problem" from lecture, where instead of finding two numbers which add to an arbitrary target, we're looking for a specific target: the maximum. We saw an algorithm that solved the movie problem in $\Theta(n)$ time, where n was the number of movies. Since the maximum is computed in $\Theta(1)$ time for a sorted array, we can use the same algorithm to solve this in $\Theta(n)$ time as well.

Problem 8. (2 points)

Suppose a and b are two numbers, with $a \leq b$. Consider the problem of counting the number of elements in an array which are between a and b; that is, the number of elements x such that $a \leq x \leq b$. You may assume for simplicity that both a and b are in the array, and there are no duplicates.

a) What is a **tight** theoretical lower bound for this problem, assuming that the array is **unsorted**? State your answer in asymptotic notation as a function of the number of elements in the array, n.

$\Theta(n)$

Solution: Since the array is in arbitrary order, we have to at least read all of the elements, taking $\Omega(n)$ time. This is a tight lower bound, because we can count the number of elements between a and b by looping through and checking whether each element is in $[a, b]$ in linear time.

b) What is a **tight** theoretical lower bound for this problem, assuming that the array is **sorted**? State your answer in asymptotic notation as a function of the number of elements in the array, n.

$\Theta(\log n)$

Solution: We can do this in worst-case $\Theta(\log n)$ time with binary search: find the index of a in worst-case $\Theta(\log n)$ time; find the index of b in worst-case $\Theta(\log n)$ time; and subtract the two indices (and add one) to get the total number of elements between a and b, inclusive.

There cannot be an algorithm that is better than $\Theta(\log n)$ in the worst case, so this is a tight lower bound. To see why, recognize that we can solve the query problem (determine True/False if a target t is in a sorted array) with any algorithm that solves this problem by setting $a = b = t$. If an algorithm solves this problem in faster than $\Theta(\log n)$, it will also solve the query problem in faster than $\Theta(\log n)$, but we saw in class that the query problem has a theoretical lower bound of $\Theta(\log n)$, so such an algorithm cannot exist.

Problem 9.

PID or Name: _____

What is the **expected** time complexity of the following function? State your answer using asymptotic notation.

```
import random

def foo(n):
    x = random.randrange(n)

    if x < 8:
        for j in range(n**3):
            print(j)
    else:
        for j in range(n):
            print(j)
```

$\Theta(n^2)$

Problem 10.

What is the **expected** time complexity of the function below? State your answer using asymptotic notation.

You may assume that `math.sqrt` and `math.log` take $\Theta(1)$ time. `math.log` computes the natural log.

```
import random
import math

def foo(n):
    # draw a number uniformly at random from 0, 1, 2, ..., n-1 in Theta(1)
    x = random.randrange(n)

    if x < math.log(n):
        for j in range(n**2):
            print(j)
    elif x < math.sqrt(n):
        print('Ok!')
    else:
        for i in range(n):
            print(i)
```

$\Theta(n \log n)$

Solution: There are three cases: Case 1) $x < \log n$; Case 2) $\log n \geq x < \sqrt{n}$; and Case 3) $x \geq \sqrt{n}$.

The probability of Case 1 is $\log n/n$. The time taken in case 1 is $\Theta(n^2)$.

The probability of Case 2 is

$$\frac{\sqrt{n}}{n} - \frac{\log n}{n} = \Theta(\sqrt{n}/n) = \Theta(1/\sqrt{n}).$$

The time taken in this case is $\Theta(1)$.

The probability of Case 3 is 1 minus the probability of the sum of the other two cases, which will simply be $\Theta(1)$:

$$1 - \Theta((\log n)/n) - \Theta(1/\sqrt{n}) = \Theta(1)$$

Therefore, the expected time is:

$$\frac{\log n}{n} \times \Theta(n^2) + \Theta(1/\sqrt{n}) \times \Theta(1) + \Theta(1) \times \Theta(n) = \Theta(n \log n) + \Theta(1/\sqrt{n}) + \Theta(n)$$
$$= \Theta(n \log n)$$

Problem 11.

State (but do not solve) the recurrence describing the runtime of the following function.

```
def foo(n):
    if n < 1:
        return 0

    for i in range(n**2):
        print("here")

    foo(n/2)
```

$$T(n) = \begin{cases} \Theta(1), & n = 1 \\ \boxed{T(n/2) + \Theta(n^2)} & n > 1 \end{cases},$$

Problem 12. (2 points)

Solve the below recurrence, stating the solution in asymptotic notation. Show your work.

$$T(n) = \begin{cases} T(n/2) + \Theta(n) & n > 1 \\ \Theta(1) & n = 1 \end{cases}$$

Solution: $\Theta(n)$.

