A Limited History of Complex Dynamics

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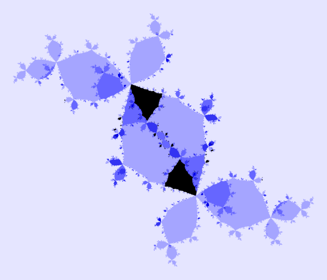
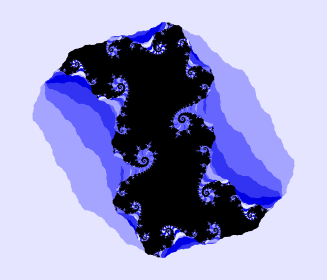
Butler University

12 April 2024





About me...

...and a shameless plug for MRC

- Junior, Mathematics, Actuarial Science, & Computer Science
- 2022, 2023 MRC Researcher
  - 2022: Dr. Krohn, Finite Projective Geometry
  - 2023: Dr. Kaschner, Fractal Geometry

MRC 2023

Outline

Function Iteration
- Motivating Examples
- Fractals

Toolbox of Tricks
- Dynamics 101
- Conjugacy
- The Mandelbrot Set

Current Work

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Basics of Function Iteration

- Consider some function, $f\colon U \to U$.
- What happens when you apply (compose) that function to the same input multiple times?

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Definition

The **orbit** of a point x is the sequence of iterates of x under f:

$$x_n = f(f(f \cdots f(x))) = (f \circ f \circ \cdots \circ f)(x) = f^n(x)$$

Motivating Example I

Question

How many “different” orbits are there?

Consider $f(x) = x^2$:

- $3 \mapsto 9 \mapsto 81 \mapsto 6561 \mapsto 43046721 \mapsto \cdots$

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- $0 \mapsto 0$

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- $0.5 \mapsto 0.25 \mapsto 0.0625 \mapsto 0.00390625 \mapsto \cdots 0$ (converges)
- $0 \mapsto 0 \mapsto 0 \mapsto \cdots 0$ (fixed point)

Motivating Example II

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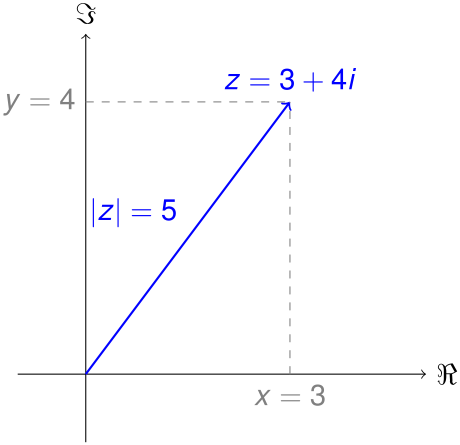
A Preview of Complex Analysis

- Complex numbers
  $\mathbb{C}: \; z = x + iy$,
  $i^2 = -1$
- "Complex Plane":
  $x + iy \leftrightarrow (x, y)$
- Each $z \in \mathbb{C}$ has a magnitude (blame: Pythagoras)
  - Provides a **distance** between two points (i.e. $|z - w|$)

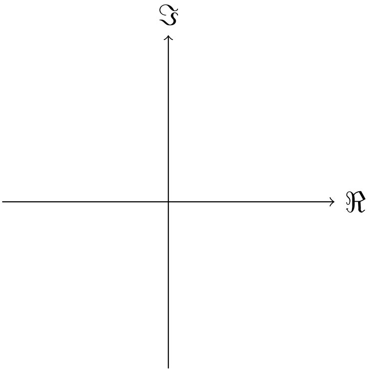


Object of Study

Definition

The **filled Julia set** is the set of points whose orbits remain bounded under iteration by f, denoted $K(f)$.

$K(z^2)$



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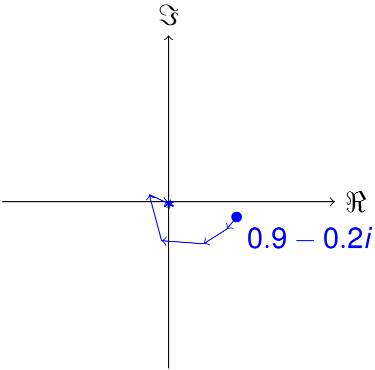

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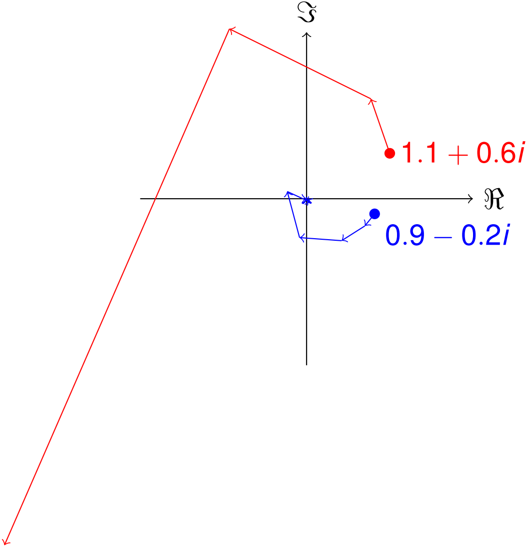

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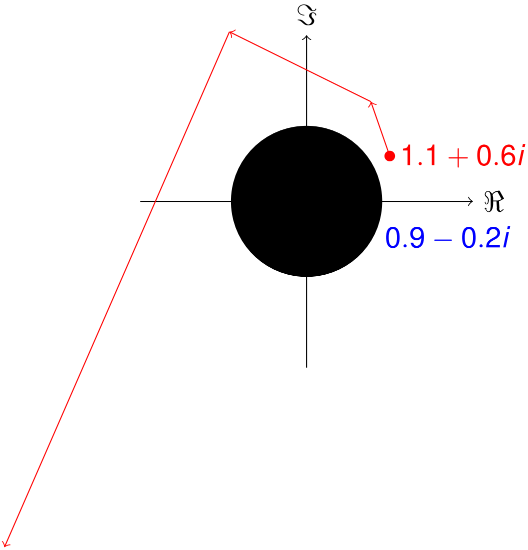
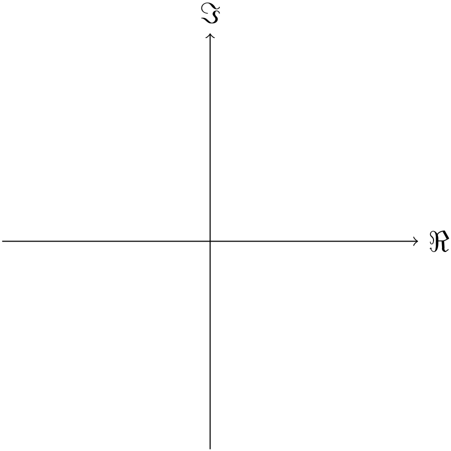


$K(z^2 - 1)$



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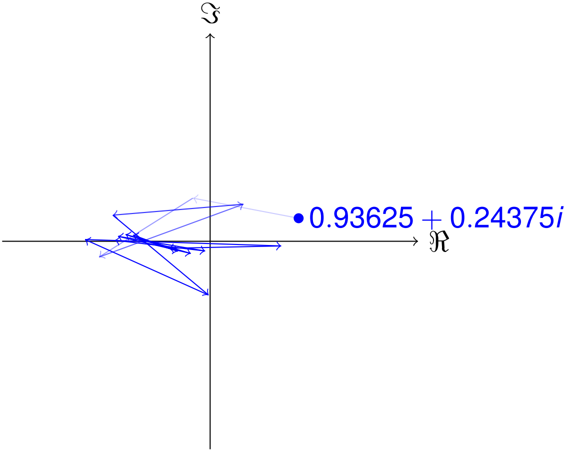

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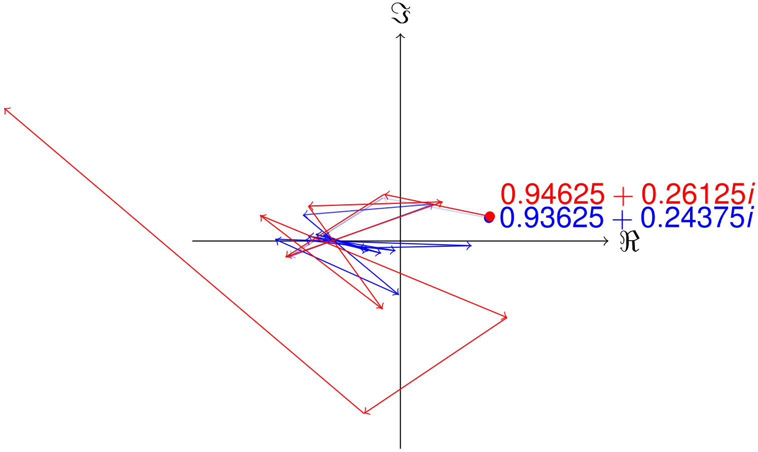

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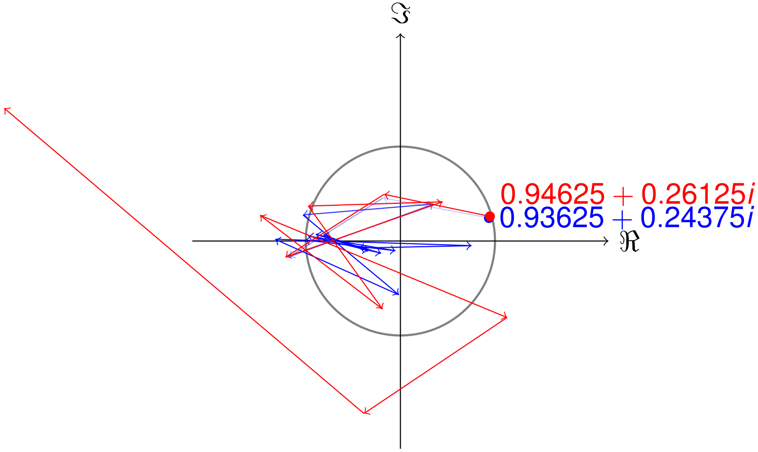

