# DSC 190

DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 1

## Today's Lecture

# Dynamic Programming

- We’ve seen that dynamic programming can lead to fast algorithms that find the optimal answer.
- Today, we’ll see one data science application: longest common substring.
- Used to match DNA sequences, fuzzy string comparison, etc.

# The Strategy

1. Backtracking solution.

2. A “nice” backtracking solution with overlapping subproblems.

3. Memoization.

# DSC 190

## DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 2

## Longest Common Subsequence

# Fuzzy String Matching

- Suppose you're doing a sentiment analysis of tweets.
- How do people feel about the University of California?
- Search for: `university of california`
- People can't spell: `uivesity of califrbia`
- How do we recognize the match?

# DNA String Matching

- Suppose you're analyzing a genome.
- DNA is a sequence of `G,A,T,C`.
- Mutations cause same gene to have slight differences.
- Person 1: `GATTACAGATTACA`
- Person 2: `GATCACAGTTGCA`

# Measuring Differences

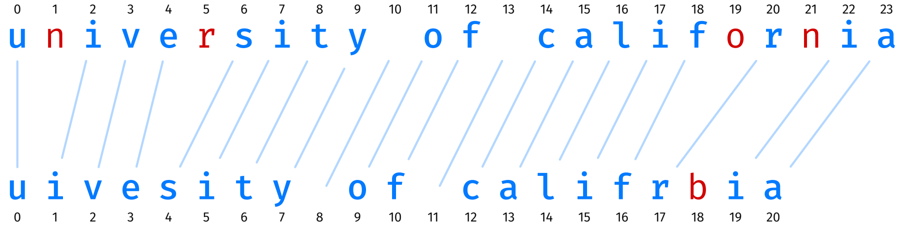
- Given two strings of (possibly) different lengths.
- Measure how similar they are.
- One approach: **longest common subsequences**.

# Common Subsequences

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| u | n | i | v | e | r | s | i | t | y | | o | f | | c | a | l | i | f | o | r | n | i | a |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| u | i | v | e | s | i | t | y | | o | f | | c | a | l | i | f | r | b | i | a |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

# Common Subsequences



# Longest Common Subsequences

- We will measure similarity by finding length of the **longest common subsequence** (LCS).
- Now: let's define the LCS..

# Subsequences

```
s a n d i e g o
s a n d i e g o   →   igo
s a n d i e g o   →   sio
s a n d i e g o   →   sadego
s a n d i e g o   →   sandiego
```

# Not Subsequences

```
s a n d i e g o
s a n d i e g o    ↛    sea
s a n d i e g o    ↛    sooo
```

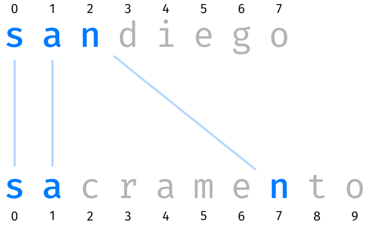
# Subsequences

- A **subsequence** of a string `s` of length $n$ is determined by a strictly monotonically increasing sequence of indices with values in $\{0, 1, \ldots, n-1\}$.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | 0 | 1 | 3 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| s | a | n | d | i | e | g | o | → | s | a | d | e | g | o |

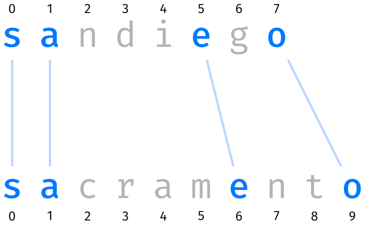
# Common Subsequences

- Given two strings, a **common subsequence** is subsequence that appears in both.



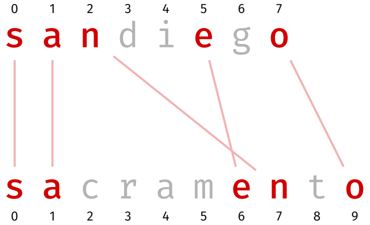
# Common Subsequences

- Given two strings, a **common subsequence** is subsequence that appears in both.



