Calculating a Feasible Parabolic Hanging Midpoint

Eleanor Olson

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Consider the following problem: Given a string of length l, let us approximate the angle that l's midpoint will create with its endpoints A and B, $(A, B \in \mathbb{R}^2)$. This l may further be defined as a scalar input k, multiplied by the distance of 2 original points. As such, we may use l throughout this problem as a stand in for:

$l = k \cdot ||M - N||$

Where M and N are points in $\mathbb{R}^2$.

To approximate the position of the point along their midpoint, let us think of the hanging function as a parabola. While this is not physically accurate, it will be easier to calcuate and can rougly approximate the physically correct catenary. We will also assume the following:

- The string of length l is perfectly inelastic
- The positions of A and B will be close enough such that $||A - B|| \leq k \cdot ||M - N||$. If this is not the case, we may consider the angle a perfect π.

To further simplify, we shall also assume that A exists at the point $(0, 0)$.

1 Possible Parabolas

First, let us define the set of all parabolas that will contain A and B. Let us define our parabola $f(x) = ax^2 + bx + c$.

$$\begin{aligned} f(A_x) &= 0 = c \\ f(B_x) &= B_y = aB_x^2 + bB_x + c \end{aligned}$$

$$\begin{aligned} B_y &= aB_x^2 + bB_x \\ b &= \frac{B_y - aB_x^2}{B_x} = \frac{B_y}{B_x} - aB_x \end{aligned}$$

$$f(x) = ax^2 + \left(\frac{B_y}{B_x} - aB_x\right)x$$

we now have an explicit equation for our parabola given we can solve for a. Now, let us determine the set of possible a's through our provided l.

2 Arc Length

We can define the arc length of our parabola as:

$$\int\limits_0^{B_x} \sqrt{1 + f'(x)^2} dx = l$$

From above, we can evaluate:

$$f'(x) = 2ax + \frac{B_y}{B_x} - aB_x$$

$$\rightarrow$$

$$\int\limits_0^{B_x} \sqrt{1 + (2ax + \frac{B_y}{B_x} - aB_x)^2}\, dx = l$$

$$-\frac{1}{4}B_x\sqrt{(-aB_x + 2ax + \frac{B_y}{B_x})^2 + 1} + \frac{1}{2}x\sqrt{(-aB_x + 2ax + \frac{B_y}{B_x})^2 + 1} + \frac{B_y\sqrt{(-aB_x+2ax+\frac{B_y}{B_x})^2+1}}{4aB_x} - \left. \frac{\ln\left(1-\frac{-aB_x+2ax+\frac{B_y}{B_x}}{\sqrt{(-aB_x+2ax+\frac{B_y}{B_x})^2+1}}\right)}{8a} + \frac{\ln\left(1+\frac{-aB_x+2ax+\frac{B_y}{B_x}}{\sqrt{(-aB_x+2ax+\frac{B_y}{B_x})^2+1}}\right)}{8a} \right|_{x=0}^{x=B_x} = l$$

Let us make this equation more readable by defining some constants:

- $m = \frac{B_y}{B_x}$
- $q = -aB_x + 2ax + m$
- $v = \sqrt{q^2 + 1}$
- $q_2 = aB_x + m$
- $v_2 = \sqrt{q_2^2 + 1}$
- $q_0 = -aB_x + m$
- $v_0 = \sqrt{q_0^2 + 1}$

$$-\frac{1}{4}B_x v + \frac{1}{2}xv + \frac{mv}{4a} + \left. \frac{\ln(1+\frac{q}{v})-\ln(1-\frac{q}{v})}{8a} \right|_{x=0}^{x=B_x} = l$$

$$-\frac{1}{4}B_x v_2 + \frac{1}{2}B_x v_2 + \frac{mv_2}{4a} + \frac{\ln(1+\frac{q_2}{v_2})-\ln(1-\frac{q_2}{v_2})}{8a} - \frac{1}{4}B_x v_0 + \frac{mv_0}{4a} + \frac{\ln(1+\frac{q_0}{v_0})-\ln(1-\frac{q_0}{v_0})}{8a} = l$$

$$\frac{1}{4}B_x v_2 + -\frac{1}{4}B_x v_0 + \frac{mv_2}{2a} + \frac{\ln(1+\frac{q_2}{v_2})-\ln(1-\frac{q_2}{v_2})+\ln(1+\frac{q_0}{v_0})-\ln(1-\frac{q_0}{v_0})}{8a} = l$$

This is a trancendental equation that must be evaluated numerically. Once we have a value for a, we may construct the full parabola. To solve the full question, we must simply plug in our values for B_x, B_y, and l, then solve for a numerically. Finally, we can plug our midpoint in and get the position, from which it is trivial to calculate its angle.