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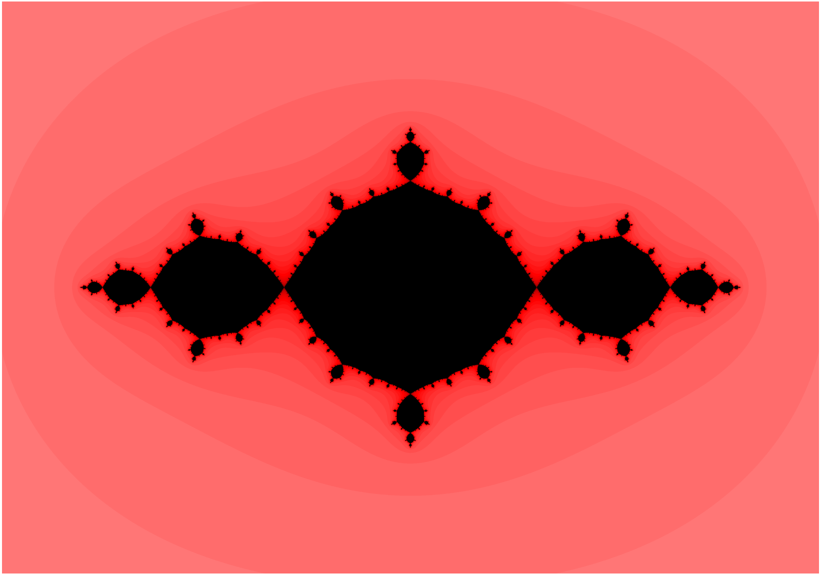

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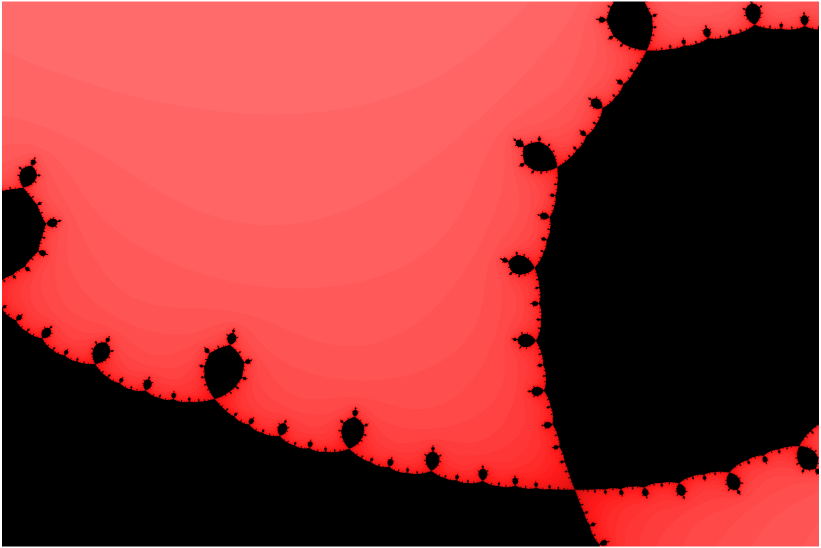
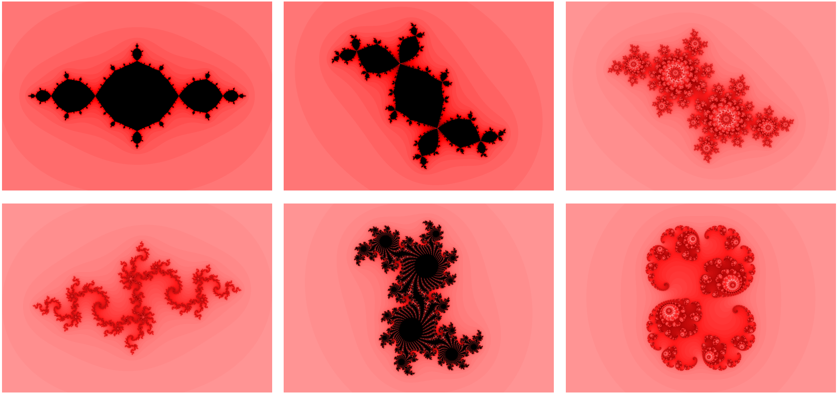


Filled Julia Sets



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Function Iteration

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Toolbox of Tricks

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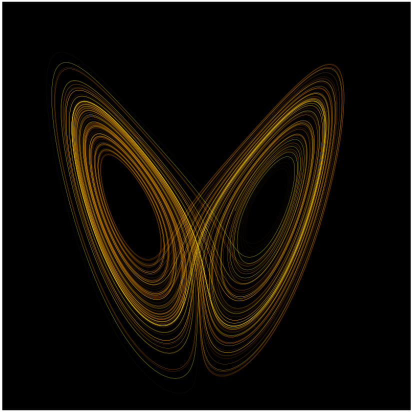
- The Mandelbrot Set

Current Work

Dynamics

- Study of mathematical or physical systems that evolve over time
- Applications to physics, biology, finance, computer engineering, etc.
- Dynamical Systems
  $\to$ Complex Dynamics
  $\to$ Discrete Dynamics



Lorenz Attractor.
Source: Wikimedia Commons.

Some History

(Left) Pierre Fatou, 1878-1929. (Right) Gaston Julia, 1893-1978. Accessed from www.quantamagazine.org.

The Dichotomy

What are we trying to answer?

Given two sufficiently close points z_0, w_0, do they exhibit roughly the same behavior?

Yes!	**No!**
$\mathcal{F}$	J
Fatou set	Julia set
Points behave roughly the same	Points do not behave roughly the same

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But what do the orbits *actually do?*

First Handy Tool

This is a hammer

Definition

A point z is called a **fixed point** of f if $f(z) = z$.

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If an orbit z_n converges to some point w, then

$$w = \lim_{n\to\infty} z_n = \lim_{n\to\infty} z_{n+1}$$

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$$\begin{aligned} w &= \lim_{n\to\infty} z_n = \lim_{n\to\infty} z_{n+1} \\ &= \lim_{n\to\infty} f(z_n) \end{aligned}$$

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If an orbit z_n converges to some point w, then

$$\begin{aligned} w &= \lim_{n\to\infty} z_n = \lim_{n\to\infty} z_{n+1} \\ &= \lim_{n\to\infty} f(z_n) = f\left(\lim_{n\to\infty} z_n\right) = f(w). \end{aligned}$$

Thus, w must be a fixed point.

Am Important Theorem

This is a saw

Theorem (Fundamental Theorem of Algebra)

A degree n polynomial of complex coefficients has exactly n roots, counting multiplicity.

Am Important Theorem

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Theorem (Fundamental Theorem of Algebra)

A degree n polynomial of complex coefficients has exactly n roots, counting multiplicity.

A byproduct of this:

a degree n complex polynomial can be factored into n linear terms

Some Calculus

This is a straight edge

Definition

The **derivative of** f **at** w

is the instantaneous rate of change of f.

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The **derivative of f at w**

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

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$$|f(z) - w|$$

distance between
$f(z)$ and w

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Suppose w is a fixed point of $f(z)$. Then

$$|f(z) - w| \qquad = |f(z) - f(w)| \qquad \approx |f'(z)| \cdot |z - w|$$

distance between $f(z)$ and w

scalar multiple of distance between z and w

Local Fixed Point Theory

This is a nail

Definition (Multiplier of a Fixed Point)

Suppose w is a fixed point of f, and let $\lambda = f'(w)$.

Local Fixed Point Theory

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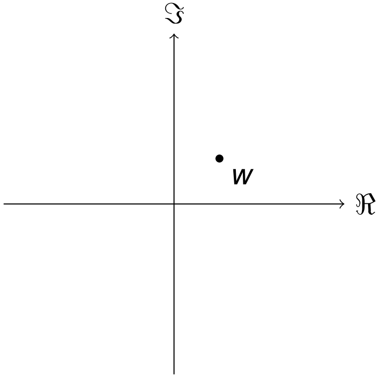
Attracting Fixed Points



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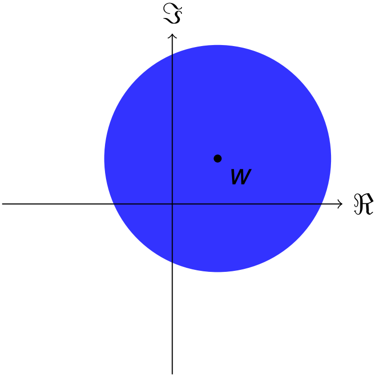
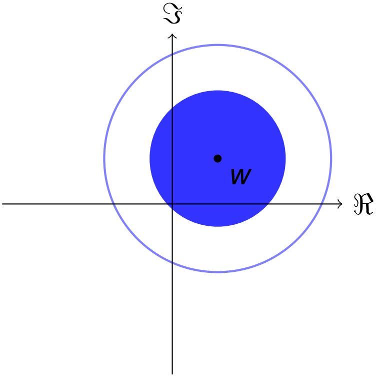


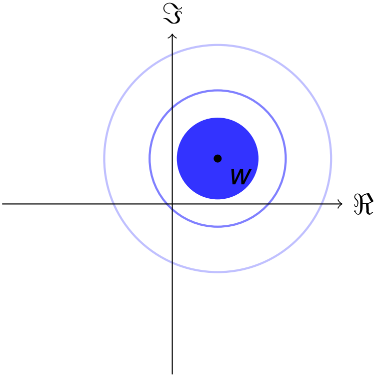
$\mathcal{B}(w, r)$

Attracting Fixed Points



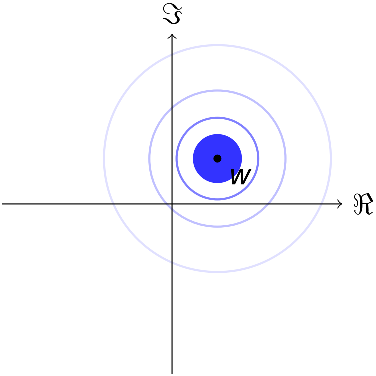
$$f(\mathcal{B}(w, r)) \subseteq \mathcal{B}(w, r)$$

Attracting Fixed Points



$$f^2(\mathcal{B}(w,r)) \subseteq f(\mathcal{B}(w,r)) \subseteq \mathcal{B}(w,r)$$

Attracting Fixed Points



$$f^3(\mathcal{B}(w,r)) \subseteq f^2(\mathcal{B}(w,r)) \subseteq f(\mathcal{B}(w,r)) \subseteq \mathcal{B}(w,r)$$

$f(z) = z^2$

Example: $f(z) = z^2$.

