Calculating a Feasible Parabolic Hanging Midpoint

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Consider the following problem: Given a string of length l, let us approximate the angle that l's midpoint will create with its endpoints A and B, $(A, B \in \mathbb{R}^2)$. This l may further be defined as a scalar input k, multiplied by the distance of 2 original points. As such, we may use l throughout this problem as a stand in for:

$l = k \cdot ||M - N||$

Where M and N are points in $\mathbb{R}^2$.

To approximate the position of the point along their midpoint, let us think of the hanging function as a parabola. While this is not physically accurate, it will be easier to calcuate and can rougly approximate the physically correct catenary. We will also assume the following:

- The string of length l is perfectly inelastic
- The positions of A and B will be close enough such that $||A - B|| \leq k \cdot ||M - N||$. If this is not the case, we may consider the angle a perfect π.

1 Possible Parabolas

First, let us define the set of all parabolas that will contain A and B. Let us define our parabola $f(x) = ax^2 + bx + c$.

$$f(A_x) = A_y = aA_x^2 + bA_x + c$$
$$f(B_x) = B_y = aB_x^2 + bB_x + c$$

$$\begin{aligned} A_y - B_y &= a(A_x^2 - B_x^2) + b(A_x - B_x) \\ &= a(A_x + B_x)(A_x - B_x) + b(A_x - B_x) \\ \frac{A_y - B_y}{A_x - B_x} &= a(A_x + B_x) + b \end{aligned}$$

$$b = \frac{A_y - B_y}{A_x - B_x} - a(A_x + B_x)$$

$$\begin{aligned} A_y &= aA_x^2 + \left(\frac{A_y - B_y}{A_x - B_x} - a(A_x + B_x)\right) A_x + c \\ c &= A_y - aA_x^2 + \left(\frac{A_y - B_y}{A_x - B_x} - a(A_x + B_x)\right) A_x \end{aligned}$$

$$f(x) = ax^2 + \left(\frac{A_y - B_y}{A_x - B_x} - a(A_x + B_x)\right) x + A_y - aA_x^2 + \left(\frac{A_y - B_y}{A_x - B_x} - a(A_x + B_x)\right) A_x$$

we now have an explicit equation for our parabola given we can solve for a. Now, let us determine the set of possible a's through our provided l.

2 Arc Length

We can define the arc length of our parabola as:

$$\int\limits_{A_x}^{B_x} \sqrt{1+f'(x)^2}dx = l$$

From above, we can evaluate:

$$f'(x) = 2ax + \left(\frac{A_y-B_y}{A_x-B_x} - a(A_x+B_x)\right)$$

$$\rightarrow$$

$$\int\limits_{A_x}^{B_x} \sqrt{1+2ax+\left(\frac{A_y-B_y}{A_x-B_x} - a(A_x+B_x)\right)}dx = l$$

$$\left(\frac{\left(-a(A_x+B_x)+2ax+\frac{A_y-B_y}{A_x-B_x}+1\right)^{\frac{3}{2}}}{3a}\right)\Bigg|_{x=A_x}^{x=B_x} = l$$

$$\frac{\left(-a(A_x+B_x)+2aB_x+\frac{A_y-B_y}{A_x-B_x}+1\right)^{\frac{3}{2}}-\left(-a(A_x+B_x)+2aA_x+\frac{A_y-B_y}{A_x-B_x}+1\right)^{\frac{3}{2}}}{3a} = l$$

To further simplify, let us set the position of A to be $(0,0)$ (this is perfectly reasonable as we are projecting into $\mathbb{R}^2$ anyways). Therefore, we have the equation

$$\frac{\left(-aB_x+2aB_x+\frac{B_y}{B_x}+1\right)^{\frac{3}{2}}-\left(-aB_x+\frac{B_y}{B_x}+1\right)^{\frac{3}{2}}}{3a} = l$$

$$\frac{\left(aB_x+\frac{B_y}{B_x}+1\right)^{\frac{3}{2}}-\left(-aB_x+\frac{B_y}{B_x}+1\right)^{\frac{3}{2}}}{3a} = l$$

Given this simplification, we may also simplify our definition of b and c to:

$b = \frac{B_y}{B_x} - a(B_x)$
$c = 0$

To properly solve this, we must simply plug in our values for B_x, B_y, and l, then solve for a numerically. Finally, we can plug our midpoint in and get the position, from which it is trivial to calculate its angle.