MA398 Actuarial Mathematics 2

Multistate Markov Models and Applications

DHW Section 8.1, part 1 – Multistate Models & Markov Processes – Discrete Case

Review/Warmup:

Let T_x be the future lifetime random variable for (x). Recall the notation…

$_tp_x =$ $S_x(t) =$

$_tq_x$

We omit the t in the case $t = 1$ year.

Relationships between v, d, δ, i:

Example 1: A fully discrete two-year term insurance on (50). The insurance pays 10 at the end of the year of death.

$p_{50} = .99$; $p_{51} = .98$; $i = .05$

Find expressions for…

a. The net single premium.

b. The level annual premium.

c. The value of $_0L$ (the future loss at time 0) if the insured dies (i) at time $t = 1/3$; (ii) at time $t = 1.5$.

d. The values of $_0V$ (cjw: illustrate the def'n and the Equiv. Pr.) and $_1V$.

Illustration 2: Multistate models.

a. Alive-dead model.

b. An insured life (x) has a whole life policy. He/she could do any of the following:

(0) Live another year and pay another premium.
(1) Live another year but let the policy lapse.
(2) Die before the next birthday.

c. (x) and (y) are married, and they buy a policy that pays

- \$B to (x) if (y) dies first and
- \$C to (y) if (x) dies first.

One could assume either that

- (x) and (y) are independent lives, or that
- T_x and T_y are not independent.

d. Multistate annuity models:

- A **joint life annuity**: payable until the first death of a group of lives (payable while the *joint status* is intact—think about our $x{:}\overline{n|}$ notation)
- A **last survivor annuity**: payable until the last death of a group of lives
- A **reversionary annuity**, which makes payments that begin on the death of a specified life, e.g. paying B_1 per year to the wife while the husband is dead but wife is alive, paying B_2 per year to the husband while the wife is dead but the husband is alive

e. A person could leave employment by retiring, getting fired, or dying. The resulting cash flows (pension, continuing health benefits, etc.) will be different in each case.

f. A person could die by "Cause 1", "Cause 2", or "other causes".

Consider the Makeham model $\mu_x = A + Bc^x$.

g. Temporary disability model: A person could be in any of three states:

(0) Alive and well
(1) Sick/Disabled
(2) Dead

An insurance policy might suspend premium billing or even provide disability income during periods of disability, plus a death benefit. So there will be differences in cash flows anytime a transition occurs between these states.

Variation: Permanent disability model.

<u>DHW (temporary) Notation</u>:

Let $\mathrm{Y}(t)$ denote the state of a model at time t.

We say that the set $\{\mathrm{Y}(t) \mid t \geq 0\}$ of random variables is a **continuous time stochastic process**.

<u>Three Permanent Assumptions for multistate models</u>

<u>(1)</u> We assume $\{\mathrm{Y}(t) \mid t \geq 0\}$ is a **Markov process** (or satisfies the Markov property):

For any states i and j and for any three points in time $0 < t_0 < t_1 < t_2$,

$$\Pr[\mathrm{Y}(t_2) = j \mid \mathrm{Y}(t_1) = i]$$

does <u>not</u> depend on $\mathrm{Y}(t_0)$.

e.g. If the Markov property is satisfied by the weather, then
Pr[Snow tomorrow | Rain today] would <u>not</u> depend on yesterday's weather.

<u>Notation</u>: $\mathrm{o}(h) \leftarrow$ any function with the property that $\lim_{h \to 0} \frac{o(h)}{h} = 0$

<u>(2)</u> We assume that

Pr[Two or more transitions within a time period of length h] = $\mathrm{o}(h)$

i.e. We may ignore Pr[two transitions within any very short time interval].

(3) For states $i \neq j$,

Pr[Transition to state j during $[x, x + t]$ | $Y(x)$ = "state i"]

is a differentiable function of t.

Assumption (3) will allow us to define a "force of transition" μ_{x+t}^{ij} for $i \neq$ j.

Permanent Notation:

a. ${}_tp_x^{ij}$ = Pr[$Y(x + t) = j$ | $Y(x) = i$]

= Pr[(x) is **in** state j at time t | (x) observed in state i at time 0]

*Note that transitions to and from "state j" are permitted during $(x, x + t)$.

b. ${}_tp_x^{ii}$ = Pr[$Y(x + t) = i$ | $Y(x) = i$]

= Pr[(x) is **in** state i at time t | (x) observed in state i at time 0]

*Note that transitions in and out of "state i" are permitted during $(x, x + t)$.

c. ${}_tp_x^{\overline{ii}}$ = Pr[$Y(x + s) = i$ for all $s \in [0, t]$ | $Y(x) = i$]

= Pr[(x) **remains** in state i throughout $[0, t]$ | (x) in state i at time 0.]

d. As always, we omit the t in the case $t = 1$ year.

Sometimes transition probabilities are collected in matrix form.

Convention: One-year transition probabilities p_x^{ij} are given in the (row-i , column-j) entry of a transition matrix.

Row and column numbering schemes sometimes begin with a "row/column 0".

Example 3: A discrete multistate example.

An insurance company issues a special fully discrete 2-year insurance to a high risk individual (x). The insurance pays a death benefit of B at the end of the year of death. You are given the following multistate model:

State 0: Alive & Well
State 1: Disabled
State 2: Dead $i = .06$

Annual transition probabilities p_{x+k}^{ij} for $k = 0, 1$ are given by the transition matrix

$$M = \begin{bmatrix} .7 & .2 & .1 \\ .1 & .6 & .3 \\ \_ & \_ & \_ \end{bmatrix}.$$

a. Find ${}_2p_x^{\overline{ii}}$ for $i = 0, 1, 2$ under the assumption that transitions occur at most once per year (we won't make that assumption for parts b, c, d).

b. Suppose that (x) is heathy at $t = 0$. Find the probability that (x) will be healthy at time $t = 2$. Also give the notation for this probability.

c. What is the largest death benefit B that can be funded by a single premium of 1000 if (x) is healthy at $t = 0$?

d. Suppose that the policy is funded by annual premiums of 1000, payable <u>only</u> if (x) is in the healthy state. What is the largest death benefit B that can be funded under this premium structure if (x) is in State 0 at time 0?

Homework 8.1 day 1:

152. An insurance company issues a special 3-year insurance to a high risk individual (x). You are given the following multi-state model:

(i) State 1: active
State 2: disabled
State 3: withdrawn
State 4: dead

Annual transition probabilities for $k = 0, 1, 2$:

i	p^{i1}_{x+k}	p^{i2}_{x+k}	p^{i3}_{x+k}	p^{i4}_{x+k}
1	0.4	0.2	0.3	0.1
2	0.2	0.5	0.0	0.3
3	0.0	0.0	1,0	0.0
4	0.0	0.0	0.0	1.0

(ii) The death benefit is 1000, payable at the end of the year of death.

(iii) $i = 0.05$

(iv) The insured is disabled (in State 2) at the beginning of year 2.

Calculate the expected present value of the prospective death benefits at the beginning of year 2.

Tip: Year 2 begins at $t = 1$ (always draw a timeline.) Answer: 439.91

Optional study problem:

151. For a multi-state model with three states, Healthy (0), Disabled (1), and Dead (2):

(i) For $k = 0, 1$:

$$p^{00}_{x+k} = 0.70$$
$$p^{01}_{x+k} = 0.20$$
$$p^{10}_{x+k} = 0.10$$
$$p^{12}_{x+k} = 0.25$$

(ii) There are 100 lives at the start, all Healthy. Their future states are independent.

Calculate the variance of the number of the original 100 lives who die within the first two years.

Tip: I smell a binomial random variable. What is the n? What is the p? Answer: $npq \approx 17$

<u>8.1 Multistate Models</u> – Day 2

<u>Recall</u>:

$${}_tp_x^{01} = \Pr[(x) \text{ is } \textbf{in} \text{ State 1 at time } t \mid (x) \text{ observed in State 0 at time 0}]$$

$${}_tp_x^{00} = \Pr[(x) \text{ is } \textbf{in} \text{ State 0 at time } t \mid (x) \text{ observed in State 0 at time 0}]$$

$${}_tp_x^{\overline{ii}} = \Pr[(x) \text{ } \underline{\textbf{remains}} \text{ in State } i \text{ throughout } [0, t] \mid (x) \text{ in State } i \text{ at time 0.}]$$

Additional Notation: $\ddot{a}_{x:\overline{n}|}^{ik}$ = EPV of the following n-year term arrangement:
$1 paid at start of any year where (x) is in state k, given a start in State i.

$A_{x:\overline{n}|}^{ik}$ = EPV of the following n-year term arrangement:
$1 paid at end of <u>each</u> year* in which a transition to State k occurs from another state occurs, given start in State i.

**This symbol is carefully defined in the continuous case by the SOA syllabus readings but not in the discrete case—in the discrete case, the only use of this symbol involves payment of a death benefit, which can occur at most one time per policy.*

<u>Example</u>: To illustrate this notation:

Consider a whole life disability income policy on a life whose current age is x. You use a model with (0) = Healthy; (1) = Temporary Disability; (2) = Dead. You make the simplifying assumption that only one transition occurs per year.

0 Healthy
1 Temp Dis
2 Dead

There is an expense that (valued at the end of a given year) is 100 for each transition into the disabled state.

The policy pays $30,000 at the start of any year if (x) is disabled at that time.

Write an expression for the EPV of the expenses and disability income payments if (x) is currently healthy.

$$\begin{tikzcd} \boxed{0} \arrow[r, shift left] \arrow[dr] & \boxed{1} \arrow[l, shift left] \arrow[d] \\ & \boxed{2} \end{tikzcd}$$

$$EPV = 100\, A_x^{01} + 30000\, \ddot{a}_x^{01}$$

Example Based on Fall 2018 LTAM Written #1

$0 \rightleftarrows 1$, $0 \to 2$, $1 \to 2$

(i) A Markov model with three states: Healthy (0), Sick (1), and Dead (2) is used to value the policy.

(ii) The annual probability transition matrix for an insured age $60+k, k=0,1,\cdots,9$ is:

"correct" / weird order

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.90-0.01k & 0.05 & 0.05+0.01k \\ 1 & 0.70-0.01k & 0.20 & 0.10+0.01k \\ 2 & 0 & 0 & 1 \end{array}$$

$k=0$ $\quad$ $k=1$

$${}_2M_{60} = \begin{bmatrix} .9 & .05 & .05 \\ .7 & .2 & .1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .89 & .05 & .06 \\ .69 & .2 & .11 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) (*2 points*) Calculate ${}_2p_{60}^{01}$.

$$= \begin{bmatrix} .8355 & .055 & .1095 \\ .761 & .075 & .164 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Consider a 50,000, fully discrete 10-year term insurance policy issued to a 60-year old.

Annual premiums of 5000 are payable only while in the healthy state (waived in the sick state).

Each year has maintenance expense of 150 at the start of the year for <u>all</u> policies in force.

k	$A^{02}_{60+k:\overline{10-k}\rvert}$	$A^{12}_{60+k:\overline{10-k}\rvert}$	$\ddot{a}^{00}_{60+k:\overline{10-k}\rvert}$	$\ddot{a}^{01}_{60+k:\overline{10-k}\rvert}$	$\ddot{a}^{10}_{60+k:\overline{10-k}\rvert}$	$\ddot{a}^{11}_{60+k:\overline{10-k}\rvert}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2	0.46667	0.49680	4.7328	0.2533	3.3340	1.4060
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Let ${}_2V^{(i)}$ denote the (conditional) EPV of future losses at time 2, given that (60) is in state i at that time. Show how to calculate ${}_2V^{(0)}$ and ${}_2V^{(1)}$.

Don't forget this

${}_2V^{(0)} = E[{}_2L \mid \text{in 0 @ } t=2] = 50000\,A^{02}_{62:\overline{8}\rvert} + 150\left[\ddot{a}^{00}_{62:\overline{8}\rvert} + \ddot{a}^{01}_{62:\overline{8}\rvert}\right] - 5000\,\ddot{a}^{00}_{62:\overline{8}\rvert}$
≈ 417.42

${}_2V^{(1)} = 50000\,A^{12}_{62:\overline{8}\rvert} + 150\left[\ddot{a}^{10}_{62:\overline{8}\rvert} + \ddot{a}^{11}_{62:\overline{8}\rvert}\right] - 5000\,\ddot{a}^{10}_{62:\overline{8}\rvert}$
≈ 8881

(c) Write equations relating (i) ${}_3V^{(0)}$, ${}_3V^{(1)}$, and ${}_2V^{(0)}$; (ii) ${}_3V^{(0)}$, ${}_3V^{(1)}$, and ${}_2V^{(1)}$.

Alive - Dead model

$({}_2V + \pi - exp)(1+i) = p_{x+2}\,{}_3V + q_{x+2}\cdot S$

$\left({}_2V^{(0)} + \underset{\pi}{5000} - \underset{exp}{150}\right)(1+i) = p^{00}_{62}\,{}_3V^{(0)} + p^{01}_{62}\,{}_3V^{(1)} + p^{02}_{62}\cdot S$

$\left({}_2V^{(1)} + \underset{\pi}{0} - 150\right)(1+i) = p^{10}_{62}\,{}_3V^{(0)} + p^{11}_{62}\,{}_3V^{(1)} + p^{12}_{62}\cdot S$

In-class practice examples:

From Spring 2019 LTAM:

4. The following probabilities have been calculated using a multiple state model with 3 states: Healthy (0), Disabled (1), and Dead (2).

x	p_x^{00}	p_x^{01}	p_x^{02}	p_x^{10}	p_x^{12}
70	0.60	0.30	0.10	0.10	0.15
71	0.50	0.30	0.20	0.10	0.25

a. Calculate the probability that a healthy life age 70 dies within 2 years.

$$p_{70}^{02} + p_{70}^{00} p_{71}^{02} + p_{70}^{01} p_{71}^{12} = .1 + (.6)(.2) + (.3)(.25) = \boxed{.295}$$

b.

You are given the following additional information:

(i) There are 100 Healthy lives, all age 70.

(ii) The future states of the 100 lives are independent.

(iii) N^d is the random variable representing the number of deaths within the first two years.

Calculate the standard deviation of N^d.

Binomial R.V. $\sqrt{100(.295)(1-.295)} = \boxed{4.56}$

From Fall 2018 LTAM:

4. For a three-state Markov chain model, you are given:

(i) The annual transition probability matrix is:

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.9 & 0.1 & 0.0 \end{pmatrix} \end{array}$$

(ii) The process is in State 1 at the start of year 1.

(iii) Transitions occur only at the end of the year.

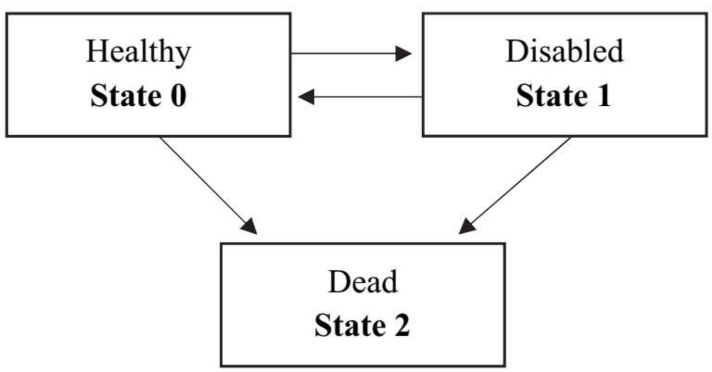
Calculate the probability that the process will be in State 1 after the transition at the end of year 3, without having transitioned to State 2.

$(.2)^2 + .2 \cdot .5 \cdot .9 + .5 \cdot .9 \cdot .2$

$\boxed{0.1880}$ ✓

From Spring 2019 LTAM:

9. A two-year term disability income insurance, issued to a Healthy life age x, offers a benefit of 25,000 at the end of each year if the policyholder is Disabled at that time.



You are given that:

$$p_{x+t}^{00} = 0.92, \quad p_{x+t}^{01} = 0.06, \quad p_{x+t}^{11} = 0.40, \quad \text{for } t = 0,1$$

Calculate the EPV of the insurance benefits at $i = 10\%$.

$25000\left(p_x^{01} v + \left(p_x^{00} p_{x+1}^{01} + p_x^{01} p_{x+1}^{11}\right) v^2\right) = \boxed{3000}$ ✓

From Fall 2018 LTAM:

16. A company issues a special insurance policy to (50) that pays 100,000 at the end of the year of death and doubles the benefit to 200,000 if the insured dies as the result of an accident.

You are given the following table of annual probabilities for this policy, where State 0 is Alive, State 1 is Death due to Accident, and State 2 is Death due to Causes Other Than Accident.

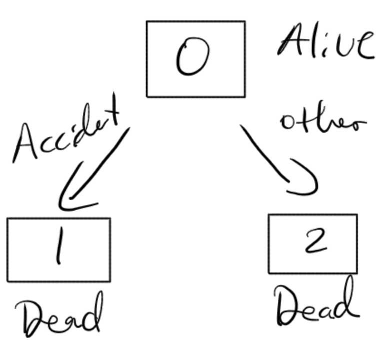
x	p_x^{00}	p_x^{01}	p_x^{02}
54	0.9905	0.0005	0.0090
55	0.9887	0.0005	0.0108
56	0.9866	0.0005	0.0129

Additionally, you are given the following information:

- $i = 0.04$
- Premiums of 1500 are paid annually.
- There are no expenses.
- The gross premium reserve at time t, for a life in State j at that time, is denoted ${}_tV^{(j)}$.
- ${}_5V^{(0)} = 2480$.

a. Draw and put English labels on a transition diagram for this multistate model.

b. Use an appropriate one-year recursion to calculate ${}_6V^{(0)}$.



$$({}_5V^{0} + \pi - exp)(1+i) = p_{55}^{00}\ {}_6V^{(0)} + \left\{ p_{55}^{01} \cdot 100000 + p_{55}^{02} \cdot 200000 \right.$$

$$(2480 + 1500)(1.04) = 0.9887\ {}_6V^{(0)} + \quad ''$$

$$\Rightarrow {}_6V^{(0)} = 2993.02$$

DHW 8.1, Day 3 – Multistate Markov models, continued – continuous-time Markov processes

Key background information for understanding transition forces in multistate models

$$\int_a^b f_{T_x}(x)\,dx$$

Consider the alive-dead model. $\underset{\text{Alive}}{\boxed{0}} \longrightarrow \underset{\text{Dead}}{\boxed{1}}$

Recall relationships between $f_{T_x}(t)$, $F_{T_x}(t)$, $S_{T_x}(t)$:

$$_tq_x = F_{T_x}(t) = Pr[T_x \le t] = 1 - Pr[T_x > t] = 1 - S_{T_x}(t) = \int_0^t f_{T_x}(x)\,dx$$

Definition/interpretation: Force of mortality: μ_x, μ_{x+t}

$$Pr[T_x \in (t, t+dt) \mid T_x > t] = \frac{F(t+dt) - F(t)}{S(t)} = \frac{f(t)\,dt}{S(t)} = \mu_{x+t}\,dt$$

Rearranging: Get formula for $f_{T_x}(t)$ – has useful interpretation *(draw timeline-style tree diagram).*

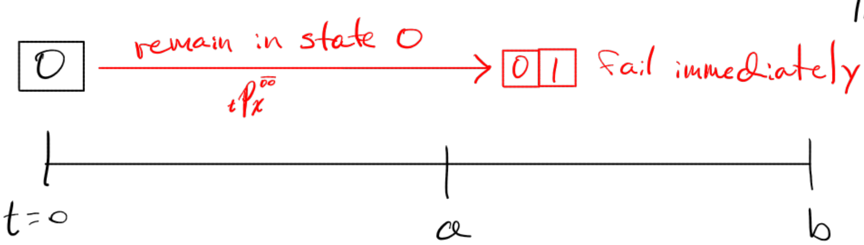
$$\Rightarrow f_{T_x}(t) = {}_tp_x\,\mu_{x+t}$$

cont. r.v.

$$\Rightarrow Pr[T_x \in (a,b)] = \int_a^b f_{T_x}(t)\,dt = \int_a^b {}_tp_x\,\mu_{x+t}\,dt$$

i.e. a sum of cases of live to $t \in (a,b)$, die w/i dt

Outline of how to get $S_x(t)$ from μ_{x+t}:

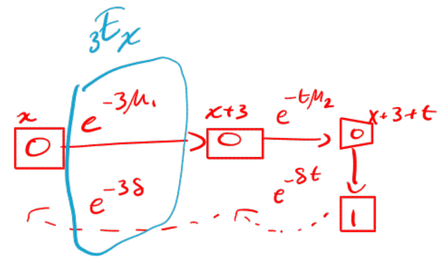


$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \Rightarrow \int \mu_{x+t}\,dt = \int \frac{f_x(t)}{S_x(t)}\,dt \overset{\text{u-sub}}{=} -\ln\left(S_x(t)\right)$$

$$\Rightarrow {}_{t_0}p_x = S(t_0) = e^{-\int_0^{t}\mu_{x+s}\,ds}$$

Hence: $f_{T_x}(t) = {}_tp_x\,\mu_{x+t}$ = ___________.

$$= e^{-\int_0^t \mu_{x+s}\,ds}\,\mu_{x+t}$$



MA 397 Review example:

Suppose that $\mu_{x+t} = \begin{cases} .03 & \text{if } t \le 3 \\ .04 & \text{if } t > 3 \end{cases}$ and $\delta = .05$.

a. Find the probability that (x) survives at least 4 years.

$${}_4p_x = e^{-\int_0^4 \mu_{x+t}\,dt} = e^{-\int_0^3 0.03\,dt}\; e^{-\int_0^1 0.04\,dt} = \boxed{e^{-0.13}}$$

$\hookrightarrow {}_3p_x \cdot p_{x+3}$

(*Remark: Survive the ".03" force for three years, then…*)

b. Find the expected present value of a whole life insurance benefit of 1 on (x), payable at the moment of death.

Here we are viewing our computation as the expected value of a ***function of*** T_x.

$$= \bar{A}^{1}_{x:\overline{3}|} + {}_3E_x\,\bar{A}_{x+3}$$

$$\bar{A}_x = \int_0^3 e^{-0.03t} e^{-0.05t}\, 0.03\,dt + e^{-0.24}\int_0^\infty e^{-0.04t} e^{-0.05t}\, 0.04\,dt$$

$$= 0.03\int_0^3 e^{-0.08}\,dt + 0.04e^{-0.24}\int_0^\infty e^{-0.09}\,dt = \frac{0.03}{-0.08}\, e^{-0.08t}\Big|_0^3 - \frac{0.04}{0.09} e^{-0.24} \cdot e^{-0.09t}\Big|_0^\infty$$

$=$ Smth

c. Find the expected present value of a whole life annuity on (x) that pays at a rate of c dollars per year.

Here we are viewing our computation as the ***sum of the expected values of a bunch of random variables, namely PV's of life-contingent payments of size*** $c \cdot dt$, *each occurring with probability* ${}_tp_x$.

$$c\,\bar{a}_x = c\,\bar{a}_{x:\overline{3}|} + c\,{}_3E_x\,\bar{a}_{x+3}$$

$$= c\int_0^3 e^{-\mu t} e^{-\delta t}\,dt + c\, e^{-3(\mu+\delta)}\int_0^\infty e^{-\mu t} e^{-\delta t}\,dt$$

for c dollars; if alive; discount; fraction of $1

d. Reminder concerning 2nd moments for the quantities in (b) and in (c). In both cases, we must find ${}^2\bar{A}_x$. *(We cannot use an integral to get the 2nd moment of the annuity pv random variable, because the random variables being summed in (c) are* *<u>not independent</u>* *and because those random variables do not individually represent possible values of the annuity itself—squaring them would* *<u>not</u>* *give us the square of a corresponding annuity value.)*

e. Find the present value of a fully continuous 3-year deferred whole-life benefit of 1 for (x).

<u>Continuous-time Markov processes/multistate models</u>

<u>Recall</u>: ${}_tp_x^{ij}$ = Pr[(x) is **<u>in</u>** state j **<u>at</u>** time t | (x) observed in state i at time 0]

${}_tp_x^{ii}$ = Pr[(x) is **<u>in</u>** state i **<u>at</u>** time t | (x) observed in state i at time 0]

${}_tp_x^{\overline{ii}}$ = Pr[(x) **<u>remains</u>** in state i throughout [0, t] | (x) in state i at time 0.]

We retain the assumptions we made earlier:

- Pr[2 transitions during [t, $t + h$]] = o(h)
- ${}_tp_x^{ij}$ is a differentiable function of t
- Markov property: For every x, t, i, and j, the transition probability ${}_tp_x^{ij}$ does <u>not</u> depend on any events/states that may have occurred prior to age (x).

<u>Definition</u>: For $j \neq i$, we define $\mu_x^{ij} = \lim_{h \to 0^+} \frac{{}_hp_x^{ij}}{h}$. error term of o(h)

<u>Intuition</u>: For small time increments h (so think of h as if it's dt), we will at times make the approximation

$${}_hp_x^{ij} \approx \mu_x^{ij} \cdot h$$

rate of transition probability per time · small moment of elapsed time

<u>Notice</u>: We only have forces of transition for $i \neq j$. There is <u>no</u> "force of staying put". To stay put, one must survive forces that try to push (x) out of the current state. That is:

$${}_tp_x^{\overline{ii}} = e^{-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij}\, ds}$$

$$= e^{-\int_0^t \mu_{x+s}^{i\tau}\, ds}$$

Remark: This splits as a product. Interpretation

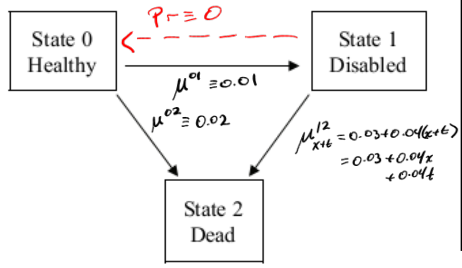
<u>Thinking about probability computations</u>:

- We may view the sum μ^{τ} of the transition forces μ^{0j} as an overall "force of exiting State 0".

- Particular example: $\mu_x = A + Bc^x$ Makeham Model

$\uparrow$ accidental $\uparrow$ growing non-accidental

Example 1 Consider the following multistate model:



Tip:

- Survival probabilities involve exponentiating.
- Probabilities of transitions involve integrals that consider various amounts of time until an instantaneous might occur—draw timeline & tell the story.

Let $\mu_{x+t}^{01} = .01$, $\mu_{x+t}^{02} = .02$, and $\mu_{x}^{12} = .03 + .04x$ be the transition intensities. Assume (x) begins in the healthy state.

a. Find the probability that (x) is still healthy in ten years.

$${}_{10}p_x^{\overline{00}} = e^{-\int_0^{10}(\mu^{01}+\mu^{02})\,dt} = e^{-0.03\cdot 10} = e^{-0.3}$$

b. Find the probability that (x) dies during the next ten years without becoming disabled first.

$\boxed{0}$ for $t_0 \in (0,10)$ $\boxed{0\to 2}$ $\boxed{2}$

$t=0$ $t=t_0$ $t=10$

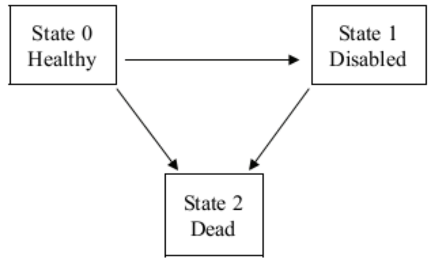
$$= \int_0^{10} {}_tp_x^{\overline{00}}\,\mu_{x+t}^{02}\,dt = \int_0^{10} e^{-\int_0^t \mu^{0\cdot}\,ds}\,\mu_{x+t}^{02}\,dt = \int_0^{10} e^{-0.03t}\,0.02\,dt$$
$$= \frac{2}{3}\left(1-e^{-0.3}\right)$$

c. Find the probability that (x) becomes disabled within ten years.

$$= \int_0^{10} {}_tp_x^{\overline{00}}\,\mu_{x+t}^{01}\,dt = \int_0^{10} e^{-0.03t}\,0.01\,dt = \frac{1}{3}\left(1-e^{-0.3}\right)$$

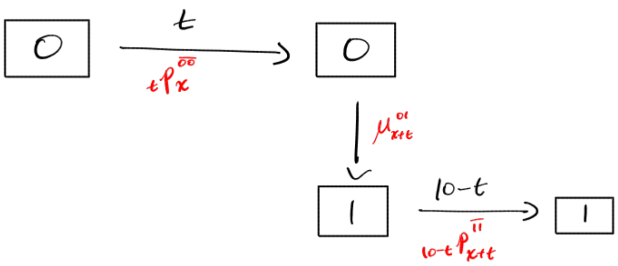
(→ ct'd)

(<u>Example 1</u>, ct'd) $\mu^{01}_{x+t} = .01$, $\mu^{02}_{x+t} = .02$, and $\mu^{12}_{x+t} = .03 + .04t$



Same forces

d. Find the probability that (*x*) becomes disabled within the next ten years and remains alive at the end of that period.



$${}_{10}p_x^{01} = \int_0^{10} {}_tp_x^{\overline{00}}\, \mu^{01}_{x+t}\, {}_{10-t}p_{x+t}^{\overline{11}}\, dt$$

μ^{12}_{x+t+s}

$$= \int_0^{10} e^{-\int_0^t \mu^{0\bullet}_{x+s}\, ds}\, \mu^{01}_{x+t}\, e^{-\int_0^{10-t} \mu^{1\bullet}_{x+s}\, ds}\, dt$$

$$= \int_0^{10} e^{-0.03t} (0.01)\, e^{-\int_0^{10-t} (0.03 + 0.04(t+s))\, ds}\, dt$$

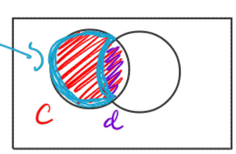
Stop for day

e. Find the probability that (*x*) becomes disabled and dies, all within the next ten years. (The easiest way <u>by far</u> is to use (c) and (d)).

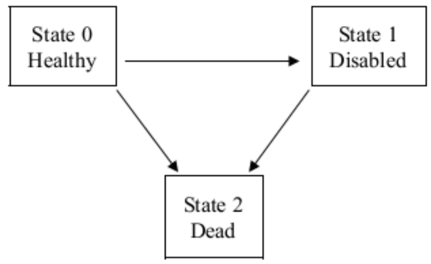
Start
29 Jan 24

Pr [(x) becomes disabled + then dies, all within 10 yrs]

$= (c) - (d)$ (see above)



<u>Example 2</u>: Consider the following multistate model



with $\mu_{x+t}^{01} = .01$, $\mu_{x+t}^{02} = .02$, and $\mu_{x+t}^{12} = .12$. Assume (x) begins in the healthy state.

a. Find the probability that (x) becomes disabled within the next ten years and remains alive at the end of that period.

$\boxed{0} \longrightarrow \boxed{1} \longrightarrow \boxed{1}$

$t=0$ ${}_tp_x^{\overline{00}}$ μ^{01} ${}_{10-t}p_{10}^{\overline{11}}$ $t=10$

Confirmed in class

Not confirmed in class, I may be wrong

$$\int_0^{10} e^{-(\overset{\mu^{01}}{0.01}+\overset{\mu^{02}}{0.02})t}\,\overset{\mu^{01}}{(0.1)}\,e^{-(\overset{\mu^{12}}{0.12})(10-t)}\,dt = \int_0^{10} e^{-1.2}(0.1)\,e^{+0.09t}\,dt = \frac{e^{-1.2}}{10(0.09)}\,e^{0.09t}\Big|_0^{10} = 0.48847$$

b. Find the probability (directly) that (x) becomes disabled and then dies within 10 years.

$t=0$ $s\in(0,10-t)$ 10

$\boxed{0} \longrightarrow \boxed{1} \longrightarrow \boxed{2}$

${}_tp_x^{\overline{00}}$ μ^{01} ${}_sp_{x+t}^{\overline{11}}$ μ^{12}

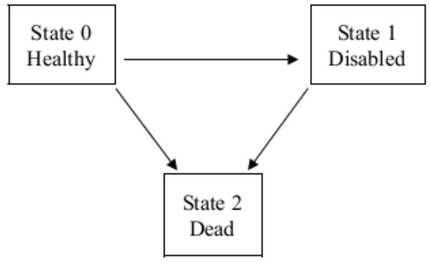
$$\int_0^{10}\left[\underbrace{e^{-(\overset{\mu^{01}}{0.01}+\overset{\mu^{02}}{0.02})t}}_{{}_tp_x^{\overline{00}}}\,\overset{\mu^{01}}{(0.1)}\int_0^{10-t}\underbrace{e^{-\overset{\mu^{12}}{0.12}s}}_{{}_sp_{x+t}^{\overline{11}}}\,\overset{\mu^{12}}{(0.12)}\,ds\right]dt$$

Have fun!

Remark: One could still do Pr[disabled by time 10] – Pr[disabled by/survives to time 10]

(→ ct'd)

(Example 2, continued) $\mu_{x+t}^{01} = .01$, $\mu_{x+t}^{02} = .02$, $\mu_{x+t}^{12} = .12$



Note to Eleanor: Kolmolgorov's Forward Diff. Eqns

c. Find the EPV of a ten-year term death benefit of B, payable upon at the moment of death of (*x*).

$$0 \xrightarrow{t} 1 \xrightarrow{s} 2 \quad \$$$

$$-t \qquad -s$$

$B\bar{A}^{02}_{x:\overline{10|}}$ has two cases: $\begin{cases} 0 \to 2 & (a) \\ 0 \to 1 \to 2 & (b) \end{cases}$

$$= \underbrace{\int_0^{10} e^{-(\mu^{01}+\mu^{02})t} \mu^{02} B e^{-\delta t}\,dt}_{(a)} + \underbrace{\int_0^{10} e^{-(\mu^{01}+\mu^{02})t} \mu^{01} \int_0^{10-t} e^{-\mu^{12}t} \mu^{12} B e^{-\delta s}\,ds\, e^{-\delta t}\,dt}_{(b)}$$

d. Find the EPV of a ten year annuity, payable continuously at a rate of C per year, with payments occurring <u>only</u> while (*x*) is in the healthy state.

$C\bar{a}^{00}_{x:\overline{10|}}$ where we are, when payable

$t=0$ $\qquad\qquad$ $t=10$

$\boxed{0} \longrightarrow \boxed{0}$ $\$ c\,dt$

$e^{-\delta t}$

$$\int_0^{10} \underbrace{e^{-(\mu^{01}+\mu^{02})t}}_{{}_tp_x^{\overline{00}}} \cdot C e^{-\delta t}\,dt = C\,\frac{\exp(-(\mu^{01}+\mu^{02}+\delta)t)}{-(\mu^{01}+\mu^{02}+\delta)}\Bigg|_0^{10}$$

e. What about $D \cdot \bar{a}^{01}_{x:\overline{10|}}$ (consider: disability payments)

$0 \qquad\qquad 10$

$\boxed{0} \xrightarrow{t \in (0,10)} \boxed{0}$

$\downarrow$

$\boxed{1} \xrightarrow{s \in (0, 10-t)} \boxed{1}$ $\$D$

$e^{-\delta(t+s)}$

$$\int_0^{10} \underbrace{e^{-(\mu^{01}+\mu^{02})t}}_{{}_tp_x^{\overline{00}}} \mu^{01} \int_0^{10-t} \underbrace{e^{-\mu^{12}t}}_{{}_tp_x^{\overline{11}}} \cdot D e^{-\delta s}\,ds\, e^{-\delta t}\,dt$$

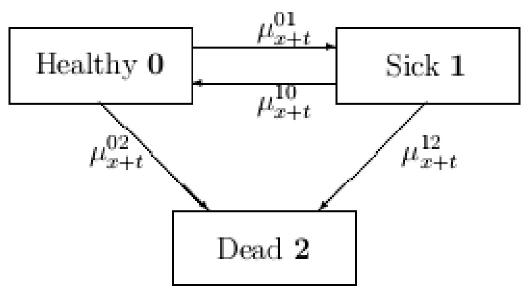
Continue to pg 23 →

Homework – Discrete (ct'd) and continuous multistate Markov models

Suggested Sample ALTAM problems (Canvas): #10b, 15ab

SOA Written Sample #5

5. (12 points) You are given the following sickness-death model and that $i = 0.05$.



The table below gives some transition probabilities calculated for this model.

x	${}_1p_x^{00}$	${}_1p_x^{01}$	${}_1p_x^{02}$	${}_1p_x^{11}$	${}_1p_x^{10}$	${}_1p_x^{12}$
60	0.96968	0.01399	0.01633	0.93300	0.04196	0.02504
61	0.96628	0.01594	0.01778	0.92477	0.04781	0.02742
62	0.96248	0.01816	0.01936	0.91552	0.05446	0.03002
63	0.95824	0.02067	0.02109	0.90514	0.06199	0.03287

(a) (3 points) Consider the three quantities under this model:

$${}_2p_x^{00} \qquad {}_2p_x^{\overline{00}} \qquad \left({}_1p_x^{00}\right)\left({}_1p_{x+1}^{00}\right)$$

Explain in words, without calculation, why these three quantities may all be different, and rank them in order, from smallest to largest.

(b) (3 points)

(i) Calculate the probability that a life who is healthy at age 60 is healthy at age 62.

(ii) Calculate the probability that a life who is healthy at age 60 is sick at age 62.

(c) (2 points) Calculate the expected present value of a three year annuity-due of 1 per year, with each payment conditional on being healthy at the payment date, for a life who is currently age 60 and is healthy.

The insurer issues a 3-year term insurance policy to a healthy life age 60. Premiums are payable annually in advance, and are waived if the policyholder is sick at the payment date. The face amount of 10,000 is payable at the end of the year of death.

(d) (4 points) Calculate the annual net premium for this insurance policy.

MA 397 Review problem:

***2.** For a whole life insurance of 1000 on (x) with benefits payable at the moment of death:

(i) The force of interest at time t, $\delta_t = \begin{cases} 0.04, & 0 < t \le 10 \\ 0.05, & 10 < t \end{cases}$

(ii) $\mu_{x+t} = \begin{cases} 0.06, & 0 < t \le 10 \\ 0.07, & 10 < t \end{cases}$

Calculate the single net premium for this insurance.

Continuous-time Markov process problems

283. For a four-state model with states numbered 0, 1, 2, and 3, you are given:

(i) The only possible transitions are 0 to 1, 0 to 2, and 0 to 3.

(ii) $\mu_{x+t}^{01} = 0.3, t \ge 0$

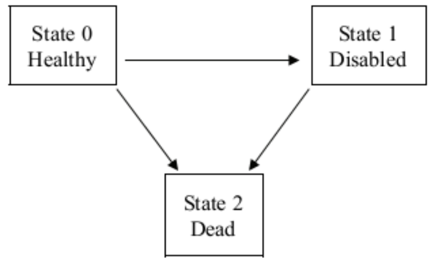
(iii) $\mu_{x+t}^{02} = 0.5, t \ge 0$

(iv) $\mu_{x+t}^{03} = 0.7, t \ge 0$

Calculate p_x^{02}

<u>From Nov. 2013 Exam MLC, #10:</u>

Consider the following multistate model



You are given the following constant forces of transition:

$$\mu^{01} = .02, \quad \mu^{02} = .03, \quad \mu^{12} = .05.$$

Calculate the conditional probability that a Healthy life on January 1, 2004 is still Healthy on January 1, 2014, given that this person is not Dead on January 1, 2014.

Answer: .83 - one way to get there results in the fraction

$$\frac{0.607}{0.607+0.121} = 0.83$$

<u>W8.2 Additional problem</u>:

Consider the setting of #10 above. (You may want to work some of these out and simply set up some of the others.)

a. Find the EPV of a whole-life fully continuous insurance of 1000. Assume $\delta = .04$.

b. Find the EPV of a 10-year term insurance of 1000. Assume $\delta = .04$.

c. Find the EPV of a fully continuous 10-year annuity of \$P per year, payable only while (x) is in the healthy state. Use $\delta = .04$.

Note that setting (b) = (c) would determine the premium for a policy with a feature that premiums are suspended during a period of disability.

d. Determine Pr[(x) becomes disabled within 10 years].
(Why is this <u>not</u> the quantity ${}_{10}p_x^{01}$?)

e. Determine Pr[(x) dies within 10 years without becoming disabled first].

f. Determine Pr[(x) is disabled and alive at time 10], that is, find ${}_{10}p_x^{01}$.

<u>DHW3e Sections 8.5-8.8</u> **Premium computation in the continuous temporary disability model**

<u>Recall</u>: Consider the alive-dead model $0 \to 1$.
Using ${}_tp_x$ notation (and giving the special constant μ case), write down integral expressions for $\bar{A}_x$ and $\bar{a}_x$.