- $0.9 \mapsto 0.81 \mapsto 0.6561 \mapsto 0.4305 \mapsto \cdots 0$.

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- $0.9 \mapsto 0.81 \mapsto 0.6561 \mapsto 0.4305 \mapsto \cdots 0$.
- $z \mapsto z^2 \mapsto z^4 \mapsto z^8 \mapsto \cdots \mapsto z^{(2^n)} \mapsto \cdots 0$ for $|z| < 1$.

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Definition

The **basin of attraction** for an attracting fixed point w

$$\mathcal{A}_w = \{z \colon \lim_{n \to \infty} f^n(z) = w\}$$

is the set of all points whose orbits converge to w.

Working Backwards

eert a si sihT

Definition

The **preimage** of a point z under f is the set of points $\{w_d\}$ such that $f(w_d) = z$. If f is a degree d polynomial, then there exists d preimages of z, counting multiplicity.

Invariance of J and $\mathcal{F}$

Proposition

The following are equivalent:

- z *is an element of* $\mathcal{F}$*;*
- $f(z)$ *is an element of* $\mathcal{F}$*;*
- $f^{-1}(z)$ *is an element of* $\mathcal{F}$*.*

Fatou and Julia sets are *totally invariant*.

A Better Tool

This is a sledgehammer..

Definition

A point z_0 is called a **degree k periodic point** of f if $f^k(z_0) = z_0$ and $z_0, z_1, z_2, \cdots, z_{k-1}$ are all distinct.

$$\begin{tikzcd}
z_0 \arrow[rr] & & z_1 \arrow[dl] \\
& z_2 \arrow[ul] &
\end{tikzcd}$$

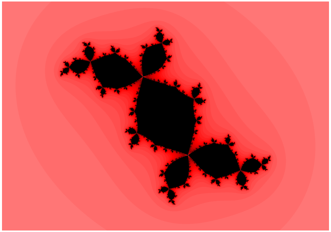
If z_0 is a degree k periodic point of f, then z_0 is a *fixed point* of f^k.

Iteration Lemma

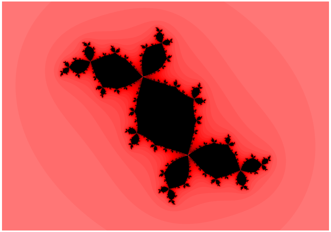
... that is really just a hammer

Lemma

For any k, the sets $\mathcal{F}(f^k)$, $J(f^k)$, and $K(f^k)$ are exactly the sets $\mathcal{F}(f)$, $J(f)$, and $K(f)$.



$K(f)$



$K(f^{2024})$

Local Fixed Point Theory

This is a bigger nail

Definition

Suppose $\{z_0, z_1, \cdots, z_{k-1}\}$ is a degree k periodic cycle of f, and let

$$\lambda = (f^k)'(z_i) = f'(z_0) \cdot f'(z_1) \cdot \ldots \cdot f'(z_{k-1})$$

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- If $|\lambda| = 1$, then $\{z_0, z_1, \cdots, z_{k-1}\}$ is an **indifferent** cycle.

Conjugate Maps

This is a box

- Can we make our lives easier?

Definition

Polynomials f and g are **conjugate** if there exists an invertible function φ such that

$$\varphi \circ f = g \circ \varphi,$$

or, equivalently,

$$f = \varphi^{-1} \circ g \circ \varphi$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbb{C} \arrow[d, "\varphi"] \\ \mathbb{C} \arrow[r, "g"] & \mathbb{C} \end{tikzcd}$$

Properties of Conjugate Maps

Properties of Conjugate Maps

- Let $f = \varphi^{-1} \circ g \circ \varphi$. Then

$$f^n = f \circ \cdots \circ f = (\varphi^{-1} \circ g \circ \varphi) \circ \cdots \circ (\varphi^{-1} \circ g \circ \varphi) = \varphi^{-1} \circ g^n \circ \varphi$$

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- Let z be fixed by f and $\varphi(z) = w$. Then

$$w = \varphi(z) = (\varphi \circ f)(z) = (\varphi \circ \varphi^{-1} \circ g \circ \varphi)(z) = (g \circ \varphi)(z) = g(w)$$

Properties of Conjugate Maps

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$$w = \varphi(z) = (\varphi \circ f)(z) = (\varphi \circ \varphi^{-1} \circ g \circ \varphi)(z) = (g \circ \varphi)(z) = g(w)$$

- Let $f'(z) = \lambda$. Then

$$\lambda = f'(w) = (\varphi^{-1} \circ g \circ \varphi)'(z) = (\varphi^{-1})(w) \cdot g'(w) \cdot \varphi'(z) = g'(w)$$

But why do we care?

All Quadratics are Conjugate to $z^2 + c$

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Let $g(z) = az^2 + bz + k$, and let $\varphi(z) = \frac{1}{a}z - \frac{b}{2a}$. Hence $\varphi^{-1}(z) = az + \frac{b}{2}$ and

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$$f(z) = (\varphi^{-1} \circ g \circ \varphi)(z) = \varphi^{-1}(g(\varphi(z)))$$

All Quadratics are Conjugate to $z^2 + c$

Let $g(z) = az^2 + bz + k$, and let $\varphi(z) = \frac{1}{a}z - \frac{b}{2a}$. Hence $\varphi^{-1}(z) = az + \frac{b}{2}$ and

$$\begin{aligned} f(z) &= (\varphi^{-1} \circ g \circ \varphi)(z) = \varphi^{-1}(g(\varphi(z))) \\ &= a\left(a\left(\frac{1}{a}z - \frac{b}{2a}\right)^2 + b\left(\frac{1}{a}z - \frac{b}{2a}\right) + k\right) + \frac{b}{2} \end{aligned}$$

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&= z^2 - bz + \frac{b^2}{4} + bz - \frac{b^2}{2} + ak + \frac{b}{2}
\end{aligned}$$

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$$

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&= z^2 + \frac{b^2}{4} - \frac{b^2}{2} + ak + \frac{b}{2} \\
&= z^2 + c
\end{aligned}
$$

Parameter Space

Consider the conjugacy classes of maps $f_c(z) = z^2 + c$:

- For what c does f_c have an attracting point?
- For what c does f_c have an attracting two-cycle?

$$\vdots$$

- For what c does f_c have an attracting k-cycle?

Attracting Fixed Points

Find c such that $f_c(z) = z^2 + c$ has a fixed point:

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Find c such that $f_c(z) = z^2 + c$ has a fixed point:

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$$f_c(z) = z^2 + c = z$$
$$z^2 - z + c = 0$$

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Find c such that $f_c(z) = z^2 + c$ has a fixed point:

$$f_c(z) = z^2 + c = z$$
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Attracting Fixed Points

Find c such that $f_c(z) = z^2 + c$ has a fixed point:

$$f_c(z) = z^2 + c = z$$
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$$a + b = 1 \qquad ab = c$$

Attracting Fixed Points

Find c such that $f_c(z) = z^2 + c$ has a fixed point:

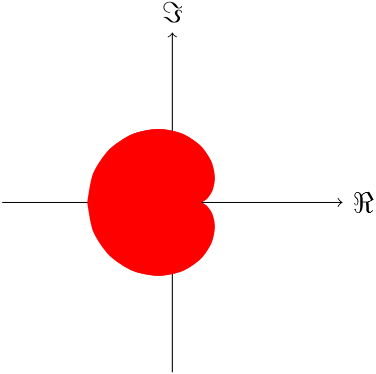
$$f_c(z) = z^2 + c = z$$
$$z^2 - z + c = 0$$
$$(z-a)(z-b) = 0$$
$$z^2 - (a+b)z + ab = 0$$
$$a + b = 1 \qquad ab = c$$

We want at least one *attracting* fixed point; so

$$|\lambda_a| = |f_c'(a)| = |2a| < 1 \rightarrow |a| < \frac{1}{2}$$

Attracting Fixed Points

$$|a| < \frac{1}{2},$$
$$f_c(a) = a^2 - c = a$$
$$\Leftrightarrow$$
$$c = a - a^2$$



Attracting Period-2 Points

Find c such that $f_c(z) = z^2 + c$ has a period-2 cycle:

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$$z^4 + 2cz^2 - z + c^2 + c = 0$$

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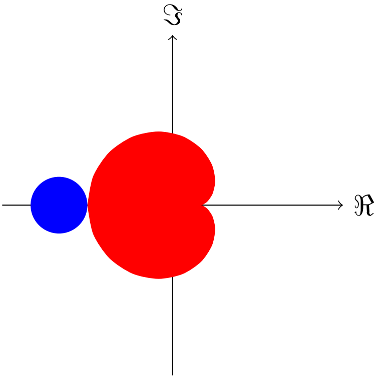
$$(z - a)(z - b)(z^2 + z + 1 + c) = 0$$

$$(z - u)(z - v) = z^2 - (u + v)z + uv$$

$$u + v = -1 \qquad uv = 1 + c$$

Attracting Period-2 Points

$$\lambda = (f_c^2)'(u) = f_c'(f_c(u))f_c'(u)$$
$$= f_c'(v)f_c'(u) = (2u)(2v) = 4uv$$
$$|\lambda| < 1 \Rightarrow |1 + c| < 1/4$$



From Kleinian groups...