Unrolling several times, we find:

$$\begin{aligned} T(n) &= T(n/2) + n \\ &= T(n/4) + \frac{n}{2} + n \\ &= T(n/8) + \frac{n}{4} + \frac{n}{2} + n \end{aligned}$$

On the kth unroll, we'll get:

$$T(n) = T(n/2^k) + \left[n + \frac{n}{2} + \frac{n}{4} + \ldots + \frac{n}{2^{k-1}}\right]$$

It will take $\log_2 n$ unrolls to each the base case. Substituting this for k, we get:

$$\begin{aligned} T(n) &= T(n/2^{\log_2 n}) + \left[n + \frac{n}{2} + \frac{n}{4} + \ldots + \frac{n}{2^{(\log_2 n)-1}}\right] \\ &= T(1) + n \underbrace{\left[1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{(\log_2 n)-1}}\right]}_{\leq 2} \end{aligned}$$

Here we've recognized the summation in square braces as a geometric sum of powers of $1/2$, which converges to 2 when there are infinitely-many terms. We've seen this sum in solving a couple of homework problems.

$$\begin{aligned} &= T(1) + cn \\ &= \Theta(1) + cn \\ &= \Theta(n) \end{aligned}$$

Problem 13.

Suppose `bar` and `baz` are two functions. Suppose `bar`'s time complexity is $\Theta(n^3)$, while `baz`'s time complexity is $\Theta(n^2)$.

Suppose `foo` is defined as below:

```
def foo(n):
    if n < 1_000:
        bar(n)
    else:
        baz(n)
```

What is the asymptotic time complexity of `foo`?

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$\Theta(n^2)$

Problem 14.

Let

$$f(n) = 5n \log n + \frac{n^3 + 5}{n + 2 + |\sin \pi n|} + n\sqrt{n}$$

Write f in asymptotic notation in as simplest terms possible.

$f(n) = \Theta($ n^2 $)$

Problem 15.

Suppose $f_1(n)$ is $O(n^2)$ and $\Omega(n)$. Also suppose that $f_2(n) = \Theta(n^2)$.
Consider the function $f(n) = f_1(n) + f_2(n)$. True or false: it must be the case that $f(n) = \Theta(n^2)$.

- ● True
- ○ False

Problem 16.

Suppose $f_1(n)$ is $O(n^2)$ and $\Omega(n)$. Also suppose that $f_2(n) = \Theta(n^2)$.
Consider the function $g(n) = f_2(n)/f_1(n)$. True or false: it must be the case that $g(n) = \Omega(n)$.

- ○ True
- ● False

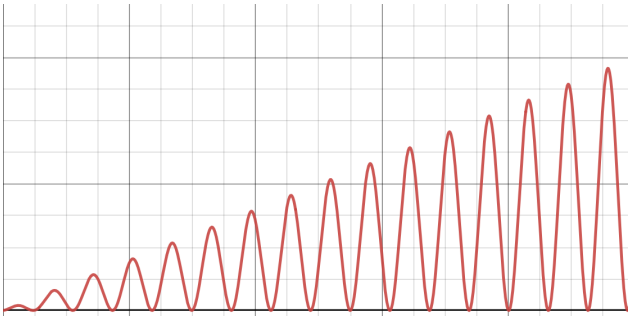
Problem 17.

Suppose $f_1(n) = \Omega(g_1(n))$ and $f_2 = \Omega(g_2(n))$. Define $f(n) = \min\{f_1(n), f_2(n)\}$ and $g(n) = \min\{g_1(n), g_2(n)\}$.
True or false: it is necessarily the case that $f(n) = \Omega(g(n))$.

- ● True
- ○ False

Problem 18.

Consider the function $f(n) = n \times (\sin(n) + 1)$. A plot of this function is shown below:



True or False: this function is $\Theta(n)$.

- ○ True
- ● False

Solution: There is no positive c such that this function is lower-bounded by cn, and so this function cannot be $\Theta(n)$.

Problem 19.

Consider again the function $f(n) = n \times (\sin(n) + 1)$ from the previous problem.

True or False: $f(n) = O(n^3)$.

- (●) True
- ( ) False

Problem 20.

Consider the iterative implementation of binary search shown below:

```
import math

def iterative_binary_search(arr, target):

    start = 0
    stop = len(arr)

    while (stop - start) > 0:
        print(arr[start])
        middle = math.floor((start + stop) / 2)
        if arr[middle] == target:
            return middle
        elif arr[middle] > target:
            stop = middle
        else:
            start = middle + 1
```

Which of the following loop invariants is true, assuming that `arr` is sorted and non-empty, and `target` is **not** in the array? Select all that apply.

- [■] After each iteration, `stop - start >= 0`.
- [ ] After each iteration, `stop - start >= 1`.
- [ ] After each iteration, `arr[start] <= target`.
- [ ] After each iteration, `arr[start] <= max(target, arr[0])`.

