Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
- **Rank of a Matrix**
- **Canonical Form of a Matrix (Echelon Form)**
- Normal Form of a Matrix
- System of Linear Equations
- Orthogonal Transformations
- Eigen Values and Eigen Vectors
- Diagonalization of Matrices
- Cayley Hamilton Theorem
- Applications to problems in Engineering

Rank of a Matrix

A matrix 'A' is said to be of **rank r,** *if there is*

(i) at least one non zero minor of the order 'r' and

(ii) every minor of order $(r+1)$ is equal to zero.

i.e. The rank of a matrix 'A' is the maximum order of its non-vanishing minor. Rank of a matrix A is denoted by $\rho(A) = r$.

Properties of a rank of matrix:

1. *If a matrix A has a non-zero minor of order r, then* $\rho(A) \geq r$.
2. *If a matrix A has all the minors of order* $(r+1)$ *is equal to zero, then* $\rho(A) \leq r$.
3. *If a matrix A of order* $m \times n$*, then the* $\rho(A) \leq$ *minimum of m and n.*
4. *Elementary transformations of a matrix do not alter the rank of a matrix.*
5. $\rho(A) = \rho(A') = \rho(A^{-1})$.

Example1: *Find the rank of the following matrices:*

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (iii) $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 8 & 9 \end{bmatrix}$

Solution: (i) We have to find the rank of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Rank of the matrix is the order of highest order non-vanishing minor of it.

Here the order of the given matrix is 2×3. So, the rank of the matrix will not be more than 2. It may be 2 if at least one minor of order 2×2 is not equal to zero.

Now consider all the minor of order 2×2 of A, we have

$$|A_1| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0; \qquad |A_2| = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 3 = 0$$

$$|A_3| = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 6 - 6 = 0$$

Since all the minors of order 2×2 of the matrix A are zero. So the rank of the matrix will be less than 2. Again 1, 2, 3 are nonzero elements of A.

Hence the rank of the given matrix is 1. i.e. $\boxed{\rho(A) = 1}$.

Solution: (ii) We have to find the rank of the matrix A.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Rank of the matrix is the order of highest order non-vanishing minor of it.

Here the order of the given matrix is 3×3. So, the rank of the matrix may be 3, if the determinant of B is not equal to zero i.e. $|B| \neq 0$.

Now consider

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(5 \times 9 - 8 \times 6) - 2(4 \times 9 - 7 \times 6) + 3(4 \times 8 - 7 \times 5)$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= 1 \times (-3) - 2 \times (-6) + 3 \times (-3)$$

$$= -3 + 12 - 9 = 0$$

Since $|B| = 0$. So, the rank of the matrix is less than 3.

Now consider $|B_1| = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

Hence the rank of the given matrix is 2. i.e. $\boxed{\rho(B) = 2}$.

Solution: (iii) We have to find the rank of the matrix A.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 8 & 9 \end{bmatrix}$$

Rank of the matrix is the order of highest order non-vanishing minor of it.

Here the order of the given matrix is 3×3. So, the rank of the matrix may be 3, if the determinant of C is not equal to zero i.e. $|C| \neq 0$.

Now consider

$$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 8 & 9 \end{vmatrix} = 1(5 \times 9 - 8 \times 2) - 2(4 \times 9 - 5 \times 2) + 3(4 \times 8 - 5 \times 5)$$

$$= 1(45 - 16) - 2(36 - 10) + 3(32 - 25)$$

$$= 1 \times (29) - 2 \times (16) + 3 \times (7)$$

$$= 29 - 32 + 21 = 18 \neq 0$$

Since $|C| = 18 \neq 0$. So, the rank of the matrix is 3.

$$\boxed{\rho(C) = 3}.$$

Elementary transformations of mAtrix

1. The interchange of i^{th} and j^{th} rows denoted by R_{ij}
2. The multiplication of each element of i^{th} row by a non zero scalar k is denoted by kR_i
3. Multiplication of every element of j^{th} row by scalar k and adding to the corresponding element of i^{th} row is denoted by $R_i + kR_j$

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Canonical or Echelon Form of a Matrix

A matrix A is said to be in echelon form if

(i) All zero rows, if any, are at the bottom of the matrix.

