subject: Foundations of Computing II

date: 26 Feb 2024

keywords: Recursion, Nonhomogeneous Relations

topic: Nonhomogenous Recurrance Relations

Announcements

Exam results released Wednesday

Read the textbook (Book page 479)

Background

Define a recurrance relation $\{a_n\}$ by

$$c_0 a_n + c_1 a_{n-1} + \dots + c_k a_{n-k} = f(n)$$

for constants $c_j \in \mathbb{R}$, some function $f(n)$

Homogenous

When $f(n)$ is identically (always) zero, $\{a_n\}$ is homogenous

Nonhomogenous

For anything else, $\{a_n\}$ non-homogenous

Only for some $f(n)$ do "nice" closed forms exist.

Types of Solveable N-H Relations

We'll focus on (for c, r non-zero):

- $f(n) = c$ — constant
- $f(n) = cn$ — linear
- $f(n) = cn^2$ — quadratic
- $f(n) = r^n$ — geometric
- $f(n) =$ combination of above — combination

We will not cover trigonometric equations

Steps

1) Write in the associated homogenous form, $\overbrace{c_0 a_n + c_1 a_{n-1} + \dots + c_k a_{n-k}}^{a_n^{(h)}} = 0$!!!
2) Get form of the particular solution, $a_n^{(p)}$, based on $f(n)$ (see pg 479)
3) Using non-homogenous relation, solve for $a_n^{(p)}$ w/ initial conditions
4) $a_n = a_n^{(h)} + a_n^{(p)}$

First-Order N.H. Example

$a_{n+1} + c a_n = f(n)$

$a_{n+1} = 2a_n + n \iff \overbrace{a_{n+1} - 2a_n}^{a_n^{(h)}} = n$

w/ $a_0 = 1$

$\longrightarrow$ Characteristic equation: $r - 2 = 0 \iff r = 2$

$\longrightarrow$ $a_n^{(h)} = c(2)^n$

$\longrightarrow$ $a_n^{(p)} = A_1 n + A_0$ (from textbook)

substitute into N.H. relation

$\longrightarrow$ $a_{n+1}^{(p)} = 2a_n^{(p)} + n = 2(A_1 n + A_0) + n$

$= 2a_1 n + 2a_0 + n$

I don't know what's going on ---

$\rightsquigarrow a_n^{(p)} = -n - 1 \longrightarrow a_n = a_n^{(h)} + a_n^{(p)} = c(2)^n - n - 1$

$1 = a_0 = c(2)^0 - 0 - 1 \Rightarrow c = 2$

$\longrightarrow a_n = 2(2)^n - n - 1 = 2^{n+1} - n - 1$

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Example 2

$a_{n+1} = a_n + n \quad , a_0 = 1$

$\longrightarrow a_{n+1} - a_n = n \rightsquigarrow r - 1 = 0 \rightarrow r = 1$

$\longrightarrow a_n^{(h)} = c \cdot 1^n = c$

$\longrightarrow a_n^{(p)} = \cancel{A_1 n + A_0} \quad A_1 n^2 + A_0 n$ (exception to table)

Solve $a_{n+1}^{(p)} = A_1(n+1)^2 + A_0(n+1)$

from def'n of relation

$= A_1(n^2 + 2n + 1) + A_0(n+1) = A_1 n^2 + A_0 n + n$

$$\Rightarrow \begin{cases} A_1 = A_1 \\ 2A_1 + A_0 = A_0 + 1 \\ A_1 = A_0 \end{cases}$$

Match coefficients

$$\Rightarrow \begin{cases} A_1 = \frac{1}{2} \\ A_0 = \frac{-1}{2} \end{cases}$$

$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)} = c + \frac{1}{2}n^2 + \frac{1}{2}n$

Yet $a_0 = 1$, thus $1 = c + \frac{1}{2}(0)^2 + \frac{1}{2}(0)$

$\Rightarrow a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$

See next page for better notes $\longrightarrow$

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topic: 10.3 Nonhomogeneous Linear Recurrance Relations

Notes outside of Class

$a_n^{(p)}$ table

$f(n)$	$a_n^{(p)}$
c	A
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
n^t	$A_t n^t + A_{t-1} n^{t-1} + \dots + A_1 n + A_0$
r^n	$A r^n$

Capital A's are unknown, not the value of a_i

$a_n^{(h)}$: "homogenous" solution
$a_n^{(p)}$: "particular" solution
$a_n = a_n^{(h)} + a_n^{(p)}$ "general" solution

To Solve N-H:

Suppose we're given a linear recurrance relation...