Solution: There was a small bug in the grading here, where originally the last option was marked as True, when it is not always true. Try, for example, the input `arr = [3, 4, 7]` and a target of `5`.

Since the last option was trickier than intended, both checking it and leaving it unchecked were marked as correct. Note, though, that the correct answer is that it is false.

Problem 21.

Consider `iterative_binary_search` from above and note the `print` statement in the `while`-loop. Suppose `iterative_binary_search` is run on the array:

```
[-202, -201, -200, -50, -20, -10, -4, -3, 0, 1, 3, 5, 6, 7, 9, 10, 12, 15, 22]
```

with target `11`.

What will be the last value of `arr[start]` printed?

10

Problem 22.

Consider the code below which claims to compute the most common element in the array, returning a pair: the element along with the number of times it appears.

```
import math

def most_common(arr, start, stop):
    """Attempts to compute the most common element in arr[start:stop]."""
    if stop - start == 1:
        return (arr[start], 1)

    middle = math.floor((start + stop) / 2)

    left_value, left_count = most_common(arr, start, middle)
    right_value, right_count = most_common(arr, middle, stop)

    if left_count > right_count:
        return (left_value, left_count)
    else:
        return (right_value, right_count)
```

You may assume that the function is always called on a non-empty array, and with `start = 0` and `stop = len(arr)`. Will this code always return the correct answer (the most common element)?

- ◯ Yes: it will always return the correct answer.
- ◯ No: it may recurse infinitely.
- ◯ No: it may try to access the array at an invalid index.
- ● No: it will run without error, but the element returned may not be the most common element in the array.

Solution: It is not true in general that the most common element in the array is the most common in the left or the right. As a simple example, take: `[1,1,1,0,0,2,2,2,0,0]`. 1 is the most common in the left half, and 2 is the most common in the right half, but 0 is the most common overall.

So even if we assume that `most_common(arr, start, middle)` finds the most common element in left half of the array, and `most_common(arr, middle, stop)` finds the most common element in the right of the array, it is not necessarily the case that the most common between them is the overall most common in the array.

You can also quickly see that the code cannot be correct because nowhere is a count greater than one ever returned.

Problem 23.

Consider the modification of `mergesort` shown below, where one of the recursive calls has been replaced by an in-place version of `selection_sort`. Recall that `selection_sort` takes $\Theta(n^2)$ time.

```
def kinda_mergesort(arr):
    """Sort array in-place."""
    if len(arr) > 1:
        middle = math.floor(len(arr) / 2)
        left = arr[:middle]
        right = arr[middle:]
        mergesort(left)
        selection_sort(right)
        merge(left, right, arr)
```

What is the time complexity of `kinda_mergesort`?

$\Theta(n^2)$

PID or Name: _____

Problem 24.

Recall the partition operation from `quickselect`. Which of the following arrays could have been partitioned at least once? Select all that apply.

- ☐ `[50, 10, 20, 30, 60, 40]`
- ■ `[20, 10, 30, 60, 40, 50]`
- ■ `[20, 50, 10, 40, 30, 60]`
- ☐ `[60, 50, 40, 30, 20, 10]`
- ■ `[10, 20, 30, 40, 50, 60]`

Problem 25.

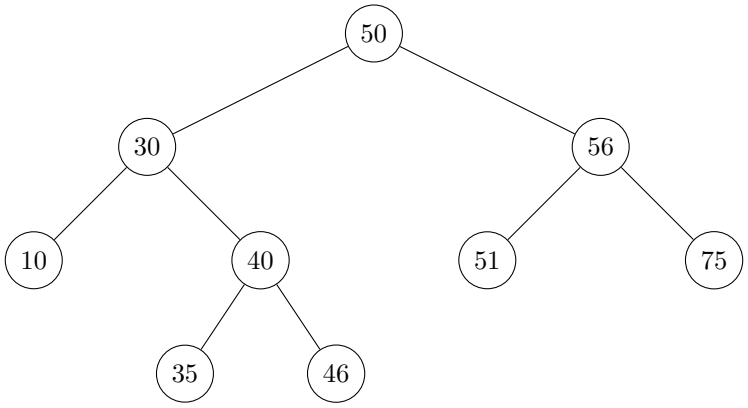
Define the **largest gap** in a collection of numbers to be the largest difference between two distinct elements in the collection (in absolute value). For example, the largest gap in $\{4, 9, 1, 6\}$ is 8 (between 1 and 9).

Suppose a collection of n numbers is stored in a **balanced** binary search tree. What is the time complexity required for an efficient algorithm to calculate the largest gap of the numbers in the BST? State your answer as a function of n in asymptotic notation.

$\Theta(\log n)$

Problem 26.

Suppose the numbers `41`, `32`, and `33` are inserted (in that order) into the below binary search tree. Draw where the new nodes will appear.



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