

DSC 140B

Representation Learning

Lecture 07 | Part 1

Change of Basis Matrices

Changing Basis

- ▶ Suppose $\vec{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)}$.
- ▶ $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ form a new, **orthonormal** basis \mathcal{U} .
- ▶ What is $[\vec{x}]_{\mathcal{U}}$?
- ▶ That is, what are b_1 and b_2 in $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$.

Exercise

Find the coordinates of \vec{x} in the new basis:

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$

$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$

$$\vec{x} = (1/2, 1)^T$$

Change of Basis

- ▶ Suppose $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ are our new, **orthonormal** basis vectors.
- ▶ We know $\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}$
- ▶ We want to write $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$
- ▶ Solution

$$b_1 = \vec{x} \cdot \hat{u}^{(1)} \qquad b_2 = \vec{x} \cdot \hat{u}^{(2)}$$

Change of Basis Matrix

- ▶ Changing basis is a linear transformation

$$\vec{f}(\vec{x}) = (\vec{x} \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (\vec{x} \cdot \hat{u}^{(2)})\hat{u}^{(2)} = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \end{pmatrix}_{\mathcal{U}}$$

- ▶ We can represent it with a matrix

$$\begin{pmatrix} \uparrow & \uparrow \\ f(\hat{e}^{(1)}) & f(\hat{e}^{(2)}) \\ \downarrow & \downarrow \end{pmatrix}$$

Example

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$

$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$

$$f(\hat{e}^{(1)}) =$$

$$f(\hat{e}^{(2)}) =$$

$$A =$$

Observation

- ▶ The new basis vectors become the **rows** of the matrix.

Example

- Multiplying by this matrix gives the coordinate vector w.r.t. the new basis.

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$

$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$\vec{x} = (1/2, 1)^T$$

Change of Basis Matrix

- ▶ Let $\hat{u}^{(1)}, \dots, \hat{u}^{(d)}$ form an orthonormal basis \mathcal{U} .
- ▶ The matrix U whose **rows** are the new basis vectors is the **change of basis** matrix from the standard basis to \mathcal{U} :

$$U = \begin{pmatrix} \leftarrow \hat{u}^{(1)} \rightarrow \\ \leftarrow \hat{u}^{(2)} \rightarrow \\ \vdots \\ \leftarrow \hat{u}^{(d)} \rightarrow \end{pmatrix}$$

Change of Basis Matrix

- ▶ If U is the change of basis matrix, $[\vec{x}]_{\mathcal{U}} = U\vec{x}$
- ▶ To go *back* to the standard basis, use U^T :

$$\vec{x} = U^T[\vec{x}]_{\mathcal{U}}$$

Exercise

Let U be the change of basis matrix for \mathcal{U} .
What is $U^T U$?

Hint: What is $U^T(U\vec{x})$?

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Lecture 07 | Part 2

Diagonalization

Matrices of a Transformation

- ▶ Let $\vec{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a linear transformation
- ▶ The matrix representing \vec{f} wrt the **standard basis** is:

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \dots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

Matrices of a Transformation

- If we use a different basis $\mathcal{U} = \{\hat{u}^{(1)}, \dots, \hat{u}^{(d)}\}$, the matrix representing \vec{f} is:

$$A_{\mathcal{U}} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- If $\vec{y} = A\vec{x}$, then $[\vec{y}]_{\mathcal{U}} = A_{\mathcal{U}}[\vec{x}]_{\mathcal{U}}$

Diagonal Matrices

- ▶ Diagonal matrices are very nice / easy to work with.
- ▶ Suppose A is a matrix. Is there a basis \mathcal{U} where $A_{\mathcal{U}}$ is diagonal?
- ▶ Yes! *If* A is symmetric.

The Spectral Theorem¹

- **Theorem:** Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.

