Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 0.2–3

0 Introduction

0.2 Introduction to differential equations

Differential Equation. A *differential equation* is an equation with a derivative in it.

Example 1.

$$\frac{d^2x}{dt^2} + x\frac{dx}{dt} = 6t$$

- What is x? dependent variable
- What is t? independent variable

$\frac{dy}{dx}$ vs y' vs $\dot{y}$

$\frac{d^2y}{dx^2}$ vs y'' vs $\ddot{y}$

$$y'' + xy' = 6x$$

- What's the difference between this differential equation and the one before it?

Higher order! Also, old (calc 1)

Solution. A *solution* for a differential equation is a function that satisfies the equation (makes the equation true). Any single solution is called a *particular solution*. The set of all solutions is called the *general solution*.

Example 2. The differential equation

$$y' = 3x^2$$

is very boring. Why?

- A particular solution is specified constants
- The general solution is unknown constants (of integration) i.e. all solutions

Why is the equation in Example 1 *much* harder to solve?

ODE: single independent variable

PDE: multiple independent variables

We will learn when and how differential equations can be solved analytically (almost never).

Barring that, we will learn how to approximate and use solutions.

[1] We should probably come up with some more specific terminology.

Who cares about these things? Right.

$$
\begin{tikzcd}
 & \text{Real world problem} \arrow[dl, "\text{abstraction}"'] & \\
\text{Mathematical Model} \arrow[rr, "\text{solve}", "\text{(approximate)}"'] & & \text{Mathematical Solution} \arrow[ul, "\text{interpretation}"']
\end{tikzcd}
$$

Example 3. $P(t) = Ce^{kt}$ is the general solution for $\dfrac{dP}{dt} = kP$. Check this.

$$\frac{dP}{dt} = Cke^{kt} = k\left(Ce^{kt}\right) = kP \checkmark$$

- What does this have to do with the flow chart above?

Example 4. Show $y = \cosh t = \frac{1}{2}(e^t + e^{-t})$ is a particular solution for $\dfrac{d^2y}{dt^2} - y = 0$ on the interval $(-\infty, \infty)$.

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(e^t + (-(-e^{-t}))\right) = \frac{1}{2}\left(e^t + e^{-t}\right) = y \Rightarrow \frac{d^2y}{dt^2} = y \checkmark$$

Example 5. For what values of r is $y = e^{rt}$ a solution for $y'' + y' - 6y = 0$?

$$r^2e^{rt} + re^{rt} - 6e^{rt} = 0$$

$$(r-2)(r+3) = 0$$

$$\boxed{r = 2, -3}$$

0.3 Classification of differential equations

Here is a terrible wall of definitions. Enjoy!

Order. The *order* of a differential equation is the order of the highest derivative that appears in the equation. More specifically,[a] a differential equation of order n is of the form

$$F\left(t, x(t), \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}\right) = 0,$$

where F is a function.

[a] or is this more generally?

Autonomous. If F (as above) is independent of t, the differential equation is called *autonomous*. Otherwise, it is called *nonautonomous*.

the independent variable

Linear and homogeneous. A differential equation of order n is called *linear* if it is of the form

$$F\left(t, x(t), \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}\right) = a_n(t)\frac{d^n x}{dt^n} + \cdots + a_1\frac{dx}{dt} + a_0 x + b(t),$$

where the a_i's and b are all functions of t. If $b(t) = 0$, then the differential equation is called *homogeneous*; otherwise, it is called *nonhomogeneous*.

"What is all this madness?" you may ask. Well, different classifications of differential equations require different techniques and strategies.

