CSE 151A - Discussion 09

Problem 1.

Given a direction $\vec{u}$, calculate the projection of $\vec{x} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ onto $\vec{u}$ and the angle between $\vec{x}$ and $\vec{u}$.

Note that in class we assumed $\vec{u}$ to be a unit vector, but that may not necessarily be the case here!

1. $\vec{u}$ is the x_1 axis
2. $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
3. $\vec{u} = \begin{pmatrix} 1.5 \\ -0.5 \\ 1 \end{pmatrix}$
4. $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

Solution: From lecture, we have been shown that the projection of $\vec{x} \in \mathbb{R}^d$ along the direction $\vec{u} \in \mathbb{R}^d$ is given by $(\vec{x} \cdot \vec{u})\vec{u}$, where $\vec{u}$ is a unit vector.
However, in the above examples, $\vec{u}$ is not a unit vector. To rectify this discrepancy, we can simply normalize $\vec{u}$ by dividing each component of $\vec{u}$ by $||\vec{u}||$, which will result in a vector of unit length in the same direction as the original $\vec{u}$.
This yields the following equation for the projection of $\vec{x}$ onto any vector $\vec{u}$: $(\vec{x} \cdot \frac{\vec{u}}{||\vec{u}||})\frac{\vec{u}}{||\vec{u}||}$
We can rewrite this more clearly as $\left(\frac{\vec{x} \cdot \vec{u}}{||\vec{u}||^2}\right)\vec{u}$ and use it to solve each projection.
We also saw in class this useful property of dot product : $\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos(\theta)$. We can rearrange this equation to solve for the angle between vectors $\vec{a}$ and $\vec{b}$ as $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}\right)$.
We will now use these two formulas to solve each problem.
We will also need $||\vec{x}|| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14} = 3.74$ for each problem, so we define it here.

1. Here $\vec{u}$ is the x_1 axis, meaning $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Since we defined this ourselves to be of unit length, we can use the original projection formula without normalization.
   $(\vec{x} \cdot \vec{u})\vec{u} = (3 \cdot 1 + (-1) \cdot 0 + 2 \cdot 0) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$
   Solving for $\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{u}}{||\vec{x}||||\vec{u}||}\right) = \cos^{-1}\left(\frac{3}{(\sqrt{14})(1)}\right) = 36.7°$
2. $\left(\frac{\vec{x} \cdot \vec{u}}{||\vec{u}||^2}\right)\vec{u} = \frac{(3 \cdot 1 + (-1) \cdot 1 + 2 \cdot 1)}{(1^2 + 1^2 + 1^2)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix}$
   $\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{u}}{||\vec{x}||||\vec{u}||}\right) = \cos^{-1}\left(\frac{4}{(\sqrt{14})\sqrt{3}}\right) = 51.89°$

3. $\left(\frac{\vec{x}\cdot\vec{u}}{||\vec{u}||^2}\right)\vec{u} = \frac{(3\cdot(1.5)+(-1)\cdot(-0.5)+2\cdot(1))}{(1.5^2+(-0.5)^2+1^2)}\begin{pmatrix}1.5\\-0.5\\1\end{pmatrix} = \frac{7}{3.5}\begin{pmatrix}1.5\\-0.5\\1\end{pmatrix} = 2\begin{pmatrix}1.5\\-0.5\\1\end{pmatrix} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$

   $\theta = \cos^{-1}\left(\frac{\vec{x}\cdot\vec{u}}{||\vec{x}||||\vec{u}||}\right) = \cos^{-1}\left(\frac{7}{(\sqrt{14})\sqrt{3.5}}\right) = \cos^{-1}(1) = 0°$

   $\vec{u}$ and $\vec{x}$ are in the same direction, so the angle between them is 0° and the projection of $\vec{x}$ onto $\vec{u}$ is just $\vec{x}$!

4. $\left(\frac{\vec{x}\cdot\vec{u}}{||\vec{u}||^2}\right)\vec{u} = \frac{(3\cdot(3)+(-1)\cdot(1)+2\cdot(-4))}{(3^2+1^2+(-4)^2)}\begin{pmatrix}3\\1\\-4\end{pmatrix} = \frac{0}{26}\begin{pmatrix}3\\1\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

   $\theta = \cos^{-1}\left(\frac{\vec{x}\cdot\vec{u}}{||\vec{x}||||\vec{u}||}\right) = \cos^{-1}\left(\frac{0}{(\sqrt{14})\sqrt{26}}\right) = \cos^{-1}(0) = 90°$

   The dot product between $\vec{u}$ and $\vec{x}$ is 0 which tells us that they are orthogonal to one another. This also means that no component of $\vec{x}$ lies on $\vec{u}$ and that the projection of $\vec{x}$ onto $\vec{u}$ is the zero vector!

Problem 2.

Given the covariance matrix $C = \begin{pmatrix}4 & 1.5\\1.5 & 1\end{pmatrix}$, calculate the variance in the direction of $\vec{u}$ for each of the following settings : $\vec{u}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{u}^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$, and $\vec{u}^{(3)} = \begin{pmatrix}2\\1\end{pmatrix}$.

If we wanted to map each data point $\vec{x}^{(i)}$ to a single feature $z_i = \vec{x}^{(i)}\cdot\vec{u}$, what choice of $\vec{u}^{(i)}$ would be best?

Solution: $\vec{u}^{(3)}$
We would like to choose the direction $\vec{u}$ in which the variance is maximized. As seen in class, this is equivalent to finding the unit vector in the same direction as $\vec{u}$ that maximizes $\vec{u}^T C\vec{u}$.
The first step here is to ensure that each of our candidate direction vectors is of unit length.
$\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ have length 1, but $\vec{u}^{(3)}$ must be rewritten as $\vec{u}^{(3)} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix}$.

Calculating the variance for each $\vec{u}^{(i)}$ yields the following :
$(\vec{u}^{(1)})^T C\vec{u}^{(1)} = (1,0)\begin{pmatrix}4 & 1.5\\1.5 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 4$
$(\vec{u}^{(2)})^T C\vec{u}^{(2)} = (0,1)\begin{pmatrix}4 & 1.5\\1.5 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1$
$(\vec{u}^{(3)})^T C\vec{u}^{(3)} = (\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})\begin{pmatrix}4 & 1.5\\1.5 & 1\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix} = 4.6$

The variance for the direction defined by $\vec{u}^{(3)}$ is the maximum among our three choices, so it should be chosen.
Another noteworthy fact is that the variance calculation with direction $\vec{u}^{(1)}$ is identical to the top-left entry of C. This is to be expected because this entry corresponds directly to the variance of our data in the x_1 direction. The same is true for x_2.
If we were not bounded to just these three options, the best direction is that of the first eigenvector of C, which in this case is the direction $\begin{pmatrix}1+\sqrt{2}\\1-\sqrt{2}\end{pmatrix}$.