${}_tp_x^{00}$ μ_{x+t}^{01}

$$\bar{A}_x = EPV(\text{Death Ben}) = \int_0^\infty {}_tp_x\, \mu_{x+t}\, e^{-\delta t}\,dt \quad \left(\overset{\mu\text{ const}}{=} \int_0^\infty e^{-\mu t} \mu e^{-\delta t}\,dt \right)$$

$0 \longrightarrow 1$

$$\bar{a}_x = EPV(\text{annuity}) = \int_0^\infty {}_tp_x\, e^{-\delta t}\,dt \quad \left(\overset{\mu\text{ const}}{=} \int_0^\infty e^{-\mu t} e^{-\delta t}\,dt \right)$$

<u>Definitions</u>: (DHW section 8.6)

start in i, end in j

Does not say anything about how you got from i to j

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t}\, {}_tp_x^{ij}\,dt = \int_{t=0}^\infty {}_tp_x^{ij} e^{-\delta t}\,dt$$

$$\ddot{a}_x^{ij} = \sum_{k=0}^\infty v^k\, {}_kp_x^{ij}$$

could be i

$\boxed{i} - \cdots \to \boxed{j} \to \boxed{k}$ $j \neq k$

Payed on every transition

$$\bar{A}_x^{ik} = \int_0^\infty \sum_{j \neq k} e^{-\delta t}\, {}_tp_x^{ij}\, \mu_{x+t}^{jk}\,dt.$$

Note: The symbol $\bar{A}_x^{ik}$ denotes the EPV of a payment of \$1 upon <u>every</u> transition into State k, given a start in state i.

DHW also introduces notations $\bar{a}_{x:\overline{n}|}^{ik}$ and $\bar{A}_{x:\overline{n}|}^{ik}$ (see DHW3e Example 8.7), which basically modify the above for n-year term policies. (In particular, the "upper 1" is dropped from above the x. There appears to be no standard way to notate endowment contracts in the multistate model notation.)

Example: Consider the temporary disability model: → Can recover!

$\boxed{0} \rightleftarrows \boxed{1}$, $\boxed{0} \to \boxed{2}$, $\boxed{1} \to \boxed{2}$

0 = healthy, 1 = critically ill, 2 = dead.

Consider a fully continuous whole-life policy that pays S = 1,000,000 benefit upon death and disability income, continuously payable at a rate of $\bar{B}$ = 50,000 per year, during periods of critical illness. This policy is sold to (40), who is healthy.

two diff. benefits

You are given the values of $\bar{A}_x^{ik}$ and $\bar{a}_x^{ik}$ for $x = 40$ and for $x = 55$, for i and $k = 0, 1, 2$.

a. Find the benefit premium $\bar{P}$, payable continuously only during healthy periods, in terms of $\bar{A}_{40}^{ik}$ and $\bar{a}_{40}^{ik}$.

EPV(Prems) = EPV(Bens)

$\bar{P}\bar{a}_{40}^{00} = 1000000\,\bar{A}_{40}^{02} + 50000\,\bar{a}_{40}^{01}$ ← Values given in table

b. Find the time-15 reserves ${}_{15}V^{(0)}$ and ${}_{15}V^{(1)}$.

(To illustrate: ${}_{15}V^{(1)}$ is the "critical illness reserve", i.e.
${}_{15}V^{(1)}$ = $E_{t=15}$[future losses past time 15| in state 1 at time 15])

${}_{15}V^{(0)} = EPV[{}_0L \mid \text{in } 0 \text{ @ } t=15] = 1000000\,\bar{A}_{55}^{02} + 50000\,\bar{a}_{55}^{01} - \bar{P}\bar{a}_{55}^{00}$

${}_{15}V^{(1)} = EPV[{}_0L \mid \text{in } 1 \text{ @ } t=15] = 1000000\,\bar{A}_{55}^{12} + 500000\,\bar{a}_{55}^{11} - \bar{P}\bar{a}_{55}^{10}$

Plan @ t=0 to use ${}_{15}p_{40}^{00} \cdot {}_{15}V^{(0)} + {}_{15}p_{40}^{01} \cdot {}_{15}V^{(1)}$ as guide to plan reserves for this policy

i.e project t=15 reserves @ t=0

Stop 29 Jan 2024

Suggested Sample ALTAM Practice: #5a(all), 5b(i, ii), 6abcd, 8ab
Assigned Reading: DHW3e Sections 1.7 – 1.7.1.
Additional HW/Practice on theNext pages of this packet →

Upcoming:

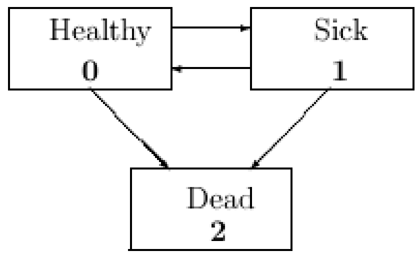
Quiz next week? Wed?

$0 \to 1$, $0 \to 2$, $1 \to 3$

<u>Practice Problem / HW</u>

SOA Written #14 (Ignore the "Thiele's Differential Equation" question for the moment):

14. (9 points) Consider the following model for income replacement insurance:



An insurer issues a whole-life income replacement policy to a healthy life age 50. The policy provides a disability income of 50,000 per year payable continuously while the life is sick. In addition the policy pays a death benefit of 200,000 at the moment of death. The policyholder pays premiums continuously, at a rate of P per year, while in the healthy state.

Assume no expenses, and an effective interest rate of 5% per year. Annuity and insurance factors for the model are given in the Table below.

(a) (2 points) Show that the annual benefit premium for the policy is 11,410 to the nearest 10.

(b) (2 points) Write down Thiele's differential equations for the net premium reserve at t for both the healthy and sick states.

(c) (2 points) Calculate the net premium reserve for a policy in force at $t = 10$, assuming the policyholder is healthy at that time.

(d) (2 points) Calculate the net premium reserve for a policy in force at $t = 10$, assuming the policyholder is sick at that time.

(e) (1 point) A colleague states that the net premium reserve in the sick state should be less than the net premium reserve in the healthy state, as the sick policyholder has, on average, paid less premium and received more benefit. Explain in words why your colleague is incorrect.

Annuity and insurance factors, 5% effective rate of interest

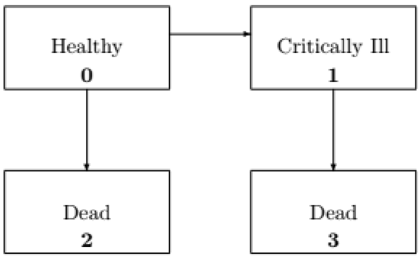
x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{11}$	$\bar{a}_x^{10}$	$\bar{A}_x^{02}$	$\bar{A}_x^{12}$
50	11.9520	1.3292	8.9808	3.2382	0.34980	0.39971
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	8.6097	1.7424	7.1596	1.7922	0.49511	0.56316

Start 31 Jan 24

Quiz next Wed

DHW3e 8.9: CII – Critical Illness Insurance models

Hardy LTAM Study Note, Example 2.5



(c) CII Model 3

Figure 3: CII Models

Recall notation:

$\bar{A}^{03}_{50\,:\,\overline{20}|}$ = EPV[$1 payment occurring whenever* a transition into (3)** occurs before time 20, given start in (0)]

*In *this particular model*, only one such future transition is possible.
**The transition is not required to be directly (0)→(3); any entrance into (3) triggers a payment.

$\bar{a}^{01}_{50\,:\,\overline{20}|}$ = EPV[annuity with payments continuously made at rate of $1/year whenever (50) is in (1), given start (0)]

(b) Use the following data to compute $\bar{a}^{00}_{50\,:\,\overline{20}|}$, $\bar{A}^{01}_{50\,:\,\overline{20}|}$, $\bar{A}^{02}_{50\,:\,\overline{20}|}$, $\bar{A}^{03}_{50\,:\,\overline{20}|}$ at $i = .05$.

x	$\bar{a}^{00}_x$	$\bar{A}^{01}_x$	$\bar{A}^{02}_x$	$\bar{A}^{03}_x$	$\bar{A}^{13}_x$
50	13.31267	0.22409	0.12667	0.14176	0.34988
60	10.17289	0.34249	0.16140	0.22937	0.47904
70	6.56904	0.49594	0.18317	0.36019	0.62237

t	${}_{20-t}p^{00}_{50+t}$	${}_{20-t}p^{01}_{50+t}$	${}_{20-t}p^{02}_{50+t}$	${}_{20-t}p^{03}_{50+t}$	${}_{20-t}p^{11}_{50+t}$
0	0.68222	0.15034	0.13788	0.02956	0.66485
10	0.75055	0.13135	0.09943	0.01867	0.75283

This is similar to using
${}_{20}E_{50} = v^{20} \cdot {}_{20}p_{50}$
in the alive-dead model. Be careful to consider all of the states at age 70 from which transitions to the "target state" 3 are (eventually) possible (e.g. 0→…→ 3).

$$\bar{a}^{00}_{50:\overline{20}|} = \bar{a}^{00}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{a}^{00}_{70} \approx 11.6236$$

≈ 0.09657

$$\bar{A}^{01}_{50:\overline{20}|} = \bar{A}^{01}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{A}^{01}_{70}$$

← for term life, give whole life policy and take away stuff after policy ends

$$\bar{A}^{02}_{50:\overline{20}|} = \bar{A}^{02}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{A}^{02}_{70}$$

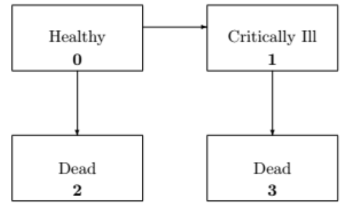
≈ 0.07957

requires a path of 0→1→3

$$\bar{A}^{03}_{50:\overline{20}|} = \bar{A}^{03}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{A}^{03}_{70} - {}_{20}p^{01}_{50}\, v^{20}\, \bar{A}^{13}_{70} \approx 0.01388$$

consider all t=20 states from which a transition into 3 is still possible.

(b, continued)
So we get…



Calculated on previous page

$$\bar{a}^{00}_{50:\overline{20|}} = \bar{a}^{00}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{a}^{00}_{70} = 11.6236$$

$$\bar{A}^{01}_{50:\overline{20|}} = \bar{A}^{01}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{A}^{01}_{70} = 0.09657$$

$$\bar{A}^{02}_{50:\overline{20|}} = \bar{A}^{02}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{A}^{02}_{70} = 0.07957$$

$$\bar{A}^{03}_{50:\overline{20|}} = \bar{A}^{03}_{50} - {}_{20}p^{00}_{50}\, v^{20}\, \bar{A}^{03}_{70} - {}_{20}p^{01}_{50}\, v^{20}\, \bar{A}^{13}_{70} = 0.01388$$

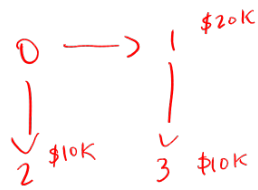
Compute the annual benefit premium π, paid only while in State 0, for the following 20-year term policies written on a healthy life aged 50:

(ii) A combined CII (critical illness insurance) and life insurance policy that pays $20,000 on CII diagnosis and $10,000 on death.

EPV[Prems] = EPV[Bens]

$$\pi\, \bar{a}^{00}_{50:\overline{20|}} = 20000\, \bar{A}^{01}_{50:\overline{20|}} + 10000\left[A^{02}_{50:\overline{20|}} + A^{03}_{50:\overline{20|}}\right]$$

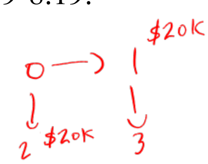
$$\Rightarrow \pi = 246.56$$



(iii) An accelerated death benefit CII policy that pays $20,000 immediately on the earlier of CII diagnosis and death. (To clarify: completely separate example/policy from (ii))
*Also: Chronic illness rider on life policy, cf. DHW3e Ex. 8.9-8.19.

$$\pi\, \bar{a}^{00}_{50:\overline{20|}} = 20000\, A^{01}_{50:\overline{20|}} + 20000\, A^{02}_{50:\overline{20|}}$$

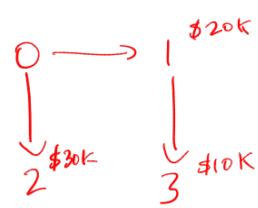
$$\Rightarrow \pi = 303.07$$



(iv) (For class to try:) A partly accelerated death benefit policy, which pays

383.47

- $20,000 on CII diagnosis,
- $30,000 if the policyholder dies without a CII claim, and
- $10,000 if the policyholder dies after a CII claim.



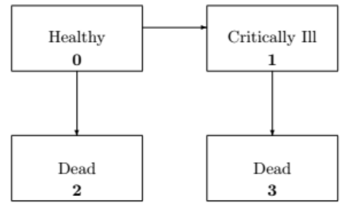
$$\pi\, \ddot{a}^{00}_{50:\overline{20|}} = 20000\, \bar{A}^{01}_{50:\overline{20|}} + 10000\, \bar{A}^{03}_{50:\overline{20|}} + 30000\, \bar{A}^{02}_{50:\overline{20|}}$$

$$\Rightarrow \pi = 383.4698372$$

(c) (For class:) Use the table to compute $\bar{a}^{00}_{60\,:\,\overline{10}|}$, $\bar{A}^{01}_{60\,:\,\overline{10}|}$, $\bar{A}^{02}_{60\,:\,\overline{10}|}$, $\bar{A}^{03}_{60\,:\,\overline{10}|}$, $\bar{A}^{13}_{60\,:\,\overline{10}|}$ at $i = .05$. Be especially careful with $\bar{A}^{03}_{60\,:\,\overline{10}|}$.

To do

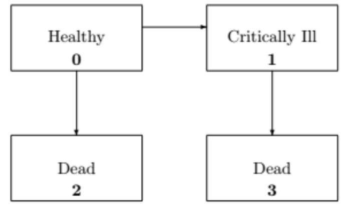
x	$\bar{a}^{00}_x$	$\bar{A}^{01}_x$	$\bar{A}^{02}_x$	$\bar{A}^{03}_x$	$\bar{A}^{13}_x$
50	13.31267	0.22409	0.12667	0.14176	0.34988
60	10.17289	0.34249	0.16140	0.22937	0.47904
70	6.56904	0.49594	0.18317	0.36019	0.62237



ans on pg 29

t	${}_{20-t}p^{00}_{50+t}$	${}_{20-t}p^{01}_{50+t}$	${}_{20-t}p^{02}_{50+t}$	${}_{20-t}p^{03}_{50+t}$	${}_{20-t}p^{11}_{50+t}$
0	0.68222	0.15034	0.13788	0.02956	0.66485
10	0.75055	0.13135	0.09943	0.01867	0.75283

(c, continued)



$$\bar{a}^{00}_{60:\overline{10|}} = \bar{a}^{00}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{a}^{00}_{70} = 7.14606$$

$$\bar{A}^{01}_{60:\overline{10|}} = \bar{A}^{01}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{A}^{01}_{70} = 0.11397$$

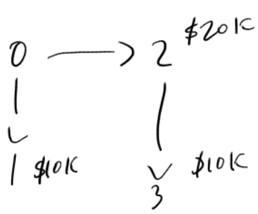
$$\bar{A}^{02}_{60:\overline{10|}} = \bar{A}^{02}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{A}^{02}_{70} = 0.07700$$

$$\bar{A}^{03}_{60:\overline{10|}} = \bar{A}^{03}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{A}^{03}_{70} - {}_{10}p^{01}_{60}\, v^{10}\, \bar{A}^{13}_{70} = 0.01322$$

$$\bar{A}^{13}_{60:\overline{10|}} = \bar{A}^{13}_{60} - {}_{10}p^{11}_{60}\, v^{10}\, \bar{A}^{13}_{70} = 0.19140$$

Find ${}_{10}V^{(0)}$ and ${}_{10}V^{(1)}$ for each of the following 20-year term policies that were written on a healthy 50-year-old:

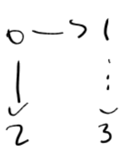
(ii) A combined CII (critical illness insurance) and life insurance policy that pays \$20,000 on CII diagnosis and \$10,000 on death with premium $\pi = 246.56$.



$${}_{10}V^{(0)} = 10000\left(\bar{A}^{02}_{60:\overline{10|}} + \bar{A}^{03}_{60:\overline{10|}}\right) + 20000\left(\bar{A}^{01}_{60:\overline{10|}}\right) - \pi\, \ddot{a}^{00}_{60:\overline{10|}} = 1419.67$$

$${}_{10}V^{(1)} = 10000\, \bar{A}^{13}_{60:\overline{10|}} \quad + \text{no premium} \qquad \approx 1914$$

(iii) An accelerated death benefit CII policy that pays \$20,000 immediately on the earlier of CII diagnosis and death. Use $\pi = 303.07$.



$${}_{10}V^{(0)} = 20000\left[\bar{A}^{01}_{60:\overline{10|}} + \bar{A}^{02}_{60:\overline{10|}}\right] - 303.07\, \bar{a}^{00}_{60:\overline{10|}} = 1653.64$$

$${}_{10}V^{(1)} = 0 \qquad (\text{no future cashflows})$$

(iv) (For class to try:) A partly accelerated death benefit policy, which pays
- \$20,000 on CII diagnosis,
- \$30,000 if the policyholder dies without a CII claim, and
- \$10,000 if the policyholder dies after a CII claim.

Use $\pi = 383.47$

1981
1914

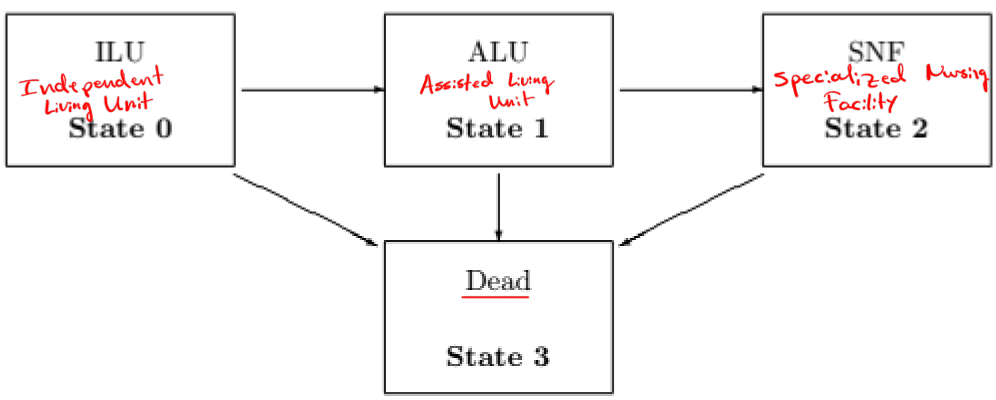
Standard abbreviation

DHW3e 8.9 Continuing Care Retirement Communities (CCRCs)
Often, a CCRC charges an entry fee, which is typically funded by the sale of a resident's home. So the entry fee may be set near the average home price in the area. The remaining costs are covered through a monthly fee.

Types of contracts: (Hardy LTAM Study Note, ∮2.4)

Some widely used CCRC contract types are described in Section 1.6. For the full lifecare (Type A) and modified lifecare (Type B) contracts, the price is expressed as a combination of entry fee and monthly fees that for Type A increase with inflation, but do not change when the resident moves between different residence categories. Type B monthly fees increase with inflation and also increase as residents move through the different categories, but the increases are less than the actual difference in cost, so there is some prefunding of the costs of the more expensive ALU and SNF facilities. Type C contracts are pay as you go, and so do not involve pre-funding, and therefore do not need actuarial modelling for costing purposes.

Hardy LTAM Study note, Example 2.6



(a) The simplified CCRC model; ILU is Independent Living Unit; ALU is Assisted Living Unit; SNF is Specialized Nursing Facility.

The actual monthly costs incurred by the CCRC, including medical care, provision of services, maintenance of buildings and all other expenses and loadings, are as follows:

Independent Living Unit:	3 500
Assisted Living Unit	6 000
Specialized Nursing Facility:	12 000

(3500; 6000; 12000 monthly costs)

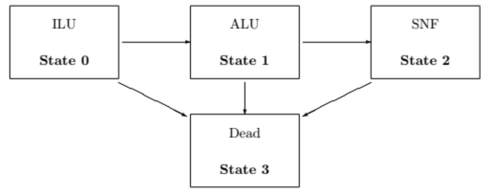
(a) Compute the level monthly fee for healthy entrants ages 65 and 70, assuming a \$200,000 entry fee and a start in State 0 (independent living unit). Assume a Type A (full lifecare) contract, i.e. the monthly fee remains the same, even if the level of care required changes.

ILU 3500/mo

ALU 6000/mo

SNF 12000/mo

x	$\ddot{a}_x^{(12)00}$	$\ddot{a}_x^{(12)01}$	$\ddot{a}_x^{(12)02}$	$A_x^{(12)03}$
65	11.6416	0.75373	0.24118	0.12824
66	11.3412	0.75157	0.25330	0.13295
67	11.0323	0.74994	0.26614	0.13778
68	10.7149	0.74877	0.27974	0.14273
69	10.3892	0.74790	0.29415	0.14780
70	10.0554	0.74720	0.30944	0.15298



Reminder: $\ddot{a}_{65}^{(12)0j}$ *is EPV of monthly payments at rate of \$1 / year (so* $\$\frac{1}{12}$ */ month) when in (j).*

EPV[Prems] = EPV[Bens, costs, expenses, etc]

$$\underbrace{200000}_{\text{entrance fee}} + \underbrace{\pi_{65}}_{\text{monthly premium}}\left[12\,\ddot{a}_{65}^{(12)00} + 12\,\ddot{a}_{65}^{(12)01} + 12\,\ddot{a}_{65}^{(12)02}\right] = \underbrace{3500}_{\text{ILU}}(\underset{\uparrow\ \text{months}}{12}\,\ddot{a}_{65}^{(12)00}) + \underbrace{6000}_{\text{ALU}}(12\,\ddot{a}_{65}^{(12)01}) + \underbrace{12000}_{\text{SNF}}(12\,\ddot{a}_{65}^{(12)02})$$

$$\Rightarrow \pi_{65} \approx 2492.4 \qquad \text{similarly, } \pi_{70} \approx 2404.9$$

(b) (For class to try:) Calculate a revised monthly fee for healthy 65-year old entrants, assuming 70% of the entry fee is refunded at the month of death

$\pi_{65} = 2610.8$
$\pi_{70} = 2565.6$

$$200000 + \pi_{65}\left[12\,\ddot{a}_{65}^{(12)00} + 12\,\ddot{a}_{65}^{(12)01} + 12\,\ddot{a}_{65}^{(12)01}\right] = 3500(12\,\ddot{a}_{65}^{(12)00}) + 6000(12\,\ddot{a}_{65}^{(12)01}) + 12000(12\,\ddot{a}_{65}^{(12)02}) + 140000\,\bar{A}_{65}^{(12)03} \leftarrow \text{entrance fee refund}$$

$$\Rightarrow \pi_{65} = 2610.81633$$

similarly,

$$\pi_{70} = 2565.54872$$

(c) (Modified from book) Suppose 20% of entrants are age 65 and the rest are age 70. Find a suitable monthly fee which is not age-dependent if (i) no refund of entry fee; (ii) 70% refund of entry fee upon death.

Idea: weighted avg $\quad .2(\pi_{65}) + .8(\pi_{70}) = \begin{cases} a)\ \ .2\cdot 2492 + .8\cdot 2404 \\ b)\ \ .2\cdot 2410 + .8\cdot 2565 \end{cases}$

$\hookrightarrow \mathbb{E}[\text{PV Bens}] = \mathbb{E}[\mathbb{E}[\text{PV Bens} \mid \text{age of entrant}]]$

<u>**HW:**</u> **Read** Sections 1.7 and 1.9 of DHW3e (on Canvas).
<u>**Suggested ALTAM Practice:**</u> #1c, 10cdef, 11abc, 12abcd (read 1.9.2), 13, 16a

12 Feb 24
Continued

DHW3e ∮18.6 Statistical estimation of forces of transition in a multistate model

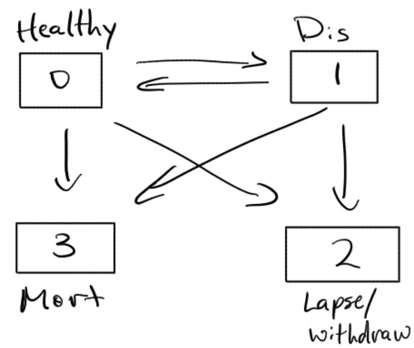
Main fact: Under the assumption that μ_{x+s}^{ij} is constant for $x \in \mathbb{Z}, 0 < s < 1$: $i \neq j$

While not shown here, maximum likelihood estimates turn out to be based on exact exposure for the time spent in each state. For those between ages x and $x+1$ (which can be generalized for periods of other than one year), let T_i be the total time policyholders are observed in state i and d_{ij} be the number of observed transitions from state i to state j. Then $\hat{\mu}_x^{ij} = d_{ij}/T_i$. Similarly $\widehat{\mathrm{Var}}(\hat{\mu}_x^{ij}) = d_{ij}/T_i{}^2$.

(Klugman et al *Loss Models* 5e) !! eww

Example (Klugman et al 5e Example 12.28)

Consider five policyholders who, between ages 50 and 51, are observed to do the following (decimals are fractions of a year):



- Disabled at age 50, dies at age 50.27.
- Healthy at age 50, disabled at age 50.34, dies at age 50.78.
- Healthy at age 50, surrendered at age 50.80.
- Purchases policy (healthy) at age 50.31, healthy at age 51.
- Healthy at age 50, disabled at 50.12, healthy at 50.45, dies at age 50.91.

Calculate the maximum likelihood estimates of the transition intensities.

Data:

1: $t=50$ $1 \xrightarrow{0.27} 3$ 50.27
2: 50 $0 \xrightarrow{0.34} 1 \xrightarrow{0.44} 3$ (50.34, 50.78)
3: 50 $0 \xrightarrow{0.8} 2$ 50.8
4: 50.31 $0 \xrightarrow{0.69} 0$ 51
5: 50 $0 \xrightarrow{.12} 1 \xrightarrow{.33} 0 \xrightarrow{.46} 3$ (50.12, 50.46, 50.91)

T_0 = time in 0 = $.34 + .8 + .69 + .12 + .46 = 2.41$

T_1 = time in 1 = $.27 + .44 + .33 = 1.04$

MLE of μ^{ij} — Red: # transitions from $i \to j$

$$\hat{\mu}^{01} = \frac{2}{2.41} = 0.83 \qquad \hat{\mu}^{10} = \frac{1}{1.04} = 0.96$$

$$\hat{\mu}^{02} = \frac{1}{2.41} = 0.41 \qquad \hat{\mu}^{12} = \frac{0}{1.04} = 0$$

$$\hat{\mu}^{03} = \frac{1}{2.41} = 0.41 \qquad \hat{\mu}^{13} = \frac{1}{1.04} = 0.96$$

Special case: Alive-dead model under constant force assumption:
$\mu_x \approx$ (#deaths)÷(total wait time)

Related estimation: "**Actuarial estimate**" $q_x \approx$ (#deaths)÷(total wait time + .5 × #deaths)

Go to page 44 →

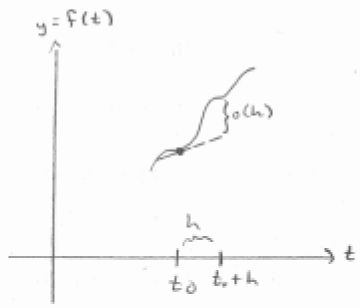
Start
7 Feb 2024

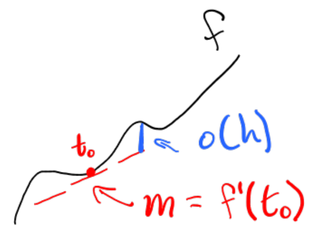
→ Projecting ${}_tP_x^{ij}$'s

DHW ∮8.4-8.5 Temporary disability model and Kolmogorov Forward Equations

Background info:

1. Another way to look at the derivative.





linear approximation

$$f'(t_0) = \lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h}$$

or for small $h \neq 0$:

$$f'(t_0)\cdot h \approx f(t_0+h) - f(t_0)$$

or

$$\underbrace{f(t_0+h) - f(t_0)}_{\text{chg }(f)} = \underbrace{f'(t_0)}_{\text{linear mult. of}} \cdot \underbrace{h}_{\text{chg in } t} + \underbrace{o(h)}_{\text{plus error}}$$

The notation o(h) represents some error term that disappears as h→0. (In fact, o(h) is small enough that the ratio o(h)/h → 0 as h → 0).

$$\lim_{h\to 0} \frac{o(h)}{h} \to 0$$

2. Euler's method. Recall the definition $\mu^{ij}_{x+t} = \lim_{h\to 0^+} \frac{{}_hp^{ij}_{x+t}}{h}$ for $i \neq j$.
This is basically a derivative, so we can get a useful approximation.

$${}_hp^{ij}_{x+t} = \mu^{ij}_{x+t}\cdot h + o(h)$$

discard error o(h) to get

$$\boxed{{}_hp^{ij}_{x+t} \approx \mu^{ij}_{x+t}\cdot h}$$

Euler's approximation

"dt"

Example 3: (DHW Example 8.5 pp.246-247)

Consider a disability income model with states
0 – alive, well 1 – disabled (possibly temporarily) 2 – dead

a. Draw a quick transition diagram.

no time passed

b. Starting point: ${}_0p_{60}^{00} =$ $\underline{1}$ and ${}_0p_{60}^{01} =$ $\underline{0}$. "boundary conditions"

c. Kolmogorov's forward equations (and Euler's method).
Suppose that we have calculated the transition probabilities ${}_tp_{60}^{ij}$ up to some time t.
Suppose that the μ_{60+t}^{ij} are known and suppose that h is small, say $h = \frac{1}{12}$ (one month).

Then *(describe in English, then fill in the notation, then get in terms of* μ_{60+t}^{ij}*):*
Also give a formula for and $\frac{d}{dt}\,{}_{t+h}p_{60}^{01}$.

(i) $${}_{t+h}p_{60}^{00} = \underbrace{{}_tp_{60}^{00}}_{\text{already in 0 @ 60+t}} \cdot \underbrace{\left(1 - {}_hp_{60+t}^{01} - {}_hp_{60+t}^{02}\right)}_{\text{then do not transition}} + \underbrace{{}_tp_{60}^{01}}_{\text{in "wrong" state @ 60+t}} \cdot \underbrace{{}_hp_{60+t}^{10}}_{\text{then correct transition}}$$

$f(t+h) - f(t)$

think: $f(t) = {}_tp_{60}^{00}$

$\Rightarrow f'(t) = \lim_{h \to 0} \dfrac{{}_{t+h}p_{60}^{00} - {}_tp_{60}^{00}}{h}$

$$= {}_tp_{60}^{00}\left(1 - \mu_{60+t}^{01}\,h - \mu_{60+t}^{02}\,h\right) + {}_tp_{60}^{01}\,\mu_{60+t}^{10}\,h + o(h)$$

- By rearranging this equation, we can compute $\frac{d}{dt}\,{}_tp_{60}^{00}$:

$$\frac{d}{dt}\left({}_tp_{60}^{00}\right) = \lim_{h \to 0} \frac{{}_{t+h}p_{60}^{00} - {}_tp_{60}^{00}}{h} = {}_tp_{60}^{00}\left(-\mu_{60+t}^{01} - \mu_{60+t}^{02}\right) + {}_tp_{60}^{01}\,\mu_{60+t}^{10} + \cancel{\frac{o(h)}{h}} \to 0$$

- By discarding the $o(h)$ term in (i), get Euler's approximation to ${}_{t+h}p_{60}^{00}$.

$${}_{t+h}p_{60}^{00} \overset{\text{def}}{\approx} {}_tp_{60}^{00}\left(1 - \mu_{60+t}^{01}\,h - \mu_{60+t}^{02}\,h\right) + {}_tp_{60}^{01}\,\mu_{60+t}^{10}\,h$$

(ii) $_{t+h}p_{60}^{01} = {}_tp_{60}^{01}\left(1 - {}_hp_{60+t}^{10} - {}_hp_{60+t}^{12}\right) + {}_tp_{60}^{00}\ {}_hp_{60+t}^{01}$

"Markov Equation"

replace transitions w/ $\mu \cdot h$

error term

$$= {}_tp_{60}^{01}\left(1 - \mu_{60+t}^{10}h - \mu_{60+t}^{12}h\right) + {}_tp_{60}^{00}\mu_{60+t}^{01}h + o(h)$$

- By rearranging this equation, we can compute $\frac{d}{dt}\,{}_tp_{60}^{01}$:

get difference quotient

$$\frac{d}{dt}\left({}_tp_{60}^{01}\right) \xleftarrow{\text{def'n}} \lim_{h\to 0}\frac{{}_{t+h}p_{60}^{01} - {}_tp_{60}^{01}}{h} = -{}_tp_{60}^{01}\mu_{60+t}^{10} - {}_tp_{60}^{01}\mu_{60+t}^{12} + {}_tp_{60}^{00}\mu_{60+t}^{01} + \cancel{\frac{o(h)}{h}}^{\to 0}$$

"Kolmogorov forward equations"

- By discarding the $o(h)$ term, we get the Euler method approximation to ${}_{t+h}p_{60}^{01}$.

$${}_{t+h}p_{60}^{01} \approx {}_tp_{60}^{01}\left(1 - \mu_{60+t}^{10}h - \mu_{60+t}^{12}h\right) + {}_tp_{60}^{00}\mu_{60+t}^{01}$$

"Euler Approx"

(iii) $_{t+h}p_{60}^{02} =$

$$= \underbrace{{}_tp_{60}^{02}}_{\text{in 2 @ 60+t}}\underbrace{(1)}_{\text{remain dead}} + \underbrace{{}_tp_{60}^{00}}_{\text{not dead}}\underbrace{\mu_{60+t}^{02}}_{\text{die}} + \underbrace{{}_tp_{60}^{01}}_{\text{not dead}}\underbrace{\mu_{60+t}^{12}}_{\text{die}}$$

Markov

$$\rightsquigarrow \frac{d}{dt}\left[{}_tp_{60}^{02}\right] = {}_tp_{60}^{00}\mu_{60+t}^{02} + {}_tp_{60}^{01}\mu_{60+t}^{12}$$

Kolmogorov

Similar process for Euler approx

<u>Example</u>, continued.

Suppose that the forces of transition are given by

$$\mu_x^{01} = a_1 + b_1 \exp\{c_1 x\},$$
$$\mu_x^{10} = 0.1\,\mu_x^{01},$$
$$\mu_x^{02} = a_2 + b_2 \exp\{c_2 x\},$$
$$\mu_x^{12} = \mu_x^{02},$$

, where $a_1 = 4 \times 10^{-4}$, $b_1 = 3.4674 \times 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \times 10^{-4}$, $b_2 = 7.5858 \times 10^{-5}$, $c_2 = 0.087498$.

Then we can compute (just by plugging these constants in):

Yeesh use a spreadsheet

t	μ_{60+t}^{01}	μ_{60+t}^{02}	μ_{60+t}^{10}	μ_{60+t}^{12}
0	0.01420	0.01495	0.00142	0.01495
$\frac{1}{12}$	0.01436	0.01506	0.00144	0.01506

c. Show how to find ${}_{\frac{1}{12}}p_{60}^{00}$ and ${}_{\frac{1}{12}}p_{60}^{01}$.

$${}_{\frac{1}{12}}p_{60}^{00} \approx \underbrace{{}_0p_{60}^{00}}_{1}\left[1 - \overbrace{\tfrac{1}{12}\cdot\underbrace{(.01420)}_{\mu_{60}^{01}}}^{\approx 0+\frac{1}{12}p_{60}} - \tfrac{1}{12}\underbrace{(.01495)}_{\mu_{60}^{02}}\right] + \underbrace{{}_0p_{60}^{01}}_{0} \times {}_{\frac{1}{12}}p_{60}^{10} = \#$$

$${}_{\frac{1}{12}}p_{60}^{01} \approx \underbrace{{}_0p_{60}^{01}}_{0}\left[1 - {}_{0+\frac{1}{12}}p_{60}^{10} - {}_{0+\frac{1}{12}}p^{12}\right] + \underbrace{{}_0p_{60}^{00}}_{1} \cdot \overbrace{\underbrace{(.01420)}_{\approx\, {}_{\frac{1}{12}}p_{60}^{01}}}^{\mu_{60}^{01}} \overset{h}{\left(\tfrac{1}{12}\right)} = \#$$

d. Having found ${}_{\frac{1}{12}}p_{60}^{00}$ and ${}_{\frac{1}{12}}p_{60}^{01}$ in (c) , show how to find ${}_{\frac{2}{12}}p_{60}^{00}$ and ${}_{\frac{2}{12}}p_{60}^{01}$.

$${}_{\frac{2}{12}}p_{60}^{00} = \underbrace{{}_{\frac{1}{12}}p_{60}^{00}}_{\text{from (c)}}\left[1 - \mu_{60\frac{1}{12}}^{01} \times \overset{h}{\tfrac{1}{12}} - \mu_{60\frac{1}{12}}^{02} \times \overset{h}{\tfrac{1}{12}}\right] + \underbrace{{}_{\frac{1}{12}}p_{60}^{01}}_{\text{from (c)}} \times \mu_{60\frac{1}{12}}^{10} \times \left(\tfrac{1}{12}\right)$$

$${}_{\frac{2}{12}}p_{60}^{01} = \overbrace{{}_{\frac{1}{12}}p_{60}^{01}}\left[1 - \mu_{60\frac{1}{12}}^{10} \times \tfrac{1}{12} - \mu_{60\frac{1}{12}}^{12} \times \tfrac{1}{12}\right] + {}_{\frac{1}{12}}p_{60}^{00} \times \mu_{60\frac{1}{12}}^{01} \times \tfrac{1}{12}.$$

<u>Example 4</u> (From LTAM Study Note, Sec. 1.2) – Long Term Disability Insurance

Independently ...
Eating
bathing
walking
dressing
toileting
Continence (bowel function)

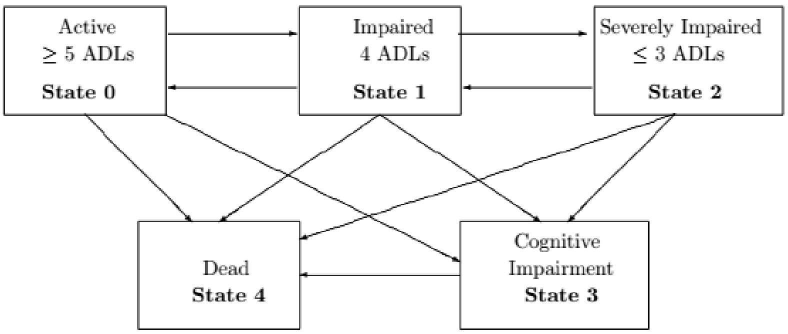


Figure 2: Example of an LTC insurance model.

(ADL = Activities of Daily Living.)

ALTAM question free bee

Example 2.2

Write down the Kolmogorov forward equations for all the probabilities for a life age x, currently in State 2, for the model in Figure 2, and give boundary conditions. Assume the usual assumptions for Markov multiple state models apply.

$$ {}_{t+h}p_x^{20} = \left\{ {}_tp_x^{20} - {}_tp_x^{20}\left(\mu_{x+t}^{01} + \mu_{x+t}^{03} + \mu_{x+t}^{04}\right)\cdot h + {}_tp_x^{21}\cdot\mu_{x+t}^{10}\,h + \underbrace{o(h)}_{\text{delete for Euler approx.}} \right\} $$

Solution 2.2

$$\frac{d}{dt}\,{}_tp_x^{20} = {}_tp_x^{21}\,\mu_{x+t}^{10} - {}_tp_x^{20}\left(\mu_{x+t}^{01} + \mu_{x+t}^{03} + \mu_{x+t}^{04}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{21} = {}_tp_x^{20}\,\mu_{x+t}^{01} + {}_tp_x^{22}\,\mu_{x+t}^{21} - {}_tp_x^{21}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{22} = {}_tp_x^{21}\,\mu_{x+t}^{12} - {}_tp_x^{22}\left(\mu_{x+t}^{21} + \mu_{x+t}^{23} + \mu_{x+t}^{24}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{23} = {}_tp_x^{20}\,\mu_{x+t}^{03} + {}_tp_x^{21}\,\mu_{x+t}^{13} + {}_tp_x^{22}\,\mu_{x+t}^{23} - {}_tp_x^{23}\,\mu_{x+t}^{34}$$

$$\frac{d}{dt}\,{}_tp_x^{24} = {}_tp_x^{20}\,\mu_{x+t}^{04} + {}_tp_x^{21}\,\mu_{x+t}^{14} + {}_tp_x^{22}\,\mu_{x+t}^{24} + {}_tp_x^{23}\,\mu_{x+t}^{34}$$

For boundary conditions, we have ${}_0p_x^{22} = 1$ and ${}_0p_x^{2j} = 0$ for $j \neq 2$.

Suggested ALTAM Practice: #1ab, 7abc, 10a, 18b(i), 23bc

8.8 Policy Values and Thiele's Differential Equation.

Helpful calculus fact:

- Let $f(t) = e^t$. The Taylor expansion about $t = 0$ is…

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots \underset{\text{near } t=0}{\approx} 1 + t \quad \text{which we are!}$$

- Throwing away nearly all of the terms and looking at $t = \pm\delta h$, we get (for small h)...