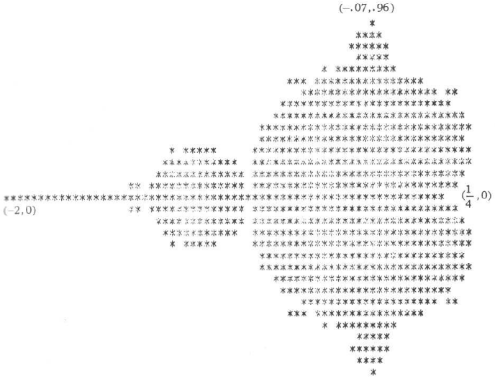
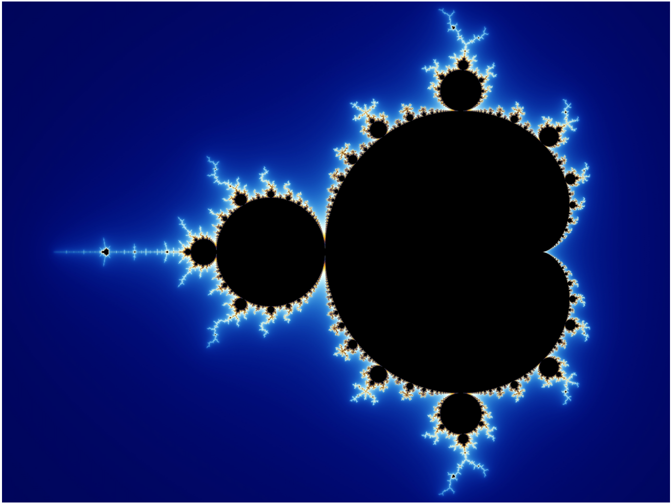


Fig. 2. The set of C's such that $f(z) = z^2 + C$ has a stable periodic orbit.

R. Brooks and P. Matelski, 1981.

The dynamics of 2-generator subgroups of $PSL(2, C)$.

...to internet fame



The Mandelbrot Set. Accessed from Wikimedia Commons.

The Old and The New

- **Douady-Hubbard** ('82): M is connected [4].
  - Mandelbrot Locally Connected (MLC) conjectured
- **Sullivan** ('85): Classification Theorem [7].
  - There exist only hyperbolic cycles, parabolic cycles, Siegel disks, or Herman rings.
- **Hubbard** ('93): If MLC, then $\mathcal{H} = \operatorname{int} M$ and $M = \overline{\mathcal{H}}$ [5].
- **Douady** ('94): $K(f)$ is not continuous with respect to f [3].

Outline

Function Iteration
- Motivating Examples
- Fractals

Toolbox of Tricks
- Dynamics 101
- Conjugacy
- The Mandelbrot Set

Current Work

A brief thread through history

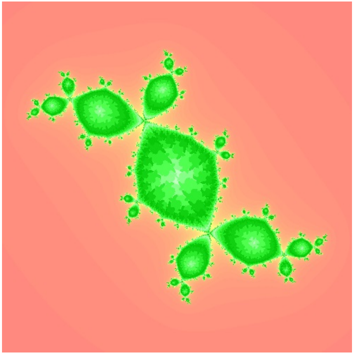
2012 [1] Boyd & Schulz: $f_n(z) = z^n + c$.

Geometric limits of Julia sets

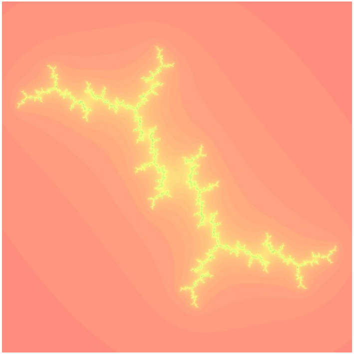
Let $f_n \colon \mathbb{C} \to \mathbb{C}$ by

$$f_n(z) = z^n + c,$$

- where $n \geq 2$ is an integer, and
- $c \in \mathbb{C}$ is a complex parameter.



$f_{2,-0.12+0.75i}$



$f_{2,-0.15+i}$

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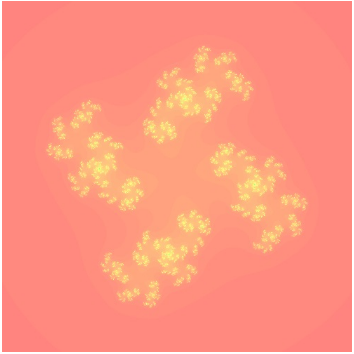

$f_{4,-0.12+0.75i}$

$f_{4,-0.15+i}$

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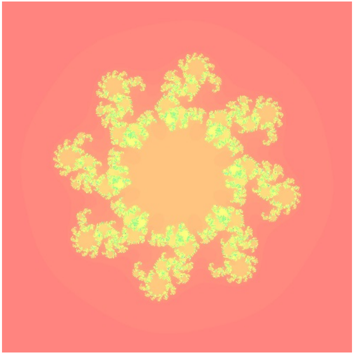
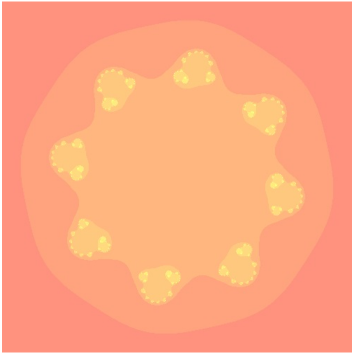


$f_{8,-0.12+0.75i}$



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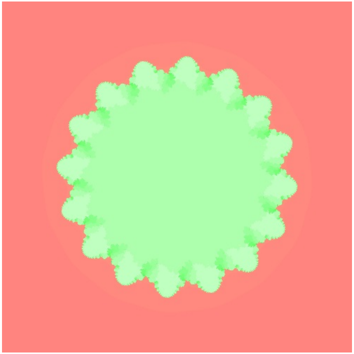
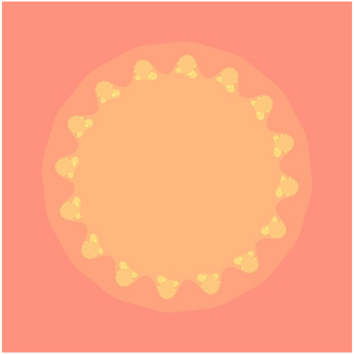


$f_{16,-0.12+0.75i}$



$f_{16,-0.15+i}$

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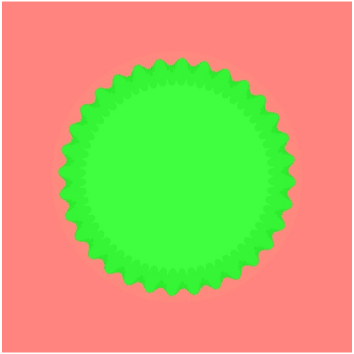
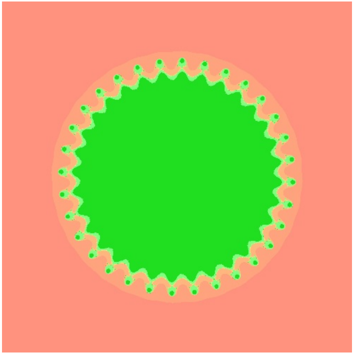


$f_{32,-0.12+0.75i}$



$f_{32,-0.15+i}$

Geometric limits of Julia sets

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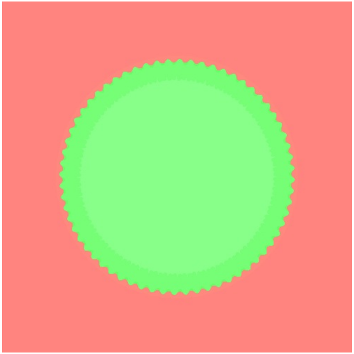
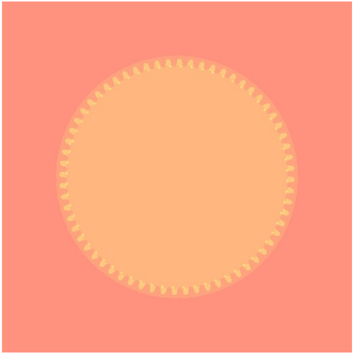


$f_{64,-0.12+0.75i}$



$f_{64,-0.15+i}$

Geometric limits of Julia sets

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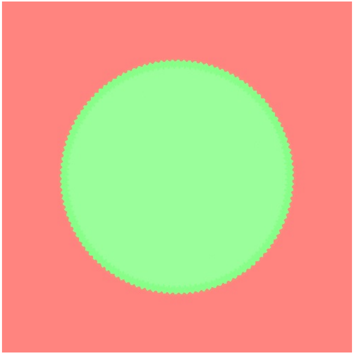
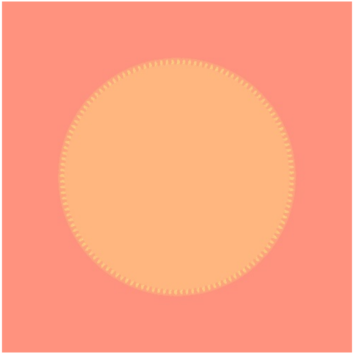


$f_{128,-0.12+0.75i}$



$f_{128,-0.15+i}$

Boyd-Schulz

$$f_{n,c}(z) = z^n + c$$

Theorem (Boyd-Schulz, 2012 [1])

Let $c \in \mathbb{C}$. Using the Hausdorff metric,

(1) *If $c \in \mathbb{C} \backslash \overline{\mathbb{D}}$, then $\lim\limits_{n\to\infty} K(f_{n,c}) = S_0 = \{|z| = 1\}$.*

(2) *If $c \in \mathbb{D}$, then $\lim\limits_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}} = \{|z| \leq 1\}$.*

(3) *If $c \in S^1$, then if $\lim\limits_{n\to\infty} K(f_{n,c})$ exists, it is contained in $\overline{\mathbb{D}}$.*

Boyd-Schulz

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(3) *If $c \in S^1$, then if $\lim_{n\to\infty} K(f_{n,c})$ exists, it is contained in $\overline{\mathbb{D}}$.*

(3) was further improved in [6] (2015).

A brief thread through history

2012 [1] Boyd & Schulz: $f_n(z) = z^n + c$.

2015 [6] Kaschner, Romero, & Simmons: $f_n(z) = z^2 + c$.

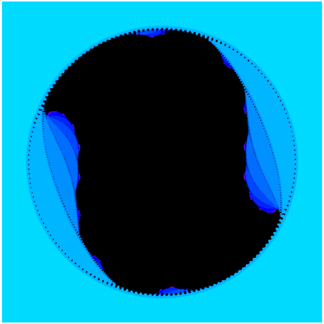
2020 [2] Brame & Kaschner: $f_n(z) = z^n + q(z)$.