¹for symmetric matrices

Eigendecomposition

- ▶ If A is a symmetric matrix, we can pick d of its eigenvectors $\hat{u}^{(1)}, \dots, \hat{u}^{(d)}$ to form an orthonormal basis.
- ▶ Any vector \vec{x} can be written in terms of this **eigenbasis**.
- ▶ This is called its **eigendecomposition**:

$$\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)} + \dots + b_d \hat{u}^{(d)}$$

Matrix in the Eigenbasis

- ▶ **Claim:** the matrix of a linear transformation \vec{f} , written in a basis of its eigenvectors, is a **diagonal** matrix.
- ▶ The entries along the diagonal will be the **eigenvalues**.

Why?

$$A_{\mathcal{U}} = \begin{pmatrix} \begin{matrix} \uparrow \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} \\ \downarrow \end{matrix} & \begin{matrix} \uparrow \\ [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} \\ \downarrow \end{matrix} & \begin{matrix} \uparrow \\ \cdots \\ \downarrow \end{matrix} & \begin{matrix} \uparrow \\ [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow \end{matrix} \end{pmatrix}$$

- ▶ $\vec{f}(\hat{u}^{(1)}) = \lambda_1 \hat{u}^{(1)}$, so $[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} = (\lambda_1, 0, \dots, 0)^T$.
- ▶ $\vec{f}(\hat{u}^{(2)}) = \lambda_2 \hat{u}^{(2)}$, so $[\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} = (0, \lambda_2, \dots, 0)^T$.
- ▶ ...

Matrix Multiplication

- ▶ We have seen that matrix multiplication evaluates a linear transformation.
- ▶ In the standard basis:

$$\vec{f}(\vec{x}) = A\vec{x}$$

- ▶ In another basis:

$$[\vec{f}(\vec{x})]_{\mathcal{U}} = A_{\mathcal{U}}[\vec{x}]_{\mathcal{U}}$$

Diagonalization

- ▶ Another way to compute $\vec{f}(x)$, starting with \vec{x} in the standard basis:
 1. Change basis to the eigenbasis with U .
 2. Apply \vec{f} in the eigenbasis with the diagonal $A_{\mathcal{U}}$.
 3. Go *back* to the standard basis with U^T .
- ▶ That is, $A\vec{x} = U^T A_{\mathcal{U}} U \vec{x}$. It follows that $A = U^T A_{\mathcal{U}} U$.

Spectral Theorem (Again)

- ▶ **Theorem:** Let A be an $n \times n$ *symmetric* matrix. Then there exists an orthogonal matrix U and a diagonal matrix Λ such that $A = U^T \Lambda U$.
- ▶ The *rows* of U are the eigenvectors of A , and the entries of Λ are its eigenvalues.
- ▶ U is said to **diagonalize** A .

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Representation Learning

Lecture 07 | Part 3

Dimensionality Reduction

High Dimensional Data

- ▶ Data is often high dimensional (many features)
- ▶ Example: Netflix user
 - ▶ Number of movies watched
 - ▶ Number of movies saved
 - ▶ Total time watched
 - ▶ Number of logins
 - ▶ Days since signup
 - ▶ Average rating for comedy
 - ▶ Average rating for drama
 - ▶ ⋮

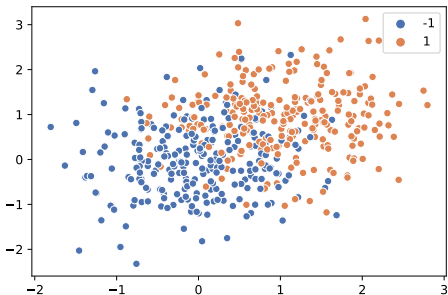
High Dimensional Data

- ▶ More features can give us more information
- ▶ But it can also cause problems
- ▶ **Today:** how do we reduce dimensionality without losing too much information?