  Accumulation factor $= e^{\delta h} \approx$ $1 + \delta h$

  Amount of interest earned on \$X during $[t, t + h] \approx$ $X(1 + \delta h)$

- In a similar manner, consider accumulated value of premium paid continuously at rate $\bar{P}$ per year during $[t_0, t_0 + h]$:

$$\bar{P}\,\bar{s}_{\overline{h|}} = \int_{t=t_0}^{t_0+h} \bar{P}e^{\delta(h-t)}\,dt = \bar{P}\,e^{\delta h}\int_{t=t_0}^{t_0+h} e^{-\delta t}\,dt = \bar{P}\,e^{\delta h}\left(\frac{e^{-\delta(t_0+h)} - e^{-\delta t_0}}{-\delta}\right)$$

$$= \bar{P} \cdot \frac{e^{-\delta t_0} - e^{-\delta(t_0-h)}}{-\delta}$$

$\updownarrow$ *Taylor approx. from above*

$$\approx \bar{P} \cdot \frac{-\delta t_0 + \delta(t_0 - h)}{-\delta} = \underline{\qquad\qquad}$$

SKIP

Intuitively, this makes sense. Premium paid at a rate of $\bar{P}$ per year for a small fraction h of the year should roughly add up to $\bar{P}h$. The amount of premium paid and amount of elapsed time h are too small for interest to have much of an effect.

Recall: If h is small, we can approximate ${}_hp_{x+t}^{ij} \approx$ $\mu_{x+t}^{ij} \cdot h$ for $i \neq j$. ← The use of this approximation is called Euler's method.

Definition: The t^{th} State i reserve ${}_tV^{(i)} =$ $\mathbb{E}_t[\text{future losses} \mid \text{in } i \text{ @ } t]$.

$$0 \rightleftarrows 1, \quad 0 \searrow 2 \swarrow 1$$

<u>Thiele's Differential Equation(s)</u> in the disability income model.

<u>Example 8.7</u> "Disability income model." $(x) = (40)$ purchases the following $n = 20$-year term insurance: A premium is paid at rate $\bar{P}$ per year <u>only</u> while the individual is healthy. A benefit is paid at a continuous rate $\bar{B}$ per year to the insured during any period of disability. The death benefit is S. In terms of transition force functions μ^{ij}_{x+t} and a constant force of interest δ, we can compute reserves using a "backwards" recursion:

- Start with boundary conditions:
  For 20-year term insurance, you set each ${}_{20}V^{(i)}$ = <u>0</u> ; for a 20-year endowment insurance, set each ${}_{20}V^{(i)}$ = <u>endow benefit</u>.

<u>Q1</u>: Let h be a short time-step prior to some future time $t \le 20$.
What change occurs in the "healthy reserve" over $[t-h, t]$?

(Change in reserve) = (think: change in reserve "savings acct.")

1. Account grows due to interest earned during $[t-h, t]$ on healthy reserve during $(t-h, t)$. Based on ${}_tV^{(0)}$, this amt. of interest is approximately…
2. Account grows due to addition of premium pd at rate $\bar{P}$ per yr during $[t-h, t]$. We saw that $\bar{P}\,\bar{s}_{\overline{h|}} \approx \bar{P} \cdot h$
3. The account is reduced by the expected cost of setting up ${}_tV^{(1)}$ if needed and/or by paying the death benefit if needed. These costs are offset, however, by the "saved up amount" ${}_tV^{(0)}$:

interest on acct

in state 0, thus get slice of premium

if transition, cost, offset

if die, benefit, offset

$$ {}_tV^{(0)} - {}_{t-h}V^{(0)} = {}_tV^{(0)}\delta h + \bar{P}h - \mu^{01}_{x+t}h\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{x+t}\left(S - {}_tV^{(0)}\right) + o(h)$$

<u>Note</u>: This looks like $f(\underbrace{t}_{\text{later time}}) - f(\underbrace{t-h}_{\text{earlier time}})$. Thus, we can compute the derivative of ${}_tV^{(0)}$ by dividing by h and taking $\lim_{h\to 0}$.

$$\frac{d}{dt}\left[{}_tV^{(0)}\right] = {}_tV^{(0)}\delta + \bar{P} - \mu^{01}_{x+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{x+t}\left(S - {}_tV^{(0)}\right)$$

Thiele's Backward Eqns

<u>Approximating reserves</u> – Starting from known values for ${}_tV^{(i)}$ at time t, we use the Euler's approximation (⇔ discard $o(h)$) to find the values for the next-earlier reserve ${}_{t-h}V^{(i)}$. It's easiest to do this by subtracting in the opposite order in Q1 above.

$$ {}_{t-h}V^{(0)} - {}_tV^{(0)} = {}_tV^{(0)}\delta h + \bar{P}h + \mu^{01}h\left({}_tV^{(1)} - {}_tV^{(0)}\right) + \mu^{02}h\left(S - {}_tV^{(0)}\right)$$

Solve for ${}_{t-h}V^{(0)}$

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$$ {}_{t-h}V^{(0)} - {}_tV^{(0)} = \underbrace{\delta h\, {}_tV^{(0)}}_{1} + \underbrace{\bar{P}h}_{2} - \underbrace{\mu^{01}_{x+t}\cdot h\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{x+t}h\left(S - {}_tV^{(0)}\right)}_{3} + \underbrace{o(h)}_{4} $$

$$ \lim_{h\to 0} \frac{{}_tV^{(0)} - {}_{t-h}V^{(0)}}{h} = \frac{d\,{}_tV^{(0)}}{dh} = \text{(negative above, divide everything by } h) $$

note the time states

$$ \lim_{h\to 0}\left(\frac{o(h)}{h}\right) = 0 $$

$$ {}_{t-h}V^{(1)} - {}_tV^{(1)} = \underbrace{\delta h\, {}_tV^{(1)}}_{1} - \underbrace{\bar{B}h}_{2} - \underbrace{\mu^{10}_{x+t}h\left({}_tV^{(0)} - {}_tV^{(1)}\right) - \mu^{12}_{x+t}\left(S - {}_tV^{(1)}\right)}_{3} + \underbrace{o(h)}_{4} $$

1) Grow current account at interest
2) Plus premium income / less costs of current state
3) If state change, offset cost of new reserve w/ old reserve
4) error terms

$$ \frac{d}{dt}\,{}_tV^{(0)} = -\delta\,{}_tV^{(0)} - \bar{P} + \mu^{01}_{x+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) + \mu^{02}\left(S - {}_tV^{(0)}\right) $$

$$ \frac{d}{dt}\,{}_tV^{(1)} = -\delta\,{}_tV^{(1)} + \bar{B} + \mu^{10}_{x+t}\left({}_tV^{(0)} - {}_tV^{(1)}\right) + \mu^{12}_{x+t}\left(S - {}_tV^{(1)}\right) $$

Go to page 32

Q2: Let h be a short time-step prior to some future time $t \le 20$.
What change occurs in the "disabled-state reserve" over $[t-h, t]$?

(Change in reserve) = (think: change in reserve "savings acct.")

1. Account grows due to interest earned during $[t-h, t]$ on healthy reserve during$(t\text{-}h, t)$. Based on ${}_tV^{(1)}$, this amt. of interest is approximately…

2. Account declines due to disability income benefit pd at rate $\bar{B}$ per yr during $[t-h, t]$. Use approximation for $\bar{B}\,\bar{s}_{\overline{h}|}$…

3. The account is reduced by the expected cost of setting up ${}_tV^{(0)}$ if needed and/or by paying the death benefit if needed. These costs are offset, however, by the "saved up amount" ${}_tV^{(0)}$:

Note: This again looks like $\underbrace{f(t)}_{\text{later time}} - \underbrace{f(t-h)}_{\text{earlier time}}$. We can compute the derivative of ${}_tV^{(1)}$ by dividing by h and taking $\lim_{h\to 0}$.

Approximating reserves: Starting from known values for ${}_tV^{(i)}$ at time t, we use the Euler's approximation (⇔ discard $o(h)$) to find the values for the next-earlier reserve ${}_{t-h}V^{(i)}$. It's easiest to do this by subtracting in the opposite order in Q2 above.

To generalize:

${}_tV^{(i)} - {}_{t-h}V^{(i)}$ = (increase acct by ≈ amt of interest, earned during $(t-h, t)$) then (add approx.. premium or , subtract approx. annuity income benefit paid) then (if a change occurs $i \to j$, set up the proper reserve ${}_tV^{(j)}$ or pay death benefit, offsetting the cost using the reserved amount ${}_tV^{(i)}$)

To approximate ${}_{t-h}V^{(i)}$, rewrite this equation in the form ${}_{t-h}V^{(i)} - {}_tV^{(i)} \approx$ …(each term's ± sign changes)… and solve for ${}_{t-h}V^{(i)}$

- You are given (or can compute) all of the values μ_{40}^{ij}, and you know δ and the premium and death benefit amounts.
- For 20-year term insurance, you set each ${}_{20}V^{(i)}$ = ___ ; for a 20-year endowment insurance, set each ${}_{20}V^{(i)}$ = ___.
- To find ${}_{19.75}V^{(0)}$, use the equation…
- To find ${}_{19.75}V^{(1)}$, use the equation…
- To find ${}_{19.50}V^{(0)}$, use the equation…
- To find ${}_{19.50}V^{(1)}$, use the equation…
- Et cetera. Of course, you would implement this using a spreadsheet.

HW "Thiele's Differential Equation":

1. Learn to write down and explain (in English) the "change-in-reserve" version of Thiele's differential equations in the temporary disability model. This will form a quiz problem later.

2. Learn how to do the same for the Kolmogorov differential equations ("change-in-${}_tp_x^{00}$" , etc.).

3. Learn how to go from the equations mentioned in 1 & 2 above to the derivative forms $\frac{d}{dt}\, {}_tV^{(0)}$ and $\frac{d}{dt}\, {}_tp_x^{00}$, etc.

Suggested Practice from ALTAM Sample Packet "Thiele's Diff. Eq": #8c, 16b
Add'l practice on the following pages.
↓

<u>Homework – Kolmogorov equations</u>: Do problem "W8.5", below. Optionally: Read DHW sections 8.1-8.6.

W8.5: Consider the model (0 = Alive & well / 1 = temporary disabled / 2 = dead) from today's lecture.

Assume constant transition forces

$$\mu_{60+t}^{01} = .01$$

$$\mu_{60+t}^{02} = .02$$

$$\mu_{60+t}^{10} = .03$$

$$\mu_{60+t}^{12} = .04 \quad .$$

Approximate ${}_{\frac{1}{12}}p_{60}^{00}$, ${}_{\frac{1}{12}}p_{60}^{01}$, ${}_{\frac{2}{12}}p_{60}^{00}$, and ${}_{\frac{2}{12}}p_{60}^{01}$ by using Euler's method and the Kolmogorov forward equations.. Maintain as much precision as possible throughout your computations. I'll post a solution soon.

Also try this SOA Written problem – Thiele's differential equation in the alive-dead model (more-or-less). (Treat "no cash value on lapse" as saying ${}_tV^{(\text{lapsed})} = 0$. ${}_tV$ means ${}_tV^{\text{alive,in-force policy}}$)

19. (6 points) An insurer issues fully continuous 10-year term insurance policies with face amount 100,000 to lives age 50. Level gross premiums of 300 per year are payable continuously throughout the term of the contract. There is no cash value on lapse.

Gross premium reserves are calculated on the following basis:

Mortality:	$\mu_x = Bc^x$, where $B = 10^{-5}$, $c = 1.1$
Lapse:	Lapse transition intensity is 0.05 per year.
Interest:	4% per year compounded continuously
Expenses:	50 per year, incurred continuously.

(a) (2 points)

(i) Write down Thiele's differential equation for ${}_tV$ for this contract.

(ii) Write down a boundary condition for Thiele's differential equation.

(b) (4 points) Using a time step of $h = 0.2$ years, estimate ${}_{9.6}V$.

ans: ${}_{9.8}V \approx 10.90$; ${}_{9.6}V \approx 20.44$

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DHW3e Ch. 9: Multiple Decrement Theory (1) Introduction

Definition: A multiple-decrement model is a multistate model of the following form:

Once you leave 0, you're done.

← exits "multiple exit model"

Multiple decrement models have their own notation, and so does DHW.

Notation: multiple decrement table

$\ell_x^{(\tau)}$ = # lives aged x on a mult. decr. table (tau τ)

${}_t d_x^{(j)}$ = # exits via j for lives ages $\in (x, x+t)$

${}_t q_x^{(j)}$ = Pr[(x) exits via j by age x+t]

${}_t q_x^{(\tau)}$ = total probability of decrement $= \sum_{j=1}^{n} {}_t q_x^{(j)}$

${}_t p_x^{(\tau)}$ = compliment of ${}_t q_x^{(\tau)}$, Pr["survival in 0"]

Dr. Wilson cannot recall ever using this

${}_{k|t} q_x^{(j)}$ = Pr[(x) survives k in 0, exits via j within subsequent t years]

When t = 1, *we write* $d_x^{(j)}$, $q_x^{(j)}$, $p_x^{(\tau)}$ *and* ${}_{k|}q_x^{(j)}$.

Relationships between ℓ's d's q's p's.

<u>Example 1</u>: In the following table, decrement 1 is accidental death, and Decrement 2 is death by non-accidental causes. It's could also be indicative abbreviations, eg w= withdraw r= retire ⋮

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$ Accidental	$d_x^{(2)}$ Non-Accidental
60	100	2	1
61	97	3	2
62	92	1	6
63	85	8	2
64	75		

$100-2-1=97$

$97-3-2=$

a. Fill in as much of the table as possible.

b. Compute ${}_2q_{60}^{(2)}$, ${}_2d_{60}^{(2)}$ ${}_{2|}q_{60}^{(2)}$, $p_{61}^{(\tau)}$

$${}_2q_{60}^{(2)} = \frac{d_{60}^{(2)}+d_{60}^{(2)}}{\ell_{60}^{(\tau)}} = \frac{2+1}{100} = 0.03$$

$${}_{2|}q_{60}^{(2)} = \frac{d_{62}^{(2)}}{\ell_{60}^{(\tau)}} = 0.06$$

$$p_{61}^{(\tau)} = \frac{\ell_{62}^{(\tau)}}{\ell_{61}^{(\tau)}} = \frac{92}{97}$$

c. A special **4-year** fully discrete policy pays 100 for decrement by cause 1 and 200 for decrement by cause 2 at the end of the year of decrement. Write an equation that could be used to find the level annual net premium π.

EPV[Prem] = EPV[Bens]

$$\pi\left(1+\underbrace{\frac{97}{100}}_{p_{60}^{(\tau)}}v+\underbrace{\frac{92}{100}}_{{}_2p_{60}^{(\tau)}}v^2+\underbrace{\frac{85}{100}}_{{}_3p_{60}^{(\tau)}}v^3\right) = 100\left(\underbrace{\frac{2}{100}}_{q_{60}^{(1)}}v+\underbrace{\frac{3}{100}}_{{}_{1|}q_{60}^{(1)}}v^2+\underbrace{\frac{1}{100}}_{{}_{2|}q_{60}^{(1)}}v^3+\underbrace{\frac{8}{100}}_{{}_{3|}q_{60}^{(1)}}v^4\right)$$

$$+200\left(\underbrace{\frac{1}{100}}_{q_{60}^{(2)}\ldots}v+\frac{2}{100}v^2+\frac{6}{100}v^3+\frac{2}{100}v^3\right)$$

get π w/ i

d. In (c), compute E[$_2$L | K$_{60}$ ≥ 2], where K$_{60}$ is the curtate future lifetime for (60).

That is, compute $_2$V.

There's only 1 state (0) you can reserve from

valued @ t=2

$${}_2V = \left[\begin{array}{c}100\left(\frac{1}{92}v+\frac{8}{92}v^2\right)\\+\\200\left(\frac{6}{92}v+\frac{2}{92}v^2\right)\end{array}\right]-\pi\left(1+\frac{85}{92}v\right)$$

$$=100A_{62:\overline{2|}}^{01}+200A_{62:\overline{2|}}^{02}-\pi(\ \downarrow\ )$$

Dr. Wilson: this notation is only sometimes used w/r/t multiple decr. models

Of course, we can study non-discrete situations as well.

"old" notation

<u>Definitions</u>: $\mu^{(j)}_{x+t}$ = Force of decrement $= \mu^{0j}_{x+t}$

(So Pr[immediate exit by cause j | survival of all decrements until age $x+t$] $\approx \mu^{(j)}\,dt$)

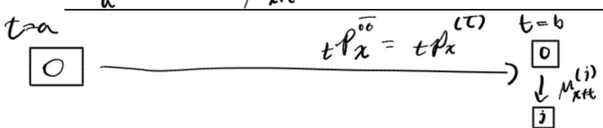
$\mu^{(\tau)}_{x+t} = \sum_j \mu^{(j)}_{x+t}$

<u>Fact</u>: In a multiple-decrement model, we have

Pr[(x) survives all decrements throughout $[0, t]$] $= e^{-\int_0^t \mu^{(\tau)}_{x+s}\,ds} = e^{-\int_0^t \mu^{(1)}_{x+s}\,ds} \cdot \ldots \cdot e^{-\int_0^t \mu^{(n)}_{x+s}\,ds}$

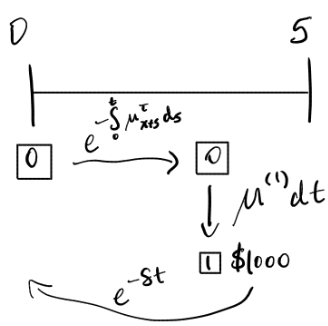
Pr[Decrement by cause j occurs between $t = a$ and $t = b$ | alive at age x]

$= \int_a^b e^{-\int_0^s \mu^{(\tau)}_{x+s}\,ds}\,\mu^{(j)}_{x+t}\,dt$ *(draw timeline)*



<u>Example 2</u>: A fully continuous 5-year term policy on (x) pays 1000 for decrement by cause 1 and 2000 for decrement by cause 2.

Given constant forces of decrement and interest $\mu^{(1)}_{x+t} = \mu_1$, $\mu^{(2)}_{x+t} = \mu_2$, and δ, find the expected present value for this insurance.



Same idea for decrement 2

$$EPV = 1000\int_0^5 e^{-\int_0^s \mu_1\,ds}\,\mu_1 e^{-\delta t}\,dt + 2000\int_0^5 e^{-\int_0^s \mu_2\,ds}\,\mu_2 e^{-\delta t}\,dt$$

$$= 1000\,\mu_1\left(\frac{1-e^{-(\mu_1+\mu_2+\delta)5}}{\mu_1+\mu_2+\delta}\right) + 2000\,\mu_2\left(\frac{1-e^{-(\mu_1+\mu_2+\delta)5}}{\mu_1+\mu_2+\delta}\right)$$

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HW Ch.9 SOA (below) #58, 206, 70, 179, 178, 135 (can split into cases and use "${}_{10}E_x^{(\tau)}$"),
Optional study: #232 (${}_2V$ = E[future loss at time 2, valued at time 2]) , 167, 105

58. XYZ Paper Mill purchases a 5-year special insurance paying a benefit in the event its machine breaks down. If the cause is "minor" (1), only a repair is needed. If the cause is "major" (2), the machine must be replaced.

Given:

(i) The benefit for cause (1) is 2000 payable at the moment of breakdown.

(ii) The benefit for cause (2) is 500,000 payable at the moment of breakdown.

(iii) Once a benefit is paid, the insurance is terminated.

(iv) $\mu_t^{(1)} = 0.100$ and $\mu_t^{(2)} = 0.004$, for $t > 0$

(v) $\delta = 0.04$

Calculate the expected present value of this insurance.

Answer: 7841

***206.** Michael, age 45, is a professional motorcycle jumping stuntman who plans to retire in three years. He purchases a three-year term insurance policy. The policy pays 500,000 for death from a stunt accident and nothing for death from other causes. The benefit is paid at the end of the year of death.

You are given:

(i) $i = 0.08$

(ii)

x	$l_x^{(\tau)}$	$d_x^{(-s)}$	$d_x^{(s)}$
45	2500	10	4
46	2486	15	5
47	2466	20	6

where $d_x^{(s)}$ represents deaths from stunt accidents and $d_x^{(-s)}$ represents deaths from other causes.

(iii) Level annual net premiums are payable at the beginning of each year.

Calculate the annual net premium.

Answer: 921

70. For a special fully discrete 3-year term insurance on (55), whose mortality follows a double decrement model:

(i) Decrement 1 is accidental death; decrement 2 is all other causes of death.

(ii)

x	$q_x^{(1)}$	$q_x^{(2)}$
55	0.002	0.020
56	0.005	0.040
57	0.008	0.060

(iii) $i = 0.06$

(iv) The death benefit is 2000 for accidental deaths and 1000 for deaths from all other causes.

(v) The level annual gross premium is 50.

(vi) ${}_1L$ is the prospective loss random variable at time 1, based on the gross premium.

(vii) K_{55} is the curtate future lifetime of (55).

Calculate $\mathrm{E}\left[{}_1L \middle| K_{55} \geq 1\right]$.

Answer: 16.72

179. Kevin and Kira are modeling the future lifetime of (60).

(i) Kevin uses a double decrement model:

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
60	1000	120	80
61	800	160	80
62	560	–	–

(ii) Kira uses a multi-state model:
(a) The states are 0 (alive), 1 (death due to cause 1), 2 (death due to cause 2).
(b) Her calculations include the annual transition probabilities.

(iii) The two models produce equal probabilities of decrement.

Calculate $p_{61}^{00} + p_{61}^{01} + p_{61}^{10} + p_{61}^{11}$.

Answer: 1.90

***178.** A special whole life insurance of 100,000 payable at the moment of death of (x) includes a double indemnity provision. This provision pays during the first ten years an additional benefit of 100,000 at the moment of death for death by accidental means.

You are given:

(i) $\mu_{x+t}^{(\tau)} = 0.001, \quad t \geq 0$

(ii) $\mu_{x+t}^{(1)} = 0.0002, \quad t \geq 0$, is the force of decrement due to death by accidental means.

(iii) $\delta = 0.06$

Calculate the single net premium for this insurance.

Answer: 1789.06

***135.** For a special whole life insurance of 100,000 on (x), you are given:

(i) $\delta = 0.06$

(ii) The death benefit is payable at the moment of death.

(iii) If death occurs by accident during the first 30 years, the death benefit is doubled.

(iv) $\mu_{x+t}^{(\tau)} = 0.008, \quad t \geq 0$

(v) $\mu_{x+t}^{(1)} = 0.001, \quad t \geq 0$, is the force of decrement due to death by accident.

Calculate the single net premium for this insurance.

Answer: 13044

***232.** For a fully discrete 4-year term insurance on (40), who is subject to a double-decrement model:

(i) The benefit is 2000 for decrement 1 and 1000 for decrement 2.

(ii) The following is an extract from the double-decrement table for the last 3 years of this insurance:

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
41	800	8	16
42	–	8	16
43	–	8	16

(iii) $v = 0.95$

(iv) The net premium is 34.

Calculate ${}_2V$, the net premium reserve at the end of year 2.

(Note: Recall that ${}_2$V = E[future loss at time 2, valued at time 2])

Answer: 11.12

167. (50) is an employee of XYZ Corporation. Future employment with XYZ follows a double decrement model:

(i) Decrement 1 is retirement.

(ii) $\mu_{50+t}^{(1)} = \begin{cases} 0.00 & 0 \le t < 5 \\ 0.02 & 5 \le t \end{cases}$

(iii) Decrement 2 is leaving employment with XYZ for all other causes.

(iv) $\mu_{50+t}^{(2)} = \begin{cases} 0.05 & 0 \le t < 5 \\ 0.03 & 5 \le t \end{cases}$

(v) If (50) leaves employment with XYZ, he will never rejoin XYZ.

Calculate the probability that (50) will retire from XYZ before age 60.

Answer: .0689

105. For students entering a college, you are given the following from a multiple decrement model:

(i) 1000 students enter the college at $t = 0$.

(ii) Students leave the college for failure (1) or all other reasons (2).

(iii) $\mu_{x+t}^{(1)} = \mu \qquad 0 \le t \le 4$

$\mu_{x+t}^{(2)} = 0.04 \qquad 0 \le t < 4$

(iv) 48 students are expected to leave the college during their first year due to all causes.

Calculate the expected number of students who will leave because of failure during their fourth year.

Answer: 7.6

33. For a triple decrement table, you are given:

(i) $\mu_{x+t}^{(1)} = 0.3, \quad t > 0$

(ii) $\mu_{x+t}^{(2)} = 0.5, \quad t > 0$

(iii) $\mu_{x+t}^{(3)} = 0.7, \quad t > 0$

Calculate $q_x^{(2)}$.

Answer: $\frac{1}{3} \cdot (1 - e^{-1.5})$

DHW2e Ch.9: Multiple Decrement Theory (2) UDD in the Multiple Decrement Table

Recall: UDD in alive-dead model. (UDD = Uniform distribution of deaths between integer ages.)

- To "assume UDD" in the alive-dead model means to assume that

$${}_sq_x = \underline{s \cdot q_x} \quad \text{for } x \in \mathbb{Z} \text{ and } s \in [0, 1].$$

- By the FTC, the density $f_{T_x}(s)$ is related to q_x:

$F_{T_x}(s) = \Pr[T_x \le s] = {}_sq_x = s \cdot q_x$ if

$\xrightarrow{d/dt} f_{T_x}(s) \equiv q_x$ for $s \in [0,1]$

- Under UDD, one has the ability to use linear interpolation between l_x's for consecutive integer ages.

eg $\ell_{53.8} = 0.8\ell_{54} + 0.2\ell_{53}$ $\quad$ $\ell_{x+s} = s\ell_{x+1} + (1-s)\ell_x$

Example 1: Consider the following life table, and do the following computations under the UDD assumption.

x	ℓ_x	d_x
60	100,000	2,000
61	98,000	3,000
62	95,000	1,000
63	94,000	

a. Find ${}_{.3}q_{61}$ using the UDD definition. Also think about: # deaths (deaths are uniformly "spread out" over each year.)

$${}_{0.3}q_{61} \overset{UDD}{=} (0.3)(q_{61}) = 0.3 \cdot \frac{3000}{98000} = \frac{\text{"30\% deaths w/i [61,62]"}}{\ell_{61}}$$

b. Find ${}_{.8}q_{61.3}$ under the UDD assumption by using linear interpolation to get the necessary ℓ_x's (which is only appropriate because of the UDD assumption)

*Take a weighted average of ℓ_{61} and ℓ_{62}, with weights 0.3 and 0.7 (= 1 – .3) used so that ℓ_{61} counts *more heavily* in the average (*Think: 61.3 is closer to 61 than to 62.*)

$$\ell_{61.3} \overset{UDD}{=} 0.3\ell_{62} + 0.7\ell_{61} = 97100$$

$$\ell_{62.1} \overset{UDD}{=} 0.1\ell_{63} + 0.9\ell_{62} = 94900$$

$$\Rightarrow {}_{0.8}q_{61.3} \overset{def}{=} \frac{\ell_{61.3+0.8}}{\ell_{61.3}} = \frac{\ell_{62.1}}{\ell_{61.3}} = \frac{94900}{97100}$$

Main point: One can choose from three common assumptions (MUDD, SUDD, constant force) in order to make a reasonable guess concerning decrement probabilities between integer ages in a multiple decrement environment. Each method has its own nice mathematical properties.

UDD in the Multiple Decrement Table (MUDD or "UDD in MDT")
UDD = Uniform distribution of **decrements** btwn integer ages.)

$x \in \mathbb{Z}$, $s \in [0,1]$

Similar to the above, we may assume ${}_sq_x^{(j)} = \underline{s \cdot q_x^{(j)}}$ whenever x is an integer and $s \in [0, 1]$.
We call this assumption **UDD in the multiple decrement table**.

Example 2: You are given the following multiple decrement table. Assume UDD in the MDT.

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
60	100,000	2,000	1,000
61	97,000	3,000	2,000
62	92,000	1,000	0

a. Compute ${}_{0.2}q_{61}^{(j)}$ for $j = 1, 2$ in two ways: using the formulas, and by drawing a little diagram showing how the "cause 1 decrements" are uniformly spread out.

$${}_{0.2}q_{61}^{(1)} \overset{MUDD}{=} 0.2\left(q_{61}^{(1)}\right) = 0.2\left(\frac{3000}{97000}\right)$$

$${}_{0.2}q_{61}^{(2)} \overset{MUDD}{=} 0.2\left(q_{61}^{(2)}\right) = 0.2\left(\frac{2000}{97000}\right)$$

b. True or false: The total probability of decrement ${}_tq_x^{(\tau)}$ can be modeled by UDD if the individual causes are modeled by UDD *in the multiple decrement table*. What does this mean about using linear interpolation to find ℓ_{x+t} between integer ages?

$${}_tq_x^{(\tau)} \overset{def}{=} {}_tq_x^{(1)} + \dots + {}_tq_x^{(n)} \overset{MUDD}{=} t\,q_x^{(1)} + \dots + t\,q_x^{(n)}$$

$$= t\left(q_x^{(\tau)}\right) \checkmark \quad \text{True}$$

c. Compute ${}_{0.2}q_{61}^{(\tau)}$ and ${}_{0.2}p_{61}^{(\tau)}$.

$${}_{0.2}q_{61}^{(\tau)} = 0.2 \cdot q_{61}^{(\tau)} = 0.2\left(\frac{3000 + 2000}{97000}\right) = \frac{1}{97}$$

$${}_{0.2}p_{61}^{(\tau)} = 1 - {}_{0.2}q_{61}^{(\tau)} = \frac{96}{97}$$

d. Compute ${}_{0.3}q_{60.9}^{(1)}$. (Plan: Start by thinking about $\ell_{60.9}$ and ${}_{0.1}d_{60.9}^{(1)}$.

$${}_{0.3}q_{60.9}^{(1)} = \frac{{}_{0.1}d_{60.9}^{(1)} + {}_{0.2}d_{61}^{(1)}}{\ell_{60.9}} \overset{MUDD}{=} \frac{0.1 \cdot d_{60}^{(1)} + 0.2 d_{61}^{(1)}}{0.9\ell_{61} + 0.1\ell_{60}} = \frac{200 + 600}{97300}$$

Single Decrement Environment / MUDD / UDD in the MDT

Consider a multiple-decrement scenario with forces of decrement given by functions

$$\mu_{x+t}^{(j)} = \mu_x^{(j)}(t).$$

We would like to consider a (hypothetical) parallel environment in which decrement (j) is the only possible decrement, with the same force function $\mu_{x+t}^{(j)}$.

Definition: We define the "single decrement survival probability" ${}_tp_x^{\prime(j)}$ by the equation

$${}_tp_x^{\prime(j)} = e^{-\int_0^t \mu_{x+s}^{j}\,ds}$$

← "prime" designates SUDD assumption

Define ${}_tq_x^{\prime(j)} = 1 - {}_tp_x^{\prime(j)}$.

Notes:

- These are also known as the "independent probabilities of decrement" (DHW terminology)
- The multiple-decrement environment probabilities ${}_tp_x^{(\tau)}$ and ${}_tq_x^{(j)}$ are called "dependent" probabilities in DHW.
- ~~DHW drops the prime and parentheses from the notation, which is problematic! We will use the SOA notation exclusively.~~

Maybe uses a ∦? Who knows

Important facts: To obtain ${}_tp_x^{(\tau)}$ from the single (!) decrement probabilities ${}_tp_x^{\prime(j)}$:

"prime", not one's

$${}_tp_x^{(\tau)} \overset{SUDD}{=} e^{-\int_0^t (\mu_{x+s}^{(1)} + \dots + \mu_{x+s}^{(n)})ds} = {}_tp_x^{\prime(1)} \cdot {}_tp_x^{\prime(2)} \cdots {}_tp_x^{\prime(n)}$$

Relationship between ${}_tq_x^{(\tau)}$ and the (multiple decrement probabilities!) ${}_tq_x^{(j)}$:

$${}_tq_x^{(\tau)} \underset{SUDD}{\overset{def}{=}} {}_tq_x^{(1)} + {}_tq_x^{(2)} + \dots + {}_tq_x^{(n)}$$

<u>Fact</u>: ${}_tq_x^{(j)} \le {}_tq_x'^{(j)}$ for any x and t. Why? (Competition – look at survival factors in integral expressions for failure probabilities.)

$$\begin{tikzcd} \boxed{0} \arrow[r] & \boxed{0} \arrow[d] \\ & \boxed{j} \end{tikzcd}$$

$$\int_0^t {}_sp_x^{(\tau)}\, \mu_{x+s}^{(j)}\, ds \le \int_0^t {}_sp_x'^{(j)}\, \mu_{x+s}^{(j)}\, ds$$

$${}_sp_x^{(1)} \cdot {}_sp_x^{(2)} \cdots {}_sp_x^{(n)} \le {}_sp_x^{(j)}$$

This means that the failure probabilities on the double decrement table and associated single decrement tables will not be the same!

<u>**Assembling single decrement probabilities into a multiple decrement table**</u>—three methods:

- Assume UDD in MDT (MUDD),
- Assume constant forces of decrement between integer ages, or
- Assume UDD in SDT (SUDD).

<u>UDD in the multiple decrement table</u> (MUDD or "UDD in MDT"):

The basic relationship that we remember for the MUDD assumption involves the failure probabilities:

a. For $x \in \mathbb{Z}$ and $t \in [0, 1]$, we have…

$${}_tq_x^{(j)} = t \cdot q_x^{(j)} \qquad {}_tq_x^{(\tau)} = t \cdot q_x^{(\tau)}$$

b. From this, we can express (for $t \in [0,1]$) $s_x^{\tau}(t) = {}_tp_x^{(\tau)}$ in terms of ${}_tq_x^{(\tau)}$:

$${}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)} = 1 - t \cdot q_x^{(\tau)}$$

c. The function ${}_tq_x^{(j)}$ = Pr[Cause (j) decrement with time-to-decrement $\le t$ | alive at age x] is analogous to a cdf (Probability that a time-to-failure $\le t$). If we differentiate, we get something like a density:

linear w/r/t t

$$\frac{d}{dt}\left[{}_tq_x^{(j)}\right] \overset{\text{MUDD}}{=} \frac{d}{dt}\left[t \cdot q_x^{(j)}\right] = q_x^{(j)}$$

d. Dividing density by survival probability gives us a force of decrement, so under MUDD we can write $\mu_{x+t}^{(j)}$ in terms of $q_x^{(j)}$ and ${}^{(\tau)}_{x}$:

$x \in \mathbb{Z} \quad t \in [0,1]$

$$\mu_{x+t}^{(j)} = \frac{\text{"density"}}{\text{"survival"}} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(j)}}$$

(Continued from prev. pg.: MUDD ⇒ $\mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{1-tq_x^{(\tau)}}$)

e. Integrate both sides for $t \in [0,1]$. Get $\frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot -\ln\left(1-q_x^{(\tau)}\right)$.

gives full year of data $\longrightarrow$ $\int_0^1 \mu_{x+t}^{(j)}\,dt = \int_0^1 \frac{q_x^{(j)}}{1-tq_x^{(j)}}\,dt \overset{\text{u-sub}}{=} -\frac{q_x^{(j)}}{q_x^{(\tau)}} \ln\left(\underbrace{1-q_x^{(\tau)}}_{p_x^{(\tau)}}\right)$

f. The LHS $\int_0^1 \mu_{x+t}^{(j)}\,dt$ looks *nearly* like $\ln\left(p_x'^{(j)}\right)$! Use (e) to find an expression for $\ln\left(p_x'^{(j)}\right)$.

$$\ln\left(p_x'^{(j)}\right) = \ln\left(e^{\int_0^1 \cdots dt}\right) = -\int \mu^{(j)}\,dt = (-1)(\quad$$

not sure what Dr. Wilson was monologue-ing about !!

g. Rearrange the equation $\ln\left(p_x'^{(j)}\right) = \frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot \ln\left(\underbrace{1-q_x^{(\tau)}}_{p_x^{(\tau)}}\right)$ into an easy-to-remember form.

$$\boxed{\frac{\ln\left(p_x'^{(j)}\right)}{\ln\left(p_x^{(\tau)}\right)} = \frac{q_x^{(j)}}{q_x^{(\tau)}}}$$

h. We can further rearrange this to get a formula for the $q_x^{(j)}$ in terms of the single-decrement probabilities.

Example 2: Under MUDD, if you know the single decrement probabilities, you know everything!

You are given, for a double-decrement scenario, that $q'^{(1)}_{50} = .1$ and $q'^{(2)}_{50} = .2$. Assume MUDD.

(So $\dfrac{\ln\left(p'^{(j)}_x\right)}{\ln\left(p^{(\tau)}_x\right)} = \dfrac{q^{(j)}_x}{q^{(\tau)}_x}$.)

$$\frac{\ln p'^{(1)}_x}{\ln p^{(\tau)}_x} = \frac{q^{(1)}_x}{q^{(\tau)}_x}$$

a. Compute the dependent probabilities $q^{(1)}_{50}$, $q^{(2)}_{50}$. *(Get .089804, .190196)*

$p'^{(1)}_{50} = 1 - q'^{(1)}_{50} = 0.9$

$p'^{(2)}_{50} = 0.8$

$p^{(\tau)}_{50} = p'^{(1)}_{50} \cdot p'^{(2)}_{50} = 0.72$

$q^{(\tau)}_{50} = 1 - 0.72 = 0.28$

$q^{(1)}_{50} \overset{\text{MUDD}}{=} \dfrac{\ln\left(p'^{(1)}_x\right)}{\ln\left(p^{(\tau)}_x\right)} \cdot q^{(\tau)}_{50} = \left(\dfrac{\ln 0.9}{\ln 0.72}\right) .28 = 0.089804$

$q^{(2)}_{50} \overset{\text{MUDD}}{=} \dfrac{\ln p'^{(2)}_x}{\ln p^{(\tau)}_x} \, q^{(\tau)}_{50} = \dfrac{\ln 0.8}{\ln 0.72} \, 0.28 = 0.190196$

Start 19 Feb 24

b. Compute ${}_{.4}q^{(1)}_{50.6}$. *Tip: Make a double-decrement table.* *(Using $\ell_{50} = 100$, get $\ell_{50.6} = 83.2$)*

MUDD gives UDD individually along each exit

x	ℓ_x	$d^{(1)}_x$
50	1000	
50.6	832	
51	720	

sum

1) $p^{(\tau)}_{50} = .9 \cdot .8 = .72$

2) UDD linear interpolation: $.6\,\ell_{51} + .4\,\ell_{50} = \ell_{50.6} = 832$

3) Count exits in mult. decr. $q^{(1)}_{50} \cdot 1000 = 89.804$

UDD → 40% exits occur $\in [50.6, 51] \to 0.4 \cdot q^{(1)}_{50}$

i.e. ${}_{0.4}q^{(1)}_{50.6} = \dfrac{0.4\, d^{(1)}_{50.6}}{\ell_{50.6}} = \boxed{\dfrac{0.4(89.804)}{832}}$

Source: DHW3e p.373

Comparison of notations

(DHW)

Table 9.4 *Summary of multiple decrement model notation.*

	AMLCR	USA and Canada	UK and Australia
Dependent survival probability	${}_tp_x^{00}$	${}_tp_x^{(\tau)}$	${}_t(ap)_x$
Dependent transition probability	${}_tp_x^{0j}$	${}_tq_x^{(j)}$	${}_t(aq)_x^j$
Dependent total transition probability	${}_tp_x^{0\bullet}$	${}_tq_x^{(\tau)}$	${}_t(aq)_x$
Independent transition probability	${}_tq_x^{*(j)}$	${}_tq_x^{\prime(j)}$	${}_tq_x^j$
Independent survival probability	${}_tp_x^{*(j)}$	${}_tp_x^{\prime(j)}$	${}_tp_x^j$
Forces of transition	μ_{x+t}^{0j}	$\mu_x^{(j)}(t)$	μ_{x+t}^j
Total force of transition	$\mu_{x+t}^{0\bullet}$	$\mu_x^{(\tau)}(t)$	$(a\mu)_{x+t}$
Multiple Decrement Table:			
Active lives	l_x	$l_x^{(\tau)}$	$(al)_x$
Decrements	$d_x^{(j)}$	$d_x^{(j)}$	$(ad)_x^j$

Homework for UDD in the Multiple Decrement Table:

187. For a double decrement table, you are given:

Age	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
40	1000	60	55
41	–	–	70
42	750	–	–

Each decrement is uniformly distributed over each year of age in the double decrement table.

Calculate $q_{41}^{\prime(1)}$.

Answer: .0766

224. A population of 1000 lives age 60 is subject to 3 decrements, death (1), disability (2), and retirement (3). You are given:

(i) The following independent rates of decrement:

x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
60	0.010	0.030	0.100
61	0.013	0.050	0.200

(ii) Decrements are uniformly distributed over each year of age in the multiple decrement table.