More geometric limits of Julia sets

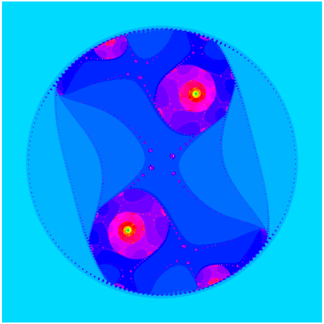
Let $f_n\colon \mathbb{C} \to \mathbb{C}$ by

$$f_n(z) = z^n + q(z),$$

- where $n \geq 2$ is an integer, and
- q is a fixed degree d polynomial.



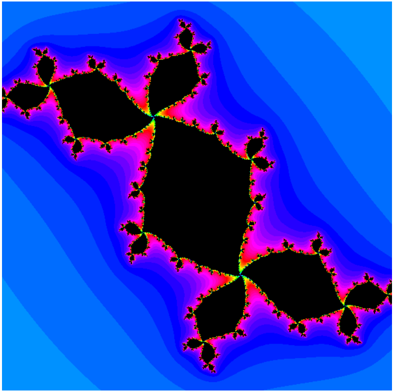
$f_{200,z^2+0.25+0.25i}$



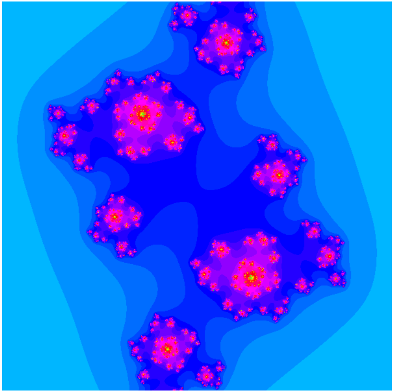
$f_{200,z^2+0.45+0.25i}$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 4$$



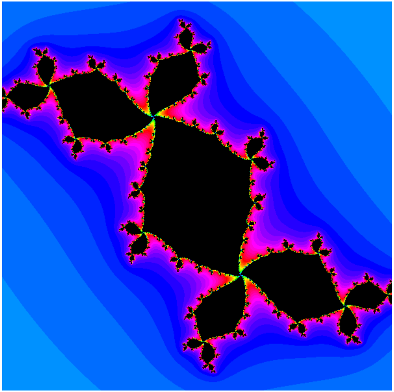
$K(q)$



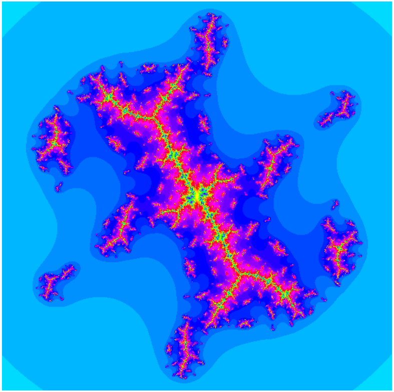
$K(f_{4,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 8$$



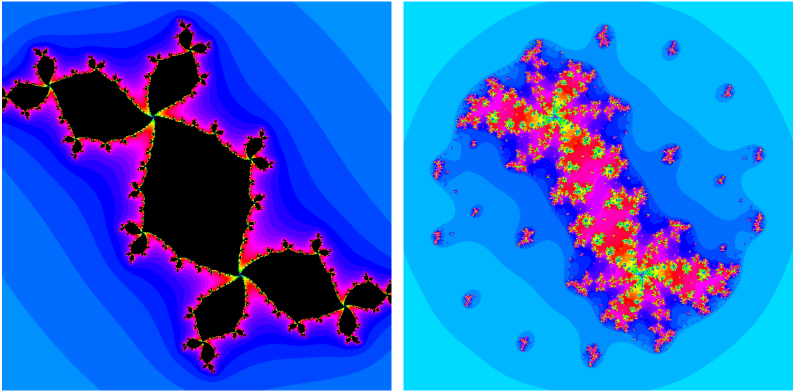
$K(q)$



$K(f_{8,q})$

Rabbit in a cage

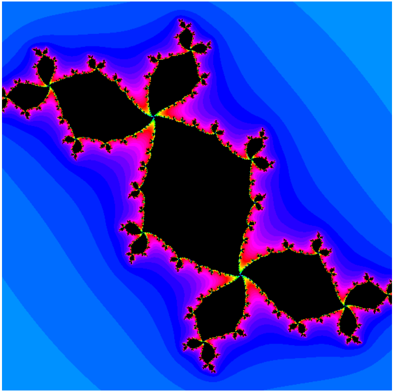
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 16$$



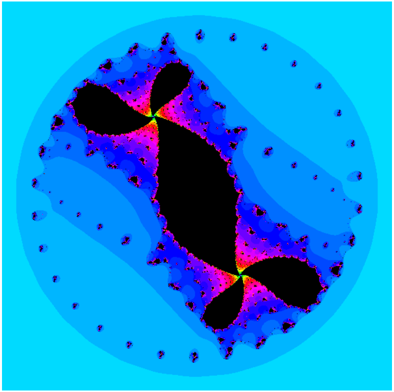
$K(q)$ $\qquad$ $K(f_{16,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 32$$



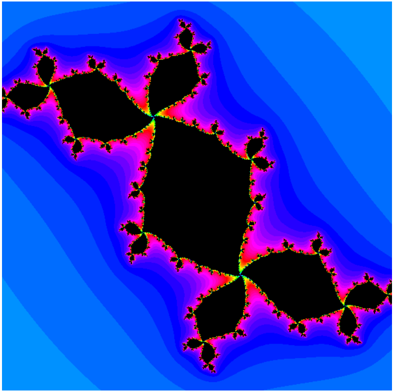
$K(q)$



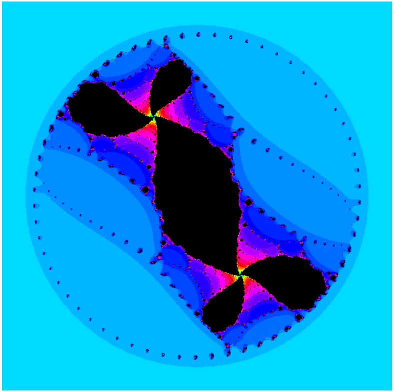
$K(f_{32,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 64$$



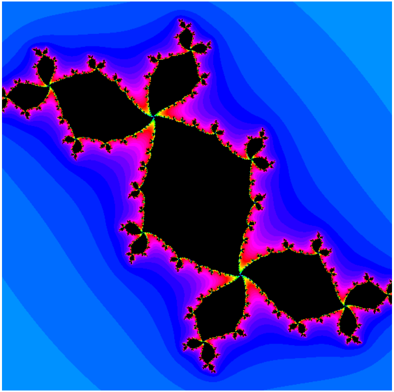
$K(q)$



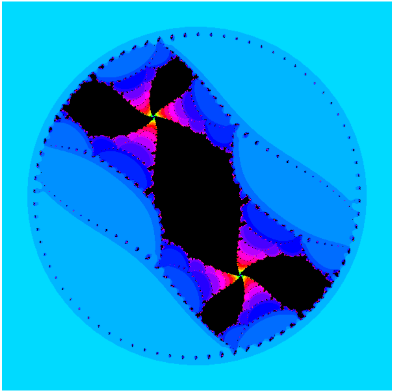
$K(f_{64,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 80$$



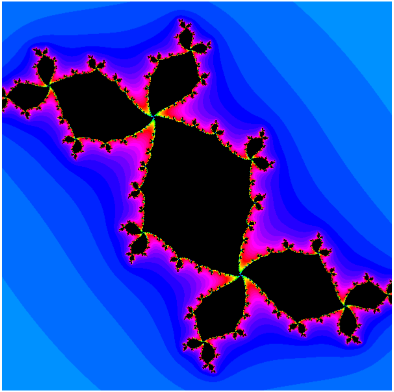
$K(q)$



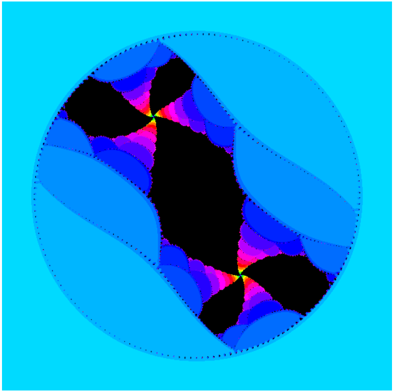
$K(f_{80,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 180$$



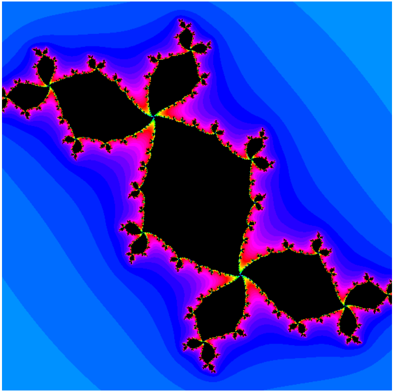
$K(q)$



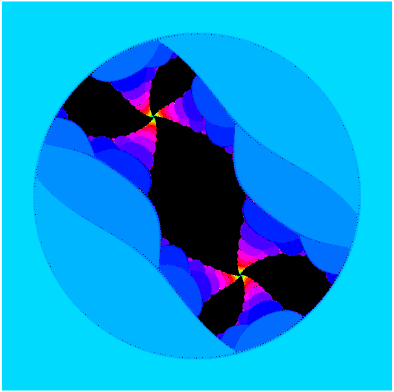
$K(f_{180,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 360$$



