# CALCULUS II: INTEGRATION TECHINQUES
# PART I

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## 1. Basic Properties of Integration

Much like differentiation, integration respects some of the basic operations of arithmetic. For functions $f$, $g$, and constants $a$, $b$, note that

$$\int (af + bg)\mathrm{d}x = a\int f\mathrm{d}x + b\int g\mathrm{d}x,$$

namely ***integration splits over addition*** and ***constants can be factored out***. This property is known as *linearity*, and is studied more deeply in linear algebra and other higher math courses.

## 2. U-Substitution

From the chain rule for derivatives, given functions $f$ and $g$ of a variable $x$ we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Thus, it stands to follow that for the correct choice of $u = g(x)$, the chain rule shows us that $\mathrm{d}u = g'(x)\mathrm{d}x$, and hence

$$\int f'(g(x))g'(x)\mathrm{d}x = \int f'(u)\mathrm{d}u.$$

To correctly apply $u$-substitution, these three steps are required:

(1) Identify $u$.
(2) Compute $\mathrm{d}u$, the derivative of $u$, in terms of $\mathrm{d}x$.
(3) Solve for $\mathrm{d}x$ in terms of $\mathrm{d}u$.

Once $u$ and $\mathrm{d}x$ are found, we will directly substitute them into our integrand. This should simplify the integral and allow us to compute the antiderivative.

**Example 1.** Evaluate $\displaystyle\int \frac{2\mathrm{d}x}{2x-1}$.

*Solution.* Note that the denominator of the integrand is a degree one polynomial, thus we will set $u$ equal to the denominator, differentiate and solve:

$$u = 2x + 1 \quad \mathrm{d}u = 2\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{2}.$$

Hence, we can substitute these values into the original integral and evaluate:

$$\int \frac{2\mathrm{d}x}{2x-1} = \int \frac{2\frac{\mathrm{d}u}{2}}{u} = \int \frac{\mathrm{d}u}{u} = \ln|u| + C = \ln|2x-1| + C$$

which is our desired result. ∎

**Example 2.** Evaluate $\int xe^{x^2}\mathrm{d}x$.

*Solution.* Set $u = x^2$, thus $\mathrm{d}u = 2x\mathrm{d}x$ and $\mathrm{d}x = \frac{1}{2x}\mathrm{d}u$. Our integrand becomes

$$\int \frac{x}{2x}e^u\mathrm{d}u = \frac{1}{2}\int e^u\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

∎

**Example 3.** Evaluate $\int 3x^2\sin(x^3+\pi)\mathrm{d}x$.

*Solution.* Set $u = x^3 + \pi$, thus $\mathrm{d}u = 3x^2\mathrm{d}x$ and $\mathrm{d}x = \frac{\mathrm{d}u}{3x^2}$. Our integrand becomes

$$\int \frac{3x^2}{3x^2}\sin(u)\mathrm{d}u = \int \sin(u)\mathrm{d}u = -\cos(u) + C = -\cos(x^3+\pi) + C$$

∎

## 3. Integration by Parts

From the product rule for derivatives, given two functions $u$ and $v$ of a variable $x$ we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u\mathrm{d}v + v\mathrm{d}u.$$

Thus, it stands to follow that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[uv] &= u\mathrm{d}v + v\mathrm{d}u\\ u\mathrm{d}v &= \frac{\mathrm{d}}{\mathrm{d}x}[uv] - v\mathrm{d}u\\ \int u\mathrm{d}v &= \int\left(\frac{\mathrm{d}}{\mathrm{d}x}[uv] - v\mathrm{d}u\right)\\ \int u\mathrm{d}v &= uv - \int v\mathrm{d}u\end{aligned}$$

This last equation is know as ***integration by parts***.

When performing integration by parts, refer the the following acronym as a rule of thumb to determine which term to *differentiate*: `ILATE`.

I: *inverse trig*, e.g. $\arccos(x)$, $\arctan(x)$, etc.
L: *logarithms*, e.g. $\log(x)$, $\ln(x)$
A: *algebraic*, e.g. polynomials
T: *trigonometric*, e.g. $\cos(x)$, $\sec(x)$, etc.
E: *exponentials*, e.g. $e^x$.