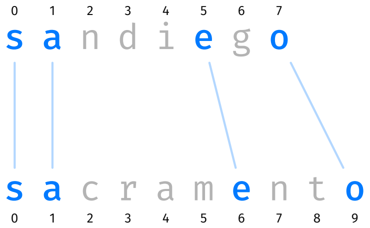
# Not Common Subsequences

- The lines cannot overlap.



# Longest Common Subsequences

- A **longest common subsequence** (LCS) between two strings is a common subsequence that has the greatest length out of all common subsequences.



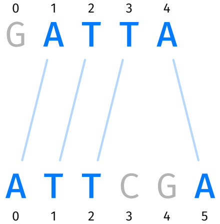
### Main Idea

The longer the LCS, the more similar the two strings.

# Common Subsequences, Formally

- Our backtracking solution will build a common subsequence piece by piece.
- How can we represent the idea of “lines between letters” more formally?

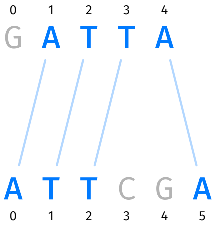
# Matching




| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

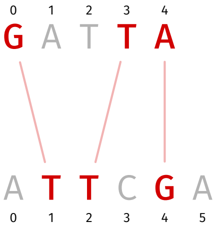
# Matching

- A **matching** between strings $a$ and $b$ is a set of $(i, j)$ pairs.
- Each $(i, j)$ pair is interpreted as "`a[i]` is paired with `b[j]`".
- Example: $\{(1,0),(2,1),(3,2),(4,5)\}$



# Invalid Matchings

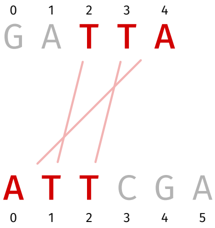
- Not all matchings represent common subsequences!
- Example: $\{(0,1),(3,2),(4,4)\}$:



# Invalid Matchings

- Not all matchings represent common subsequences!

- Example: $\{(4, 0), (2, 1), (3, 2)\}$:

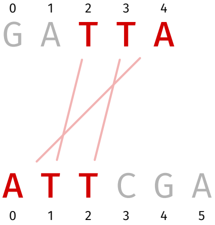


# Valid Matchings

- We'll say a matching $M$ is **valid** if:
  - `a[i] == b[j]` for every pair $(i, j)$; and
  - there are no "crossed lines"

# "Crossed Lines"

- Suppose $(i, j)$ and $(i', j')$ are in the matching.
- "Crossed lines" occur when either:
  - $i < i'$ but $j > j'$; or
  - $i > i'$ but $j < j'$.

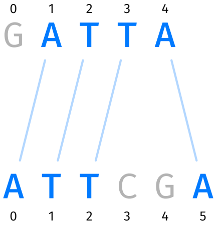


# Valid Matchings

- We'll say a matching $M$ is **valid** if:
  - `a[i] == b[j]` for every pair $(i, j)$; and
  - there are no "crossed lines". that is, for every choice of distinct pairs $(i, j), (i', j') \in M$:

$$i < i' \text{ and } j < j' \qquad \text{or} \qquad i > i' \text{ and } j > j'$$

- Example: $\{(1,0),(2,1),(3,2),(4,5)\}$



# DSC 190

## Data Structures & Algorithms

Lecture 12 | Part 3

## Step 01: Backtracking

# Road to Dynamic Programming

- We’ll follow same road to a DP solution as last time.
- **Step 01: Backtracking solution.**
- Step 02: A “nice” backtracking solution with overlapping subproblems.
- Step 03: Memoization.

# Backtracking

- We'll build up a matching, one pair at a time.
- Choose an arbitrary pair, `(i, j)`.
  - Recursively see what happens if we **do** include `(i, j)`.
  - Recursively see what happens if we **don't** include `(i, j)`.
- This will try **all valid matchings**, keep the best.