Calculate the expected number of people who will retire before age 62.

Answer: 265.63

36. For a double decrement table, you are given:

(i) $q_x'^{(1)} = 0.2$

(ii) $q_x'^{(2)} = 0.3$

(iii) Each decrement is uniformly distributed over each year of age in the double decrement table.

Calculate ${}_{0.3}q_{x+0.1}^{(1)}$.

Answer: .053

DHW2e Ch. 9: Multiple Decrement Theory (3) UDD in the Single Decrement Tables

Assume that we have uniform distribution of decrements in every single decrement table. How can we relate the single-decrement probabilities ${}_tp'^{(j)}_x$, ${}_tq'^{(j)}_x$ to the multiple decrement probabilities ${}_tp^{(j)}_x$, ${}_tq^{(j)}_x$?

a. The basic UDD relationship (***in the single-decrement context***) is that for $t \in [0, 1]$ and $x \in \mathbb{Z}$, …

$$ {}_tq'^{(j)}_x = t \cdot q'^{(j)}_x \quad \text{for } x \in \mathbb{Z},\ t \in [0,1] $$

b. If we temporarily consider the single decrement environment to be an alive-dead model, then we get the following cdf and density for the time-to-failure (for $t \in [0,1]$):

$$ f^{(j)}_x(t) = \frac{d}{dt} F_x(t) \equiv q'^{(j)}_x \qquad t \in [0,1] $$

The formulas in (b) hold, even though they arise from a hypothetical context.

c. Remaining in the single-decrement context, we recall that a density for a time-to-failure/ future lifetime variable can be written as a product of survival and force of decrement:

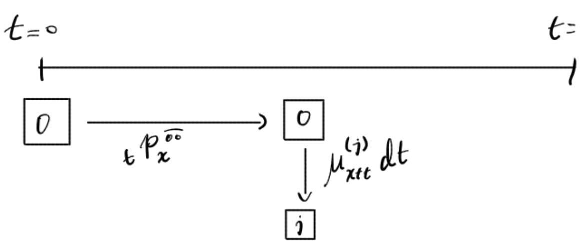
$$ \mu^{(j)}_{x+t} = \frac{f^{(j)}_x(t)}{{}_tp'^{(j)}_x} \rightsquigarrow f_x(t) = {}_tp'^{(j)}_x \cdot \mu^{(j)}_{x+t} $$

We'll keep our eyes peeled for a chance to use (c).

d. Back to the *multiple decrement* context—all forces $\mu^{(j)}_{x+t}$ are operating. (Note: We will not have UDD here if we have UDD in the single decrement context.)

Write down an integral to compute…

$q^{(j)}_x$ (= Pr[(x) decrements by (j) within 1 year]) .



$$ q^{(j)}_x = \int_0^1 {}_tp^{(\tau)}_x \, \mu^{(j)}_{x+t} \, dt $$

survive everything

$$ = \int_0^1 \overbrace{{}_tp'^{(1)}_x \cdots {}_tp'^{(j)}_x \cdots {}_tp'^{(n)}_x}^{\text{survive everything}} \, \mu^{(j)}_{x+t} \, dt $$

(b) (c)

$$ f^{(j)}_x(t) = q'^{(j)}_x $$

$$ \Longrightarrow q^{(j)}_x \overset{\text{SUDD}}{=} q'^{(j)}_x \int_0^1 \prod_{k \neq j} \left(1 - t\, q'^{(k)}_x\right) dt $$

e. When there are only two decrements, the relationship

$$q_x^{(j)} = q'^{(j)}_x \int_0^1 \left[\prod_{i \neq j} (1 - t \cdot q'^{(i)}_x) \right] dt$$

becomes…

$$q_x^{(j)} = q'^{(j)} \int_0^1 (1 - t\, q'^{(i)}_x)\, dt \quad \text{for } i \neq j$$

$$\text{e.g. } q^{(1)} = q'^{(1)} \int_0^1 (1 - t\, q'^{(2)}_x)\, dt = q'^{(1)} \left(t - \frac{t^2}{2} q'^{(2)}_x \right) \Big|_{t=0}^{1} \to q'^{(1)}_x \left(1 - \frac{1}{2} q'^{(2)}_x \right)$$

(How to remember it.)

In a sense, the competition from exit 2 reduces Pr[exit from 1] by a factor of $\frac{1}{2}$ in a single decr. environment

Example 1: Suppose that $p'^{(1)}_x = .9$ and $p'^{(2)}_x = .8$. Compute $q_x^{(1)}$ under the assumption of UDD in each single decrement table. → SUDD

$$q_x^{(1)} = q'^{(1)} \left(1 - \frac{1}{2} q'^{(2)} \right) = 0.1 \left(1 - \frac{1}{2}(0.2) \right) = 0.09$$

Competing discrete and continuous decrements and SUDD: *If decrement #1 has SUDD and decrement #2 occurs only at one discrete point in the year, then decrement #1 also has UDD in the multiple decrement table until the time at which decrements by cause #2 occur.*

Example 2: Suppose that decrement 1 is death and decrement 2 is retirement,

$$p'^{(1)}_{60} = p'^{(1)}_{61} = .9\ ,$$

and

$$p'^{(2)}_{60} = p'^{(2)}_{61} = .8\ .$$

Assume that deaths are uniformly distributed between integer ages in the associated single decrement table and that all retirements happen immediately before age 61 or immediately before age 62.

SUDD

Find ${}_2q^{(1)}_{60}$ and ${}_2q^{(2)}_{60}$. *Tip: Keep track of #deaths & retirements as you go.*

x | $\ell_x^{(\tau)}$ can be any fixed scalar, powers of 10 are nice

60 | 100

↓ 10% die

Moment just before retirement → $60.\bar{9}$ | 90

↓ 20% retire

61 | 72

↓ 10% die

$61.\bar{9}$ | 64.8

↓ 20% retire

62 | 51.84

x	$\ell_x^{(\tau)}$	$d^{(1)}$	$d^{(2)}$
60	100	10	18
61	72	7.2	12.96
62	51.84		

$${}_2q^{(1)}_{60} = \frac{10+7.2}{100} = .172$$

$${}_2q^{(2)}_{60} = \frac{18+12.96}{100} = .3096$$

Example 3: Suppose that decrement 1 is death and decrement 2 is retirement,

$p'^{(1)}_{60} = .9$, ← SUDD

and

$p'^{(2)}_{60} = .8$. ← only @ $t=60.6$

Assume that deaths are uniformly distributed between integer ages in the associated single decrement table and that all retirements happen at age 60.6.

a. Find $q^{(2)}_{60}$. *Tip: Keep track of #deaths & retirements as you go.*

(Note: Although there is a way to find $q^{(1)}_{60}$ by determining #deaths after age 60.6, that method is somewhat more technical and is not covered by DHW. The procedure is covered by MQR 5e Ex. 13.9 & solution on p.112 of DHW4e solution manual. An alternate method is shown in (b) below.)

x	$l^{(\tau)}_x$		
60	100		
$60.5\bar{9}$	94	← $.6(0.9)(100)$	(6 exits)
60.6	75.2	← $.8(94)$	(18.8 exits)
61	????	← UDD does not apply	

Womp Womp

b. Find $p^{(\tau)}_{60}$ and use it to find $q^{(1)}_{60}$.

$p^{(\tau)}_{60} = 1 - q^{(\tau)}_{60}$

$(0.9)(0.8) = 1 - \left(q^{(1)}_{60} + q^{(2)}_{60} \right)$

.72

$q^{(1)}_{60}$: solve

$q^{(2)}_{60}$: 0.188

$q^{(1)}_{60} = 0.092$

Study problem: (This reviews the earlier UDD in MDT / MUDD topic from previous handout.)

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
60	100,000	2,000	1,000
61	97,000	3,000	2,000
62	92,000	1,000	0

Assuming UDD over each year of age in the multiple-decrement table, compute

a. $_{.5}q_x^{(2)}$ for $x = 60$ and for $x = 61$.

b. $_{.5}q_{60.6}^{(2)}$

c. $_{.5}p_{60.6}^{(\tau)}$

Suggested Practice from ALTAM Packet: #2

Homework problems for UDD in Single Decrement Table:

1. This is from p.26 of the DHW1e supplement (published long ago), rewritten in our notation.

   Show that with three decrements, the SUDD assumption yields the following relationship:

$$q_x^{(1)} = q'^{(1)}_x \left(1 - \tfrac{1}{2}(q'^{(2)}_x + q'^{(3)}_x) + \tfrac{1}{3}q'^{(2)}_x q'^{(3)}_x\right)$$

   (Obviously, one can then derive analogous expressions for $q_x^{(2)}$ and $q_x^{(3)}$ in the same way—you don't need to waste time doing this).

2. Do the following from the SOA problem set:

SUDD in each decrement:

234. For a triple decrement table, you are given:

(i) Each decrement is uniformly distributed over each year of age in its associated single decrement table.

(ii) $q_x'^{(1)} = 0.200$

(iii) $q_x'^{(2)} = 0.080$

(iv) $q_x'^{(3)} = 0.125$

Calculate $q_x^{(1)}$.

Answer: .1802

SUDD/discrete: Do SOA #42, 83

42. For a double decrement table where cause 1 is death and cause 2 is withdrawal, you are given:

(i) Deaths are uniformly distributed over each year of age in the single-decrement table.

(ii) Withdrawals occur only at the end of each year of age.

(iii) $l_x^{(\tau)} = 1000$

(iv) $q_x^{(2)} = 0.40$

(v) $d_x^{(1)} = 0.45\ d_x^{(2)}$

Calculate $p_x'^{(2)}$.

Answer: .512

83. For a double decrement model:

(i) In the single decrement table associated with cause (1), $q'^{(1)}_{40} = 0.100$ and decrements are uniformly distributed over the year.

(ii) In the single decrement table associated with cause (2), $q'^{(2)}_{40} = 0.125$ and all decrements occur at time 0.7.

Calculate $q^{(2)}_{40}$.

Answer: .11625

DHW3e Ch. 9: Multiple Decrement Theory (4) Constant Force Between Integer Ages

Constant force assumption between integer ages – an alternative (to MUDD or SUDD) way to estimate survival between integer ages

Suppose that the forces of decrement $\mu_{x+t}^{(j)}$ are constants (say $\mu^{(j)}$) for every $t \in [0, 1]$.

(Note: Different constants will be needed for different age intervals $[x, x+1]$: we can write $\mu_x^{(j)}$ if necessary to distinguish these from each other.)

1. Show that $\frac{{}_tq_x^{(j)}}{{}_tq_x^{(\tau)}} = \frac{\mu^{(j)}}{\mu^{(\tau)}}$ for every $t \in [0, 1]$.

Consider $\frac{{}_tq_x^{(j)}}{{}_tq_x^{(\tau)}} = \frac{\int_0^t {}_sp_x^{\overline{00}} \mu^{(j)} ds}{\int_0^t {}_sp_x^{\overline{00}} \mu^{(\tau)} ds} = \frac{\mu^{(j)} \int_0^t {}_sp_x^{\overline{00}} ds}{\mu^{(\tau)} \int_0^t {}_sp_x^{\overline{00}} ds} = \frac{\mu^{(j)}}{\mu^{(\tau)}}$ ✓

2. Setting $t = 1$, (1) says: $\frac{q_x^{(j)}}{q_x^{(\tau)}} = \frac{\mu^{(j)}}{\mu^{(\tau)}}$.

3. So $\frac{{}_tq_x^{(j)}}{{}_tq_x^{(\tau)}} = \frac{q_x^{(j)}}{q_x^{(\tau)}}$. Rearrange to get ${}_tq_x^{(j)} =$ $q_x^{(\tau)} \cdot \frac{{}_tq_x^{(j)}}{{}_tq_x^{(\tau)}} = q_x^{(\tau)} \left(\frac{\mu^{(j)}}{\mu^{(\tau)}}\right)$

Interpretation: Within any period of constant decrement forces,

${}_tq_x^{(j)}$ is the fraction of ${}_tq_x^{(\tau)}$ given by arranging the appropriate ***one-year*** decrement probabilities into a fraction.

4. The total survival probability ${}_tp_x^{(\tau)}$ (and hence, the complementary probability ${}_tq_x^{(\tau)}$) is easy to get under constant force interage assumption from the one-year probability $p_x^{(\tau)}$

$${}_tq_x^{(\tau)} = 1 - {}_tp_x^{(\tau)} = 1 - e^{-\mu^{(\tau)}t} = 1 - \underbrace{\left(e^{-\mu^{(\tau)} \cdot 1}\right)}_{\text{one-year prob.}}{}^{t} = 1 - \left(p_x^{(\tau)}\right)^t$$

5. Hence, we can use ${}_tq_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot {}_tq_x^{(\tau)}$ to compute ${}_tq_x^{(j)}$ more-or-less straight off of a life table.

Constant forces of decrement between integer ages

$$\Leftrightarrow \quad {}_tq_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot {}_tq_x^{(\tau)} \quad \text{and} \quad {}_tp_x^{(\tau)} = \left(p_x^{(\tau)}\right)^t, \; t \in [0,1], x \in \mathbb{Z}$$

$\rightarrow \ell_{51} = 75, \; d_{50}^{(\tau)} = 5$

Example: Suppose that $\ell_{50} = 80$, $d_{50}^{(1)} = 3$, $d_{50}^{(2)} = 2$. Compute ${}_{0.2}q_{50}^{(1)}$ under the assumption of constant forces of decrement between integer ages. *Note that we can perform this computation without finding the forces of decrement!*

$${}_{.2}q_{50}^{(1)} = \frac{q_{50}^{(1)}}{q_{50}^{(\tau)}} \cdot {}_{.2}q_{50}^{(\tau)} = \frac{3/80}{5/80}\left(1-(p_{50})^{.2}\right) = \frac{3}{5}\left(1-\left(\frac{75}{80}\right)^{.2}\right)$$

Recall: Under the constant force inter-age assumption: $\frac{\mu_x^{(j)}}{\mu_x^{(\tau)}} = \frac{q_x^{(j)}}{q_x^{(\tau)}}$

(Proof: Evaluate the integrals for $q_x^{(j)}$ and $q_x^{(\tau)}$, and then factor out constant scalars.)

Theorem: Under constant force between integer ages, we get the same formula for ${}_tp_x'^{(j)}$ as we had under MUDD ($x \in \mathbb{Z}$, $t \in [0,1]$).

Proof: Write down the definition of $p_x'^{(j)}$ and use fact (above) about the ratios of forces.

$$p_x'^{(j)} \stackrel{\text{def}}{=} e^{-\int_{s=0}^{1} \mu^{(j)}\, ds} = e^{-\mu^{(j)}} = e^{-\frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot \mu^{(\tau)}} = \left[e^{-\mu^{(\tau)}}\right]^{\frac{q_x^{(j)}}{q_x^{(\tau)}}} = \left[p_x^{(\tau)}\right]^{\frac{q_x^{(j)}}{q_x^{(\tau)}}}$$

$$\text{So } \ln\left[p_x'^{(j)}\right] = \ln\left[\left(p_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}\right] = \frac{q_x^{(j)}}{q_x^{(\tau)}} \ln\left(p_x^{(\tau)}\right)$$

$$\Rightarrow \quad \frac{\ln p_x'^{(j)}}{\ln p_x'^{(\tau)}} = \frac{q_x^{(j)}}{q_x^{(\tau)}} \quad \text{(Same relationship as MUDD!)}$$

<u>Example</u>: You are given, for a double-decrement scenario, that $q'^{(1)}_{50} = .1$ and $q'^{(2)}_{50} = .2$.

a. Compute the dependent probabilities $q^{(1)}_{50}$, $q^{(2)}_{50}$ under the constant force assumption. (Use theorem and previous example.)

$p'^{(1)}_{50} = .9 \quad p'^{(2)}_{50} = .8 \quad \longrightarrow \quad q^{(1)}_{50} = \frac{\ln .9}{\ln .72}(.28)$

$\rightarrow p^{(\tau)}_{50} = .72 \rightarrow q^{(\tau)}_{50} = .28 \qquad q^{(2)}_{50} = \frac{\ln .8}{\ln .72}(.28)$

b. Outline how to find ${}_{.4}q^{(1)}_{50.6}$.

${}_{.4}q^{(1)}_{50.6} \overset{\mu\text{ const}}{\equiv} {}_{.4}q^{(1)}_{50} = \frac{q^{(1)}_{50}}{q^{(\tau)}_{50}}\, {}_{.4}q^{(\tau)}_{50} = \frac{q^{(1)}_{50}}{q^{(\tau)}_{50}}\left(1 - {}_{.4}p^{(\tau)}_{50}\right) = \frac{q^{(1)}_{50}}{q^{(\tau)}_{50}}\left(1 - \left(p^{(\tau)}_{50}\right)^{0.4}\right)$

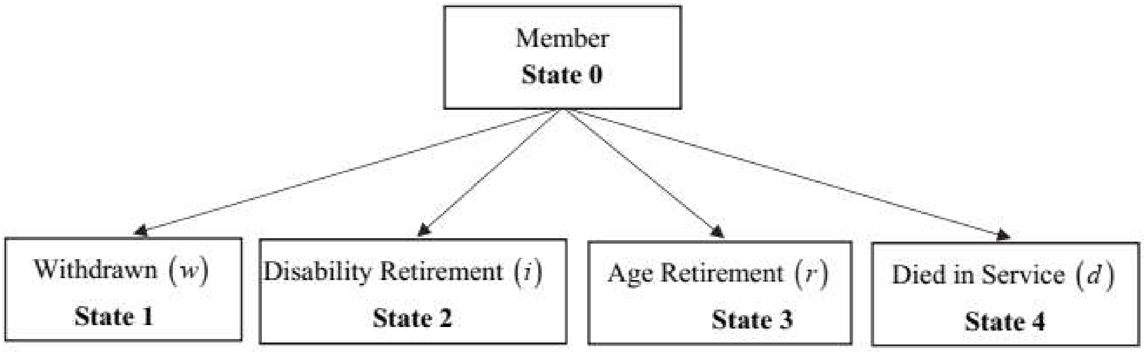
<u>Summary</u> – Under the assumption of constant forces of decrement between integer ages:

- ${}_tq^{(j)}_x = \frac{q^{(j)}_x}{q^{(\tau)}_x} \cdot {}_tq^{(\tau)}_x$ and ${}_tp^{(\tau)}_x = \left(p^{(\tau)}_x\right)^t$, $t \in [0,1], x \in \mathbb{Z}$
- $\frac{\ln\left(p'^{(j)}_x\right)}{\ln\left(p^{(\tau)}_x\right)} = \frac{q^{(j)}_x}{q^{(\tau)}_x}$

Homework problem – From Spring 2019 LTAM

Suggestion: Use the multiple decrement $q^{(j)}$ notation. The w_x, i_x, r_x, d_x are four decrements, so think $d^{(1)}{}_x$, etc. How does the constant force assumption play out in a multiple decrement table?

5. A pension plan uses the following multiple decrement model:



You are given the following information:

(i) All transitions are modeled assuming constant forces of transition between integer ages.

(ii) The following excerpt from the multiple decrement table:

x	l_x	w_x	i_x	r_x	d_x
61	58,622	3,201	812	28,460	413
62	25,736	--	--	--	--

Calculate ${}_{0.6}p_{61}^{01}$.

Summary of Main Facts – Interage Assumptions

- $p_x^{(\tau)} = p_x'^{(1)} \cdot \cdots \cdot p_x'^{(n)}$ and $q_x^{(\tau)} = q_x^{(1)} + \cdots + q_x^{(n)}$

UDD in multiple decrement table:

- $\dfrac{\ln\left(p_x'^{(j)}\right)}{\ln\left(p_x^{(\tau)}\right)} = \dfrac{q_x^{(j)}}{q_x^{(\tau)}}$
- We do get the usual UDD facts in the multiple decrement environment:
  For $x \in \mathbb{Z}$ and $s \in [0, 1]$: ${}_sq_x^{(j)} = s \cdot q_x^{(j)}$; ${}_sq_x^{(\tau)} = s \cdot q_x^{(\tau)}$; linear interpolation between ℓ_x and ℓ_{x+1} .

UDD in single decrement table:

- $q_x^{(j)} = q'^{(j)}_x \int_0^1 \left[\prod_{i \neq j} \left(1 - t \cdot q'^{(i)}_x\right) \right] dt$
- For a double decrement model, this simplifies to $q_x^{(1)} = q'^{(1)}_x \times (1 - \frac{1}{2} q_x'^{(2)})$

  (Keep a fraction of $q'^{(1)}_x$ by removing ½ of the competing single-decrement probability, which is what you might expect to lose due to introducing competition that behaves under UDD in its SDT)

- We do not get UDD facts in the multiple decrement environment.

Competing discrete and continuous decrements and SUDD:

- *If decrement #1 has SUDD and decrement #2 occurs only at one discrete point in the year, then decrement #1 also has UDD in the multiple decrement table until the time at which decrements by cause #2 occur.*

Constant forces between integer ages:

- ${}_tq_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot {}_tq_x^{(\tau)}$ and ${}_tp_x^{(\tau)} = \left(p_x^{(\tau)}\right)^t$, $t \in [0,1], x \in \mathbb{Z}$.
- (These allow you to compute ${}_tq_x^{(j)}$ straight from multiple decrement table without knowing the forces of decrement.)
- $\dfrac{\ln\left(p_x'^{(j)}\right)}{\ln\left(p_x^{(\tau)}\right)} = \dfrac{q_x^{(j)}}{q_x^{(\tau)}}$
- Also true that $\frac{q_x^{(j)}}{q_x^{(\tau)}} = \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}}$ (proof: write integrals for the q's and factor out constant $\mu^{(\ldots)}$'s)

Start 21 Feb 24

Quiz Monday!

idea: married couple, business partners, etc

DHW 3e Ch.10 – Multiple Life Functions

Let T_x and T_y denote the future lifetime random variables for (x) and (y).

$x{:}y$ = joint status of x and y

$x{:}y$ ends on first failure of x or y

Definition: $T_{xy} = T_{x:y}$ and $T_{\overline{xy}} = T_{\overline{x:y}}$.

$T_{x:y}$ = time until first failure

$T_{\overline{x:y}}$ = last survivor status (time until last failure)

$T_{w:x:y:z} = T_{wxyz}$ (time until first failure of 4 statuses)

$T_{\overline{w:x:y:z}} = T_{\overline{wxyz}}$ (time until last failure of 4)

Two cases can happen:

- If $T_x < T_y$, then $T_{xy} = T_x$, $T_{\overline{xy}} = T_y$
- If $T_y < T_x$, then $T_{xy} = T_y$, $T_{\overline{xy}} = T_x$

Useful conclusion:

$$T_x + T_y = T_{xy} + T_{\overline{xy}} \implies \underbrace{\mathbb{E}[T_x]}_{\mathring{e}_x} + \underbrace{\mathbb{E}[T_y]}_{\mathring{e}_y} = \underbrace{\mathbb{E}[T_{xy}]}_{\mathring{e}_{xy}} + \underbrace{\mathbb{E}[T_{\overline{xy}}]}_{\mathring{e}_{\overline{xy}}}$$

In fact, for any function, $f(T_x) + f(T_y) = f(T_{xy}) + f(T_{\overline{xy}})$

Notation:

$${}_tq^{\,1}_{\,x\,y} = {}_tq^{\,1}_{\,x\,:\,y} = \Pr\left[\begin{smallmatrix} T_x < T_y \\ T_x \le t \end{smallmatrix}\right] = \Pr[\,(x) \text{ fails first before time } t\,]$$

$${}_tq^{\;\;2}_{\,x\,y} = {}_tq^{\;\;2}_{\,x\,:\,y} = = \Pr[T_x < T_y \le t] = \Pr[\,(y) \text{ fails second, before time } t\,]$$

$${}_tp_{\,xy} = {}_tp_{\,x:y} = \Pr\left[\begin{smallmatrix} T_x > t \\ T_y > t \end{smallmatrix}\right] = \Pr[\text{both survive } t]$$

~~${}_tp_{\overline{xy}} = {}_tp_{\overline{x:y}}$~~

$${}_tq_{\overline{xy}} = {}_tq_{\overline{x:y}} = \Pr[T_{\overline{xy}} < t] = \Pr[T_x < t \text{ and } T_y < t]$$

Independent lives case: We say that (x) and (y) are independent lives if T_x and T_y are independent random variables. In this case: $x \perp\!\!\!\perp y \iff x$ indep. y

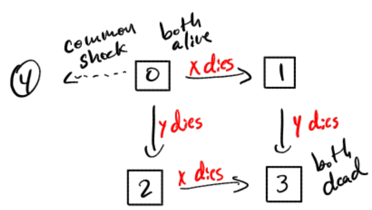
(For independent lives:) ${}_tp_{\,xy} \overset{x \perp\!\!\!\perp y}{=} {}_tp_x \cdot {}_tp_y$

(For independent lives:) ${}_tq_{\,\overline{xy}} \overset{x \perp\!\!\!\perp y}{=} {}_tq_x \cdot {}_tq_y$

Not all joint statuses involve independent lives:

- Common shock (Lilly and James Potter)
- Common lifestyle (Tonks and Lupin)
- Broken heart syndrome (Severus Snape?)

Change in survivability due to loss of a loved one, etc



Insurance and annuity notation

EPV[benefit of 1], payable at end of year of the first death of (50) and (60), is denoted $A_{50:60}$.

EPV[benefit of 1], payable at end of year of the first death of (50) and (60) if the death occurs within n years, is denoted $A^{\;1}_{50:60:n}$. (Be careful: $A_{50:60:\overline{n}|}$ denotes EPV of an endowment-insurance.) $A^{\;1}_{50:60:\overline{n}|}$

EPV[annuity due of 1], payable at the beginning of any year in which both (50) and (60) are alive at year's beginning, is denoted $\ddot{a}_{50:60}$.

EPV[annuity due of 1], payable at the beginning of any year in which both (50) and (60) are alive at year's beginning, until the end of an n-year period, is denoted $\ddot{a}_{50:60:\overline{n}|}$.

Excerpt from SULT at $i = 5\%$

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\vert}$	$A_{x:\overline{10}\vert}$	$\ddot{a}_{x:\overline{20}\vert}$	$A_{x:\overline{20}\vert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60
70	91,082.4	0.010413	12.0083	0.42818	0.21467	7.6491	0.63576	11.1109	0.47091	0.73295	0.50994	0.17313	70

x	$\ddot{a}_{xx}$	A_{xx}	${}^2A_{xx}$	$\ddot{a}_{x:x:\overline{10}\vert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\vert}$	x
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
60	13.2497	0.36906	0.16555	7.8080	11.2220	0.46562	0.24895	7.5110	60
70	9.9774	0.52488	0.30743	7.2329	7.7208	0.63234	0.42760	6.4497	70

(Lives are assumed to be independent in the joint life portion of the SULT.)

Example 4: Use the SULT at $i = 5\%$ (above). In each case, compute from the life table and give an English description.

a. $\boxed{A_{50:60}}$ and $\ddot{a}_{50:60}$.

$$A_{50:60} = 0.32048 \qquad \ddot{a}_{50:60} = 14.2699$$

b. $\boxed{A_{\overline{50:60}}}$ and $\ddot{a}_{\overline{50:60}}$. Use the fact $A_{50:60} + A_{\overline{50:60}} = A_{50} + A_{60}$

Pay on (50) death
+ Pay on (60) death
− Pay on first death

$$A_{\overline{50:60}} = A_{50} + A_{60} - A_{50:60} = 0.18931 + 0.29028 - 0.32048$$

$$\ddot{a}_{\overline{50:60}} = \ddot{a}_{50} + \ddot{a}_{60} - \ddot{a}_{50:60} = 17.0245 + 14.9041 - 14.2699$$

Pay \$1/yr until (50) death
+ Pay \$1/yr until (60) death
− Pay \$1/yr until first death

c. $\ddot{a}_{50:60:\overline{10}|}$. $= 7.9044$

Pay \$1/yr until either dies OR 10 yrs pass

<u>Example 4</u>:

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\rvert}$	$A_{x:\overline{10}\rvert}$	$\ddot{a}_{x:\overline{20}\rvert}$	$A_{x:\overline{20}\rvert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60
70	91,082.4	0.010413	12.0083	0.42818	0.21467	7.6491	0.63576	11.1109	0.47091	0.73295	0.50994	0.17313	70

x	$\ddot{a}_{xx}$	A_{xx}	${}^2A_{xx}$	$\ddot{a}_{x:x:\overline{10}\rvert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\rvert}$	x
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
60	13.2497	0.36906	0.16555	7.8080	11.2220	0.46562	0.24895	7.5110	60
70	9.9774	0.52488	0.30743	7.2329	7.7208	0.63234	0.42760	6.4497	70

(Lives are assumed to be independent in the joint life portion of the SULT.)

d. Let ${}_nE_{x:y}$ denote (i) the EPV of an endowment benefit of \$1, payable at time t if the joint status $x{:}y$ is surviving at that time; (ii) equivalently, ${}_nE_{x:y} = {}_np_{x:y} \cdot v^n$. Show how to compute ${}_nE_{x:y}$ from ${}_nE_x$ and ${}_nE_y$ in the independent lives case.

Note to self - Eleanor observation

$${}_nE_{xy} = {}_np_{xy}\, v^n = {}_np_x\, {}_np_y\, v^n = {}_np_x\, v^n\, {}_np_y = {}_nE_x\, {}_np_y \overset{??}{=} \frac{{}_nE_x\, {}_nE_y}{v^n} = {}_nE_x\, {}_nE_y (1+i)^n$$

oh boy, Dr. Wilson never thought of this and loves it!

e. Compute $A_{\overset{1}{50:50}:\overline{20}\rvert}$ and $\ddot{a}_{50:50:\overline{20}\rvert}$ assuming independent lives.

$$A_{\overset{1}{50:50}:\overline{20}\rvert} = A_{50:50} - {}_{20}E_{50:50}\, A_{70:70}$$
$$= A_{50:50} - {}_{20}E_{50}\, {}_{20}p_{50}\, A_{70:70} = A_{50:50} - {}_{20}E_{50} \cdot {}_{20}E_{50} \cdot v^{-20} \cdot A_{70:70}$$
$$= .24669 - (.34824)^2 (1.05)^{20} (.52488)$$

$$\ddot{a}_{50:50:\overline{20}\rvert} = \ddot{a}_{50:50} - {}_{20}E_{50:50}\, \ddot{a}_{70:70}$$
$$= \ddot{a}_{50:50} - {}_{20}E_{50} \cdot {}_{20}E_{50} \cdot v^{-20}\, \ddot{a}_{70:70}$$

We get the same geometric series relationships between $A_{50:50:\overline{n}\rvert}$ and $\ddot{a}_{50:50:\overline{n}\rvert}$ (as well as the continuous version). Note that the status subscripts must match.

$$\ddot{a}_{50:50:\overline{n}\rvert} = \frac{1 - A_{50:50:\overline{n}\rvert}}{1-v}$$

Endowment, not term insurance

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\vert}$	$A_{x:\overline{10}\vert}$	$\ddot{a}_{x:\overline{20}\vert}$	$A_{x:\overline{20}\vert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60
70	91,082.4	0.010413	12.0083	0.42818	0.21467	7.6491	0.63576	11.1109	0.47091	0.73295	0.50994	0.17313	70

x	$\ddot{a}_{xx}$	A_{xx}	${}^2A_{xx}$	$\ddot{a}_{xx:\overline{10}\vert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\vert}$	x
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
60	13.2497	0.36906	0.16555	7.8080	11.2220	0.46562	0.24895	7.5110	60
70	9.9774	0.52488	0.30743	7.2329	7.7208	0.63234	0.42760	6.4497	70

(Lives are assumed to be independent in the joint life portion of the SULT.)

<u>Example 5</u>: Use the SULT (above):

a. Consider the following 10-year deferred annuity on independent lives Harry (50) and Sally (60).

- Level annual premiums of π are payable for at most 10 years, and only while both Harry and Sally are alive.
- There are no annuity income payments to Harry and Sally during the first 10 years.
- After 10 years, the annuity pays 50,000 per year as long as at least one of Harry or Sally is living.

naïve method

Compute π.

$$\pi\,\ddot{a}_{50:60:\overline{10}|} = \left({}_{10}E_{\overline{50:60}}\right) 50000\,\ddot{a}_{\overline{60:70}}$$

Consider the individual life, take away duplicate payments

$$\pi\,\ddot{a}_{50:60:\overline{10}|} = {}_{10}E_{50}(50000)\,\ddot{a}_{60} + {}_{10}E_{60}(50000)\,\ddot{a}_{70} - {}_{10}E_{50:60}(50000)\,\ddot{a}_{60:70}$$

$$= 50000\left({}_{10}E_{50}\,\ddot{a}_{60} + {}_{10}E_{60}\,\ddot{a}_{70} - {}_{10}E_{50}\,{}_{10}E_{60}\,v^{-n}\,\ddot{a}_{60:70}\right)$$

Solve for π

b. Consider a multistate model in which (0) = both alive; (1) = Harry alive, Sally dead; (2) = Sally alive, Harry dead; (3) = both dead. $\Pr[0 \to 3 \mid X \perp\!\!\!\perp Y] = 0$

Show how to compute ${}_5V^{(0)}$ and ${}_5V^{(1)}$. Give an expression for the expected present value at time 0 (money valued at time 0) of the time-5 reserve.

$${}_5V^{(0)} = 50000\left({}_5E_{55}\,\ddot{a}_{60} + {}_5E_{65}\,\ddot{a}_{70} - {}_5E_{55:65}\,\ddot{a}_{60:70}\right) - \pi\,\underbrace{\ddot{a}_{55:65:\overline{5}|}}_{\ddot{a}_{55:65} - {}_5E_{55:65}\,\ddot{a}_{60:70}}$$

<u>Layering techniques:</u>

c. Rework (a) but assume that the annuity income is paid at a rate of 80,000 per year while both parties are alive but only at 50,000 per year if only one of Harry or Sally is alive on the payment date.

if both alive, you've paid \$100k, but only need \$80k

$$\pi\,\ddot{a}_{50:60:\overline{10}|} = 50000\left({}_{10}E_{50}\,\ddot{a}_{60} + {}_{10}E_{60}\,\ddot{a}_{70}\right) - 20000\,{}_{10}E_{60:70}\,\ddot{a}_{60:70}$$

or $\quad (a) + {}_{10}E_{50:60}\,\ddot{a}_{60:70}\cdot 30000$

Reversionary annuities: $\ddot{a}_{x|y}$ = EPV[1 unit/yr payable to (y) after (x) has died, payable while (y) is alive].

(Strategy: Pretend you pay (y) throughout lifetime, and remove any payments made while both alive.)

$\ddot{a}_{x|y} = \underset{\text{y alive}}{\ddot{a}_y} - \underset{\text{both alive}}{\ddot{a}_{xy}}$

(Example 5, ct'd)

x	l_x	q_x	$\ddot{a}_x$	A_x	^2A_x	$\ddot{a}_{x:\overline{10}\vert}$	$A_{x:\overline{10}\vert}$	$\ddot{a}_{x:\overline{20}\vert}$	$A_{x:\overline{20}\vert}$	$_5E_x$	$_{10}E_x$	$_{20}E_x$	x
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60
70	91,082.4	0.010413	12.0083	0.42818	0.21467	7.6491	0.63576	11.1109	0.47091	0.73295	0.50994	0.17313	70

x	$\ddot{a}_{xx}$	A_{xx}	$^2A_{xx}$	$\ddot{a}_{x:x:\overline{10}\vert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	$^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\vert}$	x
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
60	13.2497	0.36906	0.16555	7.8080	11.2220	0.46562	0.24895	7.5110	60
70	9.9774	0.52488	0.30743	7.2329	7.7208	0.63234	0.42760	6.4497	70

(Lives are assumed to be independent in the joint life portion of the SULT.)

d. Harry is (50); Sally is (60); their lives are independent.
Find the expected present value of a reversionary annuity that pays 20,000 per year to Harry at the start of any year in which Harry is alive but Sally is dead.

$$20000\,\underset{S|H}{\ddot{a}_{60|50}} = 20000\left(\underset{H}{\ddot{a}_{50}} - \underset{H:S}{\ddot{a}_{50:60}}\right)$$

Stop 21 Feb 24

Homework problems for DHW3e Ch. 10 – Joint life and last survivor benefits

Suggested practice from ALTAM Sample Problems (Canvas): #18, 20ab

Fall 2018 LTAM #14

14. Joe, age 65, and his wife Lucy, age 55, purchase a special 10-year deferred annuity policy with the following premium and benefit terms:

- Level annual premiums are payable for at most 10 years, while both Joe and Lucy are alive.
- There are no annuity payments during the first 10 years.
- After 10 years, at the start of each year the annuity pays:
  - 100,000 if both Joe and Lucy are alive at the payment date.
  - 55,000 if only one of them is alive at the payment date.

You are given the following assumptions:

(i) Joe and Lucy have independent future lifetimes.

(ii) Mortality follows the Standard Ultimate Life Table.

(iii) $i = 0.05$

Calculate the annual net premium.

Spring 2019 LTAM #12

12. For two lives, both age 50, you are given:

(i) Mortality follows the Standard Ultimate Life Table.

(ii) The future lifetimes are independent.

(iii) $i = 0.05$

Calculate $\ddot{a}_{\overline{50:50}:\overline{20|}}$.

Spring 2018 Exam MLC #3

3. A couple, both age 65, have the option to receive one of the following:

- A life annuity of F per year, payable at the beginning of each year while at least one is alive.
- A lump sum of 100,000 if both lives survive 5 years.

You are given:

(i) $i = .06$;

(ii) ${}_5E_{65} = .65623$

(iii) $\ddot{a}_{65} = 9.8969$; $\ddot{a}_{65:65} = 7.8552$

(iv) Their future lifetimes are independent.

(v) The actuarial present values of the payments under the two options are equal.

Compute F. *(answer: 4827.)*

SOA Sample Problems (MLC/LTAM) ***You may have solved this problem earlier (Ch. 8)**

194. For multi-state model of an insurance on (x) and (y):

(i) The death benefit of 10,000 is payable at the moment of the second death.

(ii) You use the states:
State 0 = both alive
State 1 = only (x) is alive
State 2 = only (y) is alive
State 3 = neither alive

(iii) $\mu^{01}_{x+t:y+t} = \mu^{02}_{x+t:y+t} = 0.06, t \geq 0$

(iv) $\mu^{03}_{x+t:y+t} = 0, t \geq 0$

(v) $\mu^{13}_{x+t:y+t} = \mu^{23}_{x+t:y+t} = 0.10, t \geq 0$

(vi) $\delta = 0.04$

Calculate the expected present value of this insurance on (x) and (y).

answer: 5357

Spring 2019 Exam LTAM Written #4

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\rvert}$	$A_{x:\overline{10}\rvert}$	$\ddot{a}_{x:\overline{20}\rvert}$	$A_{x:\overline{20}\rvert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
40	99,338.3	0.000527	18.4578	0.12106	0.02347	8.0863	0.61494	12.9935	0.38126	0.78113	0.60920	0.36663	40
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60

x	$\ddot{a}_{xx}$	A_{xx}	${}^2A_{xx}$	$\ddot{a}_{x:x:\overline{10}\rvert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\rvert}$	x
40	17.6283	0.16055	0.03909	8.0649	16.5558	0.21163	0.06275	8.0337	40
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
60	13.2497	0.36906	0.16555	7.8080	11.2220	0.46562	0.24895	7.5110	60

4. (*11 points*) Pat and Robin, each age 40, buy a fully discrete, last survivor insurance with a sum insured of 100,000.

You are given:

(i) Premiums are payable while at least one life is alive, for a maximum of 20 years.

(ii) Mortality of each follows the Standard Ultimate Life Table (SULT).

(iii) $i = 0.05$

(iv) With independent future lifetimes, $\ddot{a}_{40:40:\overline{20}\rvert} = 12.9028$.

(a) (*2 points*) Show that the annual net premium assuming that the future lifetimes are independent is 620 to the nearest 10. You should calculate the value to the nearest 1.

(b) (*1 point*) State two reasons why couples may have dependent future lifetimes.

#4 Continued, next page

The insurer decides that premiums and reserves for this policy will be determined using a mortality model incorporating dependency.

You are given the following information about this model:

(i) The future lifetimes for the first 20 years are not independent.

(ii) If both lives survive 20 years, it is assumed that the future lifetimes from that time will be independent, and will follow the Standard Ultimate Life Table.

(iii) The mortality of each of Pat and Robin, individually, follows the Standard Ultimate Life Table, whether the other is alive or dead.

(iv) $\ddot{a}_{40:40:\overline{10}|} = 8.0703$, $\ddot{a}_{40:40:\overline{20}|} = 12.9254$, ${}_{20}E_{40:40} = 0.35912$, ${}_{10}E_{50:50} = 0.59290$

(v) $A_{\overline{50:50}} = 0.13441$

(vi) ${}_{10}p_{40:40} = 0.9866$, ${}_{10}p_{\overline{40:40}} = 0.9980$

Use the dependent mortality model for the rest of this question.