Example 6. Classify the following differential equations:

- $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y - \sin t = 0$

  Non-autonomous : not indep of t
  Linear : l.c. of $\frac{d^n x}{dt^n}$
  2^{nd} order
  non homogeneous : $h(t) \neq 0$

- $y'' + yy' = 0$

  2^{nd} order
  Autonomous
  non linear

Eg: $y' = yx$ Problem w/ integrating directly

Eg $y' = xe^x$ or $\frac{dy}{dx} = xe^x$

$Y = \int xe^x dx = \boxed{xe^x - e^x + c}$ General Solution

$= e^x(x-1) + c$

+	x	e^x
−	1	e^x
+	0	e^x

↳ initial condition
$Y_0 = Y(X_0)$ to identify a particular solution

An ODE w/ initial conditions is an Initial Value Problem (IVP)

$Y(0) = 0$

$\to e^0(0-1) + c = 0 \to c = 1$

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Sections 1.1–2

1 First order ODEs

In case no one mentioned it, and *ODE* is an ordinary differential equation, which is just a differential equation with no partial derivatives (those are called PDEs). The word "ordinary" is just used to let you know that since there are no partial derivatives, you won't have to do anything too silly. While this course deals exclusively in ODEs, we maintain the right to do silly things.

1.1 Integrals as solutions

Which is easier to solve?

- $\dfrac{dy}{dx} = f(x, y)$
- $\dfrac{dy}{dx} = f(x)$

Why?

Example 1. Solve $y' = xe^x$. What do you need to identify a single particular solution?

Example 2. Solve $y' = xe^x$, $y(0) = 0$.

IVP. An *IVP*, or *initial value problem*, is an ODE with enough initial conditions to identify a single particular solution.

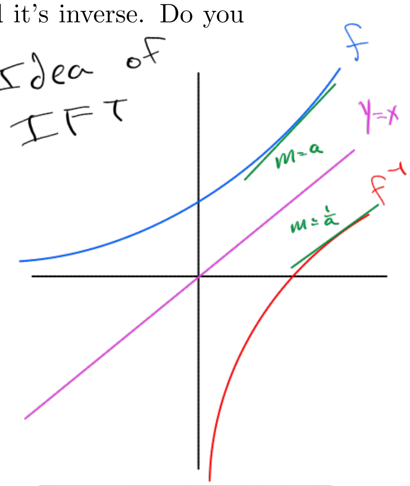
Can we solve $\dfrac{dy}{dx} = f(y)$? Why is this harder?

Here's a fun fact from Calculus 1 that will help:

> **Inverse Function Theorem.** If $y(x)$ is continuously differentiable and has a nonzero derivative at x_0, then
> $$\left(y^{-1}\right)'(y(x_0)) = \frac{1}{y'(x_0)}.$$
> That is, the derivative of the inverse at $y(x_0)$ is the reciprocal of the derivative at x_0.

This is a really neat theorem. Draw the graph of a nonlinear one-to-one function and it's inverse. Do you see why this theorem is true?

Idea of IFT



Don't forget that $\dfrac{dy}{dx} = f(y)$. When $x(y) = y^{-1}$ is differentiable, we have

$$\frac{dx}{dy} = (y^{-1})' \quad \underline{\text{and}} \quad \frac{1}{y'} = \frac{1}{f(y)}$$

(not precise, but ignore it)

$$(y' = f(y))$$

Then from the Inverse Function Theorem, we know that

IFT

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

(when y is continuously differentiable and has a nonzero derivative). Now we can just

.

See next page for Sol'n

Example 3 (Exercise 1.1.6)**.** Solve $y' = (y-1)(y+1)$, $y(0) = 3$.

$\frac{dy}{dx}$

Not integrable

$$\frac{dy}{dx} = (y-1)(y+1)$$

$$\frac{dx}{dy} = \frac{1}{(y-1)(y+1)}$$

$$\int \frac{dx}{dy}\, dy = \int \frac{1}{(y-1)(y+1)}\, dy$$

$$1 = A(y+1) + B(y-1)$$

$$B = \tfrac{-1}{2} \quad A = \tfrac{1}{2}$$

$$\frac{1}{2}\int \frac{1}{y-1}\, dy - \frac{1}{2}\int \frac{1}{y+1}\, dy$$

$$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| + c$$

Solve for y

$$2x - 2c = \ln|y+1| - \ln|y-1|$$

$$2x - 2c = \ln\left|\frac{y+1}{y-1}\right|$$

$$e^{2x-2c} = \frac{y+1}{y-1}$$

$$ye^{2x-2c} - e^{2x-2c} = y+1$$

$$y = \frac{1 + e^{2x}\cancel{e^{-2c}}\, C_0}{e^{2x}\cancel{e^{-2c}}\, C_0 - 1}$$

