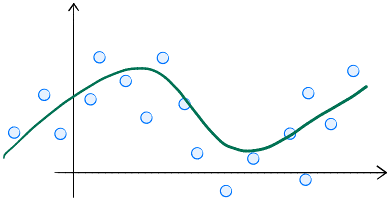


DSC 40A

Lecture 10

Least Squares Regression, pt. V

Last Time

- How do we make predictions using multiple features?
- Assume a linear prediction rule:

$$\begin{aligned} H(x_1, \dots, x_d) &= w_0 + w_1 x_1 + w_2 x_2 + \dots + w_d x_d \\ &= \text{Aug}(\vec{x}) \cdot \vec{w} \end{aligned}$$

- We found the normal equations:

$$X^T X \vec{w} = X^T \vec{y}$$

- Solving the normal equations for $\vec{w}$ gives the best-fitting prediction rule.

Today

- Interpreting the results.
- How do we fit prediction rules like $H(x) = w_2 x^2 + w_1 x + w_0$?
- Least squares **classification**.

Interpreting $\vec{w}$

- With d features, $\vec{w}$ has $d + 1$ entries.
- w_0 is the **bias**.
- $w_1, \ldots, w_d$ each give the **weight** of a feature.

$$H(\vec{x}) = w_0 + w_1 x_1 + \ldots + w_d x_d$$

- Sign of w_i tells us about relationship between ith feature and outcome.

Example: Predicting Sales

- For each of 26 stores, we have:
  - net sales,
  - size (sq ft),
  - inventory,
  - advertising expenditure,
  - district size,
  - number of competing stores.
- Goal: predict net sales given size, inventory, etc.
- To begin:

$$H(\text{size}, \text{competitors}) = w_0 + w_1 \times \text{size} + w_2 \times \text{competitors}$$

Discussion Question

What will be the sign of w_1 and w_2?

48 A) $w_1 = +, \quad w_2 = -$

14 B) $w_1 = +, \quad w_2 = +$

C) $w_1 = -, \quad w_2 = -$

D) $w_1 = -, \quad w_2 = +$

$$H(\text{size}, \text{competitors}) = w_0 + w_1 \times \text{size} + w_2 \times \text{competitors}$$

(DEMO)

Discussion Question

Which has the greatest effect on the outcome?

A) size:	$w_1 = 16.20$
B) inventory:	$w_2 = 0.17$
C) advertising:	$w_3 = 11.53$
D) district size:	$w_4 = 13.58$
E) competing stores:	$w_5 = -5.31$

Which features are most “important”?

- **Not necessarily** the feature with largest weight.
- Features are measured in different units, scales.
- We should **standardize** each feature.

Standard Units

- To standardize (z-score) a feature, subtract mean, divide by standard deviation.

- Example: 10, 20, -30, 5, 15
  - Mean: 4
  - Standard Dev: $\sqrt{\frac{1}{5}\sum(x_i - \bar{x})^2} \approx 17.7$
  - Standardized:

$$\frac{10-4}{17.7} = 0.34, \quad \frac{20-4}{17.7} = 0.90, \quad \frac{-30-4}{17.7} = -1.92,$$

$$\frac{5-4}{17.7} = 0.06, \quad \frac{15-4}{17.7} = 0.62$$

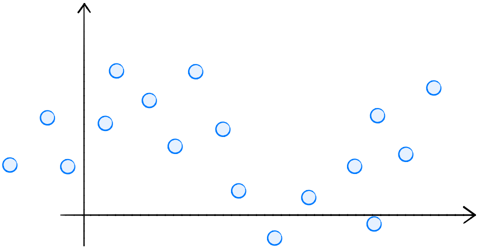
Standard Units

- Standardize each feature (store size, inventory, etc.) separately.
- No need to standardize outcome (net sales).
- Solve normal equations. The resulting $w_0, w_1, \ldots, w_d$ are called the **standardized regression coefficients**.
- They can be directly compared to one another.

(DEMO)

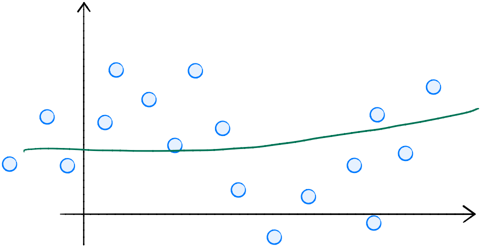
Fitting Non-Linear Patterns

- Fit a 4th-order polynomial to the data:



- We know how to fit rules of the form $H(x) = w_1 x^4 + w_0$.
  - Define $z_i = x_i^4$.
  - Use $w_1 = \frac{\sum(z_i - \bar{z})(y_i - \bar{y})}{\sum(z_i - \bar{z})^2}$ and $w_0 = \bar{y} - w_1 \bar{z}$.