(c) (*3 points*)

(i) Show that $A_{40:40} = 0.158$ to the nearest 0.001. You should calculate the value to the nearest 0.0001.

(ii) Show that ${}_{10}E_{40:40} = 0.606$ to the nearest 0.01. You should calculate the value to the nearest 0.0001.

(iii) Show that $\ddot{a}_{50:50:\overline{10}|} = 8.02$ to the nearest 0.01. You should calculate the value to the nearest 0.0001.

<u>Hint</u>: For question (c)-(i), use given info (iv) for the 20-year period and SULT for the subsequent to assemble the whole life annuity factor $\ddot{a}_{40:40}$. Then use the relationship between $\ddot{a}_{40:40}$ and $A_{40:40}$.

(d) (*1 point*) Show that the annual net premium is 645 to the nearest 5. You should calculate the value to the nearest 0.1.

(e) (*3 points*) Let ${}_kL$ denote the net future loss random variable at time k for the insurance.

(i) Calculate $E\left[{}_{10}L\right]$ given that only Pat is alive at time 10.

(ii) Calculate $E\left[{}_{10}L\right]$ given that both Pat and Robin are alive at time 10.

(iii) Calculate $E\left[{}_{10}L\right]$ given that at least one of Pat and Robin is alive at time 10.

<u>**Hint**</u> for (iii):
$P(at\ least\ one\ alive) = {}_{10}p_{\overline{40:40}} = P(exactly\ one\ alive) + P(both\ alive)$
Use this to get

$$E[{}_{10}L \mid at\ least\ one\ alive] = \\ = \frac{E[{}_{10}L \mid exactly\ one\ alive] \cdot P(exactly\ one\ alive) + E[{}_{10}L \mid both\ alive] \cdot P(both\ alive)}{P(exactly\ one\ alive) + P(both\ alive)}$$

Answers in (e): (i) 13738.59; (ii) 8223.25; (iii) 8286.25

DHW3e Ch. 10, Continued Practice with joint/last survivor benefits

*These problems use the SULT, which assumes i = .05. (**Table and solutions** appears after practice problems.)*

1. a. A fully discrete whole life insurance pays 1000 upon the death of the last survivor of independent lives aged (x) and (y), who are both age 30 when the insurance is written. Level premiums are paid while both (x) and (y), survive. Compute the premium.

$$\pi \ddot{a}_{30:30} = 1000 A_{\overline{30:30}} = 1000\left(A_{30} + A_{30} - A_{30:30}\right) = 1000\left(0.07698 + 0.07698 - 0.10369\right)$$

$$18.8224\pi \longrightarrow = 50.27$$

$$\Rightarrow \pi = 2.6708$$

b. At time 6, both (x) and (y) are still living. Compute ${}_6V$, the time-6 reserve.

$${}_6V = 1000 A_{\overline{36:36}} - \pi \ddot{a}_{36:36}$$

$$= 1000\left(0.10101 + 0.10101 - 0.13480\right) - 2.6708\left(18.1693\right) = 18.69426943$$

2. a. Find the EPV of a fully discrete whole life insurance that pays 1000 upon the first death of independent lives (36) and (46).

$$1000 A_{36:46}$$

b. Find the EPV of a fully discrete whole life insurance that pays 1000 upon the first death of independent lives (46) and (56).

$$1000 A_{46:56}$$

c. Compute ${}_{10}E_{36:46}$ to five decimal places. Also: Show how you could compute ${}_3E_{36:46}$ using ℓ_x's .

$${}_{10}E_{36:46} = {}_{10}E_{36}\ {}_{10}E_{46}\ \underset{(1.05)^{10}}{v^{-10}} = 0.6024$$

$${}_3E_{36:46} = \frac{\ell_{39}}{\ell_{36}} \cdot \frac{\ell_{49}}{\ell_{46}} \cdot v^{10}$$

d. Find the EPV of a fully discrete 10-year term life insurance that pays 1000 upon the first death of independent lives (36) and (46) during the term of the policy.

$$1000 A_{36:46} - {}_{10}E_{36:46} A_{46:56}$$

3. Find the EPV of a fully discrete reversionary annuity that pays 1000 per year to (40) after (30) has died (and pays nothing to (30) after (40) has died). (Assume independent lifetimes.)

   Hint: Go ahead and pay (40) 1000 per year while (40) is alive. Then remove 1000 per year whenever…

4. Find the EPV of the following special fully discrete reversionary annuity that pays
   - 1000 per year to (40) after (30) has died, and
   - 2000 per year to (30) after (40) has died.

   (Assume independent lifetimes.)

Excerpt from SULT @ i = 5%

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\vert}$	$A_{x:\overline{10}\vert}$	$\ddot{a}_{x:\overline{20}\vert}$	$A_{x:\overline{20}\vert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
26	99,843.8	0.000280	19.6499	0.06429	0.00841	8.0978	0.61439	13.0491	0.37862	0.78236	0.61191	0.37354	26
27	99,815.9	0.000287	19.5878	0.06725	0.00900	8.0974	0.61441	13.0474	0.37869	0.78233	0.61183	0.37334	27
28	99,787.2	0.000296	19.5228	0.07034	0.00964	8.0970	0.61443	13.0455	0.37878	0.78229	0.61174	0.37310	28
29	99,757.7	0.000305	19.4547	0.07359	0.01033	8.0966	0.61445	13.0434	0.37888	0.78224	0.61163	0.37284	29
30	99,727.3	0.000315	19.3834	0.07698	0.01109	8.0961	0.61447	13.0410	0.37900	0.78219	0.61152	0.37254	30
31	99,695.8	0.000327	19.3086	0.08054	0.01192	8.0956	0.61450	13.0384	0.37913	0.78213	0.61139	0.37221	31
32	99,663.2	0.000341	19.2303	0.08427	0.01281	8.0949	0.61453	13.0354	0.37927	0.78206	0.61124	0.37183	32
33	99,629.3	0.000356	19.1484	0.08817	0.01379	8.0943	0.61456	13.0320	0.37943	0.78199	0.61108	0.37141	33
34	99,593.8	0.000372	19.0626	0.09226	0.01486	8.0935	0.61460	13.0282	0.37961	0.78190	0.61090	0.37094	34
35	99,556.7	0.000391	18.9728	0.09653	0.01601	8.0926	0.61464	13.0240	0.37981	0.78181	0.61069	0.37041	35
36	99,517.8	0.000412	18.8788	0.10101	0.01727	8.0916	0.61468	13.0192	0.38004	0.78170	0.61046	0.36982	36
37	99,476.7	0.000436	18.7805	0.10569	0.01863	8.0905	0.61474	13.0138	0.38029	0.78158	0.61020	0.36915	37
38	99,433.3	0.000463	18.6777	0.11059	0.02012	8.0893	0.61480	13.0078	0.38058	0.78145	0.60990	0.36841	38
39	99,387.3	0.000493	18.5701	0.11571	0.02173	8.0879	0.61486	13.0011	0.38090	0.78130	0.60957	0.36757	39
40	99,338.3	0.000527	18.4578	0.12106	0.02347	8.0863	0.61494	12.9935	0.38126	0.78113	0.60920	0.36663	40
41	99,285.9	0.000565	18.3403	0.12665	0.02536	8.0846	0.61502	12.9850	0.38167	0.78094	0.60879	0.36558	41
42	99,229.8	0.000608	18.2176	0.13249	0.02741	8.0826	0.61511	12.9754	0.38212	0.78072	0.60832	0.36440	42
43	99,169.4	0.000656	18.0895	0.13859	0.02963	8.0804	0.61522	12.9647	0.38263	0.78048	0.60780	0.36307	43
44	99,104.3	0.000710	17.9558	0.14496	0.03203	8.0779	0.61534	12.9526	0.38321	0.78021	0.60721	0.36159	44
45	99,033.9	0.000771	17.8162	0.15161	0.03463	8.0751	0.61547	12.9391	0.38385	0.77991	0.60655	0.35994	45
46	98,957.6	0.000839	17.6706	0.15854	0.03744	8.0720	0.61562	12.9240	0.38457	0.77956	0.60581	0.35809	46

x	$\ddot{a}_{xx}$	A_{xx}	${}^2A_{xx}$	$\ddot{a}_{x:x:\overline{10}\vert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\vert}$	x
30	18.8224	0.10369	0.01917	8.0844	18.1212	0.13709	0.03001	8.0747	30
31	18.7253	0.10832	0.02052	8.0833	17.9924	0.14322	0.03227	8.0724	31
32	18.6238	0.11315	0.02198	8.0821	17.8579	0.14962	0.03472	8.0698	32
33	18.5176	0.11821	0.02357	8.0807	17.7176	0.15630	0.03736	8.0669	33
34	18.4066	0.12350	0.02529	8.0792	17.5713	0.16327	0.04022	8.0636	34
35	18.2905	0.12902	0.02716	8.0774	17.4187	0.17054	0.04331	8.0600	35
36	18.1693	0.13480	0.02919	8.0755	17.2597	0.17811	0.04664	8.0559	36
37	18.0426	0.14083	0.03138	8.0733	17.0941	0.18600	0.05023	8.0513	37
38	17.9104	0.14713	0.03375	8.0708	16.9217	0.19421	0.05410	8.0461	38
39	17.7723	0.15370	0.03632	8.0680	16.7423	0.20275	0.05827	8.0403	39
40	17.6283	0.16055	0.03909	8.0649	16.5558	0.21163	0.06275	8.0337	40
41	17.4782	0.16771	0.04209	8.0614	16.3619	0.22086	0.06756	8.0264	41
42	17.3217	0.17516	0.04533	8.0575	16.1607	0.23044	0.07273	8.0182	42
43	17.1586	0.18292	0.04882	8.0531	15.9518	0.24039	0.07827	8.0090	43
44	16.9888	0.19101	0.05258	8.0481	15.7353	0.25070	0.08420	7.9986	44
45	16.8122	0.19942	0.05663	8.0426	15.5109	0.26139	0.09056	7.9870	45
46	16.6284	0.20817	0.06098	8.0363	15.2787	0.27244	0.09734	7.9740	46
47	16.4374	0.21727	0.06567	8.0293	15.0385	0.28388	0.10459	7.9594	47
48	16.2390	0.22671	0.07070	8.0215	14.7903	0.29570	0.11232	7.9431	48
49	16.0331	0.23652	0.07609	8.0126	14.5341	0.30790	0.12054	7.9248	49
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
51	15.5982	0.25723	0.08806	7.9916	13.9979	0.33344	0.13858	7.8815	51
52	15.3690	0.26814	0.09468	7.9792	13.7180	0.34676	0.14842	7.8559	52
53	15.1318	0.27944	0.10175	7.9653	13.4304	0.36046	0.15885	7.8272	53
54	14.8867	0.29111	0.10929	7.9496	13.1352	0.37451	0.16986	7.7953	54
55	14.6336	0.30316	0.11732	7.9321	12.8328	0.38891	0.18148	7.7596	55
56	14.3725	0.31559	0.12586	7.9125	12.5233	0.40365	0.19372	7.7199	56

(*This table applies to joint statuses for independent lives.*)

SOLUTIONS to DHW3e Ch. 10, Continued Practice with joint/last survivor benefits

These problems use the SULT, which assumes $i = .05$. (Table and solutions appears after practice problems.)

1. a. A fully discrete whole life insurance pays 1000 upon the death of the last survivor of independent lives aged (x) and (y), who are both age 30 when the insurance is written. Level premiums are paid while both (x) and (y), survive. Compute the premium.

$$\pi\, \ddot{a}_{30:30} = 1000\, A_{\overline{30:30}}$$

$$\pi\, \ddot{a}_{30:30} = 1000\,(A_{30} + A_{30} - A_{30:30})$$

$$\pi = 2.67$$

b. At time 6, both (x) and (y) are still living. Compute ${}_6V$, the time-6 reserve.

$${}_6V = 1000\, A_{\overline{36:36}} - \pi \cdot \ddot{a}_{36:36}$$

$$= 1000\,(A_{36} + A_{36} - A_{36:36}) - \pi \cdot \ddot{a}_{36:36}$$

$$= 18.70$$

2. a. Find the EPV of a fully discrete whole life insurance that pays 1000 upon the first death of independent lives (36) and (46).

$$1000\, A_{36:46} = 178.11$$

b. Find the EPV of a fully discrete whole life insurance that pays 1000 upon the first death of independent lives (46) and (56).

$$1000\, A_{46:56} = 272.44$$

c. Compute ${}_{10}E_{36:46}$ to five decimal places for independent lives (36, 46).
Also: Show how you could compute ${}_3E_{36:46}$ using ℓ_x's .

$${}_{10}E_{36:46} = \Big(\overset{{}_{10}E_{36}}{\cancel{{}_{10}E_{36:46}}}\Big)\Big(\overset{{}_{10}E_{46}}{\cancel{{}_{20}E_{26:46}}}\Big) \div v^{10} = .60240; \qquad {}_3E_{36:46} = \frac{\ell_{39}}{\ell_{36}} \cdot \frac{\ell_{49}}{\ell_{46}} \cdot v^{3}$$

d. Find the EPV of a fully discrete 10-year term life insurance that pays 1000 upon the first death of independent lives (36) and (46) during the term of the policy.

$$1000\, A_{36:46} - {}_{10}E_{36:46} \cdot 1000\, A_{46:56} = 13.99$$

3. Find the EPV of a fully discrete reversionary annuity that pays 1000 per year to (40) after (30) has died (and pays nothing to (30) after (40) has died). (Assume independent lifetimes.)

Hint: Go ahead and pay (40) 1000 per year while (40) is alive. Then remove 1000 per year whenever…

" $1000\,\ddot{a}_{30|40}$ "

$$1000\,\ddot{a}_{40} - 1000\,\ddot{a}_{30:40}$$
$$=$$
$$1000\,[18.4578 - 18.1212]$$
$$=$$
$$336.6$$

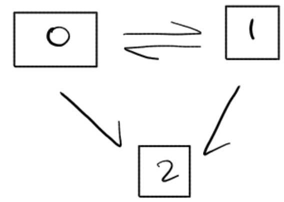
4. Find the EPV of the following special fully discrete reversionary annuity that pays
   - 1000 per year to (40) after (30) has died, and
   - 2000 per year to (30) after (40) has died.

   (Assume independent lifetimes.)

$$\left.\begin{array}{c} 1000\,[\ddot{a}_{40} - \ddot{a}_{30:40}] \\ + \\ 2000\,[\ddot{a}_{30} - \ddot{a}_{30:40}] \end{array}\right\} = \begin{array}{c} 336.6 \\ + \\ 2524.4 \end{array}$$

$$= \longrightarrow = 2861$$

MA 398 Midterm Exam Review



Problem #1 will appear exactly as below on the midterm exam.

1. Kolmogorov forward equation: Consider a temporary disability model with states

   0 – alive, well 1 – temporarily disabled 2 – dead.

   Suppose that we have calculated the transition probabilities ${}_tp_{60}^{0j}$ up to some time t. Suppose that all values for μ_{60+t}^{ij} are known and suppose that h is a small time increment.

   a. Write out the Kolmogorov forward equations: $\underbrace{{}_{t+h}p_{60}^{00} = {}_tp_{60}^{00}}_{\text{Right place}}\underbrace{\left(1 - {}_hp_{60+t}^{01} - {}_hp_{60+t}^{02}\right)}_{\text{Dont leave}} + \underbrace{{}_tp_{60}^{01}}_{\text{Wrong place}}\underbrace{\left({}_hp_{60+t}^{10}\right)}_{\text{Correct transition}}$ ✱

   $${}_{t+h}p_{60}^{00} = \ldots + o(h)\,; \qquad {}_{t+h}p_{60}^{01} = \ldots + o(h).$$

   $$= {}_tp_{60}^{00}\left(1 - \mu_{60+t}^{01}h - \mu_{60+t}^{02}h\right) + {}_tp_{60}^{01}\mu_{60+t}^{10}h + o(h)$$

   Check that you have labeled all forces (μ's) both with **superscripts** indicating the type of transition **and with subscripts** indicating the age at which the forces are operating.

   b. Be able to give an English interpretation for the probability represented by the factors in each term in (a). ← Might be omitted

   c. What does it mean to use Euler's method in conjunction with the equations in (a)? ignore o(h)

   d. Give the boundary conditions ${}_0p_{60}^{00} = \underline{\ 1\ }$ and ${}_0p_{60}^{01} = \underline{\ 0\ }$ that allow a starting point for the recursion. $t=0$ ${}_0p_{60}^{02} = 0$

   e. Give a formula for $\frac{d}{dt}\,{}_tp_{60}^{00} = \lim_{h\to 0}\dfrac{{}_{t+h}p_{60}^{00} - {}_tp_{60}^{00}}{h}$ take ✱, subtract ${}_tp_{60}^{00}$, divide by h, take limit

   $$= -{}_tp_{60}^{00}\mu_{60+t}^{01} - {}_tp_{60}^{00}\mu_{60+t}^{02} + {}_tp_{60}^{01}\mu_{60+t}^{10}$$

   (You don't need to show work in (e) unless you want to. Recall that you can rearrange (a) so that taking $\lim_{h\to 0}$ gives the desired result.)

2. An insurance company issues a <u>2-year term</u> insurance to a high risk individual (x). You are given a three-state model with p_{x+k}^{ij} = M_{ij} (= the (row-i, column-j) entry of M) for $k = 0, 1, 2$ where

(i) $M = \begin{bmatrix} .8 & .1 & .1 \\ .2 & .3 & .5 \\ 0 & 0 & 1 \end{bmatrix}$. (States are 0 - healthy, 1 – critically ill, 2 - dead.)

(ii) The death benefit is 1000, payable at the end of the year of death.

(iii) The insured healthy at time 0.

(iv) The discount factor v is a constant.

a. Write an expression in terms of v for the expected present value of the death benefit.

$$1000\Big(\underbrace{0.1v}_{0\to 2} + \underbrace{(0.8)(0.1)v^2}_{0\to 0\to 2} + \underbrace{(0.1)(0.5)v^2}_{0\to 1\to 2}\Big)$$

b. Find ${}_3p_x^{00}$.

$$\underbrace{(0.8)^3}_{00} + \underbrace{(0.8)(0.1)(0.2)}_{010} + \underbrace{(0.1)(0.2)(0.8)}_{100} + \underbrace{(0.1)(0.3)(0.2)}_{110}$$

$$= 0.55$$

Solutions for #2-7 are on the pages following these problems.

3. Consider a multistate model with three states:

0 – healthy
1 – permanently disabled
2 – dead

(x) is in State 0 at time 0. The forces of transition μ^{01}, μ^{02}, and μ^{12} are constant and represent the only possible transitions (i.e. no transition 1→0 is possible).

a. Find the probability that (x) remains in state 0 for two years. Simplify fully.

$$e^{-2(\mu^{01}+\mu^{02})} \quad e^{-2\mu^{01}} e^{-2\mu^{02}} = \left(e^{-\mu^{01}} e^{-\mu^{02}}\right)^2$$

b. Find the probability that (x) dies within two years without becoming disabled first. Simplify fully.

$$\int_0^2 e^{-t(\mu^{01}+\mu^{02})} \mu^{02}\, dt = \frac{\mu^{02} e^{-2(\mu^{01}+\mu^{02})}}{-\mu^{01}-\mu^{02}}$$

c. Set up an integral expression (do <u>not</u> simplify this time) for the probability that (x) becomes disables by time 2 and is still alive at time 2.

That is, find an expression for ${}_2p_x^{01}$.

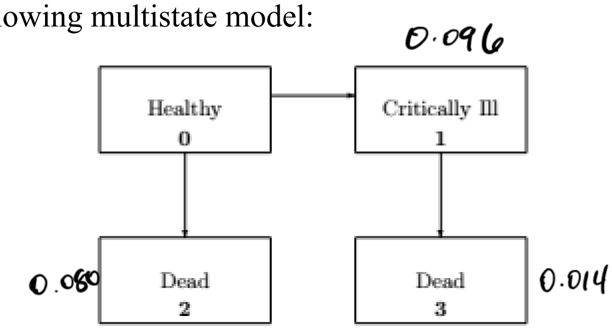
$$\int_0^2 e^{-t(\mu^{01}+\mu^{02})} \mu^{01} e^{-(2-t)\mu^{12}}\, ds\, dt$$

d. Set up an integral expression (do <u>not</u> simplify this time) for the expected present value of a 2-year term insurance on (x) with death benefit of \$1. (Sum of integrals.)

$$\int_0^2 e^{-t(\mu^{01}+\mu^{02})} \mu^{02} e^{-\delta t}\, dt \; +$$

$$\int_0^2 \int_0^{2-s} e^{-t(\mu^{01}+\mu^{02})} \mu^{01} e^{-s(\mu^{12})} \mu^{12} e^{-\delta(t+s)}\, ds\, dt$$

4. Consider the following multistate model:



a. You are given that

$$\bar{A}^{01}_{50:\overline{20}|} = .096$$

$$\bar{A}^{02}_{50:\overline{20}|} = .080$$

$$\bar{A}^{03}_{50:\overline{20}|} = .014$$

Compute the EPV of a fully continuous 20-year term partially accelerated death benefit, written on a healthy 50-year old, that pays…

- \$10,000 upon critical illness diagnosis,
- \$20,000 upon a death that follows a CI diagnosis,
- \$30,000 upon a death that is not preceded by CI diagnosis.

Simplify to the nearest cent.

$$10000(0.096) + 20000(0.014) + 30000(0.080) = 3640$$

b. You are given the following values:

$\bar{A}^{01}_{60} = .34$; $\quad \bar{A}^{01}_{70} = .50$;

$\bar{A}^{03}_{60} = .23$ $\quad \bar{A}^{03}_{70} = .36$ $\quad \bar{A}^{13}_{70} = .62$

${}_{10}p^{00}_{60} = .75$; $\quad {}_{10}p^{01}_{60} = .15$; $\quad i = .05$

Compute each value to the nearest .001:

(i) $\bar{A}^{01}_{60:\overline{10}|}$ $\quad \bar{A}^{01}_{60:\overline{10}|} = \bar{A}^{01}_{60} - {}_{10}p^{00}_{60}\, v^{10} \bar{A}^{01}_{70} = 0.110$

(ii) $\bar{A}^{03}_{60:\overline{10}|}$ $\quad \bar{A}^{03}_{60:\overline{10}|} = \bar{A}^{03}_{60} - {}_{10}p^{00}_{60}\, v^{10} \bar{A}^{03}_{70} - {}_{10}p^{01}_{60}\, v^{10} \bar{A}^{13}_{70} = 0.007$

Remember that there are two states from which a 70-year old can reach State 3.

5. Consider the following facts concerning a double-decrement model:

$\ell^{\tau}_{60} = 1{,}000$
$q'^{(1)}_{60} = .04$
$q'^{(2)}_{60} = .1$

Assume UDD in the multiple decrement table, so that

$$\frac{\ln\left(p'^{(j)}_{x}\right)}{\ln\left(p^{(\tau)}_{x}\right)} = \frac{q^{(j)}_{x}}{q^{(\tau)}_{x}} \quad \text{for } j = 1, 2;\ x \in \mathbb{Z}\,.$$

Compute ${}_{.2}q^{(2)}_{60.7}$ to four decimal places.

$$q^{\tau}_{60} = 1-\left(1-q'^{(1)}\right)\left(1-q'^{(2)}\right) = 0.136$$

$$q^{(2)}_{60} = \frac{\ln(0.9)}{\ln(0.864)}\,0.136 = 0.0989$$

$$1000\ {}_{.2}q^{(2)}_{x} = 19.6043$$

$$\ell_{60.7} = .7(864) + .3(1000) \approx 904.8$$

$${}_{.2}q^{(2)}_{60.7} = \frac{19.6043}{904.8} = 0.021667$$

6. Consider the following facts concerning a triple-decrement model:

$q'^{(1)}_{x} = .05$
$q'^{(2)}_{x} = .10$
$q'^{(3)}_{x} = .15$

Assume UDD in each single decrement table, so that

$$q^{(j)}_{x} = q'^{(j)}_{x} \int_0^1 \left[\, \prod_{i \neq j} \left(1 - t \cdot q'^{(i)}_{x}\right) \right] dt$$

for $j = 1, 2, 3$. Compute $q^{(3)}_{x}$ to four decimal places.

$$q^{(3)}_{x} = q'^{(3)}_{x} \int_0^1 \left(1-tq'^{(1)}\right)\left(1-tq'^{(2)}\right)dt = .15\int_0^1 (1-.05t)(1-0.1t)\,dt = 0.139$$

7. Use SULT (below) at $i = 5\%$ to find each EPV. Assume that Moe and Johnny, who own a popular Butler hangout, have independent lives.

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\vert}$	$A_{x:\overline{10}\vert}$	$\ddot{a}_{x:\overline{20}\vert}$	$A_{x:\overline{20}\vert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60
70	91,082.4	0.010413	12.0083	0.42818	0.21467	7.6491	0.63576	11.1109	0.47091	0.73295	0.50994	0.17313	70

x	$\ddot{a}_{xx}$	A_{xx}	${}^2A_{xx}$	$\ddot{a}_{x:x:\overline{10}\vert}$	$\ddot{a}_{x:x+10}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\vert}$	x
50	15.8195	0.24669	0.08187	8.0027	14.2699	0.32048	0.12929	7.9044	50
60	13.2497	0.36906	0.16555	7.8080	11.2220	0.46562	0.24895	7.5110	60
70	9.9774	0.52488	0.30743	7.2329	7.7208	0.63234	0.42760	6.4497	70

a. Calculate ${}_{10}E_{50:60}$ and ${}_{20}E_{50:60}$.

$${}_{10}E_{50:60} = {}_{10}E_{50}\,{}_{10}E_{60}\cdot v^{-10} = 0.56724 \qquad {}_{20}E_{50:60} = 0.2726493$$

b. Find EPV of a fully discrete 10-year deferred annuity due that pays 100,000 per year until the first death of Moe (50) and Johnny (60).

$$100000\,{}_{10}E_{50:60}\cdot \ddot{a}_{60:70} = 636558.5026$$

c. Find EPV of a fully discrete annuity due on Moe (50) and Johnny (60) that pays 100,000 per year as long as at least one of them is alive.

$$100000\left(\ddot{a}_{50} + \ddot{a}_{60} - \ddot{a}_{50:60}\right) = 1765870$$

d. Find the EPV of a fully discrete reversionary annuity that pays 100,000 per year to Johnny (60) if Moe has previously died, while Johnny is alive.

$$100000\left(\ddot{a}_{60} - \ddot{a}_{50:60}\right) = 63420$$

e. Find the EPV of a fully discrete 20-year term insurance policy that pays 100,000 upon the first death of Moe (50) and Johnny (60).

$$100000\left(A_{50:60} - {}_{20}E_{50:60}\,A_{70:80}\right) = 14807.29324$$

f. Find the EPV of a fully discrete whole life insurance policy that pays 100,000 upon the last death of Moe (50) and Johnny (60).

$$100000\left(A_{50} + A_{60} - A_{50:60}\right) = 15911$$

SOLUTIONS to Exam Review Problems

1. Covered in lecture notes.

2.

(a)

$$1000 * \left[.1v + (.8)(.1)v^2 + (.1)(.5)v^2 \right]$$

(b) ${}_3p_x^{00}$:

$$\left.\begin{array}{lll}
t=0 & & t=3 \\
0 \to 0 \to 0 \to 0 & \leftarrow & (.8)^3 \\
 & & + \\
0 \to 0 \to 1 \to 0 & \leftarrow & (.8)(.1)(.2) \\
 & & + \\
0 \to 1 \to 0 \to 0 & \leftarrow & (.1)(.2)(.8) \\
 & & + \\
0 \to 1 \to 1 \to 0 & \leftarrow & (.1)(.3)(.2)
\end{array}\right\} +$$

$${}_3p_x^{00} = .55$$

3. a.

(↑ could work 2b using matrix multiplication)

$$e^{-(\mu^{01}+\mu^{02})(2)}$$

b.

$$\int_{t=0}^{2} e^{-(\mu^{01}+\mu^{02})t}(\mu^{02})\,dt = \frac{-\mu^{02}e^{-(\mu^{01}+\mu^{02})t}}{\mu^{01}+\mu^{02}}$$

$$= \frac{\mu^{02}\left(1-e^{-(\mu^{01}+\mu^{02})(2)}\right)}{\mu^{01}+\mu^{02}}$$

c.

$$\int_{t=0}^{2} e^{-(\mu^{01}+\mu^{02})t} * \mu^{01} * e^{-\mu^{12}(2-t)}\,dt$$

#3, continued

d.

$$\int_{t=0}^{2} e^{-(\mu^{01}+\mu^{02})t}(\mu^{02})\cdot e^{-\delta t}\,dt$$
$$+$$
$$\int_{t=0}^{2}\int_{s=0}^{2-t} e^{-(\mu^{01}+\mu^{02})t}\cdot\mu^{01}\cdot e^{-\mu^{12}s}\cdot\mu^{12}\cdot e^{-\delta(t+s)}\,ds\,dt$$

4. a.

$$10000\,\bar{A}^{01}_{50:\overline{20}|} + 20000\,\bar{A}^{03}_{50:\overline{20}|} + 30000\,\bar{A}^{02}_{50:\overline{20}|}$$
$$= 3640$$

b.

(i) $\bar{A}^{01}_{60:\overline{10}|} = \bar{A}^{01}_{60} - {}_{10}p^{00}_{60}\cdot v^{10}\cdot\bar{A}^{01}_{70} \approx .10978 \quad (.110)$

(ii) $\bar{A}^{03}_{60:\overline{10}|} = \bar{A}^{03}_{60} - {}_{10}p^{00}_{60}\cdot v^{10}\cdot\bar{A}^{03}_{70} - {}_{10}p^{01}_{60}\cdot v^{10}\cdot\bar{A}^{13}_{70}$
$\approx .007149$
$(\approx .007)$

Problem 5:

5. Consider the following facts concerning a double-decrement model:

$\ell_{60}^{(\tau)} = 1{,}000$

$q'^{(1)}_{60} = .04$ $\quad p'^{(1)}_{60} = .96$ $\quad \Longrightarrow \quad p^{(\tau)}_{60} = (.9)(.96) = .864$

$q'^{(2)}_{60} = .1$ $\quad p'^{(2)}_{60} = .9$ $\quad q^{(\tau)}_{60} = 1 - .864 = .136$

Assume UDD in the multiple decrement table, so that

$$\frac{\ln(p'^{(j)}_x)}{\ln(p^{(\tau)}_x)} = \frac{q^{(j)}_x}{q^{(\tau)}_x} \text{ for } j = 1, 2;\ x \in \mathbb{Z}. \longrightarrow q^{(2)}_{60} = q^{(\tau)}_{60} \cdot \frac{\ln(p'^{(2)}_{60})}{\ln(p^{(\tau)}_{60})}$$

$$= .136 \times \frac{\ln(.9)}{\ln(.864)}$$

$$= .0980$$

Compute ${}_{.2}q^{(2)}_{60.7}$ to four decimal places.

$\ell_{60}^{(\tau)} = 1000$

② $\longrightarrow \ell_{60.7} = .3\,(\overset{\ell_{60}^{(\tau)}}{1000}) + .7\,(\overset{\ell_{61}^{(\tau)}}{864})$

$= 904.8$

① $\rightarrow \ell_{61} = p^{(\tau)}_{60} \times \ell^{(\tau)}_{60} = 864$

③ Expect # failures by (2) in $[60, 61]$

is $.0980 \times 1000 = 98.0$

And under UDD (in MDT),

failures expected in .2 of that year is

$.2 \times \underbrace{.0980}_{q^{(2)}_{60}} \times \underbrace{1000}_{\ell^{(\tau)}_{60}} = 19.6$

④ ${}_{.2}q^{(2)}_{60.7} = \dfrac{19.6}{904.8}$

$\approx .02166$

Problem 6:

$$q^{(3)}_x = q'^{(3)} \int_0^1 (1 - t \cdot q'^{(1)}_x)(1 - t \cdot q'^{(2)}_x)\, dt$$

$$= .15 \int_0^1 (1 - .05t)(1 - .1t)\, dt$$

$$= .15 \int_0^1 (1 - .15t + .005t^2)\, dt$$

$$= .15 \left(t - \frac{.15t^2}{2} + \frac{.005t^3}{3} \right) \Big|_0^1$$

$$= .139$$

Problem 7:

a.

$${}_{10}E_{50:60} = {}_{10}E_{50} \cdot {}_{10}E_{60} \div v^{10} = .56724 \; ; \quad {}_{20}E_{50:60} = {}_{20}E_{50} \cdot {}_{20}E_{60} \div v^{20} = .27265$$

b. Find EPV of a fully discrete 10-year deferred annuity due that pays 100,000 per year until the first death of Moe (50) and Johnny (60).

$${}_{10}E_{50:60} \times 100{,}000\, \ddot{a}_{60:70} = 636{,}556.73$$

c. Find EPV of a whole life annuity due on Moe (50) and Johnny (60) that pays 100,000 per year as long as at least one of them is alive.

$$100000\, \ddot{a}_{\overline{50:60}} = 100000\left[\ddot{a}_{50} + \ddot{a}_{60} - \ddot{a}_{50:60}\right] = 1{,}765{,}870$$

d. Find the EPV of a reversionary annuity that pays 100,000 per year to Johnny (60) if Moe has previously died, while Johnny is alive.

$$100{,}000\left[\ddot{a}_{60} - \ddot{a}_{50:60}\right] = 63{,}420$$

e. Find the EPV of a fully discrete 20-year term insurance policy that pays 100,000 upon the first death of Moe (50) and Johnny (60).

$$100000\, A^{1}_{50:60:\overline{20|}} = 100000\left[A_{50:60} - {}_{20}E_{50:60} \times A_{70:80}\right] = 14807.25$$

f. Find the EPV of a ~~20-year term~~ whole life insurance policy that pays 100,000 upon the last death of Moe (50) and Johnny (60).

$$100000\, A_{\overline{50:60}} = 100000\left[A_{50} + A_{60} - A_{50:60}\right]$$

$$= 15911$$

DHW 3e 11.1-11.3 – Introduction to Pension Mathematics

Two main categories of employer-sponsored pension plans:

DB – Defined Benefit : employer guarantee pension income (e.g. 1.5% final salary per year employed)
↳ as annual income post retirement
→ Employer bears all risk

DC – Defined Contribution : Plan specifies amt of contribution, e.g. match
E.g. 401(k), 403(b) → Employee bears all risk

Want a pension to replace some of employment income. Some issues:

1. How to define the amount of income you're planning to replace (various definitions of final salary)

   e.g. salary earned in final year of employment, or average of final 3 years' annual salaries, or average of yearly earnings over all years of employment

   ~60-70% is typical / target

2. What percentage of income should be replaced?

   Definition: Replacement ratio $= \dfrac{\text{Pension Income in Year post-Ret}}{\text{Final (average) Salary}}$

3. How to project (from early part of employee's career) the future salary near retirement

   Salary scale.

   Let x_0 represent the age at which employment begins.
   A salary scale is a set of values s_y for ages $y \geq x_0$ such that…

   $$\frac{s_y}{s_{y'}} = \frac{\$\text{ salary earned during age } [y, y+1)}{\$\text{ salary earned during age } [y', y'+1)}$$

   (Think of the values from the salary scale as numerators and denominators that you would use to assemble ratios of salaries comparing various years' earnings.)

   **Note that the subscripts indicate the age at the beginning of the year in question.*

   **Another note: The only thing that matters about a salary scale is the ratios between pairs of values. The individual values themselves may or may not have any real-life interpretation.*

<u>One main point:</u> Get your hands on the <u>total salary paid</u> during a particular year and the endpoints (usually ages) of the year during which that salary was paid. Watch out *rate* of salary—translate into total paid and year endpoints ASAP.

<u>Example 1</u> (Based on DHW 2e Example 10.2 / DHW 1e Example 9.1; "assumption ii")

Consider the following salary scale:

Table 9.1. *Salary scale for Example 9.1.*

x	s_x	x	s_x	x	s_x	x	s_x
30	1.000	40	2.005	50	2.970	60	3.484
31	1.082	41	2.115	51	3.035	61	3.536
32	1.169	42	2.225	52	3.091	62	3.589
33	1.260	43	2.333	53	3.139	63	3.643
34	1.359	44	2.438	54	3.186	64	3.698
35	1.461	45	2.539	55	3.234		
36	1.566	46	2.637	56	3.282		
37	1.674	47	2.730	57	3.332		
38	1.783	48	2.816	58	3.382		
39	1.894	49	2.897	59	3.432		

$$\frac{\text{Salary }[60,61)}{\text{Salary }[34,35)} = \frac{s_{60}}{s_{34}}$$

$$\frac{x}{75000} = \frac{s_{60}}{s_{64}}$$

a. An employee earns \$75,000 during the year prior to her 35th birthday. Find the amount of salary earned by the employee during the year prior to her 61st birthday, assuming she remains employed.

$$\frac{75000 \cdot s_{60}}{s_{34}} = 192\,273.73 \approx 192000$$

Professionally Round to ~10k or ~1k as these are estimates. In class, be exact so Dr. W knows we're correct

b. A pension plan for this company defines the "final average salary" for the pension benefit to be the average salary in the three years before retirement. Compute this quantity for the employee in (a) assuming that she retires at age 64.

61 62 63 64

$$\frac{75000}{3}\left(\frac{s_{61}}{s_{34}} + \frac{s_{62}}{s_{34}} + \frac{s_{63}}{s_{34}}\right) = \frac{75000}{3s_{34}}\left(s_{61}+s_{62}+s_{63}\right) = 198086.83 \approx 198000$$

c. Assume that the pension plan sets a target replacement ratio of 70% and that the employee in (b) retires at age 64*. If the replacement ratio target has been met (on the basis of the final average salary defined <u>for this plan</u> & computed in (b)), how much pension income will she receive during the first year of retirement, assuming she survives?

$$70\%(\text{final avg salary}) = 0.7(198086.83) = 138,661$$

**I chose age 64 so that we are presented with a choice in which line of the table to use: s_{63} or s_{64}. (DHW asks about age 65.)*

d. Suppose now that salaries are always adjusted halfway between ages ("six months before the valuation date" (DHW)).

Consider an employee who earns salary **at a rate of** \$100,000 per year at exact age 55. What is the predicted final average salary in the three years to age 65 (sic, DHW)?

Key!

54 54.5 55 55.5 56

$\underbrace{\qquad\qquad}_{\$100000 \text{ earned}}$

Approx $s_{54.5} \approx \frac{1}{2}(s_{54}+s_{55})$

$$\frac{100000}{3s_{54.5}}\left(s_{62}+s_{63}+s_{64}\right)$$

Careful!!!: Amount earned during a fixed year vs. *rate* of salary at an instant in time.

Example 2: (DHW 2e Example 10.2, under "assumption i")

a. Consider an employee whose salaries are always adjusted halfway between ages (DHW: "six months before the valuation date").

A member aged exactly 35 **earns $75000 during the year to the valuation date** (sic). The final average salary defined by the pension plan is the average salary in the three years before retirement.

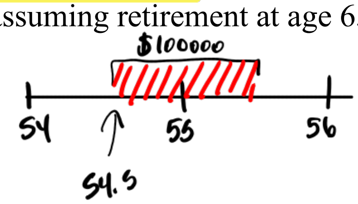
i.e. expect 4% raises year-to-year

Calculate the predicted final salary using the salary scale $s_y = 1.04^y$ assuming that she retires at age 65.

$$\frac{Sal[64,65]}{Sal[34,35]} = \frac{s_{64}}{s_{34}} = 1.04^{\overbrace{64-34}^{30}}$$

$$\to \frac{75000}{3}\left(1.04^{30} + 1.04^{29} + 1.04^{28}\right) = 234,018$$

b. Consider another pension plan member, exact age 55 at the valuation date, who was paid salary **at a rate of $100,000 per year at that time**. ***At this company***, all salary adjustments occur 6 months before the valuation date. Calculate his predicted final average salary assuming retirement at age 65.



let $y = 1.04$

$$\frac{100000}{3\, s_{54.5}}\left(s_{64} + s_{63} + s_{62}\right) = \frac{100000}{3}\left(y^{9.5} + y^{8.5} + y^{7.5}\right)$$

$$\approx 139638.78$$

Example 3: DHW 2e, Example 10.3.

The current **annual salary rate** of an employee aged exactly 40 is \$50,000. Salaries are revised continuously. Use the salary scale given by $s_y = 1.03^y$:

approx

Salary[39.5, 40.5] ≈ 50,000

(a) Estimate the employee's salary between ages 50 and 51.

$$\frac{Sal[50,51]}{Sal[39.5,40.5]} = \frac{s_{50}}{s_{39.5}} \rightarrow 50000 \cdot 1.03^{10.5} \approx 68196$$

(b) Find the employee's **annual rate of salary** at age 51.

$$Sal[50.5, 51.5] = 50000 \cdot 1.03^{11} \approx 69212$$

DHW3e Homework problem for Sections 11.1-11.3 (Li and Ng / Actex)

30. An employee aged exactly 62 on January 1, 2010 has an annual salary rate of 75,000 on that date. Salaries are revised annually on December 31 each year. Future salaries are estimated using the salary scale given in the table below, where S_y / S_x, $y > x$ denotes the ratio of salary earned in the year of age from y to $y + 1$ to the salary earned in the year of age x to $x + 1$, for a life in employment over the entire period $(x, x + 1)$.

x	S_x
62	3.589
63	3.643
64	3.698
65	3.751

The multiple decrement table below models exits from employment:

(i) $d_x^{(1)}$ denotes retirements.