General Solution

$$e^{-2c} = C_0$$

$$y(0) = 3 \to 3 = \frac{1+C_0}{C_0 - 1} \to 3C_0 - 3 = 1 + C_0 \to C_0 = 2$$

$$y = \frac{1 + 2e^{2x}}{2e^{2x} - 1}$$

particular solution

$$y' = (y-1)(y+1) \; ; \; y(0) = 3$$

$$\frac{dy}{dx} \xrightarrow[\text{Theorem}]{\text{Inverse Function}} \frac{dx}{dy} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

Partial Fractions

$$\Rightarrow 1 = A(y+1) + B(y-1)$$

$$y \doteq 1 \Rightarrow 1 = 2A \Rightarrow A = \tfrac{1}{2}$$

$$y \doteq -1 \Rightarrow 1 = -2B \Rightarrow B = \tfrac{-1}{2}$$

$$\Rightarrow x = \int \frac{dx}{dy}\,dy = \int \frac{1}{(y-1)(y+1)}\,dy = \frac{1}{2}\int \frac{1}{y-1}\,dy - \frac{1}{2}\int \frac{1}{y+1}\,dy$$

$$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| + c$$

$$\Rightarrow 2x - 2c = \ln\left|\frac{y-1}{y+1}\right|$$

$$\Rightarrow e^{2x}\underbrace{e^{-2c}}_{C_0} = \frac{y-1}{y+1} \Rightarrow C_0 e^{2x} y + C_0 e^{2x} = y - 1$$

$$\Rightarrow (C_0 e^{2x} - 1)y = -(1 + C_0 e^{2x}) \Rightarrow y = \frac{1 + C_0 e^{2x}}{1 - C_0 e^{2x}}$$

$$\underline{y(0) = 3} \Rightarrow 3 = \frac{1 + C_0 e^{2(0)}}{1 - C_0 e^{2(0)}}$$

$$\Rightarrow 3 - 3C_0 = 1 + C_0 \Rightarrow 2 = 4C_0 \Rightarrow C_0 = \tfrac{1}{2}$$

Check: $\dfrac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = {}^{3/2}\!/_{1/2} = 3$ ✓

$$\Rightarrow y = \frac{1 + \frac{1}{2}e^{2x}}{1 - \frac{1}{2}e^{2x}}\left(\frac{2}{2}\right) = \boxed{\frac{2 + e^{2x}}{2 - e^{2x}}}$$

1.2 Slope fields

Recall that, in general, first order equations are of the form

$$y' = f(x, y),$$

where f is any function you like, depending on *both* x and y. If f depends on just one of these variables, we saw in the last section that you can just integrate to solve.

What does the equation $y' = f(x, y)$ mean? It takes x and y values and assigns (by f) a value to y', often interpreted as *slope*. That is,

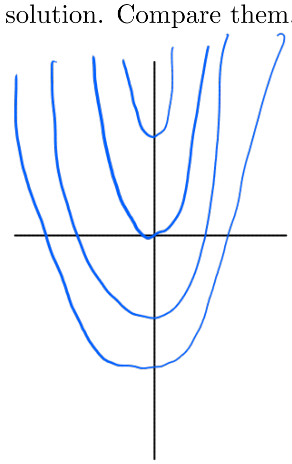
We can graph this!