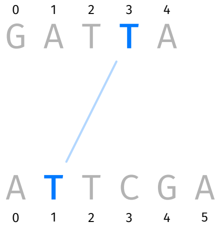
# Backtracking

```
def lcs_bt(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.arbitrary_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    best_with = ...

    # best without
    best_without = ...

    return max(best_with, best_without)
```

# Recursive Subproblems

- What is BEST(`a, b, pairs`) if we assume that `(i, j)` **is** in matching?
- If `a[i] != a[j]`:
  - Your current common substring is **invalid**. Length is zero.
  - Don't build matching further.
- If `a[i] == a[j]`:
  - Your current common substring has length one.
  - Pairs remaining to choose from: those **compatible** with $(i, j)$.
  - You find yourself in a similar situation as before.
  - Answer: 1 + BEST(`activities.compatible_with(x)`)

`pairs.compatible_with(x)`




| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

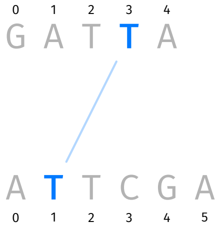
# Backtracking

```python
def lcs_bt(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.arbitrary_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    if a[i] == b[j]:
        best_with = 1 + lcs_bt(a, b, pairs.compatible_with(i, j))
    else:
        best_with = 0

    # best without
    best_without = ...

    return max(best_with, best_without)
```

# Recursive Subproblems

- What is BEST(`a, b, pairs`) if we assume that `(i, j)` **is not** in matching?
- Imagine not choosing `x`.
  - Your current common substring is empty.
  - Activities left to choose from: all except `(i, j)`.
- You find yourself in a similar situation as before.
- Answer: BEST(a, b, `pairs.without(i, j)`)

# `pairs.without(x)`




| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# Backtracking

```python
def lcs_bt(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.arbitrary_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    # assume (i, j) is in the LCS, but only if a[i] == b[j]
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt(a, b, pairs.compatible_with(i, j))

    # best without
    best_without = lcs_bt(a, b, pairs.without(i, j))

    return max(best_with, best_without)
```

# Backtracking

- This will try all **valid** matchings.
- Guaranteed to find optimal answer.
- But takes exponential time in worst case.

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## Data Structures & Algorithms

Lecture 12 | Part 4

## Step 02: A “Nicer” Backtracking Solution

# Arbitrary Sets

- In previous backtracking solution, subproblems are arbitrary sets of pairs.
- Rarely see the same subproblem twice.
- This is not good for memoization!

| | | | | |
|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) |

# Nicer Subproblems

- In backtracking, we are building a solution piece-by-piece.
- In last lecture, we saw that a careful choice of next piece led to nice subproblems.
- Let's try choosing the *last* letters from each string as the next piece of the matching.

# Last Letters

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| G | A | T | T | A |

| | | | | | |
|---|---|---|---|---|---|
| A | T | T | C | G | A |
| 0 | 1 | 2 | 3 | 4 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# Nicer Backtracking

```
def lcs_bt_nice(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.last_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt_nice(a, b, pairs.compatible_with(i, j))

    # best without
    best_without = lcs_bt_nice(a, b, pairs.without(i, j))

    return max(best_with, best_without)
```

# Subproblems

- There are two subproblems: LCS using `pairs.compatible_with(i, j)` and LCS using `pairs.without(i, j)`
- Are they “nicer”?

# `pairs.compatible_with(i, j)`

| | 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|
| | G | A | T | T | A | |

| | | | | | |
|---|---|---|---|---|---|
| A | T | T | C | G | A |
| 0 | 1 | 2 | 3 | 4 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
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| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# Nicer Subproblems

- By taking $(i, j)$ as bottom-right pair, `pairs.compatible_with(i, j)` is again rectangular.
- Easily described by its bottom-right pair, $(i - 1, j - 1)$!
- Instead of keeping set of `pairs`, just need to pass in $i$ and $j$ of last element.

```python
def lcs_bt_nice_2(a, b, i, j):
    """Solve LCS problem for a[:i], b[:j]."""
    if i < 0 or j < 0:
        return 0

    # best with
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt_nice_2(a, b, i-1, j-1)

    # best without
    best_without = ...

    return max(best_with, best_without)
```

# `pairs.without(i, j)`

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| | G | A | T | T | A |

| | A | T | T | C | G | A |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# Problem

- `pairs.without(i, j)` is **not** rectangular.
- Cannot be described by a single pair.
- But there's a fix.