(ii) $d_x^{(2)}$ denotes deaths in employment.

(iii) There are no other modes of exit.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
62	42,680	4,068	312
63	38,300	3,560	284
64	34,456	3,102	215
65	31,139	31,139	–

The employee has insurance that pays a death benefit equals to 3 times his salary at death if death occurs while employed and prior to age 65; and pays 0 otherwise. The death benefit is payable at moment of death. Assume deaths occur at mid-year.

The annual effective rate of interest is 0.06.

Calculate the actuarial present value of the death benefit.

Key Issues:

1) Data (salary in 1 year vs salary rate @ instant)

2) Computation / Answer (same issues)

DHW 3e 11.4 – Defined Contribution Retirement Plans – Setting the Contribution Rate
(Based on DHW2e Example 9.3)

Rate

Consider a male employee entering a DC pension plan at age 25, when he earns $100,000/year. The employee wants to make contributions at the end of each year at a fixed percentage of the salary rate at that time.

Assumptions:

We accidentially used 3%, but all #s use 4%

- The salary scale is given by $s_y = (1.04)^y$. Salaries are adjusted immediately after the end-of-year contributions.
- Contributions to the pension are assumed to earn a 10% return each year.
- Survival and life contingent annuity present values given as needed, below.

Example 1 Let's see what the man can buy in the way of an annuity income product if he plans to retire at age 65 and contributes 6% of the previous year's salary at the end of each year.

a. Determine the projected value in the retirement fund at age 65. (For projecting the amount that someone saves to retirement, we use the future value, not the expected future value, so that the person can evaluate what he/she can actually purchase when/if the age is actually attained.)

Salary: 100K, 100K(1.04), 100K(1.04)^2, …, 100K(1.04)^{39}

Ages: 25 (t=0), 26 (1), 27 (2), …, 64 (39), 65 (40)

DC: 100K(0.06), 100K(1.04)(0.06), …, 100K(1.04)^{38}(0.06), 100K(1.04)^{39}(0.06)

Evaluate this flow assuming 10% growth @ t=40 i.e. age 65

↳ grows to $0.06(100000)(1.1)^{39}$

$$= 0.06(100000)(1.1)^{39}\left(\frac{1-x^{40}}{1-x}\right) \quad \text{for } x=\frac{1.03}{1.1} \rightarrow \$4\,054\,823$$

Think back to geometric variable annuities from FM

b. Assume that $\ddot{a}_{65}^{male} = 10.5$ and determine the amount of annual pension income that can be purchased at age 65 with the fund from (a).

$$\left(\text{Value of pension fund}\right) = P \cdot \ddot{a}_{65}^{male}$$

$\rightarrow P = \$385,317$ given each year

c. What is the replacement rate in (b), if we use the final year's salary as the "final salary"?

$$\frac{Sal[65,66)}{Sal[64,65)} = \frac{385317}{100000 \cdot 1.03^{39}} \approx 83.5\%$$

Very good!

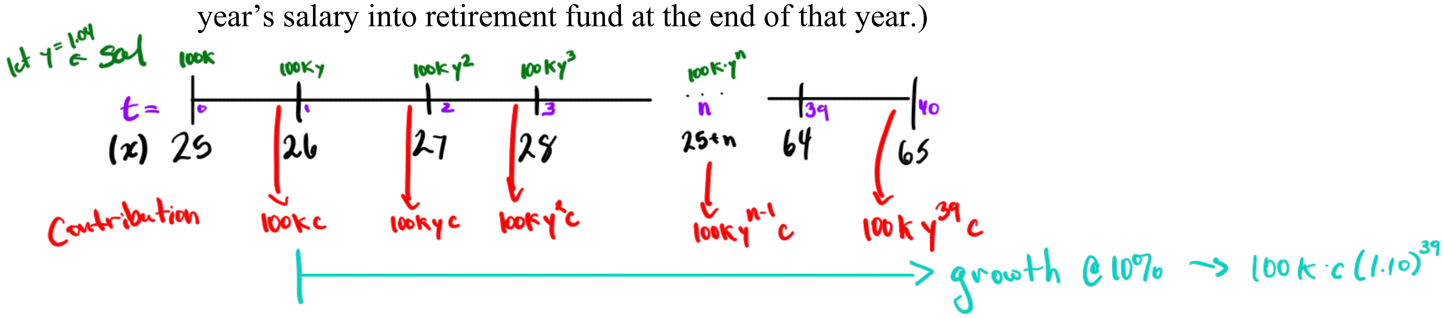
Example 2: Same man, age 25, earning at \$100,000 during first year, $s_y = (1.04)^y$ with adjustments occurring immediately after the end of the year. Contributions earn investment income of 10% annually.

a. Project the man's salary during his final year of employment.

retire @ (65) → want $Sal[64,65) = \$100000\,(1.04)^{39}$

$= 461\,636.60$

b. Find an expression for the value at age 65 of the man's retirement fund if he contributes at a rate of c at the end of each year. (For example, if $c = 6\%$, then he puts 6% of his previous year's salary into retirement fund at the end of that year.)



Geometrically increasing annuity

first term

common ratio

terms

$$\text{Proj. fund} = 100000\,c\cdot(1.10)^{39}\cdot\frac{1-\left(\frac{1.04}{1.1}\right)^{40}}{1-\frac{1.04}{1.1}} = \boxed{67{,}430{,}391\,c}$$

c. The man plans at age 65 to purchase both

(i) A life annuity payable annually during his lifetime, with target replacement ratio 65% of his final year's salary, and

(ii) A reversionary annuity for his wife (who will be 61 when he retires at 65) at 60% of his pension income (so she receives 60% of his pension income if he dies first).

Determine the price of these products at age 65 if

$$\ddot{a}_{65}^{male} = 10.5, \quad \ddot{a}_{61}^{female} = 13.9, \quad \ddot{a}_{65:61}^{m:f} = 10.0$$

i) $0.65(461637)\,\ddot{a}_{65}^{male} = 3{,}150{,}673$

replacement   final sal   annuity

ii) $0.6\,(461637)\,\underbrace{\ddot{a}^{m|f}_{65|61}}_{=\ddot{a}^{f}_{61}-\ddot{a}^{f:m}_{61:65}} = 702{,}150$

joint status: order does not matter

d. Determine c by setting accumulated value of contributions at age 65 equal to the cost of the retirement package. (DHW notes that c depends only on the desired replacement ratios and not on the starting salary rate.)

$67430391c = 3150673 + 702150 \rightarrow c = 0.0571$

$= 5.71\%$

Assigned Reading: DHW3e, Sections 1.6, 1.10 (including subsections; this is on Canvas.)

Practice Problem for DHW3e 11.4

Suppose the 25-year-old man from Example 2 decides to contribute at a rate of 5.5% at the end of each year. His first year's salary ([25,26]) is still $100,000.

a. His salary raises turned out to be 5% each year, but investment income on his retirement fund was less favorable and was only 8% per year.

Determine the size of the retirement fund at age 65.

$$\text{Total fund} = \underset{\text{starting sal}}{100000}\,\underset{\text{fund growth to ret.}}{(1.08)^{39}}\,\underset{\text{contrib. rate}}{(0.055)}\,\underset{\text{annuity}}{\left(\frac{1-\left(\frac{1.05}{1.08}\right)^{40}}{1-\frac{1.05}{1.08}}\right)}$$

$$= 2692164.344$$

Skip

b. When it's time to purchase annuities, the prices of the annuities have increased because the interest rate is lower than was anticipated. Prices are based upon

$$\ddot{a}_{65}^{male} = 11.35, \quad \ddot{a}_{61}^{female} = 15.47, \quad \ddot{a}_{65:61}^{m:f} = 10.75$$

Determine the price of each of these products:

(i) A life annuity paying $1 per year for a man aged 65.

(ii) A reversionary annuity paying $.6 per year to the wife (age 61) after the man has died, if she is still living.

c. The man uses the pension fund in (a) to purchase this package:
(i) A life annuity paying level payments P every year while he is alive, and
(ii) A reversionary annuity paying 0.6P to his wife after he has died.
What is the largest P that he can afford?

$$2692164.34 = P\ddot{a}_{65}^{m} + 0.6P\ddot{a}_{65|61}^{m|f}$$

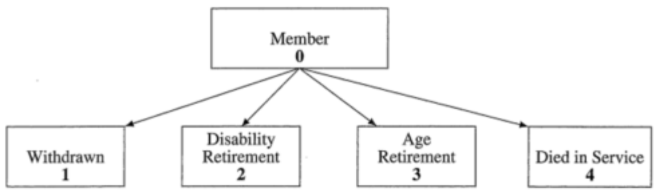
$$= P(11.35 + 0.6(15.47 - 10.75))$$

$$\rightarrow P = 189829.67$$

DHW3e 11.5 – 11.9 – Funding Defined Benefit (DB) Plans *part 1*
Service Table, Accrued Actuarial Liability, TUC vs. PUC, Normal Cost

Key: w_x = # withdrawals; i_x = # disability retirements; r_x = # age retirements; d_x = # deaths

The Service Table (DHW2e Figure 10.1 & Table 10.2 = Appendix D.4)



Standard Service Table

x	l_x	w_x	i_x	r_x	d_x	x	l_x	w_x	i_x	r_x	d_x	
35	218,833.9	10,665.3	213.3	0	83.5	51	114,572.5	2,266.1	113.3	0	150.9	
36	207,871.8	10,130.9	202.6	0	83.6	52	112,042.2	2,215.9	110.8	0	162.8	
37	197,454.7	9,623.1	192.5	0	84.0	53	109,552.7	2,166.5	108.3	0	176.0	
38	187,555.1	9,140.6	182.8	0	84.7	54	107,101.9	2,117.8	105.9	0	190.5	
39	178,147.0	8,681.9	173.6	0	85.7	55	104,687.7	2,069.9	103.5	0	206.4	
40	169,205.8	8,246.0	164.9	0	86.9	56	102,307.9	2,022.6	101.1	0	223.9	
41	160,707.9	7,831.8	156.6	0	88.5	57	99,960.2	1,976.0	98.8	0	243.2	
42	152,631.0	7,438.0	148.8	0	90.5	58	97,642.2	1,929.9	96.5	0	264.4	
43	144,953.7	7,063.7	141.3	0	92.7	59	95,351.5	1,884.3	94.2	0	287.6	
44	137,656.1	6,707.9	134.2	0	95.3	60	93,085.4	0	0	27,925.6	0	Exact Age
45	130,718.7	2,586.1	129.3	0	99.7	60	65,159.8	0	61.9	6,187.6	210.4	
46	127,903.5	2,530.4	126.5	0	106.2	61	58,699.9	0	55.7	5,573.1	211.5	
47	125,140.4	2,475.6	123.8	0	113.4	62	52,859.6	0	50.2	5,017.5	212.7	
48	122,427.6	2,421.8	121.1	0	121.4	63	47,579.3	0	45.2	4,515.2	213.9	
49	119,763.2	2,369.0	118.5	0	130.3	64	42,805.0	0	40.6	4,061.0	215.1	
50	117,145.5	2,317.1	115.9	0	140.1	65	38,488.3	0	0	38,488.3	0	Exact Age

$w_x \rightarrow$ withdrawals; $i_x \rightarrow$ disability; $r_x \rightarrow$ retirements; $d_x \rightarrow$ deaths

<u>Example 1</u>: *A warmup example adapted from Spring 2018 Exam MLC.*

as only one piece of the package

XYZ offers a pension plan that <u>includes a</u> lump sum death-in-service benefit, payable immediately on death:

- 10,000 for each full year of service on death-in-service between ages 63 and 64.
- 15,000 for each full year of service on death-in-service between ages 64 and 65.

You are given:

- Deaths are assumed to occur half-way through the year of age (simplifying assumption).
- Decrements follow the ALTAM Pension Service Table.
- $i = .05$
- *(Included on exam, perhaps unnecessarily:)* The traditional unit credit funding method is used.

think reserving, but in the context of pension mathematics

a. What is the <u>actuarial liability</u> for this death benefit for an employee who is 50 years old with 10 years of service?

Why 10? Work beyond (50) has not been completed yet, so you only consider service to-date ("pensionable service")

$$AL_{50} = \underbrace{\frac{d_{63}}{\ell_{50}}}_{\text{Pr. death}} \cdot \underbrace{10000(10)}_{\text{D.B.}} \underbrace{v^{13.5}}_{\text{payable @ moment of death}} + \frac{d_{64}}{\ell_{50}} 15000(10)\, v^{14.5} = 230.256$$

(Values on prev. page)

b. Consider the employee from (a); if the employee remains in service at age 51, what will be the actuarial liability at that time? Also find $EPV_{50}[AL_{51}]$.

$$AL_{51} = \frac{d_{63}}{\ell_{51}} 10000(11)\, v^{12.5} + \frac{d_{64}}{\ell_{51}} 15000(11)\, v^{13.5} = 271.918$$

Discount to (50)

$$EPV_{50}[AL_{51}] = \frac{\ell_{51}}{\ell_{50}} \left[\frac{d_{63}}{\ell_{51}} \cdot 10000(11) v^{13.5} + \frac{d_{64}}{\ell_{51}} \cdot 15000(11) v^{14.5} \right]$$

$$EPV_{50}\left[\begin{array}{l}\text{add'l benefit}\\ \text{accrual during ages}\\ [50,51) \text{ beyond } AL_{50}\end{array}\right] = \frac{d_{63}}{\ell_{50}} \cdot 10000(1)\, v^{13.5} + \frac{d_{64}}{\ell_{50}} \cdot 15000(1)\, v^{14.5}$$

i.e. topoff beyond growth @ interest for one year further

one more year of pensionable service

NC_{50}

= "Normal Contribution" | "Normal Cost"

c. Note the element of this situation that's fundamentally different from our previous life insurance reserving:

- The pension plan's liability for the benefit (and the need to reserve for that liability) is gradually accruing as the employee completes more years of service.
- The quantity AL_{50} does not include a dollar amount for the additional benefit that the employee will earn during [50, 51], because the employee hasn't earned it yet by working for another year.

To address this, the employer/pension plan makes a contribution at age 50 called the <u>normal contribution</u> or <u>normal cost</u>.

$$AL_{50} + NC_{50} = EPV_{50}\left[\begin{array}{c}\text{benefits to be paid out}\\ \text{mid-year in [50, 51], at the}\\ \text{value to which they will have}\\ \text{accrued}\end{array}\right] + EPV_{50}[AL_{51}]$$

($EPV_{50}[AL_{51}]$ is my notation: $EPV_{50}[AL_{51}] = p_{50}^{00} \cdot v \cdot AL_{51}$.)

Notice:

$NC_{50} = EPV_{50}$[<u>additional accrual of benefit liability</u> (beyond AL_{50}) during [50, 51]]

d. Consider what happens if this employee remains alive and in service to age 63, at which time he'll have accrued 23 years of service.

$$AL_{63.0} = \frac{d_{63}}{l_{63}} 10000(23)v^{0.5} + \frac{d_{64}}{l_{63}} 15000(23)v^{1.5}$$

consider add'l benefit accrued during (63, 64)

mid-year exit assumption

$$AL_{63.0} + NC_{63} = \underbrace{\frac{d_{63}}{l_{63}} 10000(23.5)v^{0.5}}_{EPV_{63}\left[\substack{\text{accrued benefit}\\ \text{for mid-year exits}}\right]} + \underbrace{\frac{d_{64}}{l_{63}} 15000(24)v^{1.5}}_{EPV_{63}[AL_{64}]}$$

But also: $$NC_{63} = \frac{d_{63}}{l_{63}} 10000(0.5)v^{0.5} + \frac{d_{64}}{l_{63}} 15000(1)v^{1.5}$$

Next: Peek at Example 2, but then return to the terminology on this page.

DB Terminology:

- Final salary plan / final average salary plan → Type of d.b plan
- Accrual rate ← % of salary/yr ↳ % of salary (as calculated) per year of service

- Actuarial Liability / Accrued Actuarial Liability

  AL_t = EPV[accrued benefits] → AL_t should be reserved in connection *only* with those benefits already accrued by past employee service.

  ↓

- Two methods/approches to compute AL_t → 2 issues: (1) Final salary; (2) # years accrued @ accrual rate

  - TUC – Traditional Unit Credit method (also called Current Unit Credit method).

    reserve using current salary

    - Uses salary/salaries from most recently completed year(s) as the "final salary" on which actuarial liability for pension benefits is based.
    - Uses the number years of service in employment completed by time/age t to determine the number of years to which the accrual rate applies in the AL_t calculation.

      (Liability has only accrued for years actually worked thus far.)

  - PUC – Projected Unit Credit method

    reserve using projected final salary

    - Uses projected salary (or projected final average salary) to each possible retirement date as the "final salary" on which actuarial liability for pension benefits is based.
    - Uses the number years of service in employment completed by time/age t to determine the number of years to which the accrual rate applies in the AL_t calculation.

      (Liability has only accrued for years actually worked thus far.)

- Normal cost. aka Normal Contribution

As with our previous example:

- The pension plan's liability for the retirement benefit (and the need to reserve for that liability) is gradually accruing as the employee completes the next year of service.
- Members still employed at the end of the year will have accrued an additional year of service.
- At certain ages, retirements may happen midway through the upcoming year—For members retiring mid-year, only a ½ year of accrual (at the accrual rate) will occur in the year of retirement..
- The quantity AL_{50} does not include a dollar amount for these additional accrued benefits, because the employee hasn't earned it yet by working beyond age 50.

add

- In TUC, current salaries must also be adjusted year-to-year

To address this, the employer/pension plan makes a contribution at age 50 called the <u>normal contribution</u> or <u>normal cost</u>. NC_{50} is defined by

$$AL_{50} + \underset{\substack{\uparrow \\ \text{funds add'l liability that} \\ \text{will accrue during } [50,51]}}{NC_{50}} = EPV_{50}[\text{ben's paid out mid-year during } [50, 51]] + EPV_{50}[AL_{51}],$$

where we define (for convenience) $EPV_{50}[AL_{51}] = p_{50}^{00} \cdot v \cdot AL_{51}$

This means that

$$NC_{50} = EPV_{50}[\textbf{\textit{additional}} \text{ (beyond } AL_{50}\text{) benefits that will accrue during } [50, 51]\,]$$

accrual rate

Example 2. (Adapted from DHW2e Example 10.10)

- Martin Ellingham (50) has 20 years past service on the valuation date.
- His salary in the previous year was $50,000.
- He is enrolled in a final salary DB plan that provides an annual pension benefit of 1.5% of the salary earned in the final year of employment per year of service.
- The pension benefit is a life annuity payable annually in advance
- There is no benefit due on death in service

Assumptions:

- Retirements follow the SOA Standard Service Table
- Retirements between integer ages are assumed to occur mid-year.
- The interest rate is $i = 5\%$
- Salaries increase at 4% / year.
- Mortality in retirement follows SULT.

x	$\ddot{a}_x$
60	14.9041
60.5	14.7766
61.5	14.5176
62.5	14.2506
63.5	13.9757
64.5	13.6931
65	13.5498

Calculate the accrued actuarial liability for this member's pension benefits using

(a) traditional unit credit (TUC) funding; (b) projected unit credit (PUC) funding

Solution

a. TUC Method:

$r_{60.0} = r_{60}^{exact}$

TUC

$$AL_{50} = \frac{\overbrace{r_{60.0}}^{27925.6}}{l_{50.0}} \cdot v^{10} \cdot \Big[\overbrace{50{,}000 \times .015(20)}^{TUC} \Big] \ddot{a}_{60}$$

$$+$$

$$\frac{\overbrace{r_{60^+}}^{6187.6}}{l_{50.0}} \cdot v^{10.5} \cdot \Big[50{,}000 \times .015(20) \Big] \ddot{a}_{60.5}$$

$$+$$

$$\vdots$$

$$+$$

$$\frac{\overbrace{r_{64}}^{\text{age } 64.5:\ 4061}}{l_{50.0}} \cdot v^{14.5} \cdot \Big[50{,}000 \times .015(20) \Big] \ddot{a}_{64.5}$$

$$+$$

$$\frac{\overbrace{r_{65}}^{\text{age } 65.0:\ 38488.3}}{l_{50}} \cdot v^{15} \cdot \Big[50{,}000 \times .015(20) \Big] \ddot{a}_{65}$$

retire in midyear

$r_{65.0} = r_{65}^{exact}$

= 90430

$$AL_{50} = \sum_{\substack{\text{possible} \\ \text{retirement} \\ \text{ages}}} \frac{\#\text{retires}}{\#\text{lives}} \left[\begin{array}{l} \text{current sal} \\ \times \text{ Accrual Rate} \\ \times \text{ Years in Service} \end{array} \right] \times \ddot{a}_{\text{ret. age}}$$

What is paid out? Yearly pension until death

annuity

Example 2, *continued*

- Martin Ellingham (50) has 20 years past service on the valuation date.
- His salary in the previous year was \$50,000.
- He is enrolled in a final salary DB plan that provides an annual pension benefit of 1.5% of the salary earned in the final year of employment per year of service.
- The pension is benefit is a life annuity payable annually in advance
- There is no benefit due on death in service

x	$\ddot{a}_x$
60	14.9041
60.5	14.7766
61.5	14.5176
62.5	14.2506
63.5	13.9757
64.5	13.6931
65	13.5498

Assumptions:

- Retirements follow the SOA Standard Service Table
- Retirements between integer ages are assumed to occur mid-year.
- The interest rate is $i = 5\%$
- Salaries increase at 4% / year.
- Mortality in retirement follows SULT.

Solution, continued.

b. PUC method

increase final Salary

PUC

~~Due~~

$$AL_{50} = \frac{\overbrace{r_{60.0}}^{27{,}925.6}}{l_{50.0}} \cdot v^{10} \cdot \Big[\overbrace{\cancel{50{,}000}}^{\text{TUC}} \cdot \underbrace{.015(20)}_{\text{still 20}} \Big]\ddot{a}_{60}$$

$50000(1.04)^{9}$ ← PUC →

$s_5 = 1.04^5$

$$\frac{Sal[59,60]}{Sal[49,50]} = \frac{1.04^{59}}{1.04^{49}}$$

$$+ \frac{\overbrace{r_{60^+}}^{6187.6}}{l_{50.0}} \cdot v^{10.5} \cdot \Big[\cancel{50{,}000} \times \underbrace{.015(20)}_{\text{still 20}}\Big]\ddot{a}_{60.5}$$

$50000(1.04)^{9.5}$

$$+ \quad \vdots$$

$$+ \frac{\overbrace{r_{64}}^{\text{age }64.5:\ 4061}}{l_{50.0}} \cdot v^{14.5} \cdot \Big[\cancel{50{,}000} \times .015(20)\Big]\ddot{a}_{64.5}$$

$50000(1.04)^{14.5}$

$$+ \frac{\overbrace{r_{65}}^{\text{age }65.0:\ 38488.3}}{l_{50}} \cdot v^{15} \cdot \Big[\cancel{50{,}000} \times .015(20)\Big]\ddot{a}_{65}$$

$50000(1.04)^{15}$

If doing this IRL, use a spreadsheet

Already accounts for future Salary

=147,569

c. Normal cost for PUC method (b):

$$\boxed{NC_{50} = \underbrace{\tfrac{1}{20}}_{\text{Years in Service}} AL_{50}}$$

Think AL_{50} as having 20 pieces from 20 years of service

→ Accrue one more year of service

think → $EPV_{50}(AL_{50}) = \frac{21}{20} AL_{50}$

d. Normal cost for TUC method (a):

AL_{50} → "20 pieces"

$$EPV_{50}[AL_{51}] = \underbrace{\tfrac{21}{20}}_{\text{Extra Year}} \cdot \overbrace{1.04}^{\text{Salary Growth}} \cdot AL_{50} \quad \underset{\text{for NC}}{\overset{\text{solve}}{\leadsto}} \quad AL_{50} + NC_{50} = EPV(AL_{51}) + \cancel{EPV\left(\text{accrued bens awarded in } [50, 51)\right)}$$

No Retirements

$$\boxed{NC_{50} = \left(\tfrac{21}{20} \cdot 1.04 - 1\right) AL_{50}}$$

This page intentionally blank to facilitate facing-page examples.

More Examples – Reserving for DB Plans

Example 3. Consider the following excerpt from a pension service table.

x	ℓ_x	r_x
45	100,000	0
46	99,990	0
⋮		0
64 (exact)		5,000
64		4,000
65 (exact)	40,000	40,000

The symbols r_x^{exact} for 64 (exact) and 65 (exact) denote expected numbers of age-retirements at exact ages 64 & 65.

For $x = 64$ (without the "exact" notation), r_x denotes the expected number of retirements during the year, which are assumed to occur at age 64.5.

You are given:

Data: (i) A 45-year old employee has a final-salary-style defined benefit pension plan allowing age-retirement only at exact ages 64, 64.5, or 65.

(ii) She earns 100,000 during [44, 45].

(iii) She has accumulated 15 years of pensionable service by age 45.

(iv) The plan has an accrual rate of 2.5%, that is, the annual pension benefit is defined to be 2.5% of final salary per year of service. There is no penalty for retiring at age 64 or 64.5. Pension benefits are paid monthly in advance.

(v) Future salaries are projected using the salary scale $s_y = 1.03^y$.

(vi) The EPV of annuities due payable monthly at a rate of 1 per year at various ages are $\ddot{a}_{64}^{(12)} = 13.9$; $\ddot{a}_{64.5}^{(12)} = 13.7$; $\ddot{a}_{65}^{(12)} = 13.5$.

Compute the actuarial liability (i.e. reserve for pension benefit liability) and normal contribution at age 45 using the **traditional unit credit** approach (that is, base your calculation on the most recent salary and the number of years of pensionable service actually completed by age 45.)

Use $i = .05$.

$$AL_{45} = \frac{r_{64}^{ex}}{\ell_{45}} v^{19}(15 \cdot 2.5\%)(100000)\,\ddot{a}_{64}^{(12)}$$

$$+ \frac{r_{64.5}}{\ell_{45}} v^{19.5}(15 \cdot 2.5\%)(100000)\,\ddot{a}_{64.5}^{(12)}$$

$$+ \frac{r_{65}^{ex}}{\ell_{45}} v^{20}(15 \cdot 2.5\%)(100000)\,\ddot{a}_{65}^{(12)} = 94570.28$$

Example. TUC

$$AL_{45} = \frac{\overbrace{5000}^{(r_{64},\text{ exact})}}{\underbrace{100{,}000}_{l_{45}}} \times v^{19} \times \underbrace{.025\,\overbrace{(100{,}000)}^{TUC}\,\overbrace{(15)}^{\text{accrued yrs service}}}_{(\text{annual amt})}\ \underbrace{\ddot{a}^{(12)}_{64}}_{(\text{pd over 12 months})}$$

$$+$$

$$\frac{\overbrace{4000}^{(r_{64},\text{ mid-yr})}}{100{,}000} \times v^{19.5} \times .025\,\overbrace{(100{,}000)}^{TUC}(15)\,\ddot{a}^{(12)}_{64.5}$$

$$+$$

$$\frac{40{,}000}{100{,}000} \times v^{20} \times .025\,\overbrace{(100000)}^{TUC}(15)\,\ddot{a}^{(12)}_{65}$$

$$= 94{,}570.28$$

$$AL_{46} = \frac{5000}{99990} \times v^{18} \times .025\,\overbrace{(103{,}000)}^{Sal[45,46]}(16)\,\ddot{a}^{(12)}_{64}$$

$$+$$

$$\frac{4000}{99990} \times v^{18.5} \times .025\,(103{,}000)(16)\,\ddot{a}^{(12)}_{64.5}$$

$$+$$

$$\frac{40{,}000}{99990} \times v^{19} \times .025\,(103{,}000)(16)\,\ddot{a}^{(12)}_{65}$$

$$= 109{,}107.20$$

EPV[AL_{46}] =

$$\frac{\overbrace{5000}^{(r_{64},\text{ exact})}}{\underbrace{100{,}000}_{l_{45}}} \times v^{19} \times .025\,\overbrace{(103{,}000)}^{Sal[45,46]}(16)\,\ddot{a}^{(12)}_{64}$$

$$+$$

$$\frac{\overbrace{4000}^{(r_{64},\text{ mid-yr})}}{100{,}000} \times v^{19.5} \times .025\,(103{,}000)(16)\,\ddot{a}^{(12)}_{64.5}$$

$$+$$

$$\frac{40{,}000}{100{,}000} \times v^{20} \times .025\,(103{,}000)(16)\,\ddot{a}^{(12)}_{65}$$

} = 103901

(Or get $EPV_{45}[AL_{46}]$ = 103901 via shortcut…

$$EPV_{45}[AL_{46}] = \left(\frac{16}{15} \cdot 1.03\, AL_{45}\right)$$

+ other bens ? 0

…NC_{45} = 9330.95)

<u>Example 4</u>. Consider the following excerpt from a pension service table.

x	ℓ_x	r_x
45	100,000	0
46	99,990	0
⋮		0
64 (exact)		5,000
64		4,000
65 (exact)	40,000	40,000

The symbols r_x for 64 (exact) and 65 (exact) denote expected numbers of age-retirements at exact ages 64 & 65.

For $x = 64$ (without the "exact" notation), r_x denotes the expected number of retirements during the year, which are assumed to occur at age 64.5.

You are given:

<u>Data</u>: (i) A 45-year old employee has a final-salary-style defined benefit pension plan allowing age-retirement only at exact ages 64, 64.5, or 65.

(ii) She earns 100,000 during [44, 45].

(iii) She has accumulated 15 years of pensionable service by age 45.

(iv) The plan has an accrual rate of 2.5%, that is, the annual pension benefit is defined to be 2.5% of final salary per year of service. There is no penalty for retiring at age 64 or 64.5. Pension benefits are paid monthly in advance.

(v) Future salaries are projected using the salary scale $s_y = 1.03^y$.

(vi) The EPV of annuities due payable monthly at a rate of 1 per year at various ages are $\ddot{a}_{64}^{(12)} = 13.9$; $\ddot{a}_{64.5}^{(12)} = 13.7$; $\ddot{a}_{65}^{(12)} = 13.5$.

Compute the actuarial liability (i.e. reserve for pension benefit liability) and normal cost at age 45 using the **projected unit credit** approach (that is, base your calculation on the projected salary in the year of retirement and the number of years of pensionable service actually completed by age 45.)

Use $i = .05$.

Example. PUC $\dfrac{r^{ex}_{64}}{\ell_{45}}$

$$AL_{45} = \frac{5000}{100{,}000} \times v^{19} \times .025\,\overbrace{(100{,}000 \times 1.03^{19})}^{\text{PUC}}\,\overbrace{(15)}^{\text{accrued-by-45 yrs of svce.}}\,\ddot{a}^{(12)}_{64}$$

accrual rate, final sal, pensionable service

$\dfrac{r_{64}}{\ell_{45}}$

$$+ \frac{4000}{100{,}000} \times v^{19.5} \times .025\,(100000 \times 1.03^{19.5})\,(15)\,\ddot{a}^{(12)}_{64.5}$$

$\dfrac{r^{exact}_{65}}{\ell_{45}}$ accrual rate, final sal, pensionable service

$$+ \frac{40{,}000}{100{,}000} \times v^{20} \times .025\,(100000 \times 1.03^{20})\,(15)\,\ddot{a}^{(12)}_{65}$$

$$= 170{,}051.6$$

$$AL_{46} = \frac{5000}{99{,}990} \times v^{18} \times .025\,\overbrace{(103{,}000 \times 1.03^{18})}^{SAL(63,64)}\,(16)\,\ddot{a}^{(12)}_{64}$$

$$+ \frac{4000}{99{,}990} \times v^{18.5} \times .025\,(103{,}000 \times 1.03^{18.5})\,(16)\,\ddot{a}^{(12)}_{64.5}$$

$$+ \frac{40000}{100000} \times v^{19} \times .025\,(103{,}000 \times 1.03^{19})\,(16)\,\ddot{a}^{(12)}_{65}$$

$$= 190476.8$$

$$EPV_{45}[AL_{46}] = \frac{5000}{100{,}000} \times v^{19} \times .025\,\overbrace{(103{,}000 \times 1.03^{18})}^{SAL(63,64)}\,(16)\,\ddot{a}^{(12)}_{64}$$

$$+ \frac{4000}{100{,}000} \times v^{19.5} \times .025\,(103{,}000 \times 1.03^{18.5})\,(16)\,\ddot{a}^{(12)}_{64.5}$$

$$+ \frac{40{,}000}{100{,}000} \times v^{20} \times .025\,(103{,}000 \times 1.03^{19})\,(16)\,\ddot{a}^{(12)}_{65}, \quad \} = 181338.33$$

Or use shortcut to get $EPV_{45}[AL_{46}]$. (Will get $NC_{45} = 11336.74$)

$$NC_{45} = \frac{1}{15} AL_{45}$$

From Spring 2019 LTAM

19. Abby, who is age 45 on 1/1/2019, is a member of a defined benefit pension plan. The retirement benefit, payable annually at the start of the year, is 1.5% of her three-year final average salary for each year of service.

FAS

Final Average Salary

You are given the following assumptions:

$\text{Sal}(65) = 1.015^{19} \cdot 65000$

(i) Abby's salary in 2019 is 65,000, and she has 15 years of service as of 1/1/2019.

(ii) Her salary is expected to increase by 2.5% in each year on January 1.

(iii) Retirement occurs only at age 65.

(iv) No benefits are payable to lives who exit the plan before age 65.

(v) ${}_{20}p_{45}^{(\tau)} = 0.29$ everyone exact retires @ 65 $= \frac{r_{65}}{\ell_{45}}$

(vi) $i = 0.06$

(vii) $\ddot{a}_{65} = 9.897$

Calculate the normal contribution for Abby for the year beginning 1/1/2019, using the Projected Unit Credit method.

$$AL_{45} = 0.29(1.06)^{-20}\left(0.015(15)(FAS)\right)\ddot{a}_{65}$$

$$FAS = \frac{1}{3}(65000)\left(1.025^{19} + 1.025^{18} + 1.025^{17}\right)$$

$NC = \frac{1}{15} AL_{45}$ (since salary is already accounted for)

Remarks – Adapted from 2018 Spring MLC Written #6:

6. (*9 points*) A defined benefit pension plan with two members, Finn and Oscar, provides for a pension benefit paid as a monthly whole life annuity-due. The annual pension benefit is 1.7% of the final one-year's salary for each year of service.

A partial list of assumptions:

(iii) There are no withdrawals from the plan other than by death or retirement.

(vi) Salaries increase every year on January 1. Future salary increases are assumed to be 2% per year. $S_y = 1.02^y$

(vii) On January 1, 2018, Finn is 25 years old. He is a new employee with no past service. His salary in 2018 is 60,000.

(viii) On January 1, 2018, Oscar is 64 years old and has 29 years of service. His salary in 2017 was 95,000 and in 2018 is 100,000.

Think about:

(i) Without further calculation, state with reasons whether the Normal Cost under the Projected Unit Credit (PUC) method will be greater or less than the TUC for Finn.

(ii) Without further calculation, state with reasons whether the Normal Cost under the PUC will be greater or less than the TUC for Oscar.

[SOA's explanations, slightly modified:]

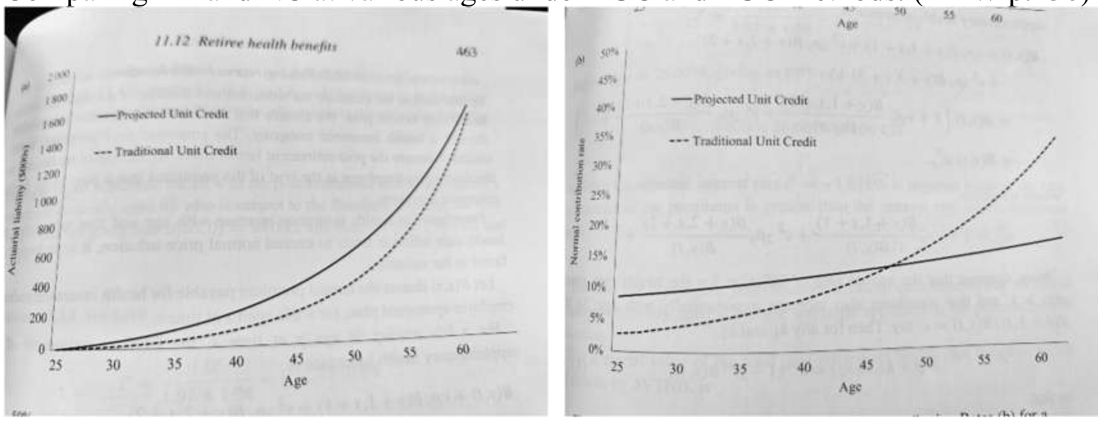
(i) Finn is the newer employee. The PUC will have the greater Normal Cost because the NC under the PUC method includes prefunding the future salary increases. Finn has many years of service ahead, so prefunding these salary increases will significantly impact (increase) the normal cost.

(ii) Oscar is nearing retirement. The TUC will have the greater Normal Cost for Oscar: All past accrued liability (and that's many years' worth for Oscar!) must be adjusted from one year to the next according to *current* salary increases.

(The PUC prefunds future salary increases, but since Oscar has little time left in employment, this cost is small compared with the TUC's salary upgrade for all previously accrued pension liability.)

See graphs for comparison (on next page)

Comparing AL and NC at various ages under PUC and TUC methods. (DHW p.436)



Illustrating the Normal Cost for Standard DB Plans

death + injury + withdraw

xxx

Consider the following excerpt from a pension service table.

x	ℓ_x	r_x	$d_x + i_x + w_x$
55	60,000	0	
⋮		0	
63 (exact)	50,000	5,000	0
63	45,000	3,000	1,000
64	41,000	1,000	1,000
65 (exact)	39,000	39,000	

Alice is 55 and earned \$100,000 during [54, 55]. She has 15 years of service.

Bob is 63 and declines to retire at exact age 63.0. He earned \$200,000 in [62, 63], has 25 years of service.

They are enrolled in a final salary plan with accrual rate 2.5%. Use $s_y = 1.04^y$ and $i = 5\%$.

We want to find AL and NC for Alice and for Bob under each method.

Annuities valued using $\ddot{a}_{63}^{(12)} = 13.9$; $\ddot{a}_{63.5}^{(12)} = 13.8$; $\ddot{a}_{64.5}^{(12)} = 13.6$; $\ddot{a}_{65}^{(12)} = 13.5$.