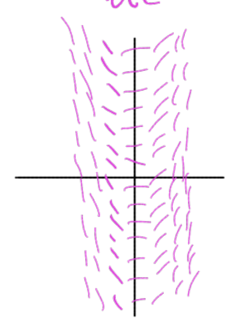
Example 4. Let $y' = 2x$. Plot the slope field by hand and find the general solution. Compare them.

$y = x^2 + C$ as general solution

ODE relate slope to "indep. and dep fxn values"

$\frac{dy}{dx} = f(x,y)$





Google "bluffton slope field" and plot a slope field by way of internet.

Example 5. Plot a slope field (via computer) for $y' = x/y$. Beware computers.

What's wrong here?

Problems!
infinite slope?
Multiple solns?

Example 6. Plot a slope field (via computer) for $y' = 2\sqrt{|y|}$. Beware intuition.

What's wrong here?

Given a problem, there are two basic questions:

1.
2.

$$\frac{dy}{dx} = f(x,y) \text{ is ODE}$$

> **Picard's Theorem.**[a] If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to the IVP
> $$y' = f(x,y), \quad y(x_0) = x_0$$
> exists near x_0 and is unique.
>
> [a] Also commonly referred to as the Fundamental Theorem of Existence and Uniqueness (FEU)

Example 7. $x' = x^{1/3}$, $x(0) = 0$ is a sufficiently simple-looking IVP, right? Show $x = 0$ is a solution, and for any nonnegative real α,

$$x(t) = \begin{cases} \left(\frac{2}{3}t\right)^{3/2}, & |t| < \alpha \\ 0, & t \le -\alpha, \alpha \le t \end{cases}$$

is also a solution. There are an uncountable number of solutions to this IVP.

What is happening here?

$$f(t,x) = x^{1/3} \qquad \frac{\partial f}{\partial x} = \frac{1}{3} x^{-2/3} \leftarrow$$

not defined @ $x=0$, which is the initial condition

Example 8. Show $y' = 1 + y^2$, $y(0) = 0$ has a unique solution $y = \tan x$ on $(-\pi/2, \pi/2)$.

$$\left.\begin{aligned} \frac{dy}{dx} &= \sec^2 x = 1 + \tan^2 x \quad \checkmark \\ y(0) &= \tan(0) = 0 \quad \checkmark \end{aligned}\right\} \text{soln to IVP}$$

Picard's Theorem

$$f(x,y) = 1 + y^2$$

$$\frac{\partial f}{\partial y} = 2y \quad \text{Continuous everywhere!}$$

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Sections 1.3

Recall in Section 0.2–3 we agreed $\frac{dy}{dx} = f(x, y)$ tends to be harder than $\frac{dy}{dx} = f(x)$, That doesn't mean they are impossible.

> **Seperable Equation.** An first order ODE is *separable* if it can be written as $y' = f(x)g(y)$, where f and g are functions

Separable equations can be solved with Integration!

1.3 Separable equations

How can we manipulate $\frac{dy}{dx} = f(x)g(y)$ to solve the ODE?

Do you want to just multiply dx by both sides? What does that even mean?

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

$y = h(x)$ $\quad dy = h'(x)dx$

$\frac{1}{g(h(x))} h'(x) = f(x) \Rightarrow \int \frac{1}{g(h(x))} h'(x)dx = \int f(x)dx$

$$\boxed{\Rightarrow \int \frac{1}{g(y)} dy = \int f(x) dy}$$

Despite the wondrous power of separable equations, there is still one minor issue. What happens when we can integrate, but we can't solve for y in a reasonable way?

> **Implicit Solutions.** A solution to an ODE not of the explicit form $y = h(x)$.