The Result



- The rule $H(x) = w_1 x^4 + w_0$ **underfits** the data.
- We need a more complicated rule:

$$H(x) = w_4 x^4 + w_3 x^3 + w_2 x^2 + w_1 x + w_0$$

The Trick

- Treat x, x^2, x^3, x^4 as different features.
- Create design matrix:

$$X = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 \end{pmatrix}$$

- Solve $X^T X \vec{w} = X^T \vec{w}$ for $\vec{w}$, as usual.
- Works for more than just polynomials.

(DEMO)

Polynomial Regression

- More complicated patterns can be fit with higher-order polynomials.

- If there are n points, a $n + 1$ degree polynomial can fit them exactly.

- But for high-order polynomials, it becomes **very hard** to solve the normal equations (numerical accuracy).

Polynomial Regression with Multiple Features

- Suppose we want to fit a rule of the form:

$$
\begin{aligned}
H(\text{size}, \text{competitors}) &= w_0 + w_1 \text{size} + w_2 \text{size}^2 \\
&\quad + w_3 \text{competitors} + w_4 \text{competitors}^2 \\
&= w_0 + w_1 s + w_2 s^2 + w_3 c + w_4 c^2
\end{aligned}
$$

- Make design matrix:

$$
X = \begin{pmatrix} 1 & s_1 & s_1^2 & c_1 & c_1^2 \\ 1 & s_2 & s_2^2 & c_2 & c_2^2 \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & s_n & s_n^2 & c_n & c_n^2 \end{pmatrix}
$$

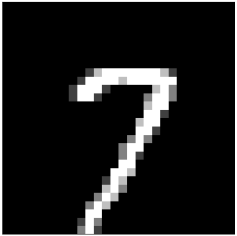
Where c_i and s_i are the competitors and size of the ith store.

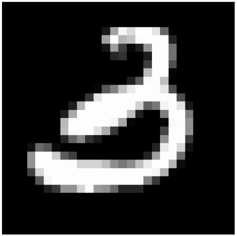
Regression vs. Classification

- **Regression**: predict a number
  - Examples: salary, store sales, height of child
- **Classification**: predict a *class*, or *group label*.
  - is this person at high risk of disease (yes/no)?
  - what type of tree is in this (pine, elm, oak, etc.)?

Binary Classification

- There are two possible classes.
- Example: handwritten digits. Is image a 7, or a 3?

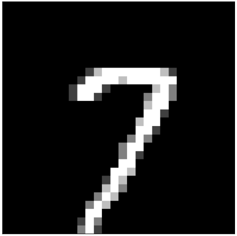




- Data: images $\vec{x}^{(i)}$, **labels** $y_i = 1$ if a seven, $y_1 = 0$ if a three.

Images as Feature Vectors

- We can pack an image into a feature vector.
- Each feature is the intensity of a particular pixel.
- Example: a 28×28 image has 784 pixels, becomes a vector in $\mathbb{R}^{784}$.



Decision Rule

- We want a rule $H(\vec{x})$ that takes in images and outputs:
  - 1 if image is a seven
  - 0 if image is a three

- We'll use a linear decision rule:

$$\begin{aligned} H(\text{image}) &= w_0 + w_1 \times (\text{pixel } 1) + \ldots + w_{784} \times (\text{pixel } 784) \\ &= \text{Aug}(\vec{x}) \cdot \vec{w} \end{aligned}$$

- Minimize MSE, same solutions:

$$R_{\text{sq}}(\vec{w}) = \frac{1}{n} \sum_{i=1}^{n} \left(\text{Aug}(\vec{x}^{(i)}) \cdot \vec{w} - \vec{y}_i \right)^2 \qquad X^T X \vec{w} = X^T \vec{y}$$

Least Squares Classification

- Our prediction $H(\vec{x})$ will not be 0 or 1 exactly.

- If $H(\vec{x}) > \frac{1}{2}$, we'll claim it is a 1; else, a 0.

(DEMO)

Least Squares Classification

- Square loss is good for regression: want $H(\vec{x})$ close to right answer.
- Not great for classification.
- If real class is 1, and $H(x) = 10$, great!
- If real class is 1, and $H(x) = -1$, not great.
- Better loss functions: hinge loss, logistic loss, etc.