1) Write it in standard form, i.e. $\underbrace{c_0 a_n + c_1 a_{n-1} + \dots + c_k a_{n-k}}_{a_n^{(h)}} = f(n)$
2) Solve $a_n^{(h)} \stackrel{\cdot}{=} 0$ ("set equal to") using old techniques (e.g. characteristic eqn)
3) Use the above table to find the form of $a_n^{(p)}$ (note exceptions)
4) Using the definition of the recurrance relation, solve for unknowns in $a_n^{(p)}$ (match coefficents)
5) Solve $a_n = a_n^{(h)} + a_n^{(p)}$ using boundary/initial conditions

Examples from Class

$a_{n+1} = 2a_n + n$, $a_0 = 1$ ☆

1) $a_{n+1} - 2a_n = n$
2) $a_n^{(h)} = a_{n+1} - 2a_n \stackrel{\cdot}{=} 0 \Rightarrow r - 2 = 0 \Rightarrow r = 2$
$\Rightarrow a_n^{(h)} = A \cdot 2^n$
3) $f(n) = n \Rightarrow a_n^{(p)} = A_1 n + A_0$
4) $a_{n+1}^{(p)} = A_1(n+1) + A_0 = A_1 n + A_1 + A_0$ (from 3)
$a_{n+1}^{(p)} = 2a_n^{(p)} + n$ (from ☆)
$= 2(A_1 n + A_0) + n = 2A_1 n + 2A_0 + n$

☆ $A_1 n + A_1 + A_0 = 2A_1 n + 2A_0 + n$
$\Rightarrow A_1 = 2A_1 + 1$ (everything w/ n)
$\hookrightarrow A_1 = -1$
$\Rightarrow A_1 + A_0 = 2A_0$ (everything else)
$\hookrightarrow A_0 = -1$
$\Rightarrow a_n^{(p)} = A_1 n + A_0 = -n - 1$

5) $a_n = a_n^{(h)} + a_n^{(p)} = C \cdot 2^n - n - 1$, yet $a_0 = 1$
$\Rightarrow 1 = C \cdot 2^0 - 0 - 1 \Rightarrow C = 2 \Rightarrow a_n = 2 \cdot 2^n - n - 1$

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Example 2

$a_{n+1} = a_n + n \ , \ a_0 = 1$

1) $a_{n+1} - a_n = n$ (standard form)

2) $a_n^{(h)} = a_{n+1} - a_n \doteq 0 \Rightarrow r - 1 = 0 \Rightarrow r = 1 \rightsquigarrow a_n^{(h)} = c \cdot (1)^n = c$

3) $f(n) = n \Rightarrow a_n^{(p)} = A_1 n + A_0$

4) $a_{n+1}^{(p)} = A_1(n+1) + A_0 = \underline{A_1 n} + A_1 + A_0$

$a_{n+1}^{(p)} = a_n^{(p)} + n = \underline{A_1 n} + A_0 + \underline{n}$

$\Rightarrow A_1 = A_1 + 1$! no solution

thus this form is wrong!

multiply by n, try again (this will happen from time to time...)

3) $a_n^{(p)} = A_1 n^2 + A_0 n$

4) $a_{n+1}^{(p)} = A_1(n+1)^2 + A_0(n+1) = A_1(n^2+2n+1) + A_0(n+1)$ (replace n w/ n+1)

$a_{n+1}^{(p)} = a_n^{(p)} + n = A_1 n^2 + A_0 n + n$ (from def'n of relation)

$$\Rightarrow \begin{cases} A_1 = A_1 & (n^2 \text{ coeff}) \\ 2A_1 + A_0 = A_0 + 1 & (n \text{ coeff}) \\ A_1 + A_0 = 0 & (\text{const coeff}) \end{cases}$$

$\Downarrow$

$A_0 = -A_1$ ↘ $2A_1 + (-A_1) = (-A_1) + 1 \Leftrightarrow A_1 = -A_1 + 1 \Rightarrow A_1 = \frac{1}{2} \Rightarrow A_0 = -\frac{1}{2}$ ✓

$\Rightarrow a_n^{(p)} = \frac{1}{2}n^2 - \frac{1}{2}n$

5) $a_n = a_n^{(h)} + a_n^{(p)} = c + \frac{1}{2}n^2 - \frac{1}{2}n \ , \ a_0 = 1$

$\Rightarrow \frac{1}{2}(0)^2 - \frac{1}{2}(0) + c = 1 \Rightarrow c = 1$

$\boxed{\Rightarrow a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1}$

not sure if it's $+\frac{1}{2}n$ or $-\frac{1}{2}n$... may have missed a minus sign !!

to be continued...