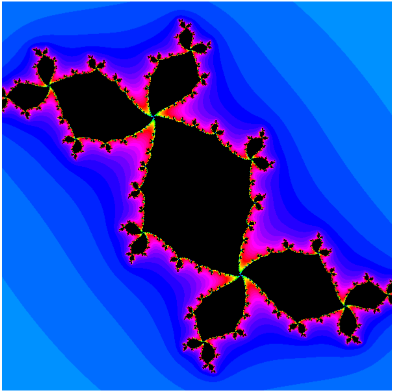
$K(q)$



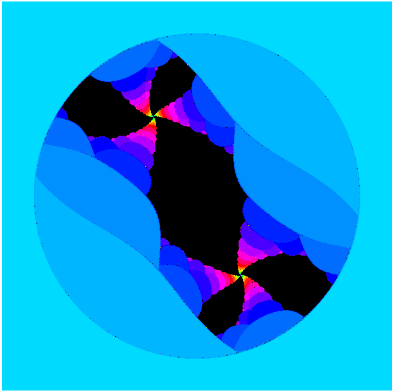
$K(f_{360,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 720$$



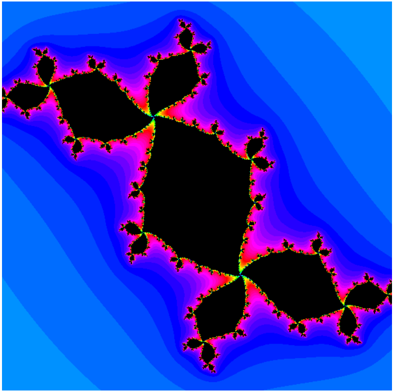
$K(q)$



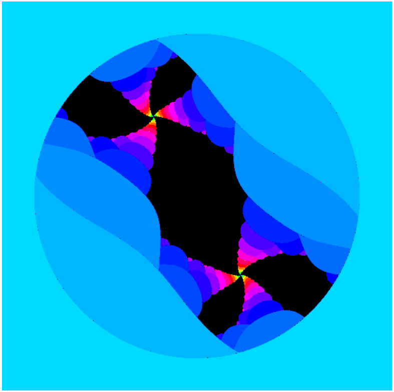
$K(f_{720,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 1800$$

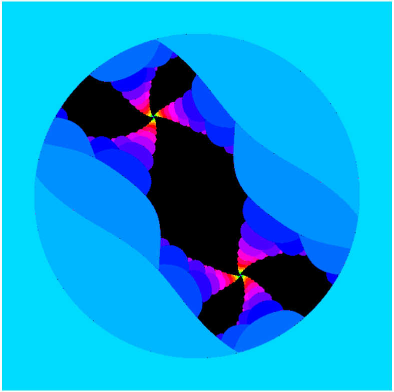


$K(q)$



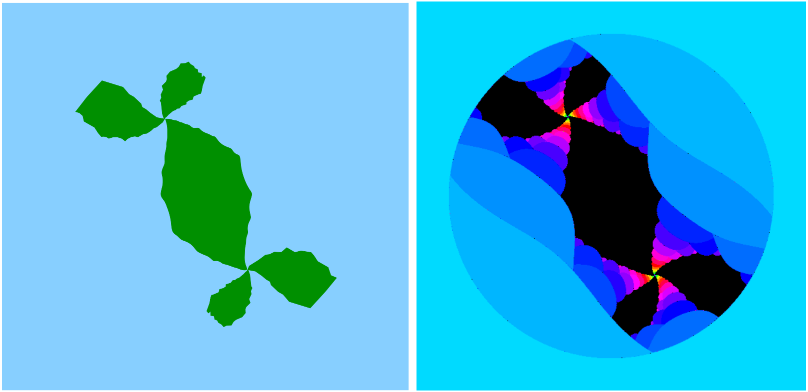
$K(f_{1800,q})$

The limit set

The limit set

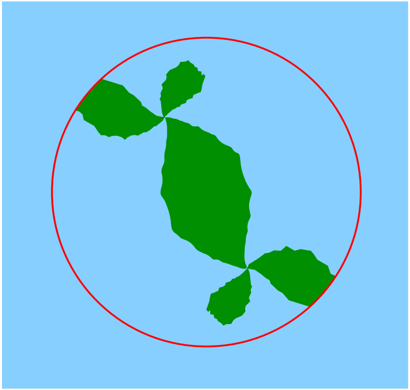
$$K_q = \bigcap_{i=0}^{\infty} q^{-i}(\bar{\mathbb{D}}) = \{z \colon q^i(z) \in \bar{\mathbb{D}}\ \forall i \geq 0\}$$

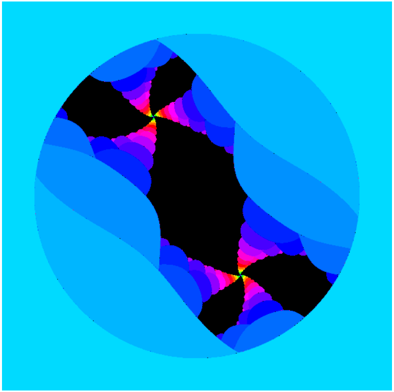


The limit set

$$K_q = \bigcap_{i=0}^{\infty} q^{-i}(\bar{\mathbb{D}}) = \{z\colon q^i(z) \in \bar{\mathbb{D}}\ \forall i \geq 0\}$$

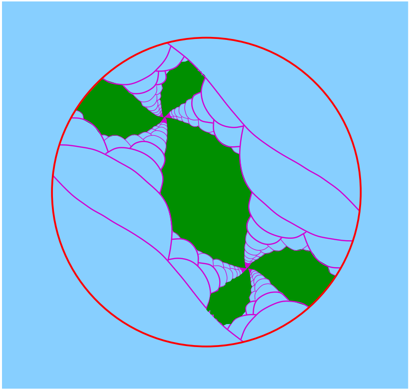
$$S_0 = \{z\colon |z| = 1\}$$

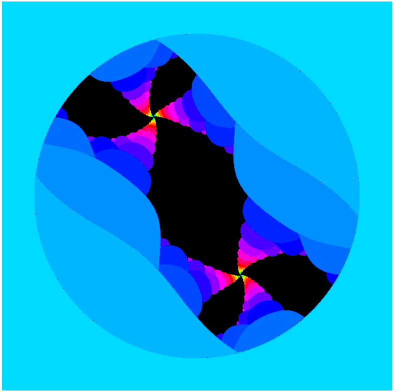




The limit set

$$
\begin{aligned}
K_q &= \bigcap_{i=0}^{\infty} q^{-i}(\bar{\mathbb{D}}) = \{z\colon q^i(z) \in \bar{\mathbb{D}}\ \forall i \geq 0\} \\
S_0 &= \{z\colon |z| = 1\} \\
S_j &= \{q^j(z) \in \partial\mathbb{D} \text{ and } q^i(z) \in \mathbb{D} \text{ for } i = 1, \ldots, j-1\}
\end{aligned}
$$



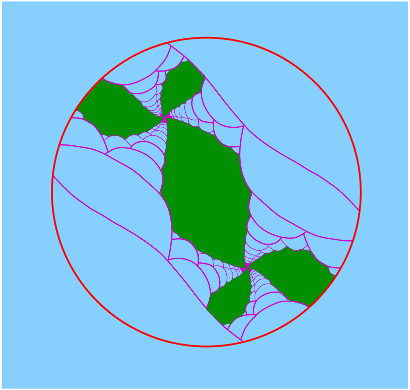


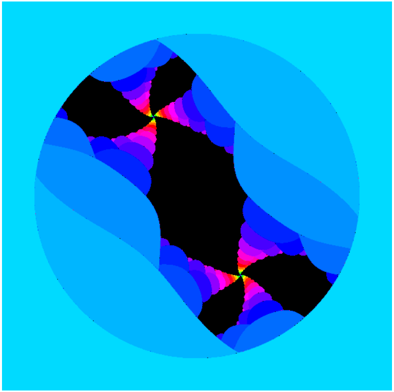
The limit set

$$K_q = \bigcap_{i=0}^{\infty} q^{-i}(\bar{\mathbb{D}}) = \{z : q^i(z) \in \bar{\mathbb{D}}\ \forall i \geq 0\}$$

$$S_0 = \{z : |z| = 1\}$$

$$S_j = \{q^j(z) \in \partial\mathbb{D} \text{ and } q^i(z) \in \mathbb{D} \text{ for } i = 1, \ldots, j-1\}$$





$$\lim_{n\to\infty} K(f_{n,q}) = K_q \cup \bigcup_{j\geq 0} S_j$$

Brame & Kaschner

$$f_n(z) = z^n + q(z)$$

Theorem (Brame-Kaschner, 2020 [2])

If $\deg q \geq 2$, q is hyperbolic, and q has no attracting fixed points in S_0, then

$$\lim_{n\to\infty} K(f_{n,q}) = K_q \cup \bigcup_{j\geq 0} S_j.$$

A brief thread through history

2012 [1] Boyd & Schulz: $f_n(z) = z^n + c$.

2015 [6] Kaschner, Romero, & Simmons: $f_n(z) = z^2 + c$.

2020 [2] Brame & Kaschner: $f_n(z) = z^n + q(z)$.

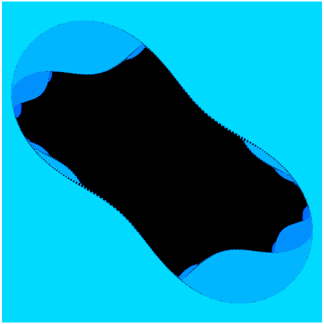
2023 Kaschner, Kapiamba, & W.: $f_n(z) = (p(z))^n + q(z)$.

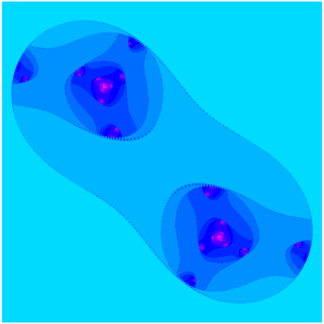
Even more geometric limits of Julia sets

Let $f_n \colon \mathbb{C} \to \mathbb{C}$ by

$$f_n(z) = (p(z))^n + q(z),$$

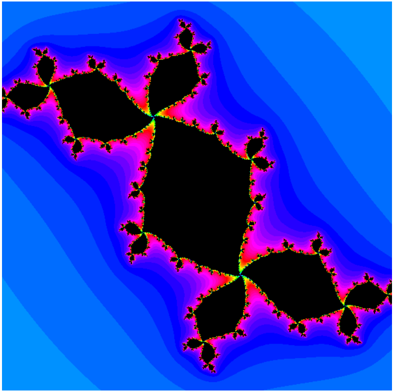
- where $n \geq 2$ is an integer, and
- p, q are fixed polynomials.