Alice, PUC Alice is 55 and earned \$100,000 during [54, 55]; she has 15 years of service.

x	sal[x-1, x]	pct raise	l_x	Years Service		i:	0.05		
55	100000	0.04	60000	15		v:	0.952381		
		Use 0 ↑ for TUC							
	Base Sal	accrual rate	years service	v^n	# ret	/ l_x	a-dots(12)		Product
63	136856.905	0.025	15	0.676839362	5000	60000	13.9		40,236.22
63.5	139567.2059	0.025	15	0.660527583	3000	60000	13.8		23,853.64
64.5	145149.8941	0.025	15	0.629073888	1000	60000	13.6		7,761.35
65	148024.4285	0.025	15	0.613913254	39000	60000	13.5		299,032.78
								AL:	370,883.99

Alice PUC

$$AL_{55} = \left(\begin{smallmatrix}\text{Age 63}\\ \text{term}\end{smallmatrix}\right) + \underbrace{\frac{3000}{60000} v^{\overset{63.5-55}{8.5}} \times \left(100000(1.04)^{8.5} \times .025 \times 15\right) \ddot{a}^{(12)}_{63.5}}_{\text{Fund accrued by 55 } EPV_{55}[\text{age 63.5 ret ben}]} + \left(\begin{smallmatrix}\text{age 64.5,}\\ \text{65 terms}\end{smallmatrix}\right)$$

Think:

$$AL_{56} = \left(\begin{smallmatrix}\text{Age 63}\\ \text{term}\end{smallmatrix}\right) + \frac{3000}{l_{56}} v^{\overset{63.5-56}{7.5}} \left(\underbrace{100000 \times 1.04}_{Sal[55,56]} (1.04)^{\overset{63.5-56}{7.5}} \times .025 \times 16\right) \ddot{a}^{(12)}_{63.5} + \left(\begin{smallmatrix}\text{age 64.5,}\\ \text{65 terms}\end{smallmatrix}\right)$$

rets w

$$AL_{55} + NC_{55} = EPV_{55}\left[\begin{smallmatrix}\text{partial yr ben}\\ \text{accruals during}\\ [55,56)\end{smallmatrix}\right] + EPV_{55}\underbrace{\left[\begin{smallmatrix}AL_{56} \text{ for those}\\ \text{who continue in DB plan}\end{smallmatrix}\right]}_{\underbrace{p^{00}_{55}}_{\frac{l_{56}}{l_{55}}} \cdot v \cdot \left[(\ldots) + \frac{3000}{l_{56}} v^{7.5} (\ldots) \ddot{a}^{(12)}_{63.5} + (\ldots)\right]}$$

$60000 \rightarrow l_{55}$

Alice PUC conclusion: $EPV\left[AL_{(56,\ 16 \text{ yrs svc})}\right] = \frac{16}{15} \cdot AL_{(55,\ 15 \text{ yrs svc.})}$

$AL_{(55,\ 15 \text{ yrs svc})} \longrightarrow NC_{55} = \frac{1}{15} AL_{(55,\ 15 \text{ yrs svc.})}$

view as "of 15 pieces"

Make one more piece in anticipation of accrual of add'l liabilities

Alice, TUC Alice is 55 and earned $100,000 during [54, 55]; she has 15 years of service.

x	sal[x-1, x]	pct raise	l_x	Years Service		i:	0.05		
55	100000	0	60000	15		v:	0.952381		
		Use 0 ↑ for TUC							
	Base Sal	accrual rate	years service	v^n	# ret	/ l_x	a-dots(12)		Product
63	100000	0.025	15	0.676839362	5000	60000	13.9		29,400.21
63.5	100000	0.025	15	0.660527583	3000	60000	13.8		17,091.15
64.5	100000	0.025	15	0.629073888	1000	60000	13.6		5,347.13
65	100000	0.025	15	0.613913254	39000	60000	13.5		202,015.83
								AL:	253,854.32

Alice TUC

$$AL_{55} = \left(\text{Age 63 term}\right) + \overbrace{\frac{3000}{60000}\, v^{\overset{(63.5-55)}{8.5}} \big(\underbrace{100{,}000}_{Sal[54,55]} \times .025 \times 15\big)\, \ddot{a}_{63.5}}^{\text{Fund accrued-by-55 } EPV_{55}[\text{age 63.5 ret. ben}]}$$

$$+ \left(\text{age 64.5, 65 terms}\right)$$

Think!

$$AL_{56} = \left(\text{Age 63 term}\right) + \frac{3000}{\ell_{56}}\, v^{\overset{63.5-56}{7.5}} \big(\underbrace{100{,}000 \times 1.04}_{Sal[55,56]} \times .025 \times 16\big)\, \ddot{a}_{63.5}$$

$$+ \left(\text{age 64.5, 65 terms}\right)$$

$$+ \left(\text{age 64.5, 65 } \cancel{\text{terms}}\right)$$

Alice TUC conclusions: $EPV_{55}\left[AL_{(56,\ 16 \text{ yrs svc})}\right] = \frac{16}{15} \times \underbrace{\frac{Sal[55,56]}{Sal[54,55]}}_{\text{"uprate salary"}} AL_{(55,\ 15 \text{ yrs})}$

AL55, 15yrs has 15 pieces. Need 1 more piece + salary uprate

$$AL_{55} + NC_{55} = \left(\text{There are no partial yr accruals}\right) + \underbrace{\frac{16}{15} \times \frac{Sal[55,56]}{Sal[54,55]} \times AL_{(55)}}_{EPV_{55}(AL_{56},\ \text{if remain in DB plan})}$$

$$NC_{55} = (\text{solve eqn.}) = \left[\left(\frac{16}{15} \times \frac{Sal[55,56]}{Sal[54,55]}\right) - 1\right] AL_{(55,\ 15 \text{ yrs})}$$

Bob, PUC Bob is 63 and declines to retire at exact age 63.0.
He earned $200,000 in [62, 63], has 25 years of service.

63 y.o. who didn't retire

x	sal[x-1, x]	pct raise	l_x	Years Service		i:	0.05		
63	200000	0.04	~~50000~~	25		v:	0.952381		
		Use 0 ↑ for TUC	45K						
	Base Sal	accrual rate	years service	v^n	# ret	/ l_x	a-dots(12)		Product
63.5	203960.7805	0.025	25	0.975900073	3000	50000	13.8		103,005.96
64.5	212119.2118	0.025	25	0.929428641	1000	50000	13.6		33,515.44
65	216320	0.025	25	0.907029478	39000	50000	13.5		1,291,297.96
								AL:	1,427,819.37

$$AL_{63} = \frac{3000}{\cancel{50000}} v^{.5} \times \left(200{,}000(1.04)^{.5} \times .025 \times 25\right) \ddot{a}_{63.5}$$
$$+$$
$$\frac{1000}{\cancel{50000}} v^{1.5} \left(200{,}000(1.04)^{1.5} \times .025 \times 25\right) \ddot{a}_{64.5}$$
$$+$$
$$\frac{39000}{\cancel{50000}} v^{2} \left(200{,}000(1.04)^{2} \times .025 \times 25\right) \ddot{a}_{65}$$

45K

Think

$$AL_{64} = (\text{No age } 63.5 \text{ term})$$
$$+$$
$$\frac{1000}{l_{64}} v^{.5} \left(200{,}000(1.04)^{.5} \times .025 \times 26\right) \ddot{a}_{64.5}$$
$$+$$
$$\frac{39000}{l_{64}} v^{1} \left(200{,}000(1.04)^{1} \times .025 \times 26\right) \ddot{a}_{65}$$

$$EPV_{63}\left(AL_{64} \text{ if remain in plan}\right) = \frac{l_{64}}{l_{63}} \cdot v \cdot AL_{64}$$

Has Same age 64.5 + 65 terms as AL_{63} except 26 yrs svc.

To get NC_{63}:
Start with
2.

Therefore

$$AL_{63} + NC_{63} = \frac{26}{25} AL_{63} - \left(\text{correction for } \tfrac{1}{2} \text{ yr accruals, ret's}\right)$$

!!!

over compensate those who leave @ 63.5

Take away extra compensation

⇒ *ct'd*

(Bob, PUC)

% people who retire

↓ @ 63.5

gave ½ year too much service

↓

$$\text{So } NC_{63} = \underbrace{\frac{1}{25} AL_{63}}_{\substack{\text{This term gives a "full piece"} \\ \text{of } AL_{63} \text{ to 63.5-retirees,} \\ \text{but they only accrue a "half piece"}}} - \underbrace{\frac{r_{63}}{l_{63}} v^{\frac{1}{2}} \times 100000(1.04)^{.5}(.025)(.5)\ddot{a}_{63.5}}_{\text{So remove the "inappropriate half piece" for the 63.5-retirees}}$$

Bob, TUC Bob is 63 and declines to retire at exact age 63.0.
He earned \$200,000 in [62, 63], has 25 years of service.

x	sal[x-1, x]	pct raise	l_x	Years Service		i:	0.05		
63	200000	0	~~50000~~	25		v:	0.952381		
		Use 0 ↑ for TUC	45K						
	Base Sal	accrual rate	years service	v^n	# ret	/ l_x	a-dots(12)		Product
63.5	200000	0.025	25	0.975900073	3000	50000	13.8		101,005.66
64.5	200000	0.025	25	0.929428641	1000	50000	13.6		31,600.57
65	200000	0.025	25	0.907029478	39000	50000	13.5		1,193,877.55
								AL:	1,326,483.78

The easiest way to handle the NC for Bob under the TUC method is to work directly with the terms of the equation that defines NC. There are other ways of breaking down sources of NC for Bob under TUC, but I would not call them "shortcuts".

$$AL_{63} = \frac{3000}{\cancel{50000}} v^{.5} * (200000 \times .025 \times 25)\,\ddot{a}_{63.5}$$

$$+ \frac{1000}{\cancel{50000}} v^{1.5} * (200000 \times .025 \times 25)\,\ddot{a}_{64.5}$$

45000

$$+ \frac{39000}{\cancel{50000}} v^{2} (200000 \times .025 \times 25)\,\ddot{a}_{65}$$

$$AL_{64} = (\text{No } 63.5 \text{ term}) + \frac{1000}{l_{64}} v^{.5} \times (200000(1.04) \times .025 \times 26)\,\ddot{a}_{64.5} + \frac{39000}{l_{64}} v^{1} \times (200000(1.04) \times .025 \times 26)\,\ddot{a}_{65}$$

$\Downarrow$ Note $\;{}_1p_{63}^{00} \times v \times \frac{l^*}{l_{64}} \times v^{n-1} = \Pr[63 \text{ retires @ } *] \cdot v^n$

$$EPV_{63}(AL_{64}) = \frac{1000}{\cancel{50000}\ l_{63}} v^{1.5} (200000(1.04) \times .025 \times 26)\,\ddot{a}_{64.5}$$

45000

$$+ \frac{39000}{\cancel{50000}} v^{2} (200000(1.04) \times .025 \times 26)\,\ddot{a}_{65}$$

45000

Correct

$$EPV_{63}\left(\begin{matrix}\text{accrued value} \\ \text{mid-yr } [63,64) \\ \text{ret's}\end{matrix}\right) = \frac{3000}{\cancel{50000}} v^{.5} (200000(1.04)^{.5} \times .025 \times 25.5)\,\ddot{a}_{63.5}$$

45000

Soln

$$AL_{63} + NC_{63} = EPV_{63}\left(\begin{matrix}\text{mid-yr} \\ \text{ret's} \\ \text{in } [63,64)\end{matrix}\right) + EPV_{63}\left[\begin{matrix}AL_{64} \\ \text{if rem.} \\ \text{in plan}\end{matrix}\right]$$

to get NC_{63}.

No correction Factor

Practice Problem: Actuarial Liability and Normal Cost for DB Plans

Consider the following excerpt from a pension service table.

x	ℓ_x	r_x	$d_x + i_x + w_x$
55	60,000	0	
⋮		0	
64 (exact)	50,000	5,000	0
64	45,000	4,000	1,000
65 (exact)	40,000	40,000	

The symbols r_x for 64 (exact) and 65 (exact) denote expected numbers of age-retirements at exact ages 64 & 65.

For x = 64 (without the "exact" notation), r_x denotes the expected number of retirements during the year, which are assumed to occur at age 64.5.

Data: (i) A 55-year old employee has a final-salary-style defined benefit pension plan allowing age-retirement only at exact ages 64, 64.5, or 65.

(ii) She earns 150,000 during [54, 55].

(iii) She has accumulated 25 years of pensionable service by age 55.

(iv) The plan has an accrual rate of 2.5%, that is, the annual pension benefit is defined to be 2.5% of final salary per year of service. There is no penalty for retiring at age 64 or 64.5. Pension benefits are paid monthly in advance.

(v) Future salaries are projected using the salary scale $s_y = 1.04^y$.

(vi) The EPV of annuities due payable monthly at a rate of 1 per year at various ages are $\ddot{a}_{64}^{(12)} = 13.9$; $\ddot{a}_{64.5}^{(12)} = 13.7$; $\ddot{a}_{65}^{(12)} = 13.5$.

(vii) Use $i = .05$.

1. Compute the actuarial liability and normal cost at age 55 using the **projected unit credit** approach (that is, base your calculation on the appropriate projected final salaries and the number of years of pensionable service actually completed by age 55.)

$$AL_{55} = \frac{5000}{60000} v^9 \left(0.025 \cdot 25 \cdot 150000 \cdot 1.04^9\right) \ddot{a}^{(12)}_{64}$$
$$+ \frac{4000}{60000} v^{9.5} \left(0.025 \cdot 25 \cdot 150000 \cdot 1.04^{9.5}\right) \ddot{a}^{(12)}_{64.5}$$
$$+ \frac{40000}{60000} v^{10} \left(0.025 \cdot 25 \cdot 150000 \cdot 1.04^{10}\right) \ddot{a}^{(12)}_{65}$$
$$= 944567.4447$$

2. Compute the actuarial liability and normal cost at age 55 using the **traditional unit credit** approach (that is, base your calculation on the most recent salary and the number of years of pensionable service actually completed by age 55.) (For this problem, you can say AL_{55} is "same as in #1 except ..."; do compute the values though.)

$$AL_{55} = \frac{5}{60} v^9 \left(0.025 \cdot 25 \cdot 150000\right) \ddot{a}^{(12)}_{64}$$
$$+ \frac{4}{60} v^{9.5} \left(0.025 \cdot 25 \cdot 150000\right) \ddot{a}^{(12)}_{64.5}$$
$$+ \frac{4}{6} v^{10} \left(0.025 \cdot 25 \cdot 150000\right) \ddot{a}^{(12)}_{65} = 641854.2589$$

3. Repeat #1 **except** change (i), (ii), (iii) as follows: Assume you're looking at a 64-year-old employee who had 34 years of service and earned 200,000 during [63, 64]; this particular employee chose not to retire at age 64.0.

$$AL_{64} = \frac{4}{45} \cdot v^{0.5} \cdot 0.025 \cdot 34 \cdot 200000 \cdot 1.04^{.5} \ddot{a}^{(12)}_{64.5}$$
$$+ \frac{40}{45} v^1 \cdot 0.025 \cdot 34 \cdot 200000 \cdot 1.04^1 \ddot{a}^{(12)}_{65}$$
$$= 2226605.472$$

4. Repeat #2 **except** assume you're looking at a 64-year-old employee who had 34 years of service and earned 200,000 during [63, 64]; this particular employee chose not to retire at age 64.0.

$$AL_{64} = \frac{4}{45} \cdot v^{0.5} \cdot 0.025 \cdot 34 \cdot 200000 \cdot 13.7$$
$$+ \frac{40}{45} v \cdot 0.025 \cdot 34 \cdot 200000 \cdot 13.5$$
$$= 2144890.145$$

Suggested Sample ALTAM Problem (Canvas): 42ab, 45

Solutions

#1 AL:

$$\left.\begin{array}{l} \frac{5000}{60000} \times v^{9} \times 150000(1.04)^{9} \times (.025 \times 25)\, \underbrace{\ddot{a}^{(12)}_{64}}_{13.9} \\ \qquad + \\ \frac{4000}{60000} \times v^{9.5} \times 150000\,(1.04)^{9.5}(.025 \times 25)\, \underbrace{\ddot{a}^{(12)}_{64.5}}_{13.7} \\ \qquad + \\ \frac{40000}{60000} \times v^{10} \times 150000\,(1.04)^{10}\,(.025 \times 25)\, \underbrace{\ddot{a}^{(12)}_{65}}_{13.5} \end{array}\right\} \boxed{= 944567.44}$$

#2 AL:

same as (1) but delete $(1.04)^n$ factors $\rightarrow$ $\boxed{= 641845.26}$

#3 & 4

(3) PUC

$$AL_{64} = \frac{4000}{45{,}000}\times v^{.5}\left(200{,}000\times(1.04)^{.5}\right)(.025\times 34)\underbrace{\ddot{a}^{(12)}_{64.5}}_{13.7} + \frac{40000}{45000}\times v^{1}\left(200{,}000\times(1.04)^{1.0}(.025\times 34)\right)\underbrace{\ddot{a}^{(12)}_{65}}_{13.5} \Bigg\}\ 2{,}226{,}605.47$$

Direct method:

$$NC_{64} = \frac{4000}{45{,}000}\times v^{.5}\left(200{,}000\times(1.04)^{.5}(.025\times \underbrace{.5}_{34.5-34})\right)\ddot{a}^{(12)}_{64.5} + \frac{40000}{45{,}000}\times v^{1}\left(200{,}000\times(1.04)^{1}(.025\times \overset{35-34}{1})\right)\ddot{a}^{(12)}_{65} \Bigg\}\ \begin{matrix}3029.917\\ +\,59428.57\\ \hline \boxed{62{,}458.91}\end{matrix}$$

NC_{64}, Indirect Method: Think of AL_{64} as if it has 34 segments. NC = one more (*) segment but remove [EPV of the extra "$.5\times.025\times$ Sal"] (*) that should not be included for age 64.5-ret's.

(*) By computing $\frac{1}{34}\times AL_{64}$

(*) For the complementary part [64.5, 65] not worked

(4) TUC

AL_{64} = (same as (3)) but delete $(1.04)^n$ factors

Get $AL_{64} = 2{,}144{,}890.14$

(↑ use "direct method" as in #3 -or- compute EPV_{64} of 64.5-retirements + $EPV_{64}[AL_{65}]$ and solve $AL_{64} + NC_{64} = EPV_{64}[\text{64.5-retirements}] + EPV_{64}[AL_{65}]$ for NC. If you imagine the algebra involved, you could factor and get the expression in (3)—with × 1.04 deleted—for NC.)

<u>DHW3e 11.11</u> – Retiree Health Benefits

<u>Example</u>: WilsonCorp provides its retirees with a supplemental health insurance policy to cover some of the expenses not covered by public government benefits (such as Medicare).

Consider two employees, ages 50 and 60.

(x, t_0)

$B(60,0)$ ← Value of benefits for (x) @ $t = t_0$

<u>Assumptions</u>:

- Today, at $t = 0$ (the "valuation date"), the premium for one year of coverage for a 60-year-old is 5000.
- At any time t, the premium to cover someone age $(60 + s)$ for one year is

$$(1.02)^s \times (\text{premium to cover someone age } 60),$$

i.e. @ time today, cost to insure $(x+1)$ is $1.02\cdot(\text{cost of } x)$

  that is, the cost to cover a person older than 60 for a year is 2% higher per year of age beyond 60.
- Health insurance inflation is 5% per year.
- The interest rate is $i = 6\%$.
- Mortality follows the DHW life table, and we can compute <u>level</u> annuity EPVs (e.g. $\ddot{a}_x$) at any interest rate.
- Retirements follow the service table from DHW, so that if we have ℓ_{50} and ℓ_{60}^{exact} employees, we expect $r_{60}^{exact}, \underbrace{r_{60}, r_{61}, \dots, r_{64}}_{\text{mid-year exits}}, r_{65}^{exact}$ retirements.

a. Consider the 50-year-old employee. Assuming an age 60.0 retirement for (50) (which would occur with probability r_{60}^{E}/ℓ_{50}), compute the PV at the valuation date $t = 0$ (age 50) of the cost at age 60.0 to purchase the benefits.

$$5000\underbrace{(1.05)^{10}}_{\text{health ins inflation}}\underbrace{(1.06)^{-10}}_{\text{cash flow discounting}} + p_{60}\cdot 5000\cdot\underbrace{(1.02)}_{\text{1 year older}}\underbrace{(1.05)^{11}}_{\text{+1 yr health ins inf}}\underbrace{(1.06)^{-11}}_{\text{+1 yr cf disc}} + {}_2p_{60}\cdot 5000\cdot\underbrace{1.02^2}_{\text{2 yr older}}\cdot\underbrace{1.05^{12}}_{\text{+2 yr health ins inf}}\cdot\underbrace{1.06^{-12}}_{\text{+2 yr cf disc}} + \dots$$

$B(60,10)$

$$= 5000\left(\frac{1.05}{1.06}\right)^{10}\left[1 + p_{60}\underbrace{\left(\frac{1.02\cdot 1.05}{1.06}\right)}_{v_{i^*}} + {}_2p_{60}\underbrace{\left(\frac{1.02\cdot 1.05}{1.06}\right)^2}_{v_{i^*}^2} + \dots\right]$$

v_{i^*} ← "new interest rate" → $v_{i^*}^2$

Plug into Excel, solve using modified life table

$$= 5000\left(\frac{1.05}{1.06}\right)^{10}\cdot \ddot{a}_{60@i^*} = 5000\left(\frac{1.05}{1.06}\right)^{10}(32.5209) = \$147{,}900$$

(to nearest \$)

Find i^* $\quad v_{i^*} = \dfrac{1.02\cdot 1.05}{1.06}$

$$1 + i = \frac{1.06}{1.02\cdot 1.05} \rightarrow i = \frac{1.06}{1.02\cdot 1.05} - 1 \approx -0.01027$$

(Costs increase by 2% for each add'l year of age; health insurance inflation 5%, i = 6%)

b. Repeat (a) except assuming an age 60.5 retirement for (50) (which would occur with probability r_{60}/l_{50}).

$$PV_{t=0}[\ ] = 5000(1.05)^{10.5}\underbrace{(1.02)^{0.5}}_{\frac{1}{2}\text{ yr older than 60}}(1.06)^{-10.5} + p_{60.5}\cdot 5000\cdot 1.05^{11.5}\,1.02^{1.5}\,1.06^{-11.5} + \cdots$$

$$= 5000\left(\frac{1.05}{1.06}\right)^{10.5}(1.02)^{0.5}\left[1 + p_{60.5}\,v_{i^*} + {}_2p_{60.5}\,v_{i^*}^2 + \cdots\right]$$

Some form of linear interpolation / UDD / IDK for non-integer ages

$$= 5000\left(\frac{1.05}{1.06}\right)^{10.5}(1.02)^{0.5}\,\ddot{a}_{60.5@i^*} = 5000\left(\frac{1.05}{1.06}\right)^{10.5}(1.02)^{0.5}(31.9097) = 145871$$

$$B(60.5, 10.5) = 5000(1.02)^{0.5}(1.05)^{10.5} \qquad i^* = \frac{1.06}{1.05\cdot 1.02} - 1 \quad \text{(Same as before)}$$

c. Calculate the **actuarial value of total health benefits** (AVTHB) per 50-year-old employee at the valuation date t = 0.

$$AVTHB = \frac{\overset{\text{Exact}}{r_{60}}}{l_{50}}(147900) + \frac{\overset{\text{midyear}}{r_{60}}}{l_{50}}(145871) + \frac{r_{61}}{l_{50}}(\#) + \cdots + \frac{\overset{\text{exact}}{r_{65}}}{l_{50}}(\#)$$

$$= 107169$$

Notation from the SOA study note $\boxed{\text{xx}}$ ← what Dr. Wilson thinks

If @ $t=0$, (x) is 50, then EPV of ben's, if they start @ $t=10$ age 60 is

$$5000(1.05)^{10}(1.06)^{-10} + p_{60} \cdot 5000\underset{\text{infl.}}{(1.05)^{11}}\underset{\text{age cost}}{(1.02)}\underset{v^{11}}{(1.06)^{-11}}$$

$$+ {}_2p_{60} \cdot 5000(1.05)^{12}(1.02)^2(1.06)^{-12}$$

$$+ \cdots$$

$=$

$$\underbrace{1.06^{-10}}_{v^{10}} \Big[\underbrace{5000(1.05)^{10}}_{B(60,10)} + p_{60} \cdot \overbrace{5000(1.05)^{11}(1.02)}^{B(61,11)}\overbrace{(1.06)^{-1}}^{v}$$

$$+ {}_2p_{60} \cdot \underbrace{5000(1.05)^{12}(1.02)^2}_{B(62,12)}\underbrace{(1.06)^{-2}}_{v^2} + \cdots \Big]$$

$B(x, t)$ = Cost, in t yrs, assoc. w/ purchase 1-yr ben's if age x at time t.

x = age at time t

t = # yrs from now/from valuation date

$=$

$$1.06^{-10} \times 5000(1.05)^{10} \times \overbrace{\Big[1 + p_{60}(1.05)(1.02)(1.06)^{-1} + {}_2p_{60}(1.05)^2(1.02)^2(1.06)^{-2} + \cdots \Big]}^{\ddot{a}_B(60,10)}$$

Unnecessary notation, continued:

ret. age x_r — time until retirement

$$\ddot{a}_B(\underset{x_r}{60}, \underset{\text{time until retirement}}{10}) = EPV_{60}\left[\begin{array}{l}\text{life annuity due with first pmt \$1 and}\\ \text{pmts increasing according to}\\ \text{aging + health care inflation}\end{array}\right]$$

$$\underset{\text{(in example (a))}}{=} 1 + p_{60} \times \underbrace{\frac{\overset{\text{infl}}{1.05} \times \overset{\text{age}}{1.02}}{1.06}}_{\text{discount}} + {}_2p_{60} \times \Bigg(\underbrace{\frac{\overset{\text{infl}}{1.05} \times \overset{\text{age}}{1.02}}{1.06}}_{\text{discount}}\Bigg)^2 + \cdots$$

Assemble as follows:

Expected ben. Cost @ 60, which is in 10 yrs:

$$B(60,10)\,\ddot{a}_B(60,10)$$

$$AVTHB_{50} = \frac{\overset{\text{exact}}{r_{60}}}{l_{50}} \cdot v^{10} B(60,10)\,\ddot{a}_B(60,10) + \frac{\overset{\text{midyr}}{r_{60^+}}}{l_{50}} v^{10.5} B(60.5,10.5) \times \ddot{a}_B(60.5,10.5) + \text{etc.} \Big]$$

d. Consider the 60-year-old employee, immediately prior to exact-age 60.0 retirements. Assuming an age 60.0 retirement for (60) (which would occur with probability ___), compute the PV at the valuation date $t = 0$ (age 60) of the cost at age 60.0 to purchase the benefits.

$$5000\left[1 + p_{60} \cdot \frac{1.02\times1.05}{1.06} + {}_2p_{60} \cdot \left[\frac{1.02\times1.05}{1.06}\right]^2 + {}_3p_{60}\left[\cdots\right]^3 + \cdots\right]$$

$$= 5000\left[1 + p_{60}\, v_{i*} + {}_2p_{60}\,(v_{i*})^2 + \cdots\right] \quad \text{where } v_{i*} = (1+i^*)^{-1} = \frac{1.02\times1.05}{1.06}$$

$$= 5000\, \underbrace{\ddot{a}_{60@i*}}_{32.5209} = 162604.5$$

e. Repeat (b) (retire @ age 60.5) for our employee who is 60 years old at the valuation date $t = 0$.

$$5000\frac{(1.02)^{.5}(1.05)^{.5}}{(1.06)^{.5}} + p_{60.5} \times \frac{(1.02)^{1.5}(1.05)^{1.5}}{(1.06)^{1.5}} + {}_2p_{60.5} \times [\quad]^{2.5} + \cdots$$

$$= 5000\frac{(1.02)^{.5}(1.05)^{.5}}{(1.06)^{.5}}\Big[\underbrace{1 + p_{60.5}\Big[\frac{\cdots}{\cdots}\Big]^{1} + {}_2p_{60.5}\Big[\frac{\cdots}{\cdots}\Big]^{2} + \cdots}_{\ddot{a}_{60.5@i^*} = 31.9097}\Big]$$

$$= 160374.21$$

f. Find AVTHB for a current 60-year-old employee.

Exact

$$\frac{r_{60}}{l_{60}}(162604.5) + \frac{r_{60}}{l_{60}}(160364) + \cdots = 148279$$

Funding methods

Notation: ${}_tV^h$ = actuarial liability at time t for an employee,
${}_tC^h$ = normal contribution for the year at time t

$$ {}_tV^h + C_t^h = \text{EPV of benefits for mid-year exits } + v\ {}_1p_x^{(\tau)}\ {}_{t+1}V^h$$

Health benefits cost what they cost, regardless of # years of service; so that's a little different from DB actuarial liability, where the benefit level increases in compensation for additional service in employment.

Pro rata method

Example: Use the pro-rata method with the following assumptions to calculate the accrued liability and normal contribution for (50) and for (60).

Assumptions:
- For this example, we make the simplifying (and more conservative) assumption that all retirements will occur at age 60.
- As an example, one could pro-rate by assuming the accumulated cost is spread over the 25-year period prior to age 60. (This includes employees who join the company within that 25-year period).

50 yo

$${}_0V = \frac{15}{25}AVTHB_{50} = 64301$$

$${}_1V = \frac{16}{25}AVTHB_{50}$$

$$\hookrightarrow NC_{t=0} = \frac{1}{25}AVTHB_{50} = \frac{1}{15}\,{}_0V$$

60 yo

$${}_0V = \frac{25}{25}AVTHB_{60} = 148279$$

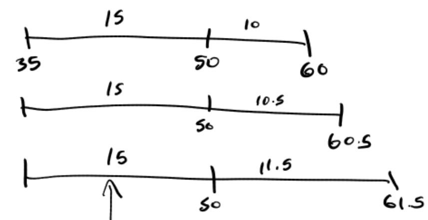
$NC = 0$ (fully funded)

<u>Linear accrual to each retirement age method.</u>

Retain the assumption that accrual over a period that begins 25 years prior to age 60.
Drop the assumption that all retirements occur by age 60, and use the service table.

Assuming (50) retires at 60.0, EPV_{50}(premiums) = 147,900
Assuming (50) retires at 60.5, EPV_{50}(premiums) = 145,871
Assuming (50) retires at 61.5, EPV_{50}(premiums) = …
…
Assuming (60) retires at 60.0, EPV_{60}(premiums) = 162,604
Assuming (60) retires at 60.5, EPV_{60}(premiums) = 160,374
Assuming (60) retires at 61.5, EPV_{60}(premiums) = …

AVTHB in different retirement scenarios



50 yo

$$ {}_0V = \frac{r_{60}^{Ex}}{\ell_{50}} \cdot \frac{15}{25}(147{,}900) + \frac{r_{60}}{\ell_{50}}\frac{15}{25.5}(145871) + \dots = 58677 $$

$$ NC_0 = \left(\text{one more piece}\right)\left(\text{View } {}_0V \text{ as 15 pieces}\right) + \left(\cancel{\text{mid year accum. bens}}\right) = \frac{1}{15}\left(58677\right) $$

60 yo

$$ {}_0V = \frac{r_{60}^{E}}{\ell_{60}}\frac{25}{25}\left(162\,604\right) + \frac{r_{60}}{\ell_{60}}\frac{25}{25.5}\left(160374\right) + \frac{r_{61}}{\ell_{60}}\frac{25}{26.5}\left(\dots\right) + \dots = \# $$

$$ NC_{60} = 0 + \frac{r_{60}}{\ell_{60}}\frac{0.5}{25.5}\left(160374\right) + \frac{1}{26.5}\frac{r_{61}}{\ell_{60}}\left(AVTHB\right)_{\text{ret @ 61.5}} + \frac{1}{27.5}\frac{r_{62}}{\ell_{60}}\left(AVTHB\right)_{\text{ret @ 62.5}} + \dots $$

r_{60}^{Ex} is already fully funded

DHW 3e: 13.1-13.2 – Profit Testing

Want to predict future profits on a policy and test profitability.

<u>Example</u>: Fully discrete 10-year term policy of $100,000 on (60) with level annual (gross) premiums $\pi = \pi^G = \$1500$. *(DHW uses P here.)*

Assumption set

<u>Profit test basis</u>: ← Used for all interest accumulations (e.g. projecting profits) <u>except</u> for determining amount of each benefit reserve.

Interest: $i = .055$
Initial expenses: $700 ("acquisition costs")
Renewal expenses: 3.5% of premiums *(including the first premium – not part of the $700.)*
Survival: $q_{60+t} = .01 + .001t$ for $t = 0, 1, \ldots, 9$
So $q_{60} = .01,\ q_{61} = .011,\ q_{62} = .012, \ldots$

Suppose that the insurer chooses to use net premium reserves as the reserving method. the following assumptions are used for reserving:

<u>Reserve basis</u>: ← Typically uses more conservative assumptions, so lower i and greater q_{x+t}'s

Interest: $i = .04$
Survival: $q_{60+t} = .011 + .001t$ for $t = 0, 1, \ldots, 9$
So $q_{60} = .011,\ q_{61} = .012,\ q_{62} = .013, \ldots$

1. Recall that the <u>net premium</u> reserve excludes consideration of expenses and profit and that it is based on the <u>net</u> premium (which forms a portion of the gross premium). Compute the net premium P *(DHW: P′)* that we will use to determine the benefit reserve.

EPV[Prems] = EPV[Bens]

$$\pi^n\left[1 + p_{60} v_{0.04} + {}_2p_{60} v^2_{0.04} + \ldots + {}_9p_{60} v^9_{0.04}\right] = 100000\left[q_{60} v_{0.04} + {}_{1|}q_{60} v^2_{0.04} + \ldots + {}_{9|}q_{60} v^9_{0.04}\right] \rightsquigarrow \pi^n = 1447.63$$

p_{60}: 1-0.011
${}_2p_{60}$: (1-0.011)(1-0.012)
${}_9p_{60}$: (1-0.011)(1-0.012)·…·(1-0.019)
q_{60}: 0.011
${}_{1|}q_{60}$: (1-0.011)0.012
${}_{9|}q_{60}$: 9 slash q 60 (1-0.011)(1-0.012)·…·(1-0.018) 0.019

2. Determine the net premium reserve for $t = 0, 1, 2$.

${}_0V^n = {}_0V = 0$ (b/c equivalence principle)

2 ways to proceed

1) Def'n of ${}_tV^n = 100000\, A^{\,1}_{60+t:\overline{10-t}|} - \pi^n \ddot{a}_{60+t}$
using reserve basis

2) Recursive Computation

$$(\underbrace{{}_0V}_{0} + \underbrace{\pi^n}_{1447.63})(\underbrace{1+i_n}_{1.04}) = \underbrace{p_{60}}_{0.089}\,{}_1V^n + \underbrace{q_{60}}_{0.011}\cdot 100000 \rightsquigarrow {}_1V^n = 410.05$$

$$(\underbrace{{}_1V}_{410.05} + \underbrace{\pi^n}_{1447.63})(\underbrace{1+i_n}_{1.04}) = p_{61}\,{}_2V^n + q_{61}\cdot 100000 \rightsquigarrow {}_2V^n = 740.88$$

t	${}_tV$	t	${}_tV$
0	0.00	5	1219.94
1	410.05	6	1193.37
2	740.88	7	1064.74
3	988.90	8	827.76
4	1150.10	9	475.45

(DHW 2e Table 12.2, p. 404)

Main idea: Profit/surplus Pr_t per policy emerging at end of year t (at time t), given that the policy is in force at time $(t-1)$, comes from…

(1) The previous year's premium payment π and benefit reserve (${}_{t\text{-}1}V$), less renewal expenses associated with that particular premium payment…

Note: DHW uses the very standard convention that acquisition costs amount to a negative profit that emerges at time $t = 0$; these are not subtracted from the premium (paid at time 0) that contributes to the profit at time $t = 1$.

"[P]rudent capital management requires us to recognize losses as early as possible…It would not generally be prudent to combine the high acquisition costs with the other first year income and outgo, as that would delay recognition of those expenses and lessen their impact" (DHW 2e p. 400).

(2) …plus interest accrued on (1)

(3) …minus expected benefits (and any associated costs) and the expected cost of setting up the next reserve at time t.

*Note carefully the conditioning implied here: The profit values are conditional on (60) being alive at time $t - 1$.

3. Compute the profit/surplus per policy emerging at $t = 0, 1, 2, 3$, given that the policy is in force at $t - 1$ for each $t = 1, 2, 3$. *For this example, the insurer/DHW has decided to set reserves equal to the net benefit reserves (other reserving choices are possible, cf. SOA Written #16a.)*

Note that we use the profit test basis/assumptions here. The reserve basis is used only for determining the amount of benefit reserves.

$t=n\text{-}1$ —— $t=1$

- Have ${}_{n-1}V$
- recieve π^G
- pay expenses @ start of year

grow @ i%

$E\left[\begin{array}{l}\to \text{pay d.b.}\\ \to \text{reserve for } {}_nV\end{array}\right] \longleftarrow$ What \$ remains is called "Pr_n" (because that's not confusing!) (profit @ t=n)

$\text{Pr}_0 = -700$ (acqusition expense)

profit basis assumptions

$\text{Pr}_1 = \big(\underbrace{{}_0V}_{0} + \underbrace{\pi^G}_{1500} - 0.035(\pi^G)\big)(1.05) - \underbrace{q_{60}}_{0.01}\underbrace{(100000)}_{\text{death ben}} - \underbrace{p_{60}}_{0.99}\underbrace{({}_1V)}_{410.05} = 121.17$

$\text{Pr}_2 = (\underbrace{{}_1V}_{410.05} + \pi^G - 0.035\pi^G)(1.05) - q_{61}(100000) - p_{61}(\underbrace{{}_2V}_{740.88}) = 126.99$

$\text{Pr}_3 = ({}_2V + \pi^G - 0.035\pi^G)(1.05) - q_{62}(100000) - p_{62}({}_3V) = 131.70$

given in table @ top of page

Pr_n assumes x alive @ $(n-1)$

t (0)	$_{t-1}V$ (2)	P (3) Premium	E_t (4) Expenses	I_t (5)	EDB_t (6) Exp. Val Death Ben	E_tV (7)	Pr_t (8) Profit Vector
0			700.00				−700.00
1	0.00	1500	52.50	82.50	1000	405.95	121.17
2	410.05	1500	52.50	102.17	1100	732.73	126.99
3	740.88	1500	52.50	120.36	1200	977.04	131.70
4	988.90	1500	52.50	134.00	1300	1135.15	135.26
5	1150.10	1500	52.50	142.87	1400	1202.86	137.61
6	1219.94	1500	52.50	146.71	1500	1175.47	138.68
7	1193.37	1500	52.50	145.25	1600	1047.70	138.41
8	1064.74	1500	52.50	138.17	1700	813.69	136.72
9	827.76	1500	52.50	125.14	1800	466.89	133.52
10	475.45	1500	52.50	105.76	1900	0.00	128.71

DHW 2e Table 12.3, p. 405 (modified from 1e: Table 11.3, page 357)

<u>Definition</u>: The vector $(Pr_0, Pr_1, Pr_2, \ldots, Pr_{10})$ of emerging profits Pr_t per policy during the year $(t-1, t)$, ***given that the policy is in force at time t – 1*** is called the **<u>profit vector</u>**.

<u>Definition</u>: The vector $(\Pi_0, \Pi_1, \Pi_2, \ldots, \Pi_{10})$ of expected profits Π_t per policy emerging during the year $(t-1, t)$, ***given <u>only</u> that the policy is in force at time 0*** is called the **<u>profit signature</u>**.

For $t = 1, \ldots, 10$, we have Π_t = <u>see below</u>.

4. Write down the profit vector and profit signature for the contract in our example. (Recall $q_{60} = .01,\ q_{61} = .011,\ q_{62} = .012, \ldots$ under profit test assumptions.)

Capital pi's

$\Pi_0 = -700$

$\Pi_1 = 121.17$

$\Pi_2 = 126.99 \cdot p_{60} = 125.72$

$\Pi_3 = 131.70 \cdot {}_2p_{60} = 128.95$

$\Pi_4 = Pr_4 \cdot {}_3p_{60} = 130.84$

underneath the hood

$E[\text{Profit}(61,62)] = E\left[E[\text{profit}(61,62) \mid \text{alive or dead}]\right]$

$= p_{60} Pr_{62} + 0$

dead case

ditto

<u>Definition</u>: The <u>expected value of future profit</u> or <u>net present value</u> (NPV) at rate r of the contract is equal to $\sum \Pi_t (v_r)^t$, where $v_r = (1 + r)^{-1}$.

5. If profits are discounted using an effective annual risk discount rate of $r = .1$, compute the NPV of the contract in our example. (74.13).

If we could earn 5.5% just by stuffing money into a bank, why would we take any risk? Need to have higher return to justify

$$NPV_{r=0.10} = -700 + \Pi_1 v_r + \Pi_2 v_r^2 + \Pi_3 v_r^3 + \ldots + \Pi_9 v_r^9$$

$$= 72.13$$

Finally, DHW begins this entire discussion by looking at the profit vector if ${}_tV = 0$ for every t, i.e. if no reserving is done at all.

6. Find the profit vector if we set ${}_tV = 0$ for every t. *n.b. This is a terrible idea!!*

(Recall $i = .055$, $q_{60} = .01$, $q_{62} = .011$, $q_{63} = .012, \ldots$ under profit test assumptions; G = 1500; acq. cost = 700; renewal expenses = 3.5% of premiums.)

Get (-700, 527.11, 427.11, 327.11, 227.11, 127.11, 27.11, -72.89, -172.89, -272.89, -372.89).

Additional examples:

***289.** For a 3-year term insurance of 1,000,000 on (60), you are given:

(i) The death benefit is payable at the end of the year of death.

(ii) $q_{60+t} = 0.014 + 0.001t$

(iii) Cash flows are accumulated at annual effective rate of interest of 0.06.

(iv) The annual gross premium is 14,500.

(v) Pre-contract expenses are 1000 and are paid at time 0.

(vi) Expenses after issue are 100 payable immediately after the receipt of each gross premium.

(vii) The reserve is 700 at the end of the first and second years.

(viii) Profits are discounted at annual effective rate of interest of 0.10.