Example 1. Solve $(1 + x)dy - ydx = 0$. $\Rightarrow (1+x)dy = ydx \Rightarrow \frac{1}{y}dy = \frac{1}{1+x}dx$

$\Rightarrow \ln|y| = \ln|1+x| + C$

$y = C_0(1+x)$

We may not want to, but we can actually solve for y for this solution. Let's do that.

$$\frac{1}{\cot y}\,dy = \frac{x}{\sec x}\,dx \Rightarrow \int \tan y\,dy = \int x\cos x\,dx$$

$$\ln|\sec y| = x\sin x + \cos x + C$$

$$\sec y = C_0 \exp(x\sin x + \cos x)$$

Example 2. Solve $\sec(x)dy = x\cot(y)dx$

Example 3. You've found a dead body! Its temperature is 88.6° F at 2am and 78.6° F at 3am. The ambient air temperature is 68.6° F from midnight to 3am. Estimate the time of death.

$$\frac{dT}{dt} = K(T - T_{air})$$

Newton's Law of Cooling

T temp (°F)

t time (h)

Find murder o' Clock

$T(0) = 98.6$ cooling began

$K > 0$

$$\frac{dT}{dt} = K(T - 68.6) \rightarrow \frac{1}{T-68.6}\,dT = -k\,dt$$

$$\ln|T - 68.6| = -kt + C$$

$$\rightarrow T - 68.6 = C_0 e^{-kt}$$

$$T = C_0 e^{-kt} + 68.6$$

$$T(0) = C_0 + 68.6 \rightarrow C_0 = 30$$

$$T(t) = 30e^{-kt} + 68.6$$

$$88.6 = T(t_0) = 68.6 + 30e^{-kt_0}$$

$$78.6 = T(t_0+1) = 68.6 + 30e^{-k(t_0+1)}$$

$$\frac{2}{3} = e^{-kt_0} \rightarrow \ln\frac{2}{3} = -kt_0$$

$$\rightarrow k = \frac{-1}{t_0}\ln\frac{2}{3}$$

$$\frac{1}{3} = e^{-k(t_0+1)} = e^{\frac{1}{t_0}\ln(\frac{2}{3})(t_0+1)}$$

$$= e^{\frac{1}{t_0}(t_0\ln\frac{2}{3} + \ln\frac{2}{3})}$$

$$= e^{\ln(\frac{2}{3}) + \frac{1}{t_0}\ln\frac{2}{3}}$$

$$= \frac{2}{3}e^{\frac{1}{t_0}\ln\frac{2}{3}}$$

$$\rightarrow \frac{1}{2} = e^{\frac{1}{t_0}\ln\frac{2}{3}}$$

$$\rightarrow \ln\frac{1}{2} = \frac{1}{t_0}\ln\frac{2}{3} \rightarrow t = \frac{\ln 2/3}{\ln 1/2} \approx 0.5849\text{ h} \sim 35\text{ min}$$

$$\boxed{\text{MURDER O'CLOCK } 1{:}25\text{ AM}}$$

See next page for Sol'n →

T = body tempurature (°F)

t = time (h)

Newton's Law of Cooling: $\frac{dT}{dt} = k(T - T_{air})$

$$\rightsquigarrow \frac{dT}{T-68.6} = k\,dt \Rightarrow \ln|T-68.6| = kt + c$$

$$\Rightarrow T - 68.6 = C_0 e^{kt}$$

$$\Rightarrow T = C_0 e^{kt} + 68.6$$

Let $t=0$ be the time the body started cooling (98.6°F)

$$\Rightarrow 98.6 = C_0 e^{k(0)} + 68.6 \Rightarrow C_0 = 30$$

We have a system of 2 unknowns and 2 variables

$$\begin{cases} 88.6 = 30e^{kt} + 68.6 \longrightarrow \frac{2}{3} = e^{kt} \\ 78.6 = 30e^{k(t+1)} + 68.6 \longrightarrow \frac{1}{3} = e^{k(t+1)} = e^{kt}e^{k} \end{cases}$$

$$\frac{1}{3} = \frac{2}{3}e^{k} \rightarrow e^{k} = \frac{1}{2} \rightarrow k = \ln\left(\frac{1}{2}\right)$$

$$\longrightarrow \frac{2}{3} = e^{(\ln\frac{1}{2})t} \rightarrow \frac{2}{3} = \left(\frac{1}{2}\right)^{t} \rightarrow t = \frac{\ln 2/3}{\ln 1/2} = 0.5849\text{ h} \sim 35\text{ min}^{*}$$

$*$ = time after start of cooling

$$\Longrightarrow \text{Murder O'Clock} = 2\text{am} - 35\text{m} = \boxed{1:25\text{ am}}$$

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Sections 1.4

Recall in the last section we looked at some *"easy"* cases of $y' = f(x, y)$. Here's a slightly less easy one.

