DSC 40A - Homework 07

Due: Friday, February 28, 2020

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope on Friday afternoon at 5:00 p.m.

Problem 1.

Let $0 < k < n$. Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Solution: Working from the definition, we have:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-1-(k-1))!}\\ &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}\end{aligned}$$

We want this to look like $n!/[k!(n-k)!]$. Our first step will be to get a $k!$ in the denominator. There is already a $k!$ in the denominator of the first term, but the second term has a $(k-1)!$. Note that $k! = k(k-1)!$. Therefore, $(k-1)! = k!/k$. Making this substitution in the second term:

$$= \frac{(n-1)!}{k!(n-1-k)!} + \frac{k(n-1)!}{k!(n-k)!}$$

Now we want to get $(n-k)!$ in the denominator. The second term already has $(n-k)!$, but the first term has $(n-k-1)!$. Noting that $(n-k)! = (n-k)(n-k-1)!$, we have that $(n-k-1)! = (n-k)!/(n-k)$. Making this substitution in the first term:

$$= \frac{(n-k)(n-1)!}{k!(n-k)!} + \frac{k(n-1)!}{k!(n-k)!}$$

The terms now have like denominators and can be added:

$$\begin{aligned}&= \frac{(n-k)(n-1)! + k(n-1)!}{k!(n-k)!}\\ &= \frac{(n-k+k)(n-1)!}{k!(n-k)!}\\ &= \frac{n(n-1)!}{k!(n-k)!}\\ &= \frac{n!}{k!(n-k)!}\\ &= \binom{n}{k}\end{aligned}$$

Problem 2.

At UC San Diego, incoming students are each randomly assigned to join one of the six colleges: Revelle, Muir, Marshall, Warren, Roosevelt, and Sixth. Suppose there are 24 incoming students, including Winona, Xanthippe, and Zelda. It cannot be assumed that all colleges will be assigned four students; since each student's assignment is random, some colleges may be assigned more students than others.

Please leave your answers as unsimplified expressions with factorials, exponents, permutations, combinations, etc.

a) All 24 incoming students get called up one at a time to be assigned to a college. How many possible orders of all 24 students have Winona as the first student called?

> **Solution:** 23!
>
> Since we know the first student is Winona, there are 23! ways to order the remaining students.

b) How many possible orders of all 24 students have Zelda, Xanthippe, and Winona (in any order) as the first three students called up to wear the sorting hat?

> **Solution:** $3! \cdot 21!$
>
> There are 3! ways to order Zelda, Xanthippe and Winona and 21! ways to order the remaining students.

c) How many ways are there to assign all 24 students to colleges such that Zelda, Xanthippe, and Winona all get assigned to Muir? Here, and in what follows, the order in which the students are assigned does not matter.

> **Solution:** 6^{21}
>
> The college of Zelda, Xanthippe and Winona is fixed and each remaining student has 6 possible colleges to be assigned to. Therefore, there are 6^{21} ways to assign students to colleges given the restrictions.

d) How many ways are there to assign all 24 students to colleges such that Zelda, Xanthippe, and Winona all get assigned to the same college?

> **Solution:** $6 \cdot 6^{21} = 6^{22}$
>
> We saw above that there are 6^{21} ways of assigning all 24 students so that Zelda, Xanthippe, and Winona are all placed in Muir.
>
> But there are also 6^{21} ways that Zelda, Xanthippe, and Winona are placed in Warren. An 6^{21} ways that they can be placed in Sixth. And so on. Hence there are in total $6 \cdot 6^{21} = 6^{22}$ ways.

e) How many ways are there to assign all 24 students to colleges such that Zelda, Xanthippe, and Winona are assigned to different colleges?

> **Solution:** $6 \cdot 5 \cdot 4 \cdot 6^{21}$
>
> Suppose the colleges assigned to Zelda, Xanthippe, and Winona are known. Then there are 6^{21} ways to assign the remaining students.
>
> There are 6 choices of college for Zelda, 5 for Xanthippe, and 4 for Winona, since each has their college assigned by sampling without replacement.
>
> Hence there are $6 \cdot 5 \cdot 4 \cdot 6^{21}$ ways.

Another approach is to first choose which three colleges can be assigned to Zelda, Xanthippe, and Winona. There are

$$\binom{6}{3} = \frac{6!}{3!3!}$$

ways of choosing these colleges. For each choice of three colleges, there are 3! different assignments of Zelda, Xanthippe, and Winona to these colleges. Hence there are

$$\frac{6!}{3!3!} \cdot 3! = \frac{6!}{3!} = 6 \cdot 5 \cdot 4$$

ways of choosing the colleges of Zelda, Xanthippe, and Winona, as we found above.

f) How many ways are there to assign all 24 students to colleges such that exactly 4 students get assigned to Warren?