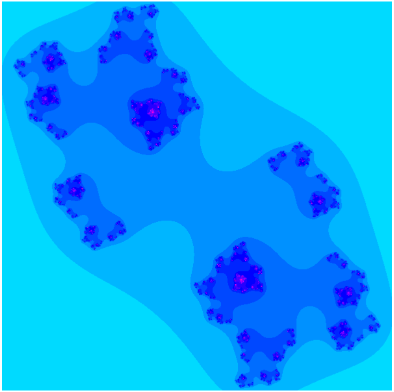


Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 4$$



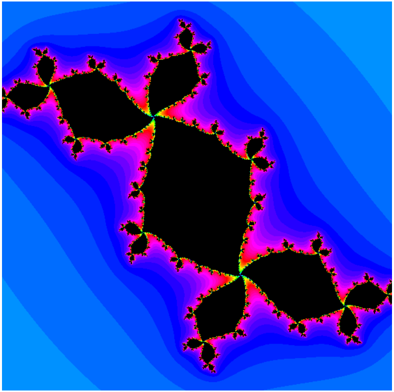
$K(q)$



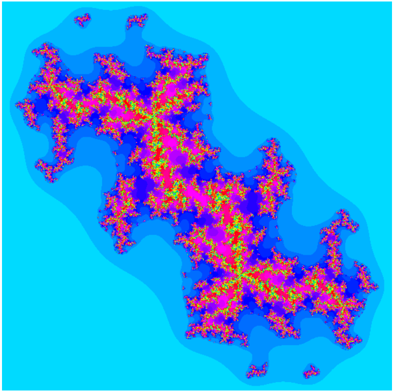
$K(f_4)$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 8$$



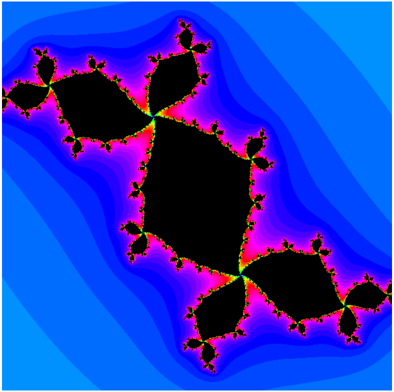
$K(q)$



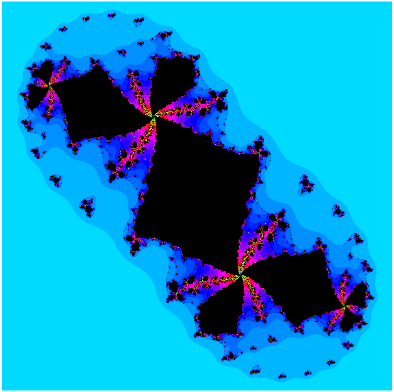
$K(f_8)$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 16$$



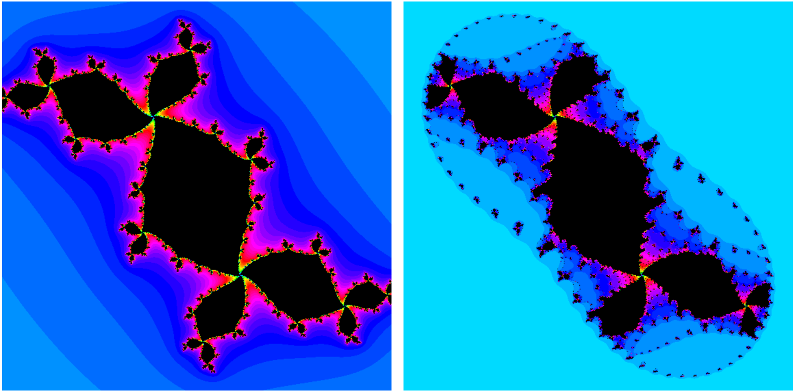
$K(q)$



$K(f_{16})$

Into the Rabbitverse

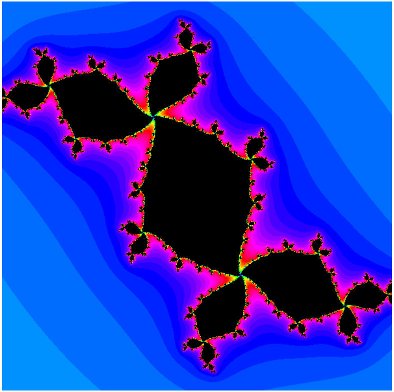
$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 32$$



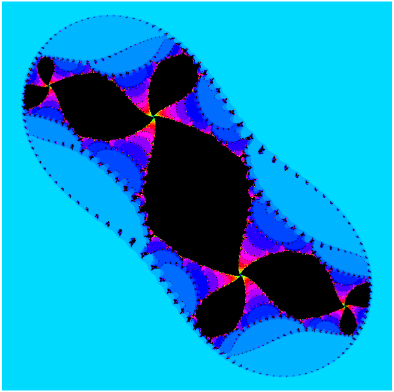
$K(q)$ $\quad$ $K(f_{32})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 64$$



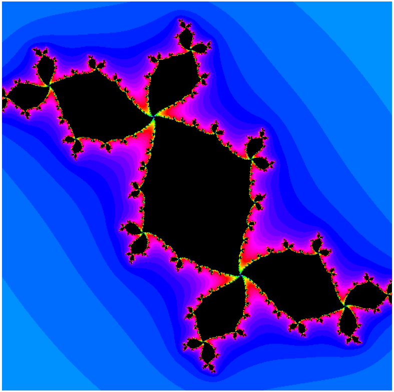
$K(q)$



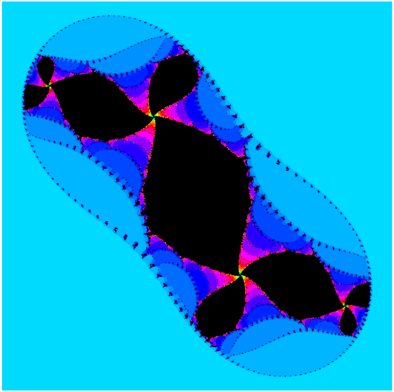
$K(f_{64})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 80$$



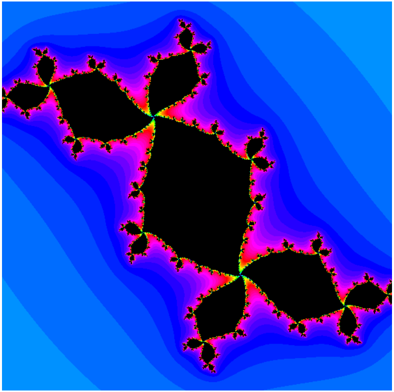
$K(q)$



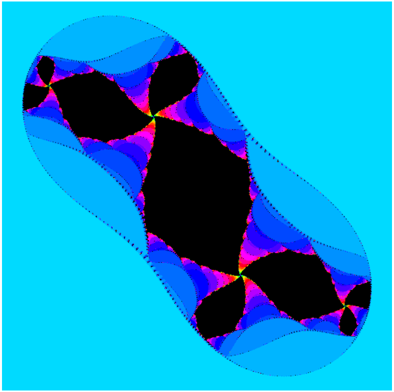
$K(f_{80})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 180$$



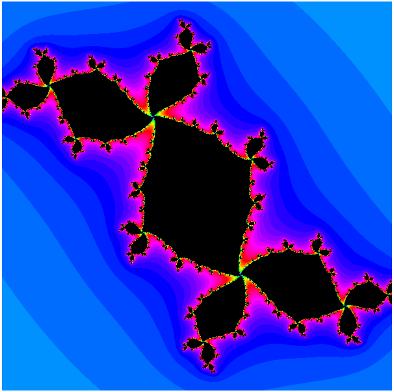
$K(q)$



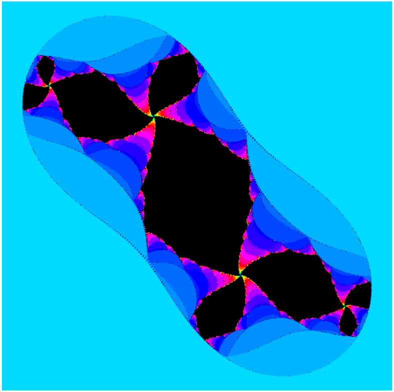
$K(f_{180})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 360$$



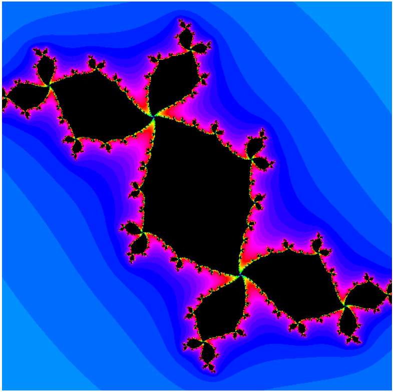
$K(q)$



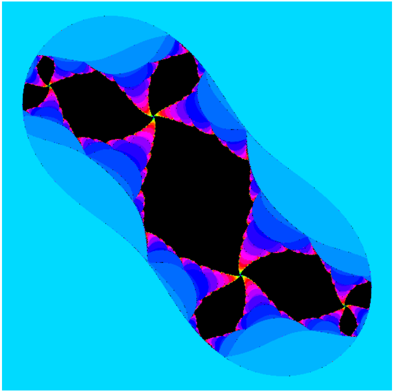
$K(f_{360})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 720$$