> **First Order Linear Equation.** An ODE of the form
> $$\frac{dy}{dx} + P(x)y = f(x)$$
> is called *first order linear.* Additionally, we call this standard form for the first order linear equation.

1.4 First Order Linear Equations

How can we solve $\dfrac{dy}{dx} + P(x)y = f(x)$?

$$q(x)\frac{dy}{dx} + p(x)y + g(x) = 0$$

1) Divide by $q(x)$

$$\rightarrow \frac{dy}{dx} + \frac{p(x)}{q(x)}y = \frac{-g(x)}{q(x)}$$

$$\rightarrow \boxed{\frac{dy}{dx} + P(x)y = f(x)}$$

standard form

Looks vaguely like a product rule

There are 5 easy steps!

1. Write in Standard Form
2. Find integrating factor

   $$r(x) = \mu(x) = e^{\int P(x)\,dx}$$
3. Multiply both sides of standard form by μ

   $$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

$$\frac{d}{dx}[\mu(x)y] = \mu'(x)y + \mu(x)y'$$
$$= P(x)\mu(x)y + \mu(x)\frac{dy}{dx}$$

4. Undo product rule

   $$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x)$$
5. Integrate!

   $$\int \frac{d}{dx}[\mu(x)y]\,dx = \int \mu(x)f(x)$$
   $$\parallel$$
   $$\mu(x)y$$

$$\rightsquigarrow y = \frac{\int \mu(x)f(x)}{\mu(x)}$$

Let's look at an example!

Example 1. Find a general solution and find an interval on which the solution is defined.

$$\frac{dy}{dx} = y + e^x \qquad \Leftrightarrow \quad y' - y = e^x$$

$$\mu(x) = e^{\int P(x)dx} = e^{\int -dx} = e^{-x}$$

$$\to \quad e^{-x}y' - e^{-x}y = \cancel{e^{-x}e^{x}}\ 1$$

$$\frac{d}{dx}\left[e^{-x}y\right] = 1 \to e^{-x}y = x + c \to \boxed{y = xe^x + ce^x}$$

Example 2. Solve $x\,dy = (x\sin(x) - y)\,dx$

$$x\frac{dy}{dx} = x\sin x - y \to xy' + y = x\sin x \to y' + \frac{y}{x} = \sin x \quad (x \neq 0)$$

$$\mu = e^{-\int \frac{1}{x}dx} = e^{\ln|x|} = x \rightsquigarrow xy' + y = x\sin x \to \frac{d}{dx}[xy] = x\sin x$$

$$\to xy = \int x\sin x\,dx \qquad \begin{array}{cc} +x & \sin x \\ -1 & -\cos x \\ 0 & -\sin x \end{array} \qquad \to xy = -x\cos x + \sin x + c$$

$$\to \boxed{y = -\cos x + \frac{\sin x}{x} + c\cdot\frac{1}{x}}$$

Example 3. Solve $y' = 2y + x(e^{3x} - e^{2x})$ given the initial condition of $y(0) = 2$.

$$y' - 2y = xe^{3x} - xe^{2x} \qquad \mu(x) = e^{-2x}$$

$$\frac{d}{dx}\left[e^{-2x}y\right] = xe^x - x \to e^{-2x}y = xe^x - e^x - \frac{1}{2}x^2 + c$$

$$\begin{array}{cc} +x & e^x \\ -1 & e^x \\ +0 & e^x \end{array}$$

$$\to \boxed{y = -\frac{1}{2}x^2e^{2x} + xe^{3x} - e^{3x} + 3e^{2x}}$$

$2 = -1 + c$

$\to c = 3$

plug back in

IVP

Example 4. Initially, 50 pounds is dissolved in a large tank holding **300 gallons of water**. A brine solution is pumped into the tank at a rate of **3 gallons per minute**, and the well-stirred solution is then pumped out at the same rate. If the concentration of the solution entering is **2 pounds per gallon**, determine the amount of salt in the tank at time t.