Solution: $\binom{24}{4} \cdot 5^{20}$

First, choose 4 of the 24 students to be assigned to Warren. Then each of the remaining 20 students has 5 colleges they could be assigned to. Combined, this gives $\binom{24}{4} \cdot 5^{20}$.

g) How many ways are there to assign all 24 students to colleges if each of the six colleges has room for only four incoming students?

Solution:

$$\binom{24}{4} \cdot \binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}$$

First pick 4 students to be put in Revelle from the original 24, then pick 4 students from the remaining 20 to be put in Muir, then 4 students from the remaining 16 to be put in Marshall, then 4 from the remaining 12 to be put in Warren, then 4 from the remaining 8 to be put in Roosevelt, and, lastly, 4 students from the remaining 4 to be put in Sixth. Notice that $\binom{4}{4} = 1$, since there are only five students left and we have to pick all of them to be in Sixth, so we don't really have any decisions to make.

h) How many ways are there to assign all 24 students to colleges if nobody is assigned to the same colleges as the person called up just before them?

Solution: $6 \cdot 5^{23}$

The first person can be assigned to any of the 6 colleges but all successive people can only be assigned to 5 colleges since they cannot be in the same colleges as the person before them. This gives $6 \cdot 5^{23}$.

i) What is the probability that Winona, Xanthippe, and Zelda are all assigned to Muir?

Solution: As found above, there are 6^{21} ways to assign all 24 students to colleges such that Winona, Xanthippe, and Zelda are in Muir.

In total, there are 6^{24} ways of assigning students to colleges.

Hence the probability is

$$\frac{6^{21}}{6^{24}}.$$

We aren't supposed to simplify the answer, but its useful to point out that this is equal to $6^{-3} = 1/216$.

j) Suppose again that the students are called up one at a time. In how many possible orders of all 24 students does Zelda get called up some time before Xanthippe? Simplify your answer as much as possible, but let factorials remain unsimplified.

Solution: $\frac{24!}{2}$

This is a good opportunity to use a symmetry argument. Consider these two disjoint events: E_1 ="Zelda is called up some time before Xanthippe" and E_2 ="Xanthippe is called up some time before Zelda". One of these events has to happen, so $P(E_1 \text{ or } E_2) = 1$. Since these events are disjoint, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$. So $P(E_1) + P(E_2) = 1$. There is no reason that Xanthippe is more likely to be called up before Zelda and *vice versa*, so $P(E_1) = P(E_2)$. Hence $P(E_1) = P(E_2) = \frac{1}{2}$.

Since $P(E_1) = |E_1|/|\Omega|$, and $\Omega = 24!$, it follows that $|E_1| = 24!/2$. That is, there are 24! orderings of the students, and Zelda appears some time before Xanthippe in exactly half of them.

Here is another way of getting the same answer. Suppose Zelda is called up first. There are then 23 places where Xanthippe can be called, and 22! ways of arranging the other students. Thus there are in total $23 \cdot 22!$ ways in which Zelda is called first and Xanthippe is called some time after.

Now suppose Zelda is called second. There are now 22 places where Xanthippe can be called (if she is to be called after Zelda), and still 22! ways of arranging the other 22 students. In total, there are $22 \cdot 22!$ ways.

If Zelda is called third, there will be $21 \cdot 22!$ ways, and so on. In general, if Zelda is called kth, then there are $(24 - k)22!$ ways in which Xanthippe can appear after Zelda.

The total number of ways is the sum:

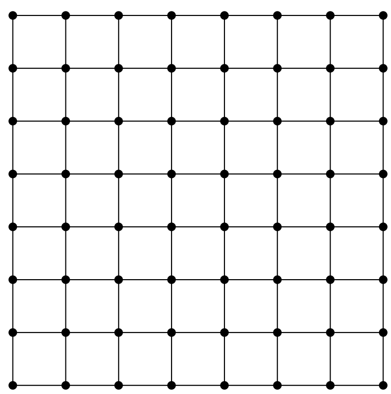
$$\begin{aligned}
\sum_{k=1}^{24} (24 - k)22! &= 22! \sum_{k=1}^{24} (24 - k) \\
&= 22! \left(\sum_{k=1}^{24} 24 - \sum_{k=1}^{24} k \right) \\
&= 22! \left(24^2 - \sum_{k=1}^{24} k \right)
\end{aligned}$$