More Features, More Problems

- ▶ Difficulties with high dimensional data:
 1. Requires more compute time / space
 2. Hard to visualize / explore
 3. The “curse of dimensionality”: it’s harder to learn

Experiment



- ▶ On this data, low 80% train/test accuracy
- ▶ Add 400 features of pure noise, re-train
- ▶ Now: 100% train accuracy, **58%** test accuracy
- ▶ **Overfitting!**

Task: Dimensionality Reduction

- ▶ We'd often like to **reduce** the dimensionality to improve performance, or to visualize.
- ▶ We will typically lose information
- ▶ Want to minimize the loss of useful information

Redundancy

- ▶ Two (or more) features may share the same information.
- ▶ Intuition: we may not need all of them.

Today

- ▶ Today we'll think about reducing dimensionality from \mathbb{R}^d to \mathbb{R}^1
- ▶ Next time we'll go from \mathbb{R}^d to $\mathbb{R}^{d'}$, with $d' \leq d$

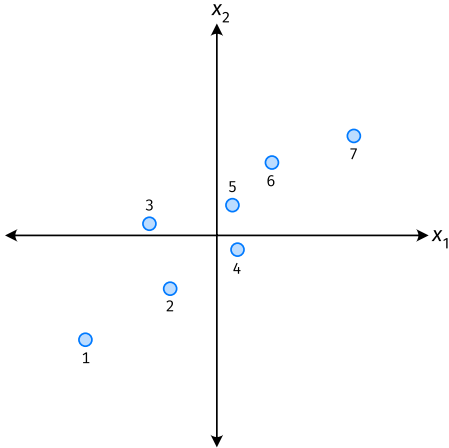
Today's Example

- ▶ Let's say we represent a phone with two features:
 - ▶ x_1 : screen width
 - ▶ x_2 : phone weight
- ▶ Both measure a phone's "size".
- ▶ Instead of representing a phone with both x_1 and x_2 , can we just use a single number, z ?
 - ▶ Reduce dimensionality from 2 to 1.

First Approach: Remove Features

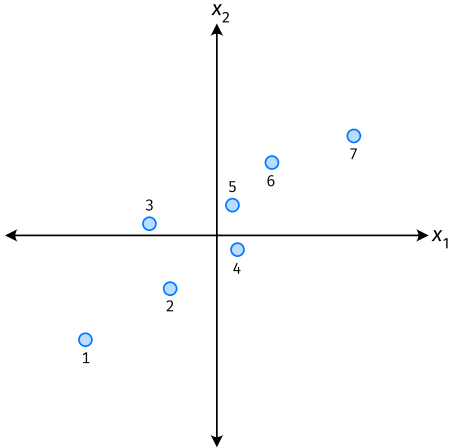
- ▶ Screen width and weight share information.
- ▶ **Idea:** keep one feature, remove the other.
- ▶ That is, set new feature $z = x_1$ (or $z = x_2$).

Removing Features



- Say we set $z^{(i)} = \vec{x}_1^{(i)}$ for each phone, i .
- Observe: $z^{(4)} > z^{(5)}$.
- Is phone 4 really “larger” than phone 5?

Removing Features



- Say we set $z^{(i)} = \vec{x}_2^{(i)}$ for each phone, i .
- Observe: $z^{(3)} > z^{(4)}$.
- Is phone 3 really “larger” than phone 4?

Better Approach: Mixtures of Features

- ▶ **Idea:** z should be a combination of x_1 and x_2 .
- ▶ One approach: linear combination.

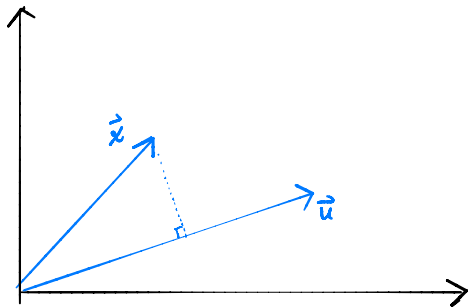
$$\begin{aligned} z &= u_1 x_1 + u_2 x_2 \\ &= \vec{u} \cdot \vec{x} \end{aligned}$$

- ▶ u_1, \dots, u_2 are the mixture coefficients; we can choose them.