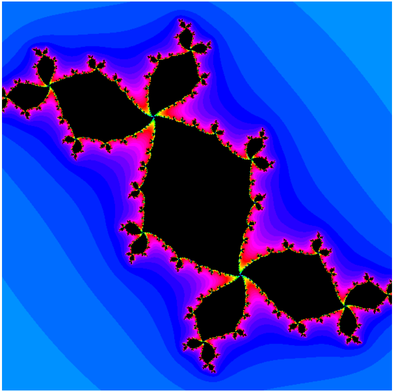
$K(q)$



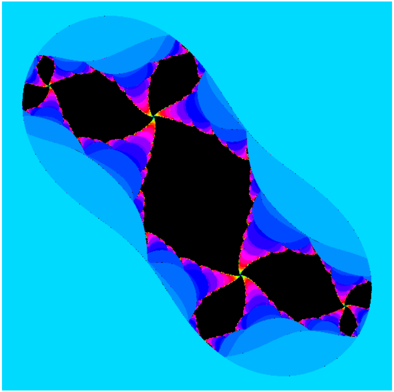
$K(f_{720})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 1800$$

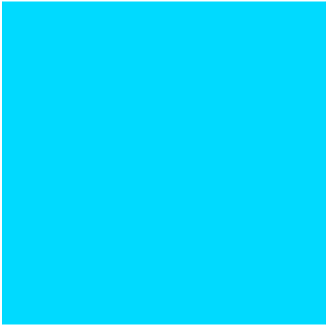


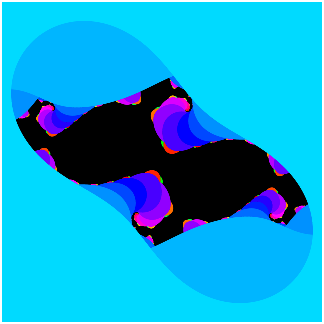
$K(q)$



$K(f_{1800})$

The trouble with Quibbles





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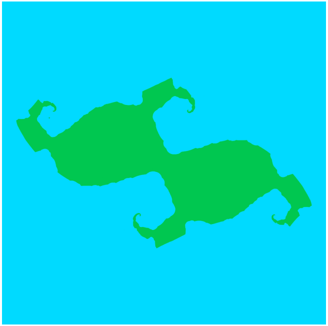
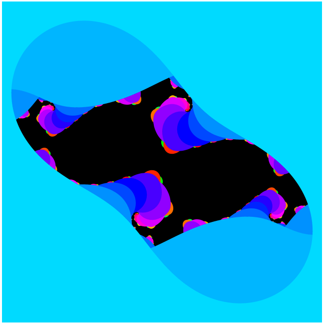
$$K_q = \bigcap_{j=0}^{\infty} q^{-j}\left(p^{-1}(\bar{\mathbb{D}})\right)$$





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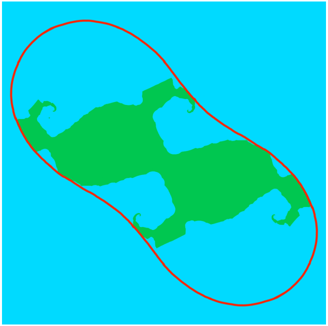
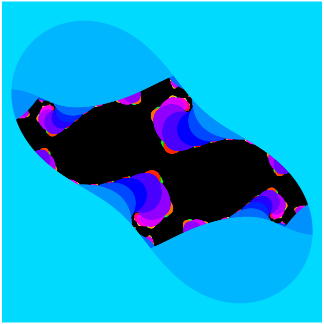
$$\mathcal{Q}_0 = \left\{p^{-1}(z) \colon |z| = 1\right\}$$





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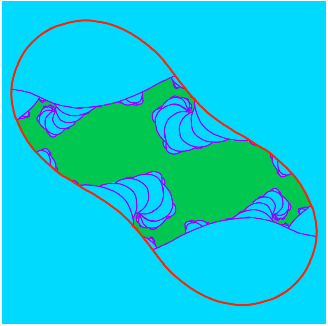

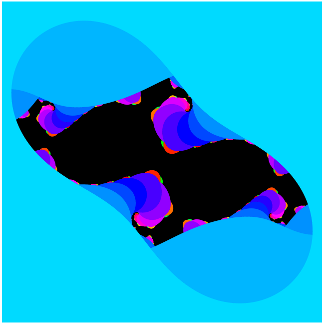

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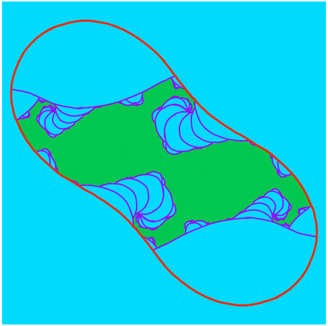
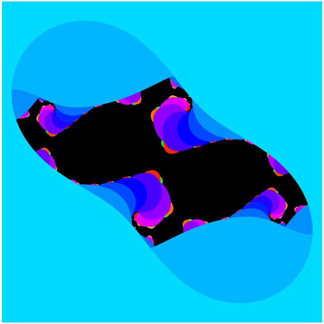




$$\lim_{n\to\infty} K(f_n) = K_q \cup \bigcup_{j\geq 0} \mathcal{Q}_j$$

Generalization

$$f_n(z) = (p(z))^n + q(z)$$

Theorem (Kaschner, Kapiamba, & W.; 2023)

If p, q are polynomials with $\deg p, q \geq 2$, and q is hyperbolic with no attracting periodic points on $\partial p^{-1}(\overline{\mathbb{D}})$, then

$$\lim_{n\to\infty} K(f_{n,p,q}) = K_q \cup \bigcup_{j\geq 0} \mathcal{Q}_j$$

A brief thread through history... and the future

2012 [1] Boyd & Schulz: $f_n(z) = z^n + c$.

2015 [6] Kaschner & Romero & Simmons: $f_n(z) = z^2 + c$.

2020 [2] Brame & Kaschner: $f_n(z) = z^n + q(z)$.

2023 Kaschner, Kapiamba, & W.: $f_n(z) = (p(z))^n + q(z)$.

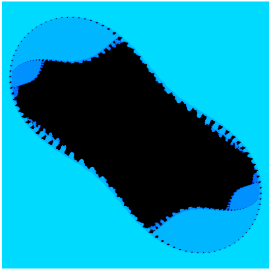
2024 Kaschner, Kapiamba, & W.: $g_n(z) = p^n(z) + q(z)$.

Current work

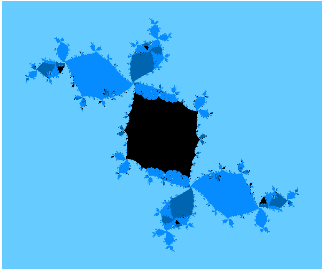
$$\begin{array}{ccc} (p(z))^n & \neq & p^n(z) \\ \text{powers} & & \text{iterates} \end{array}$$

Behold, for

- $p(z) = z^2 - 0.1 + 0.75i$,
- $q(z) = z^2 - 0.1 + 0.2i$;
- $n = 51$;



$f_n = (p(z))^n + q(z)$



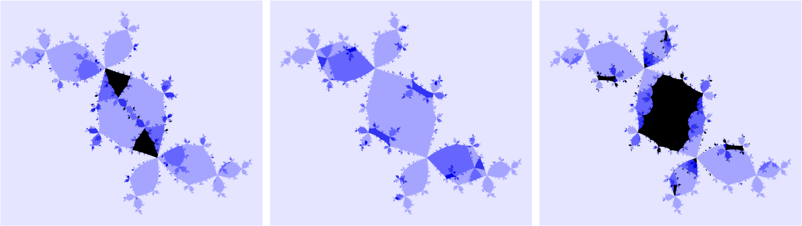
$g_n = p^n(z) + q(z)$

Immediate issues with subsequential limits

$$g_n(z) = p^n(z) + q(z)$$

$$p(z) = z^2 - 0.123 + 0.745i \quad q(z) = z^2 - 0.2 - 0.3i$$

$K(g_n)$ for $n = 49, 50, 51$.



Immediate issues with subsequential limits

$$g_n(z) = p^n(z) + q(z)$$

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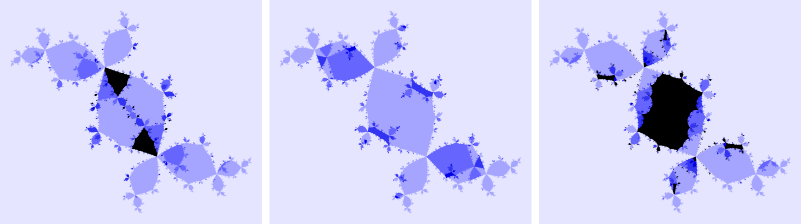

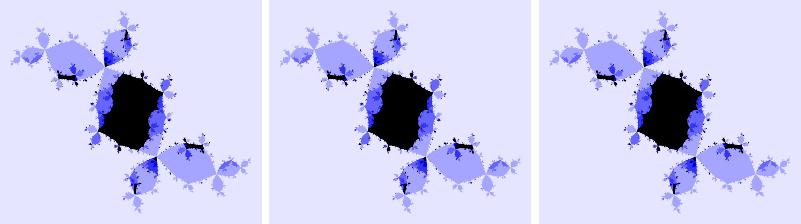

$K(g_n)$ for $n = 54, 57, 60$.

Escaping the Rabbitverse

- Suppose p is hyperbolic with periodic attracting cycle $z_1, \cdots, z_k$.
- For each n there exists $\ell \in \{1, \cdots, k\}$ such that

$$g_n(z) = p^{km+\ell} + q(z) \approx z_\ell + q(z)$$

- Define $\hat{g}(z) \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ via

$$\hat{g}(z) = \begin{cases} q(z) + \lim_{m\to\infty} p^{n_m} & z \in \mathrm{int}\, K(p) \\ p(z) & z \in \mathcal{J}(p) \\ \infty & z \in \hat{\mathbb{C}} \backslash K(p) \end{cases}$$

Major Results

Theorem

For any polynomials p, q,

$$\partial K(\hat{g}) \subseteq \liminf_{m\to\infty} K(g_{n_m}) \subseteq \limsup_{m\to\infty} K(g_{n_m}) \subseteq K(\hat{g})$$

Theorem

$\lim_{n\to\infty} K(g_{n_m}) = K(\hat{g})$ *if*

- p *hyperbolic, and*
- $\operatorname{int} K(\hat{g})$ *is comprised of attracting basins for* $\hat{g}$.

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THANK YOU!

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