Now you'd have to recognize the remaining sum as an arithmetic sum. It totals to $24(24+1)/2$. Therefore:

$$\begin{aligned}
&= 22!\left(24^2 - \frac{24(24+1)}{2}\right)\\
&= 22!\left(\frac{2\cdot 24^2}{2} - \frac{24(24+1)}{2}\right)\\
&= 22!\left(\frac{24^2-24}{2}\right)\\
&= 22!\cdot 24\cdot\left(\frac{24-1}{2}\right)\\
&= 22!\cdot 24\cdot\left(\frac{23}{2}\right)\\
&= \frac{24\cdot 23\cdot 22!}{2}\\
&= \frac{24!}{2}
\end{aligned}$$

Problem 3.

Four different points are chosen at random from a grid which is 7 units wide and 7 units tall:



What is the probability that the four points form a rectangle (with vertical and horizontal sides)?

For the purposes of this problem, the width of each of the rectangle's sides must be nonzero.

Solution: For this problem, we use as the sample space the number of sets of four points.

First we count the number of sets of four points that form a rectangle with vertical and horizontal sides. The easy way is to think of a rectangle as being determined by a pair of columns which form the vertical sides, and a pair of rows which form the horizontal sides. Therefore, the number of all possible rectangles can be expressed as $C(8,2) * C(8,2)$.

Since the total number of sets of four points is $C(64,4)$, the probability that such a choice of four points forms a rectangle is

$$\frac{C(8,2) * C(8,2)}{C(64,4)} = \frac{784}{635376} = \frac{7}{5673} = 0.0012.$$

Another way to do this problem is to think of a rectangle as being determined by a pair of opposite corners. There are 64 ways to pick one corner, since it can go at any point on the grid. Once this corner is determined, the opposite corner cannot be in the same row or column, so that eliminates $8+8-1=15$

grid points. Therefore, there are $64 - 15 = 49$ ways to pick the opposite corner, which determines the rectangle. This method counts the same rectangle in multiple ways, however. We are distinguishing between the first corner chosen and the second corner chosen, but we would get the same rectangle if we interchanged the first corner with the second corner. Furthermore, this method only uses one pair of opposite corners, and the other pair of opposite corners (chosen in either order) would also produce the same rectangle. So there are four ways to obtain each rectangle:

- choose top left corner then bottom right corner
- choose bottom right corner then top left corner
- choose top right corner then bottom left corner
- choose bottom left corner then top right corner

Therefore, the number of rectangles is $\dfrac{64 * 49}{4} = 784$, which is the same answer we had earlier. This means the probability of getting a rectangle is $\dfrac{784}{635376} = \dfrac{7}{5673} = 0.0012$ as before.

Problem 4.

One hundred Californians from four different cities (SD, LA, SF, and SJ) were asked whether they like or dislike Los Angeles. The results were distributed as follows:

City	# of People	# of Likes
SD	30	5
LA	40	21
SF	10	4
SJ	20	7

In what follows, suppose that the above data are stored in a table with one hundred rows (one row per person).

You may leave your answers unsimplified, but they should not contain $\sum$ or $\ldots$.

a) How many different ways are there to order the table's rows?

Solution: 100!

b) How many different ways are there to order the table's rows in which all rows from the same city are grouped together?

Solution: $4! \times 30! \times 40! \times 10! \times 20!$

There are 4! ways to order the groups; 30! ways of ordering the SD group, 40! ways of ordering the LA group, 10! ways of ordering the SF group, and 20! ways of ordering the SJ group.

c) How many different subsets of size 10 contain 5 people from LA and 5 people from SD?

Solution: $\binom{40}{5} \times \binom{30}{5}$

d) How many different subsets of size 10 contain exactly 2 people from SF and exactly 5 people from SJ?

Solution: $\binom{10}{2} \times \binom{20}{5} \times \binom{70}{3}$

There are $\binom{10}{2}$ ways of picking two people from SF and $\binom{20}{5}$ ways of picking 5 people from SJ. The remaining three people cannot be from either SF or SJ, and must be from LA or SD. There are therefore 70 people to choose from. Hence there are $\binom{70}{3}$ ways of picking the last three people.

e) What is the probability that a person selected at random is from LA and does not like LA?

Solution: 19/100

f) What is the probability that a person selected at random is from SD or likes LA?