How much salt is present after 50 minutes? After a long time?

$A(t)$ amount of salt @ time t (lbs)

t time (min)

$$\frac{dA}{dt} = \left(\text{rate in} - \text{rate out}\right) = \left(\frac{3\text{ gal}}{\text{min}}\cdot\frac{2\text{ lbs}}{\text{gal}} - \frac{3\text{ gal}}{\text{min}}\cdot\frac{A(t)}{300}\frac{\text{lbs}}{\text{gal}}\right) = 6 - \frac{A(t)}{100}$$

$$A' + \frac{1}{100}A = 6$$

$$\mu = e^{0.01t} \rightsquigarrow A = 600 + ce^{-\frac{t}{100}} \to c = -550$$

Begin 12 Sept

Review: - Seperable $\frac{dy}{dx} = f(x)g(y)$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

- Linear $\frac{dy}{dx} + P(x)y = f(x)$

Let $\mu(x) = e^{\int P(x)\,dx}$

then $y = \frac{1}{\mu(x)} \int \mu(x) f(x)\,dx$

"The Day of Weird Subs"

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Sections 1.5

We have learned some really neat tricks to leverage separability and linearity and solve ODEs. When all of those things fail, here's the next thing you try:

> **Homogeneous ODE**. A first order ODE is called homogeneous if it can be written as
> $$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

You may notice that this word has been used before. We give a different definition here because it made sense to someone at some point. Use context to determine which version of "homogeneous" you're dealing with.

1.5 Substitution

How can we solve $xy' + y + x = 0$ with initial condition $y(1) = 1$?

Subtract x, divide by it!

$\rightarrow y' + \frac{1}{x}y = -1$ *

We could use linear techniques ... or substitute

$v = \frac{y}{x} \Rightarrow y = xv$

$y' = v + xv'$

$v + xv' + v = -1$

$v' + \frac{2}{x}v = \frac{-1}{x}$

this eqn is ~~also~~ linear ... so we got nowhere. Do Integrating factor method on either *

$\Rightarrow y = \frac{3-x^2}{2x}$ (use IVP to solve for C)

y is our dependent variable, so get rid of them all w/ v

Substitution problems are a lot like ice cream. They come in many flavors, and if you have too many, your brain freezes.

Example 1. Solve the IVP $2yy' + 1 = y^2 + x$, $y(0) = 1$.

Not linear!

$v = y^2 \rightarrow v' = 2yy'$

Some times substitutions help

$v' + 1 = v + x$

Linear!

$\Rightarrow v' - v = x - 1$

$\mu(x) = e^{\int -dx} = e^{-x}$

$+x$	e^{-x}
-1	$-e^{-x}$
$+0$	e^{-x}

$\rightarrow v = \frac{1}{e^{-x}} \int (xe^{-x} - e^{-x})dx$

$= \frac{1}{\cancel{e^{-x}}}\left(-x\cancel{e^{-x}} \cancel{-e^{-x}} \cancel{+e^{-x}} + c\right)$

$= ce^{x} - x$

$y^2 = Ce^{x} - x$

$\Rightarrow 1 = Ce^{0} - 0$

$\Rightarrow y^2 = e^{x} - x$

You may find it helpful to know the contents of this chart:

If you see...	Try this substitution!
xy'	$v = \frac{y}{x}$
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Example 2. Bernoulli's Equation!