# Observation

- A common substring cannot have pairs both in the last row and the last column. **Crossing lines!**

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| G | A | T | T | A |

| | | | | | |
|---|---|---|---|---|---|
| A | T | T | C | G | A |
| 0 | 1 | 2 | 3 | 4 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
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| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# Consequence

- BEST(`pairs.without(i, j)`) = max {BEST(`pairs.without_row(i)`), BEST(`pairs.without_col(j)`)}

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| 0 | 1 | 2 | 3 | 4 | | | | | | | | |
| G | A | T | T | A | | | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| | | | | | | | (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| A | T | T | C | G | A | | (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| 0 | 1 | 2 | 3 | 4 | 5 | | | | | | | |
| | | | | | | | (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# Observation

- `pairs.without_row(i)` represented by subprob. $(i - 1, j)$
- `pairs.without_col(j)` represented by subprob. $(i, j - 1)$

| 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| G | A | T | T | A | |

| A | T | T | C | G | A |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) |
| (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) |
| (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) |
| (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) |
| (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) |

# “Nice” Backtracking

```python
def lcs_bt_nice_2(a, b, i, j):
    """Solve LCS problem for a[:i], b[:j]."""
    if i < 0 or j < 0:
        return 0

    # best with
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt_nice_2(a, b, i-1, j-1)

    # best without
    best_without = max(
            lcs_bt_nice_2(a, b, i-1, j),
            lcs_bt_nice_2(a, b, i, j-1)
            )

    return max(best_with, best_without)
```

# One More Observation

- This is fine, but we can do a little better.
- If `a[i] == b[j]`, we can assume $(i, j)$ is in matching – don't need to consider otherwise![1]

0 1 2 3 4
G A T T A

A T T C G A
0 1 2 3 4 5

[1]This is true we chose last pair; not true if choice was arbitrary.

# “Nicer” Backtracking

```python
def lcs_bt_nice_2(a, b, i, j):
    """Solve LCS problem for a[:i], b[:j]."""
    if i < 0 or j < 0:
        return 0

    # best with
    if a[i] == b[j]:
        # best with (i, j)
        return 1 + lcs_bt_nice_2(a, b, i-1, j-1)
    else:
        # best without (i, j)
        return max(
                lcs_bt_nice_2(a, b, i-1, j),
                lcs_bt_nice_2(a, b, i, j-1)
                )
```

# Overlapping Subproblems

- Suppose $a$ and $b$ are of length $m$ and $n$.
- There are $mn$ possible subproblems.
- Backtracking tree has exponentially-many nodes.
- We will see many subproblems over and over again!

# DSC 190

## Data Structures & Algorithms

Lecture 12 | Part 5

## Step 03: Memoization

# Backtracking

- The backtracking solutions are slow.
- `a = 'CATCATCATCATCATGAAAAAAAA'`
- `b = 'GATTACAGATTACAGATTACA'`
- “Nice” backtracking solution: 8 seconds.
- Memoized solution: 100 microseconds.

```python
def lcs_dp(a, b, i=None, j=None, cache=None):
    """Solve LCS problem for a[:i], b[:j]."""
    if i is None:
        i = len(a) - 1

    if j is None:
        j = len(b) - 1

    if cache is None:
        cache = {}

    if i < 0 or j < 0:
        return 0

    if (i,j) in cache:
        return cache[(i, j)]

    # best with
    if a[i] == b[j]:
        # best with (i, j)
        best = 1 + lcs_dp(a, b, i-1, j-1, cache)
    else:
        # best without (i, j)
        best = max(
                lcs_dp(a, b, i-1, j, cache),
                lcs_dp(a, b, i, j-1, cache)
                )

    cache[(i, j)] = best
    return best
```

# Top-Down vs. Bottom-Up

- This is the top-down dynamic programming solution.
- It takes time $\Theta(mn)$, where $m$ and $n$ are the string lengths.
- To find a bottom-up iterative solution, start with the easiest subproblem.
- What is it?