Solution: $\frac{30}{100} + \frac{37}{100} - \frac{5}{100} = \frac{62}{100}$

Here we used the general formula for the union of events: $P(\text{from SD}) + P(\text{likes LA}) - P(\text{from SD and likes LA})$

g) What is the probability that a person selected at random is from southern California (LA or SD)?

Solution: $\frac{30+40}{100}$

h) What is the probability that, if a group of five people is randomly selected, exactly one will be from SD and exactly three will like LA?

Solution: There two situations: the person from SD likes LA or they do not like SD. We'll count the number of subsets which satisfy each situation separately.

In the first situation, there are $\binom{5}{1}$ ways of picking someone from SD who likes LA, $\binom{32}{2}$ ways of picking two people not from SD who like LA, and $\binom{70-32}{2} = \binom{38}{2}$ ways of picking two people not from SD who dislikes LA.

In the second situation, there are $\binom{25}{1}$ ways of picking someone from SD who dislikes LA, $\binom{32}{3}$ ways of picking three people not from SD who like LA, and $\binom{38}{1}$ ways of picking someone not from SD who dislikes LA.

There are $\binom{100}{5}$ ways of selecting 5 people. Hence the probability is:

$$\frac{\binom{5}{1} \times \binom{32}{2} \times \binom{38}{2} + \binom{25}{1} \times \binom{32}{3} \times \binom{38}{1}}{\binom{100}{5}}$$

Problem 5.

You arrive to campus in one of three ways: on the bus, in your car, or on foot.

The buses often run late and have long lines, so when you take the bus, you have a 50% chance of being late for class. When you drive, you sometimes hit traffic or have trouble finding a parking spot, so you have a 30% chance of being late for class. When you walk to campus, you only have a 5% of arriving late. One day, you arrive late for your midterm, and your professor wonders how you got to school that day.

a) If your professor assumes that you are equally likely to use all three modes of transportation, what will the professor calculate for the probability that you took the bus on the day of your midterm?

Solution: We will use Bayes' Theorem, which says that

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

Let B be the event that you took the bus, and A be the event that you were late, so that Bayes theorem becomes

$$P(\text{bus}|\text{late}) = \frac{P(\text{late}|\text{bus})P(\text{bus})}{P(\text{late})}.$$

We are given that

$$\begin{aligned} P(\text{late}|\text{bus}) &= 0.5 \\ P(\text{bus}) &= \frac{1}{3} \end{aligned}$$

To calculate $P(\text{late})$, notice that since you take exactly one of the three modes of transportation,

$$\begin{aligned} P(\text{late}) &= P(\text{late and bus}) + P(\text{late and drive}) + P(\text{late and walk}) \\ &= P(\text{late}|\text{bus})P(\text{bus}) + P(\text{late}|\text{drive})P(\text{drive}) + P(\text{late}|\text{walk})P(\text{walk}) \\ &= 0.5 * \frac{1}{3} + 0.3 * \frac{1}{3} + 0.05 * \frac{1}{3} \\ &= 0.2833 \end{aligned}$$

Plugging in the values above gives that

$$\begin{aligned} P(\text{bus}|\text{late}) &= \frac{P(\text{late}|\text{bus})P(\text{bus})}{P(\text{late})} \\ &= \frac{0.5 * \frac{1}{3}}{0.2833} \\ &= 0.5882. \end{aligned}$$

Thus, the professor would estimate that there is approximately a 59% chance that you took the bus.

b) If your professor happens to know that you take the bus 20% of the time, drive 20% of the time, and walk 60% of the time, what will the professor calculate for the probability that you took the bus on the day of your midterm?

Solution:

We use Bayes Theorem as in part (a), except now

$$\begin{aligned} P(\text{late}|\text{bus}) &= 0.5 \\ P(\text{bus}) &= 0.2 \\ P(\text{late}) &= P(\text{late and bus}) + P(\text{late and drive}) + P(\text{late and walk}) \\ &= P(\text{late}|\text{bus})P(\text{bus}) + P(\text{late}|\text{drive})P(\text{drive}) + P(\text{late}|\text{walk})P(\text{walk}) \\ &= 0.5 * 0.2 + 0.3 * 0.2 + 0.05 * 0.6 \\ &= 0.19 \end{aligned}$$

Plugging in the values above gives that

$$\begin{aligned}
P(\text{bus}|\text{late}) &= \frac{P(\text{late}|\text{bus})P(\text{bus})}{P(\text{late})} \\
&= \frac{0.5 * 0.2}{0.19} \\
&= 0.5263.
\end{aligned}$$

Thus, the professor would estimate that there is approximately a 53% chance that you took the bus.