Normalization

- ▶ Mixture coefficients generalize proportions.
- ▶ We could assume, e.g., $|u_1| + |u_2| = 1$.
- ▶ But it makes the math easier if we assume $u_1^2 + u_2^2 = 1$.
- ▶ Equivalently, if $\vec{u} = (u_1, u_2)^T$, assume $\|\vec{u}\| = 1$

Geometric Interpretation



- ▶ z measures how much of \vec{x} is in the direction of \vec{u}
- ▶ If $\vec{u} = (1, 0)^T$, then $z = x_1$
- ▶ If $\vec{u} = (0, 1)^T$, then $z = x_2$

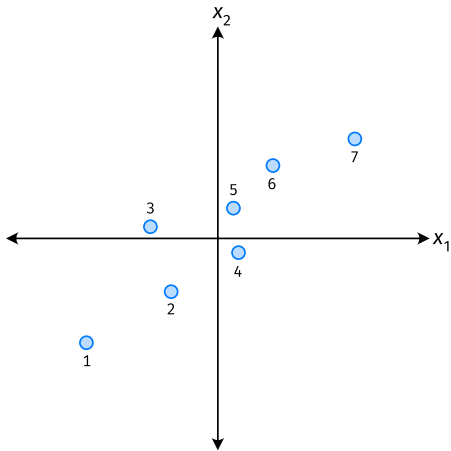
Choosing \vec{u}

- ▶ Suppose we have only two features:
 - ▶ x_1 : screen size
 - ▶ x_2 : phone thickness
- ▶ We'll create single new feature, z , from x_1 and x_2 .
 - ▶ Assume $z = u_1x_1 + u_2x_2 = \vec{x} \cdot \vec{u}$
 - ▶ Interpretation: z is a measure of a phone's size
- ▶ How should we choose $\vec{u} = (u_1, u_2)^T$?

Visualization

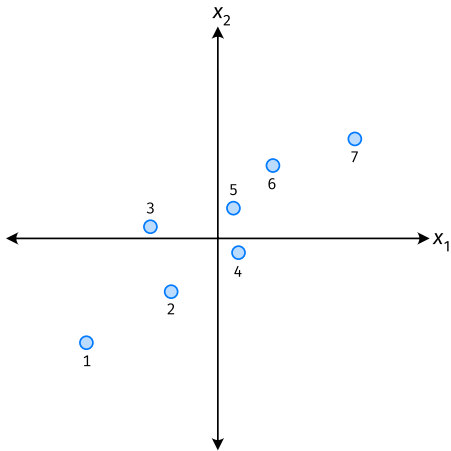
http://dsc140b.com/static/vis/pca-max_variance/

Example



- ▶ \vec{u} defines a direction
- ▶ $\vec{z}^{(i)} = \vec{x}^{(i)} \cdot \vec{u}$ measures position of \vec{x} along this direction

Example



- ▶ Phone “size” varies most along a diagonal direction.
- ▶ Along direction of “max variance”, phones are well-separated.
- ▶ **Idea:** \vec{u} should point in direction of “max variance”.

Our Algorithm (Informally)

- ▶ **Given:** data points $\vec{x}^{(1)}, \dots, \vec{x}^{(n)} \in \mathbb{R}^d$
- ▶ Pick \vec{u} to be the direction of “max variance”
- ▶ Create a new feature, z , for each point:

$$z^{(i)} = \vec{x}^{(i)} \cdot \vec{u}$$

PCA

- ▶ This algorithm is called **Principal Component Analysis**, or **PCA**.
- ▶ The direction of maximum variance is called the **principal component**.

Exercise

Suppose the direction of maximum variance in a data set is

$$\vec{u} = (1/\sqrt{2}, -1/\sqrt{2})^T$$

Let

- ▶ $\vec{x}^{(1)} = (3, -2)^T$
- ▶ $\vec{x}^{(2)} = (1, 4)^T$

What are $z^{(1)}$ and $z^{(2)}$?

Problem

- ▶ How do we compute the “direction of maximum variance”?