Linear Algebra

In class final review

There is a matrix w/ $\lambda = 1$, $\lambda = 2$, w/ eigenvectors

$\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix},$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Find P.

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$ Done :)

How to Diagonalize:

1) IF $A \in M_{n \times n}$, must have n lin. indep eigenvectors; $\vec{v}_1, \dots, \vec{v}_n$ w/ $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$
2) Def $P = [\vec{v}_1 \cdots \vec{v}_n]$
3) Def $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$A = PDP^{-1}$$

Ex $\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

$$c_A(\lambda) = \det(A - \lambda I)$$

$$= \begin{bmatrix} 1-\lambda & 0 & -1 & 3 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & -3-\lambda & 0 \\ 0 & 0 & 0 & 7-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2(-3-\lambda)(7-\lambda)$$

$\rightarrow$ E.values are $1, -3, 7$

$1 \leq$ geom mult. $\leq$ alg mult

$$\dim(V_\lambda) = \dim(\ker(A - \lambda I))$$

eigenspace

$$\operatorname{Ker}(A-I) = \left[\begin{array}{cccc|c} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = x_4 = 0$

x_1, x_2 freeeeeee

$= \text{span} \cancel{\{\vec{e}_3, \vec{e}_4\}}$

$\{\vec{e}_1, \vec{e}_2\}$

$\to$ geom mult $(\lambda = 1) = 2 =$ alg. mult ✓

$$\operatorname{Ker}(A+3\lambda) = \left[\begin{array}{cccc|c} 4 & 0 & -1 & 3 & 0 \\ 0 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = \frac{1}{4} x_3$

$x_2 = \frac{1}{4} x_3$

$x_4 = 0$

x_3 freeeeeeeeeeeeeee

$$\to \vec{V}_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

$$\ker(A-7I) = \left[\begin{array}{cccc|c} -6 & 0 & -1 & 3 & 0 \\ 0 & -6 & -1 & 1 & 0 \\ 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1/6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = \frac{1}{2} x_4$

$x_2 = \frac{1}{6} x_4$

$x_3 = 0$

$x_4 = \text{free}$

$$\vec{v}_4 = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix} \right\}$$

$\to A = PDP^{-1}$

$$= \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}^{-1}$$

$$\boxed{A \text{ is symmetric if } A = A^T}$$

A symmetric iff A orthogonally diagonalizable.

I.e. P has orthonormal columns

$$\vec{x}_1 = \begin{bmatrix} 7 \\ 7 \\ 7 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 7 \\ 7 \\ 0 \\ 7 \end{bmatrix}$$

Graham-Schmidt this!

$$\left.\begin{aligned} w_1 &= x_2 \\ w_2 &= x_3 \end{aligned}\right\} \text{ Already orthogonal!}$$

$$w_3 = -\frac{\langle x_1, w_1 \rangle}{\|w_1\|} w_1 - \frac{\langle x_1, w_2 \rangle}{\|w_2\|} w_2 + x_1$$