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \quad n \in \mathbb{R}$$

sub: $v = y^{1-n}$ NOT $n-1$, easy mistake

$v' = (1-n)y^{-n}y'$

divide by y^n

$y' + P(x)y = f(x)y^n$

$y^{-n}y' + P(x)y^{1-n} = f(x)$

$\frac{1}{1-n}v' + P(x)v = f(x)$

$v' + (1-n)P(x)v = f(x)(1-n)$ Linear!

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 1.6

Idea: we can qualitatively study autonomous equs w/o solving them!

Recall,

> **Autonomous Equations.** First order autonomous ODEs are of the form
> $$\frac{dx}{dt} = f(x)$$
> No indep variables as function input!

Also recall,

> **Newton's Law of Cooling.**
> $$\frac{dx}{dt} = -k(x - A)$$
> Autonomous

Note that $x = A$ is a constant solution to any Newton's Law of Cooling problem.

1.6 Autonomous Equations

Constant solutions for an ODE are called equilibrium solutions (or equilibria solutions if you have more than one).

Any point x_0 on the x-axis where $\frac{dx}{dt} = f(x_0) = 0$ is called a critical point. Why?

derivative is zero! See Calc. 1

> **Stability of Equilibria.**
>
> An equilibrium is *stable* (or attracting) if nearby solutions approach it as $t \to \infty$.
> *unstable* (or repelling) if nearby solutions move away from it as $t \to \infty$.
>
> Equilibria that are not stable or unstable are called *shunt* (or indifferent).

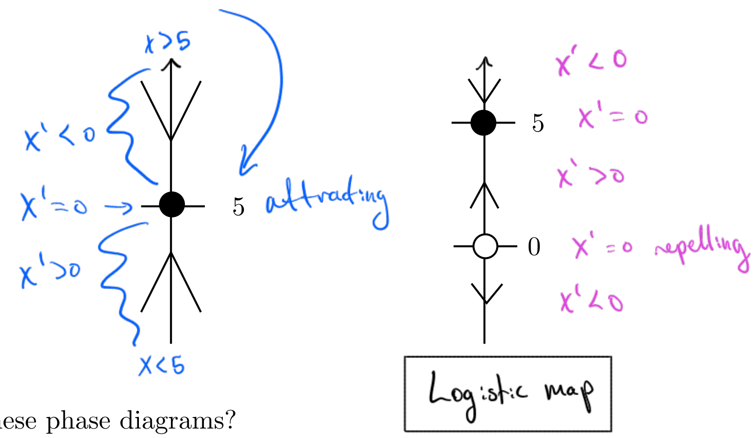
Goal: understand behavior of autonomous equs through the study of critical/equilibrium points

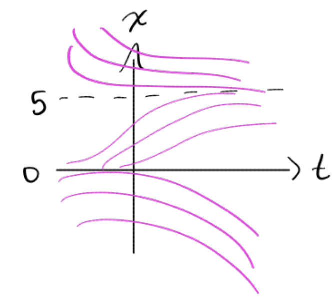
Compare the phase diagrams or phase portraits of the following ODEs equilibria.

$$x' = -0.3(x-5) \quad \text{and} \quad x' = 0.1x(5-x)$$

Think first derivative test

$x'=0$ @ $x=5$ $\qquad$ $x'=0$ @ 0 and 5





How do we construct these phase diagrams?

1.

2.

3.

4.

Example 1. Logistic growth with harvesting:

$$\underbrace{\frac{dx}{dt} = kx(M-x)}_{\text{general logistic growth}} - h \text{ where } k=1 \text{ and } M=2$$

$\hookrightarrow x' = x(2-x) - h$

$= -x^2 + 2x - h$

$\dfrac{-2 \pm \sqrt{4-4h}}{-2}$

K = growth constant

M = carrying capacity

h = harvesting parameter

$\rightarrow x_\pm = 1 \pm \sqrt{1-h}$

← Bifurcation theory!