(ii) Each leading non-zero entry in a row is to the right of the leading non-zero entry in the preceding row.

Note: The rank of a matrix in echelon form is equal to the number of non zero rows of the matrix.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad Rank(A) = 3$$

Row Echelon Form : *An Echelon form of a matrix A is called* ***Row Echelon Form*** *if all the principal diagonal elements are one (1).*

Example: *Following matrices are in the Row Echelon form or canonical form*

$$(i)\;\; A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad (ii) \quad B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (iii) \quad C = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row Reduced Echelon Form : *An Echelon form of a matrix A is called* ***Row Reduced Echelon Form*** *if all the principal diagonal elements are one (1) and all other elements are zero.*

Example: *Following matrices are in the Row Reduced Echelon form*

$$(i)\;\; A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (ii) \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (iii) \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Properties of an Echelon or Canonical Form

(i) If a matrix is reduced to its Echelon form, then the rank of a matrix is equal to the number of nonzero rows present in its Echelon form.

(ii) If a matrix is square, then the Echelon form of the matrix is same as Upper Triangular matrix.

(iii) Echelon form of a matrix is called Gauss Elimination Method which is used to solve the system of linear equations.

(iv) Row Echelon form is used to check whether the given vectors are linearly independent or linearly dependent.

(v) Row reduced Echelon form is used to find the rank and nullity of a matrix.

Example2: *Reduce the following matrices to its Echelon Form and hence find the rank.*

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ *(iii)* $B = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution: (i) We have to reduce the following matrix A to its Echelon form and also the rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon Form of the given matrix.

Hence the rank of the given matrix is equal to nonzero rows. i.e. $\boxed{\rho(A) = 2}$.

Solution: (ii)

We have to reduce the following matrix A to its Echelon form and also the rank.

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Apply R_{12}, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Apply $R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon Form of the given matrix.

Hence the rank of the given matrix is equal to nonzero rows. i.e. $\boxed{\rho(A) = 2}$.

Home Assignment

1. Find the rank of the following matrices:

(a) $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{bmatrix}$ $[Ans\!: \rho(A) = 1]$

(b) $\begin{bmatrix}1 & 3 & 4 & 3\\3 & 9 & 12 & 3\\1 & 3 & 4 & 1\end{bmatrix}$ $[Ans\!: \rho(A) = 2]$

(c) $\begin{bmatrix}1 & 2 & 3 & 0\\2 & 4 & 3 & 2\\3 & 2 & 1 & 3\\6 & 8 & 7 & 5\end{bmatrix}$ $[Ans\!: \rho(A) = 3]$

(d) $\begin{bmatrix}5 & 6 & 7 & 8\\6 & 7 & 8 & 9\\11 & 12 & 13 & 14\\16 & 17 & 18 & 19\end{bmatrix}$ $[Ans\!: \rho(A) = 2]$

(e) $\begin{bmatrix}1^2 & 2^2 & 3^2 & 4^2\\2^2 & 3^2 & 4^2 & 5^2\\3^2 & 4^2 & 5^2 & 6^2\\4^2 & 5^2 & 6^2 & 7^2\end{bmatrix}$ $[Ans\!: \rho(A) = 3]$

2. Reduce the following matrices to its Echelon form and hence find the rank.

(a) $\begin{bmatrix}1 & 4 & 5\\2 & 6 & 8\\3 & 7 & 22\end{bmatrix}$ $[Ans\!: \rho(A) = 3]$

(b) $\begin{bmatrix}1 & 3 & 4 & 3\\3 & 9 & 12 & 3\\1 & 3 & 4 & 1\end{bmatrix}$ $[Ans\!: \rho(A) = 2]$

3. *Determine the values of p such that the*

$rank\ of\ A = \begin{bmatrix}1 & 1 & -1 & 0\\4 & 4 & -3 & 1\\p & 2 & 2 & 2\\9 & 9 & p & 3\end{bmatrix}$ is 3.

$[Ans\!: p = -6]$

Questions?

Thanks