**Example 4.** Evaluate $\int x\sin(x)\mathrm{d}x$.

 

*Solution.* From `ILATE`, *algebraic* appears before *exponential*, so we will take the derivative of $x$. Hence,

$$u = x \qquad \mathrm{d}v = \sin(x)\mathrm{d}x$$
$$\mathrm{d}u = 1 \cdot \mathrm{d}x \qquad v = -\cos(x)$$

hence

$$\int x\cos(x)\mathrm{d}x = -x\cos(x) - \left(\int -\cos(x)\mathrm{d}(x)\right) = -x\cos(x) + \sin(x) + C$$

which is our final answer. □

**Example 5.** Evaluate $\int \ln(x)\mathrm{d}(x)$.

*Solution.* At first glance, there is no product here to undo with integration by parts. However, we can non-obviously let $u = \ln x$ and $v = 1\mathrm{d}x$. Hence,

$$u = \ln x \qquad \mathrm{d}v = 1\mathrm{d}x$$
$$\mathrm{d}u = \frac{1}{x} \cdot \mathrm{d}x \qquad v = x$$

and

$$\int \ln(x)\mathrm{d}x = x\ln(x) - \int x \cdot \frac{1}{x}\mathrm{d}x = x\ln(x) - \int \mathrm{d}x = x\ln(x) - x + C$$

and indeed, differentiating the above to verify our answer results in the desired $\ln(x)$. □

3.1. **Advanced Technique: Tabular Integration.** Tabular integration is a shorthand method to quickly compute an integral that would otherwise require multiple applications of individual integrations by parts. To perform tabular integration, you will set up a three-column table: the first alternates signs (positive/negative) starting with positive, the second column are the terms to differentiate (using `ILATE`), and the third column are the terms to integrate. Starting with the second column, continue taking derivates until you reach zero or repeat a value seen above. Compute the same number of antiderivatives, and bring the signs down the entire column, alternating as you go.

For instance, suppose we are asked to compute $\int x^3 \sin(x)\mathrm{d}x$. The table would start as

| $+$ | $x^3$ | $\sin(x)$ |
|---|---|---|
| | | |

we would take derivatives of $x^3$, as follows:

| $+$ | $x^3$ | $\sin(x)$ |
|---|---|---|
| | $3x^2$ | |
| | $6x$ | |
| | $6$ | |
| | $0$ | |

and then fill in the rest of the table:

| $+$ | $x^3$ | $\sin(x)$ |
|---|---|---|
| $-$ | $3x^2$ | $-\cos(x)$ |
| $+$ | $6x$ | $-\sin(x)$ |
| $-$ | $6$ | $\cos(x)$ |
| $+$ | $0$ | $\sin(x)$ |

Finally, to compute our antiderivative, we will sum together all the products of the sign and derivative in each row with the antiderivative in the *next* row, as follows:

$$\begin{aligned}\int x^3\sin(x) &= (x^3)(-\cos(x)) - (3x^2)(-\sin(x)) + (6x)(\cos(x)) - (6)(\sin(x)) + C\\ &= -x^3\cos x + x^2\sin x + 6x\cos x - 6\sin x + C.\end{aligned}$$

**Example 6.** Evaluate $\int x^5 e^x \mathrm{d}x$.

*Solution.* Using `ILATE`, we know to differentiate the polynomial $x^5$. Thus using tabular integration;

| $+$ | $x^5$ | $e^x$ |
|---|---|---|
| $-$ | $5x^4$ | $e^x$ |
| $+$ | $20x^3$ | $e^x$ |
| $-$ | $60x^2$ | $e^x$ |
| $+$ | $120x$ | $e^x$ |
| $-$ | $120$ | $e^x$ |
| $+$ | $0$ | $e^x$ |

thus our final answer is

$$\int x^5 e^x \mathrm{d}x = \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + c$$

after factoring the common exponential. Observe that this problem would normally require five seperate applications of integration by parts, that we condensed into one process. $\square$