$NPV_{10\%} = \Pi_0 + \Pi_1 v + \Pi_2 v^2 + \Pi_3 v^3$
$= -215.81$

Calculate the net present value of the policy.

assume ${}_0V = 0$

${}_1V = 700$ $\qquad \pi^G = 14500$

${}_2V = 700$

Profit signature in blue

$\Pi_0 = -1000$

$Pr_0 = -1000$

$Pr_1 = (\underbrace{{}_0V}_{0} + \underbrace{\pi^G}_{14500} - 100)(1.06) - \underbrace{q_{60}}_{0.014} \cdot 1000000 - \underbrace{p_{60}}_{0.986}\,\underbrace{{}_1V}_{700} = 573.88$ $\qquad \Pi_1 = 573.88$

$Pr_2 = ({}_1V + 14500 - 100)(1.06) - \underbrace{q_{61}}_{0.015} \cdot 1000000 - \underbrace{p_{61}}_{0.985}\,\underbrace{{}_2V}_{700} = 316.5$ $\qquad \Pi_2 = 316.5\, p_{60} = 312.39$

$Pr_3 = ({}_2V + 14500 - 100)(1.06) - 0.016 \cdot 1000000 - 0.984 \cdot \overset{0}{{}_3V} = 6.0$ $\qquad \Pi_3 = 6\,{}_2p_{60} = 5.83$

Profit Vector

***290.** For a 10-year term life insurance on (60), you are given:

(i) Mortality follows the Illustrative Life Table
(ii) The independent annual lapse rate is 0.05; lapses occur at the end of the year.
(iii) The profit vector:

Time in years (t)	Profit (Pr_t)
0	-700
1	180
2	130
3	130
4	135
5	135
6	140
7	140
8	140
9	135
10	130

Hurdle Rate/Profit Rate/etc.

(iv) Profits are discounted at an annual effective rate of 0.10.

Calculate the expected present value of future profits for a policy that is still in force immediately after the 7th year end. → i.e. @ $t=7$

(From the life table: ℓ_{60} = 8,188,074 and

x	l_x
66	7,373,338
67	7,201,635
68	7,018,432
69	6,823,367
70	6,616,155

,)

$Pr_8 = 140$ $\quad$ $Pr_9 = 135$ $\quad$ $Pr_{10} = 130$

7 $\quad$ 8 $\quad$ 9 $\quad$ 10

From Perspective of $t=7$

$\Pi_8 = Pr_8$

$\Pi_9 = Pr_9 \cdot \frac{\ell_{68}}{\ell_{67}} \cdot .95$ ← p_{67} ← don't lapse

$\Pi_{10} = Pr_{10} \cdot \frac{\ell_{69}}{\ell_{67}} \cdot .95^2$ ← ${}_2p_{67}$ ← don't lapse, twice

$$NPV_{r=10\%} = 140 + \Pi_9 v + \Pi_{10} v^2 = 314.086$$

Suggested reading: Read DHW 3e 13.1-13.5, *especially* the introductory nonmathematical material.

DHW3e Sections 13.3-13.5 – Profit Measures

Recall from last time: For an n-year contract with annual cash flows, we make the following definitions:

Definition: The vector $(Pr_0, Pr_1, Pr_2, \ldots, Pr_{10})$ of (expected) emerging profits Pr_t per policy during the year $(t-1, t)$, given that the policy is in force at time $t-1$, is called the profit vector.

You automatically assume for Pr_t that (x) alive @ $(x+t-1)$

Computing Pr_t:

$$Pr_0 = -\text{acquisition cost}$$

Sum Insured

Reserves for next year

$$Pr_t = \left({}_{t-1}V + \pi^G - \text{ann. expenses}\right)(1+i) - q_{x+(t-1)}\left(\text{Death Ben}\right) - p_{x+(t-1)}\, {}_tV$$

Definition: The vector $(\Pi_0, \Pi_1, \Pi_2, \ldots, \Pi_n)$ of expected profits Π_t per policy emerging during the year $(t-1, t)$, ***given only that the policy is in force at time 0*** is called the profit signature.

For $t = 1, \ldots, n$, we have $\Pi_t = \underline{Pr_t \cdot {}_{t-1}p_x}$.

$\dfrac{l_{x+t-1}}{l_x}$

Definition: The net present value NPV (or expected present value of future profit EPVFP) at rate r of a contract is defined by

NPV = $\underline{\sum_t \Pi_t v^t}$, where $v = (1 + r)^{-1}$.

Hurdle Rate
Risk Preference Rate
Risk Discount Rate
Required Rate of Return
all synonymous

Note: We do not assume that r is the same interest rate that was used to accumulate cash flows when computing the profit vector and profit signature. Further, these rates are usually different than the rate used for the purposes of determining net premium reserves.

Definition: The internal rate of return (IRR) is the* rate r for which the NPV of a contract is equal to zero.

*Assuming that this rate exists, in which case, it may not be unique!

Polynomials in r, may have multiple (or none) real solutions

Definition: Hurdle rate (or risk discount rate): This is a rate r, chosen by the insurer, at which to compute the NPV of the contract. The contract is deemed profitable if the IRR is higher than this rate.

multiple

none

Example 7: Given the profit signature

(-700, 121.17, 125.72, 128.95, … 119.75, 113.37)

for a ten-year term insurance, find the NPV at the hurdle rate 10%. (74.13). What information does this give the insurer about the profitability of the contract?

$$-700 + 121.17(1.1)^{-1} + 125.72(1.1)^{-2} + \cdots + 113.37(1.1)^{-10} = 74.13$$

This product/project increases our overall cash flows in a way beyond our rate of return of 10%

<u>Definition</u>: Profit margin.

$$\frac{EPV_{r\%}[\text{Profits}]}{EPV_{r\%}[\text{Revenues}]} = \frac{EPV_{r\%}[\Pi_t]}{EPV_{r\%}[\pi^G]}$$

Profit margin = $\dfrac{EPV_{r\%}[\text{Profits}]}{EPV_{r\%}[\text{Revenues}]}$, evaluated using the risk discount rate*
(or discounted exactly as instructed on an SOA exam).

*DHW says "for all calculations" (2e p.412) , by which they appear to mean…

- Use the risk discount rate to turn the profit signature into an NPV
- Use the mortality model (from profit test basis) and the *risk discount rate* to determine the APV of gross premiums. (This won't have been done yet.)
- Note that the amount of the gross premium and the profit signature were determined independently of setting a hurdle rate—in this sense, we have not really used the risk discount rate for <u>all</u> calculations.

<u>Example 8</u>: Continuing the "10-year term insurance on (60)" example.

<u>Profit Testing Basis</u>

Mortality: $q_{60} = .01,\ q_{61} = .011,\ q_{62} = .012,\ q_{63} = .013, \dots\ q_{68} = .018$

Gross premium: $1500

Profit signature:
(-700, 121.17, 125.72, 128.95, 130.84, 131.39, 130.56, 128.35, 124.76, 119.75, 113.37)
$t = 0$ 1 2 3 4 5 6 7 8 9 10

Compute the profit margin at hurdle rate 10%. (Get 74.13/9684 ≈ .00765 ≈ .77%)

$$EPV_{10\%}[\text{Profits}] = NPV_{10\%} = \sum \Pi_t v^t = 74.13$$
$$EPV_{10\%}[\text{Revenues}] = \sum_{t=0}^{9} {}_tp_x \cdot v^t \cdot 1500 = 9684$$
$\Rightarrow$ Profit Margin $= 0.77\%$

<u>Remark</u>: We can see now why the -700 is assigned to time zero. We wouldn't want to accumulate the 700 expense at one interest rate and then turn around and discount it at a different rate.

Synonyms "discounted break-even period"

<u>Definition</u>: <u>Discounted payback period</u> or <u>break-even period</u>. This is the smallest integer k of years (periods) for which $\sum_{t=0}^{k} \Pi_t \cdot v^t$ is positive. ← ***this sum is called the partial NPV; NPV(k)***

$k \le n$ = length of contract

<u>Example 9</u>: Let $v = (1.10)^{-1}$ Consider the following computations (based on the profit signature of the previous example):

$NPV(1) = -700 + 121.17v^1 = -589.85$
$NPV(2) = -700 + 121.17v^1 + 125.72v^2 = -485.95$
⋮
$NPV(8) = -700 + 121.17v^1 + \dots + 128.35v^7 + 124.76v^8 = -20.37$
$NPV(9) = -700 + 121.17v^1 + \dots + 128.35v^7 + 124.76v^8 + 119.75v^9 = 30.42$

First non-negative value for NPV(k) occurs @ time 9

This contract has a(n) __9__-year discounted payback period.

<u>Suggested practice/ALTAM Sample Problems</u>: #34 (omit 34g), 35 a.k.a "using the profit test to determine reserves"

DHW 3e Section 13.7 – Zeroization and Zeroized Reserves

Main idea:

- Typical NPV calculations use a higher interest rate than the $(1 + i)$ factor used in one-year accumulations of premium, reserves, expenses.
- Thus, reserving capital is costly and reduces NPV.
- Earlier emergence of profits increases NPV.
- We can set a reserving structure that exactly meets expected insurer liabilities near the end of a contract, thus zeroing out the Pr_t and Π_t for the t's that occur at the late stage of a contract. Payoff: More profits emerge earlier.

Sketch of Example from DHW 2e: \$100,000 discrete 10-year term policy on (60).

Gross premium: \$1500/year.
Expenses: 52.50 associated with each premium payment, and \$700 acquisition expense at $t = 0$.
Cash flows are accumulated at $i = .055$.

$q_{60} = .01, \ldots, q_{64} = .014, \ldots$

Net Premium Reserve Method

i.e. match Expected payouts with expected income @ t=0, forward recurse

An insurer could choose to do reserving based on net benefit reserves and reserve basis assumptions. We did this earlier in class, and got…

t	${}_tV$	t	${}_tV$
0	0.00	5	1219.94
1	410.05	6	1193.37
2	740.88	7	1064.74
3	988.90	8	827.76
4	1150.10	9	475.45

So, for example,

$$\mathrm{Pr}_5 = (1150.10 + 1500 - 52.50)(1.055) - q_{64}(100000) - p_{64}(1219.94) = 137.61$$

and

$$\Pi_5 = {}_4p_{60} \cdot 137.61 = 131.39 .$$

$\Rightarrow$ Profit Signature = (-700, 121.17,…, 131.39,…, 113.37)

$\Rightarrow \mathrm{NPV}_{10\%} = \sum \Pi_t v^t = 74.13$

DHW Example, continued
Gross premium: \$1500/year.
Expenses: 52.50 associated with each premium payment, and \$700 acquisition expense at $t = 0$.
Cash flows are accumulated at $i = .055$.

Suppose the insurer decides to use a different method of reserving. We'll use ${}_tV^Z$ for the time-t reserve computed via the "new" method:

> Zeroizing reserves: For as many years as possible at the termination-end of the contract, choose positive reserve amounts so that emerging profit is zero for those years. This causes profit to emerge earlier in the contract, increasing NPV.

- Start with ${}_9V^Z$ (Recall it's a 10-year policy):
  - What value for ${}_9V^Z$ would give $Pr_{10} = 0$? (Use $q_{69} = .019$.)

End of Policy

$$Pr_{10} \overset{\text{want}}{=} 0 = \left({}_9V^Z + \overset{\pi^G}{1500} - \overset{\text{Expenses}}{52.50}\right)(1.065) - \overset{q_{69}}{0.019}\left(\overset{\text{Bens.}}{100000}\right) - \overset{p_{69}}{0.981}\left(\underset{{}_{10}V^Z}{0}\right) \rightsquigarrow {}_9V^Z = 353.45$$

  - If that value for ${}_9V^Z$ is positive, keep it. Otherwise set ${}_9V^Z$ equal to zero.

- Repeat to find ${}_8V^Z$:
  - What value for ${}_8V^Z$ would give $Pr_9 = 0$? (Use $q_{68} = .018$.)

$$0 = \left({}_8V^Z + \underset{\pi^G}{1500} - \underset{\text{Expenses}}{52.50}\right)\underset{1+i}{(1.055)} - \underset{q_{68}}{0.018}\underset{\text{Bens}}{(100000)} - \underset{p_{68}}{0.982}\underset{{}_9V^Z}{(353.45)} \rightsquigarrow {}_8V^Z = 587.65 > 0$$

  - If that value for ${}_8V^Z$ is positive, keep it. Otherwise set ${}_8V^Z$ equal to zero.

- Continuing, it turns out we get the values in the table → → →

  …and these values guarantee that $Pr_4 = Pr_5 = \ldots = Pr_{10} = 0$.

Table 12.6 *Zeroized reserves.*

t	${}_tV^Z$
6	732.63
5	658.32
4	494.78
3	247.62

- Next, to find ${}_2V^Z$:
  - What value for ${}_2V^Z$ would give $Pr_3 = 0$? (Use $q_{62} = .012$.)

$$Pr_3 = 0 = ({}_2V^Z + 1500 - 52.50)(1.055) - 0.012(100000) - 0.988(247.62)$$
$$\rightsquigarrow {}_2V^Z = \underbrace{-78.17 \not> 0 \rightarrow {}_2V^Z := 0}$$

  - If that value for ${}_2V^Z$ is positive, keep it. Otherwise set ${}_2V^Z$ equal to zero.

- Compute Pr_3. $(0 + 1500 - 52.50)(1.055) - 0.012(100000) - 0.988(247.62) = 82.47 = -1.055(V)$

  grow @ $i = 5.5\%$

  negate for pos profit

- We likewise find that ${}_1V^Z$ would need to be negative to zero out Pr_2. So we set ${}_1V^Z = 0$ and obtain a nonzero Pr_2 (= 427.11). Similarly, ${}_0V^Z = 0$ and $Pr_1 = 527.11$.
  Recall $Pr_0 = -700$.

DHW Example, continued:

- Using $p_{60} = .99$, $p_{61} = .989$, $p_{62} = .988$, etc., show how to get each Π_t from the corresponding Pr_t.

$$\Pi_t = {}_{t-1}p_x \, Pr_t$$

- Compute $NPV_{10\%}$. (Get 189.31)

$$NPV_{10\%} = \sum \Pi_t v^t$$

t	Pr_t	Π_t
0	−700.00	−700.00
1	527.11	527.11
2	427.11	422.84
3	82.47	80.74
4	0.00	0.00
5	0.00	0.00
6	0.00	0.00
7	0.00	0.00
8	0.00	0.00
9	0.00	0.00
10	0.00	0.00

Emerging profit after zeroization.

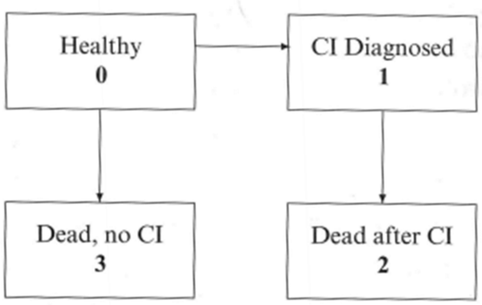
Just by changing when we reserve, due to discounting rates we more than double our money

→ Profit margin: denominator doesn't change, but numerator does by same amt

→ 0.0195 ~ 1.95%

DHW3e Section 13.9 – Profit Testing in Multistate Models

DHW3e Example 13.2



t	${}_tV^{(0)}$	${}_tV^{(1)}$
0	0	0
1	700	43 000
2	1200	43 000
3	1600	42 000
4	2000	42 000
5	2000	40 000
6	1600	38 000
7	1200	34 000
8	1000	27 000
9	500	17 000

Fully discrete 10-year partially accelerated CI and term insurance, sold to a healthy (60).

Benefits:
\$100,000 at end of year if (3) entered during year.
\$50,000 at end of year if (1) entered during the year.
\$50,000 at end of year if (2) entered during the year.

Premium: \$2500 if healthy,
0 otherwise.

Profit test assumptions:
\$250 acquisition expenses
Annual renewal expenses:
5% of each premium or
\$25 if in state 1.

Investments earn 6%/year
Risk discount rate is 12%/year

Transition probabilities – next page.

Compute: NPV, DPP, Profit Margin.

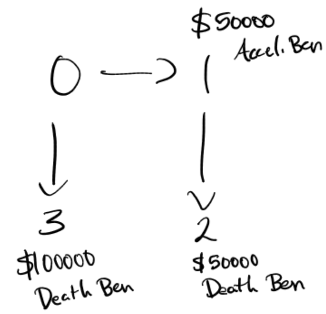
??? He zooomed through this

Recall :

$${}_3V^{(1)} \underset{\text{Gross prem. reserve}}{=} 50000\,A^{12}_{63:\overline{7}|} + 25\,\ddot{a}^{11}_{63:\overline{7}|}$$

$${}_3V^{(0)} = 100000\,A^{03}_{63:\overline{7}|} + 50000\left(A^{01}_{63:\overline{7}|} + A^{02}_{63:\overline{7}|}\right) + 0.05(2500)\,\ddot{a}^{00}_{63:\overline{7}|} + 25\,\ddot{a}^{01}_{63:\overline{7}|} - 2500\,\ddot{a}^{00}_{63:\overline{7}|}$$

t	x	p_x^00	p_x^01	p_x^02	p_x^03	p_x^11	p_x^12
0	60	0.983	0.01	0.005	0.002		
1	61	0.981	0.01	0.006	0.003	0.65	0.35
2	62	0.979	0.01	0.007	0.004	0.65	0.35
3	63	0.977	0.01	0.008	0.005	0.65	0.35
4	64	0.975	0.01	0.009	0.006	0.65	0.35
5	65	0.973	0.01	0.01	0.007	0.65	0.35
6	66	0.971	0.01	0.011	0.008	0.65	0.35
7	67	0.969	0.01	0.012	0.009	0.65	0.35
8	68	0.967	0.01	0.013	0.01	0.65	0.35
9	69	0.965	0.01	0.014	0.011	0.65	0.35



Profit Testing

$Pr_0^{(0)} = -250$ (What state of the model are we in?)

$$Pr_1^{(0)} = (0 + \overset{\Pi}{2500} - \overset{\text{Expenses}}{0.05(2500)})(1.06) - \overset{p_{60}^{00}}{0.983}\,{}_1V^{(0)} - p_{60}^{01}(50000 + {}_1V^{(1)})$$

Accelerated Benefit from Contract

$$- p_{60}^{02}(50000 + 50000 + \cancel{{}_1V^{(2)}})$$ 0, they're dead

$$- p_{60}^{03}(100000 + \cancel{{}_1V^{(3)}})$$ 0

no prem inc. expenses

$$\text{E.g. } Pr_2^{(1)} = ({}_1V^{(1)} + 0 - 25)(1.06) - p_{61}^{11}\,{}_2V^{(1)} - p_{61}^{12} \cdot 50000$$

Table 13.12 *Profit signature and NPV function for Example 13.2*

t (1)	${}_tp_x^{00}$ (2)	${}_tp_x^{01}$ (3)	$Pr^{(0)}$ (4)	$Pr^{(1)}$ (5)	Π (6)	NPV(t) (7)
0	1.00000	0.00000	−250.00	0.00	−250.00	−250.00
1	0.98300	0.01000	199.40	0.00	199.40	−71.96
2	0.96432	0.01633	252.30	103.50	249.05	126.57
3	0.94407	0.02026	203.10	753.50	208.16	274.74
4	0.92236	0.02261	39.50	−306.50	31.08	294.49
5	0.89930	0.02392	287.50	993.50	287.64	457.70
6	0.87502	0.02454	500.70	173.50	454.43	687.93
7	0.84964	0.02470	308.30	653.50	285.81	817.22
8	0.82330	0.02455	−49.50	963.50	−18.26	809.84
9	0.79614	0.02419	124.00	43.50	103.16	847.04
10	0.76827	0.02369	47.50	493.50	49.76	863.06

$$NPV_{10\%} = \sum_t \Pi_t v^t$$

$EPV_{12\%}$[Premiums] $= 2500 \sum {}_tp_x^{00} \times v^t = 14{,}655.31$

$NPV_{12\%} \div EPV_{12\%} = .0589$

Weighted Average

$\Pi_0 = Pr_0 = -\text{acq. expense}$

$\Pi_1 = Pr_1 = Pr_1^{(0)}$

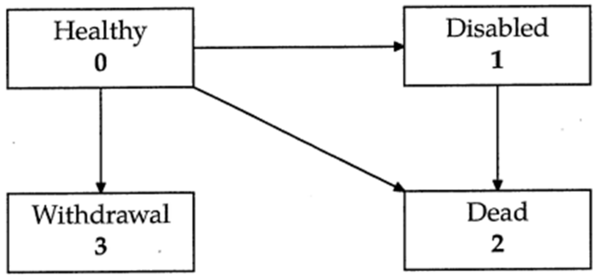
$\Pi_2 = p_{60}^{00}\,Pr_2^{(0)} + p_{60}^{01}\,Pr_2^{(1)}$

$\Pi_3 = {}_2p_{60}^{00}\,Pr_3^{(0)} + {}_2p_{60}^{01}\,Pr_3^{(1)}$

$\vdots$

(From Weishaus LTAM/ALTAM)

EXAMPLE 73D For a disability policy modeled with the following Markov chain:



you are given

(i) A benefit of 500 per year is paid at the start of the year to a person disabled at that time.

(ii) A benefit of 90% of the reserve for the healthy state at the end of the year is paid at the end of the year to someone withdrawing from that state during that year.

(iii) A benefit of 10000 is paid at the end of the year of death.

(iv) The following probabilities:

$${}_9p_x^{00} = 0.65 \qquad {}_9p_x^{01} = 0.10$$
$$p_{x+9}^{01} = 0.03 \qquad p_{x+9}^{02} = 0.01 \qquad p_{x+9}^{03} = 0.05 \qquad p_{x+9}^{12} = 0.02$$

(v) The following reserves at the end of the indicated years:

Year	Healthy	Disabled
9	1250	6400
10	1400	6580

(vi) Premium is 250, paid at the beginning of the year whether healthy or disabled.

(vii) Expenses are 5% of premium.

(viii) $i = 0.055$

Calculate Π_{10}, the tenth year profit per policy issued.

(Should get $\Pi_{10} = .65(-65.0875) + .1(-173.3375) = -59.640625$)

DHW3e, Chapter 14 -
Universal Life Insurance, Introductory Examples
(See also MQR 4e 11.5 (on which this example is based) and 16.1.)

UL Type A Example:
Consider a fully discrete universal life contract issued to (30) with level total death benefit ("Type A"), face amount 100,000.

Mortality assumptions for determining contracted cost of insurance (COI):

$$q^*_{30} = .00076\ ,\ q^*_{31} = .00081\ ,\ q^*_{32} = .00085\ ,\ q^*_{33} = .00095$$

(These q^*'s are called COI rates.) For now, we will pretend that no one surrenders their policy.

Percent-of-contribution expense factors are $r_1 = .75$ for the first year and $r_t = .01$ for subsequent years.

Fixed expense amounts of $e_1 = 100$ in first year and $e_t = 20$ in subsequent years are due at beginning of the year.

Interest is credited to the account value (on a contractual basis) at $i = i^c = .03$. ← These rates are **not**
The interest rate used to compute the cost of insurance (COI) is $i^{COI} = i^q = .03$. ← required to be equal!

> Unfortunate notation: In both MQR and DHW, t seems to represent the year number, which is one higher than the time at which the transaction occurs. The account value AV_t is the account value at the end of the t^{th} year, and thus matches the account value at time t. *Yuck!* The subscripts on G, r, e represent which contributions and expenses contribute to AV_t.

The insured makes an annual contribution of 5000. Calculate the end-of-year account value for years 1, 2, 3.

One-year recursion for **level-benefit** UL account values:

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UL Type B Example:
Consider a fully discrete universal life contract issued to (30) with death benefit equal to the **face amount plus the account value** (this arrangement defines "UL Type B").

Face amount: 100,000.

Mortality assumptions for determining contracted cost of insurance:

$$q_{30} = .00076\ ,\ q_{31} = .00081\ ,\ q_{32} = .00085\ ,\ q_{34} = .00095$$

Percent-of-contribution expense factors are $r_1 = .75$ for the first year and $r_t = .01$ for subsequent years.

Fixed expense amounts of $e_1 = 100$ in first year and $e_t = 20$ in subsequent years are due at beginning of the year.

Interest is credited to the account value (on a contractual basis) at $i = .03$.
The interest rate used to compute the cost of insurance (COI) is $i^{\text{COI}} = i^q = .03$.

The insured makes an annual contribution of 5000. Calculate the end-of-year account value for years 1, 2, 3.

One-year recursion for **variable-failure-benefit** UL account values:

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For your reference, here is an excerpt from SOA's notation document for the 2014 MLC Exam.

SOA Notation and Definitions (ALTAM 7/2023):

Universal Life

The following terminology and conventions will be used in examination questions. These are all consistent with *AMLCR*, but may differ from terminology and conventions used in practice in some cases.

(i) Account Values (AV) are calculated at discrete, regular intervals. The question will indicate the calculation period.

(ii) Premiums are paid at the start of each time interval, and benefits are paid at the end of each time interval.

(iii) Premiums are level and paid throughout the term of the policy, unless the question explicitly states otherwise.

(iv) The Cost of Insurance rate is the mortality rate used to determine the cost of insurance.

(v) The rate of interest used to discount the mortality charge may be assumed to be equal to the credited rate unless otherwise specified.

(vi) Expense charges, credited interest rates, and other account value factors are assumed to be constant throughout the term of the policy, unless the question explicitly states otherwise.

(vii) The Death Benefit at t , DB_t , is the amount of benefit that would be payable at t in the event that the insured dies in the time interval that ends at t.

(viii) The Additional Death Benefit at t , ADB_t , is the difference between the Death Benefit and the Account Value at t, i.e., $\text{ADB}_t = \text{DB}_t - \text{AV}_t$

(ix) Corridor factors set an age-dependent minimum Death Benefit at t, in terms of the Account Value at t. Specifically, given a corridor factor c, the Death Benefit at t must be at least $c \times$ the Account Value at t.

(x) If no corridor factors are stated in the question, assume the policy has none.

(xi) Type A UL policies have a fixed Death Benefit, unless the corridor factor minimum death benefit applies.

(xii) Type B UL policies have a fixed Additional Death Benefit, so the Death Benefit is the sum of the pre-specified Additional Death Benefit and the Account Value, unless the corridor factor minimum applies.

(xiii) The Cash Value of the policy is the Account Value minus a Surrender Penalty if applicable. Cash Surrender Value is an equivalent term to Cash Value and may also be used on the exam.

(xiv) Reserves are equal to Account Values unless a no-lapse guarantee applies, or unless otherwise specified.

So, in particular, what SOA calls COI is calculated as follows:

$$\text{COI} = \underbrace{q_{x+t}^{COI}}_{"COI\ \boldsymbol{rate}"} \times v_{\text{to discount mortality charge}} \times \underbrace{(\text{amt of total DB above } AV_{end})}_{ADB_{end} = (AV_{end} - DB)}$$

Universal Life, More Examples

MQR 4e 16.1 *(Modified to match SOA conventions.)*

Fixed death benefit: 100,000 ← the base death benefit is called "face amount" for either type of UL policy.
AV on April 30: 4000
May 1 contribution: 1000
Credited annual interest rate: 4.5% (effective)
Annual rate for discounting COI: 0% ← Recall that v^{COI} could be based on a different rate than the rate used for crediting interest to the AV.
Monthly expenses: fixed expense of 40 and 4% of contribution
Monthly mortality for COI: "q" = .0001
Surrender charge*: 10 per 1000 of face amount.
Outstanding loan balance: 500 on May 31

The* ***surrender charge* *can vary depending on the duration of the contract, and is likely to be highest during the early years of the contract. Also called* ***surrender penalty****.*

Find account value, cash value, and actual cash surrender value on May 31.

***296.** For two universal life insurance policies issued on (60), you are given:

(i) Policy 1 is a Type A Universal Life with face amount 100,000.
(ii) Policy 2 is a Type B Universal Life with face amount 100,000.

For each policy:

(i) Death benefits are paid at the end of the month of death.
(ii) Account values are calculated monthly.
(iii) Level monthly premiums of G are payable at the beginning of each month. Past premiums may have been different from G, and may not have been the same for both policies.
(iv) Mortality rates for calculating the cost of insurance:
  a. Follow the Illustrative Life Table.
  b. Assume UDD for fractional ages.
(v) Interest is credited at a monthly effective rate of 0.004.
(vi) The interest rate used for accumulating and discounting in the cost of insurance calculation is a monthly effective rate of 0.004.
(vii) Level expense charges of E are deducted at the beginning of each month.

At the end of the 36th month the account value for Policy 1 equals the account value for Policy 2.

Calculate the ratio of the account value for Policy 1 at the end of the 37th month to the account value of Policy 2 at the end of the 37th month.

(Use

x	l_x	$1000\, q_x$
60	8,188,074	13.76
63	7,823,879	17.88
64	7,683,979	19.52

.)

Universal Life, More Examples

<u>MQR 4e 16.1</u> *(Modified to match SOA conventions.)*

Fixed death benefit: 100,000 ← the base death benefit is called "face amount" for either type of UL policy.
AV on April 30: 4000
May 1 contribution: 1000
Credited annual interest rate: 4.5% (effective)
Annual rate for discounting COI: 0% ← Recall that v^{COI} could be based on a different rate than the rate used for crediting interest to the AV.
Monthly expenses: fixed expense of 40 and 4% of contribution
Monthly mortality for COI: "q" = .0001
Surrender charge*: 10 per 1000 of face amount.
Outstanding loan balance: 500 on May 31

**The surrender charge can vary depending on the duration of the contract, and is likely to be highest during the early years of the contract.*

Find account value, cash value, and actual cash surrender value on May 31.

***296.** For two universal life insurance policies issued on (60), you are given:

(i) Policy 1 is a Type A Universal Life with face amount 100,000.
(ii) Policy 2 is a Type B Universal Life with face amount 100,000.

For each policy:

(i) Death benefits are paid at the end of the month of death.
(ii) Account values are calculated monthly.
(iii) Level monthly premiums of G are payable at the beginning of each month. Past premiums may have been different from G, and may not have been the same for both policies.
(iv) Mortality rates for calculating the cost of insurance:
  a. Follow the Illustrative Life Table.
  b. Assume UDD for fractional ages.
(v) Interest is credited at a monthly effective rate of 0.004.
(vi) The interest rate used for accumulating and discounting in the cost of insurance calculation is a monthly effective rate of 0.004.
(vii) Level expense charges of E are deducted at the beginning of each month.

At the end of the 36th month the account value for Policy 1 equals the account value for Policy 2.

Calculate the ratio of the account value for Policy 1 at the end of the 37th month to the account value of Policy 2 at the end of the 37th month.

(Use

x	l_x	$1000\, q_x$
60	8,188,074	13.76
63	7,823,879	17.88
64	7,683,979	19.52

.)

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UL, conclusion: Corridor factor requirements and UL profit testing

Example 1: DHW 2e Example 13.5 p. 457 (1e Supplement p.45): Project the account values and cash values for a level-death benefit ("Type A") UL Policy issued to (45) with corridor factor requirement. Assume that interest is credited to the account at 4% per year.

- Face Amount \$100 000
- Death Benefit Type A with corridor factors (γ_t) applying to benefits paid in respect of deaths in the tth year, as follows:

t	1	2	3	4	5	6	7	8	9	10
γ_t	2.15	2.09	2.03	1.97	1.91	1.85	1.78	1.71	1.64	1.57

t	11	12	13	14	15	16	17	18	19	20
γ_t	1.50	1.46	1.42	1.38	1.34	1.30	1.28	1.26	1.24	1.24

- CoI based on: 120% of SSSM, 4% interest; the CoI is calculated assuming the fund earns 4% interest during the year.
- Expense Charge: 20% of first premium + \$200, 3% of subsequent premiums.
- Initial premium: \$3500.
- Surrender Penalties:

Year of surrender	1	2	3-4	5-7	≥ 8
Penalty	\$2 500	\$2 100	\$1 200	\$600	\$0

(SSSM: $q_{[45]} = .0006592$; $q_{[45]+1} = .0007974$)

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Example 2: Repeat Example 1, but change *only* the face amount of the policy. So…

Project the account values and cash values for a level-death benefit ("Type A") UL Policy issued to (45) with corridor factor requirement. Interest is credited at 4% per year.

New face amount: 5000

- Death Benefit Type A with corridor factors (γ_t) applying to benefits paid in respect of deaths in the tth year, as follows:

t	1	2	3	4	5	6	7	8	9	10
γ_t	2.15	2.09	2.03	1.97	1.91	1.85	1.78	1.71	1.64	1.57

t	11	12	13	14	15	16	17	18	19	20
γ_t	1.50	1.46	1.42	1.38	1.34	1.30	1.28	1.26	1.24	1.24

- CoI based on: 120% of SSSM, 4% interest; the CoI is calculated assuming the fund earns 4% interest during the year.
- Expense Charge: 20% of first premium + \$200, 3% of subsequent premiums.
- Initial premium: \$3500.
- Surrender Penalties:

Year of surrender	1	2	3-4	5-7	≥ 8
Penalty	\$2 500	\$2 100	\$1 200	\$600	\$0

(SSSM: $q_{[45]} = .0006592$; $q_{[45]+1} = .0007974$)

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Example 3: Back to the setting of Example 1; i.e. \$100,000 level death benefit. We computed $AV_1 = 2626.97$, $AV_2 = 6173.07$,

Profit test:

(b) Profit test the contract using the basis below. Use annual steps, and determine the NPV and DPP using a risk discount rate of 10% per year.

Assume

- Level premiums of \$3500 paid annually in advance.
- Insurer's funds earn 6% per year.
- Policyholders' accounts are credited at 4% per year.
- Surrenders occur at year ends. The surrender rate given in the following table is the proportion of in-force policyholders surrendering at each year end.

Duration at year end	Surrender Rate q^{w}_{45+t-1}
1	5%
2-5	2%
6-10	3%
11	10%
12-19	15%
20	100%.

← **Note:** DHW *should* have said $q'^{(w)}$ is the *independent* probability that a policy surrenders ("*independent* surrender rate"?), given in-force at year's end.

Note also: We have $q^{(d)} = q'^{(d)}$ as there is no competition with the withdrawal decrement during any policy year.

- Mortality follows the Standard Select Survival Model.
  (SSSM: $q^{d}_{[45]} = .0006592$; $q^{d}_{[45]+1} = .0007974$)
- Incurred expenses are:
  - Pre-contract expenses of 60% of the premium due immediately before the issue date,
  - Maintenance expenses of 2% of premium at each premium date including the first,
  - \$50 on surrender,
  - \$100 on death.
- The insurer holds reserves equal to the policyholder's account value.
- Surrender charges are $SC_1 = 2500$, $SC_2 = 2100$ for surrender charges at times $t = 1, 2$.

The insurer holds reserves equal to the policyholder's account value.

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We get the following profit vector (Pr_0 , … , Pr_{20}):

Pr_t
−2100
1067
213
268
342
⋮
1839
1938

(Recall: SSSM: $q^d_{[45]} = .0006592$; $q^d_{[45]+1} = .0007974$; $q'^w_{[45]} = .05$; $q'^w_{[45]+1} = .02$)

Show how to compute the profit signature and NPV at 10% hurdle rate.

If time: How do the recursions change for a policy with D.B. = 100,000 + AV_t?

Reading: DHW3e Sections 1.4, omitting subsection 1.4.3
HW: You may now attempt the suggested exercise at the end of this UL notes.
Suggested ALTAM Problems (Canvas): 50a, 52, 53, 54

DHW 3e 14.3.7 – UL policies with no-lapse guarantees

<u>Context</u>: Consider a fully discrete Type A policy (level death benefit = S) with annual cash flows.

Excerpt from DHW 2e, p. 446:

An additional feature of some policies is the **no-lapse guarantee**, under which the death benefit coverage continues even if the account value declines to zero, provided that the policyholder pays a pre-specified minimum premium* at each premium date.

This guarantee could apply if expense and mortality charges increase sufficiently to exceed the minimum premium. The policyholder's account value would support the cost of the death benefit until it is exhausted, at which time the no-lapse guarantee would come into effect.

*This pre-specified premium can, in some policies/contracts, be as low as 0 after the policy has been in force for a specified amount of time.

<u>Reserving</u> is the main issue that DHW considers in this context (p.462):

<u>Example</u>: DHW 2e, p. 462:

Suppose the Type A UL policy has been issued to (x) and has been in force for t years.

Suppose this policy has a no-lapse guarantee, in this particular case allowing the policyholder to cease premiums and still maintain their death benefit insurance.

Note: Such a feature usually wouldn't kick in until the policy had been in-force for a specified number of years.

$\text{EPV}_{\text{time } t}$ [Death Benefit] = $S\,A_{x+t}$.

Account value = ${}_t\text{AV}$.

- If ${}_t\text{AV} \geq S\,A_{x+t}$, then the account value is enough of a reserve.
- If ${}_t\text{AV} < S\,A_{x+t}$, then additional reserving to support the no-lapse guarantee is needed.

Define ${}_t\text{V}^{\text{No-Lapse-Guarantee}} = {}_t\text{V}^{\text{NLG}} =$ ______________________.

${}_t\text{V}^{\text{NLG}}$ is referred to as the *reserve for the no-lapse guarantee.*

Example   Same setup, but with a variation: The no-lapse guarantee expires at time n (age $x + n$).

$\text{EPV}_{\text{time } t}$ [No-Lapse Guaranteed Death Benefit] = $S\,A^{\;1}_{x+t\,:\,\overline{n-t}|}$.

Account value = ${}_t\text{AV}$.

- If ${}_t\text{AV} \geq S\,A^{\;1}_{x+t\,:\,\overline{n-t}|}$, then the account value is enough of a reserve.
- If ${}_t\text{AV} < S\,A^{\;1}_{x+t\,:\,\overline{n-t}|}$, then additional reserving needed to support the no-lapse guarantee.

Define ${}_tV^{\text{No-Lapse-Guarantee}} = {}_tV^{\text{NLG}} =$ ____________________.

Example: May 2012 MLC #27

27. For a universal life insurance policy with death benefit of 100,000 on (40), you are given:

(i) The account value at the end of year 4 is 2029.

(ii) A premium of 200 is paid at the start of year 5.

(iii) Expense charges in renewal years are 40 per year plus 10% of premium.

(iv) The cost of insurance charge for year 5 is 400.

(v) Expense and cost of insurance charges are payable at the start of the year.

(vi) Under a no lapse guarantee, after the premium at the start of year 5 is paid, the insurance is guaranteed to continue until the insured reaches age 49.

(vii) If the expected present value of the guaranteed insurance coverage is greater than the account value, the company holds a reserve for the no lapse guarantee equal to the difference. The expected present value is based on the Illustrative Life Table at 6% interest and no expenses.

Calculate the reserve for the no lapse guarantee, immediately after the premium and charges have been accounted for at the start of year 5.

Suggested ALTAM Problems (Canvas): 50a, 52, 53, 54

Suggested HW exercise:
Recall: profit margin is defined to be (NPV ÷ EPV(Premiums)).
Note: "COI rate = .022" is telling you $q_x^* = .022$ used in computing the cost of insurance.

18. (13 points) A life insurance company issues a Type B universal life policy to a life age 60. The main features of the contract are as follows.

    Premiums: 3,000 per year, payable yearly in advance.

    Expense charges: 4% of the first premium and 1% of subsequent premiums.

    Death benefit: 10,000 plus the Account Value, payable at the end of the year of death.

    Cost of insurance rate: 0.022 for $60 \leq x \leq 64$; discounted at 3% to start of policy year.

    Cash surrender values: 90% of the account value for surrenders after 2 or 3 years, 100% of the account value for surrenders after four years or more.

    The company uses the following assumptions in carrying out a profit test of this contract.

    Interest rate: 6% per year.

    Credited interest: 5% per year.

    Survival model: $q_{60+t} = 0.02$ for $t = 0, 1, 2, 3$.

    Withdrawals: None in the first three years; all contracts assumed to surrender at the end of the fourth year.

    Initial expenses: 200 pre-contract expenses.

    Maintenance expenses: 50 incurred annually at each premium date including the first.

    Risk discount rate: 8% per year.

    There are no reserves held other than the account value.

    (a) (4 points) Show that the projected final account value for an insured surrendering at the end of the fourth year is 12,365 to the nearest 5.

    (b) (7 points) Calculate the profit margin for a new policy.

    (c) (2 points) Calculate the NPV for a policy that is surrendered at the end of the second year.

SOA's Solution, next page.

18. (a)

$$AV_1 = (3,000\,(0.96) - 10,000(0.022)v_{3\%})\,(1.05) = (2,880.0 - 213.59)\,(1.05) = 2,799.73$$
$$AV_2 = (2,799.73 + 3,000(0.99) - 213.59)\,(1.05) = 5,833.95$$
$$AV_3 = (5,833.95 + 3,000(0.99) - 213.59)\,(1.05) = 9,019.87$$
$$AV_4 = (9,019.87 + 3,000(0.99) - 213.59)\,(1.05) = 12,365.10$$

(b) Profit test table: (note, numbers rounded for presentation)

t	AV_{t-1}	P	Exp	Int	EDB	ESB	EAV_t	Pr_t	Π_t	NPV(t)
0	–	–	200	–	–	–	–	-200	-200	-200
1	0	3,000	50	177	256	–	2,744	127.27	127.27	-82.16
2	2,800	3,000	50	345	317		5,717	60.76	59.54	-31.11
3	5,834	3,000	50	527	380		8,839	91.12	87.51	38.35
4	9,020	3,000	50	718	447	12,118	–	122.96	115.73	123.42

EPV of premiums is

$$3,000(1 + 0.98v_{8\%} + (0.98v_{8\%})^2 + (0.98v_{8\%})^3) = 10,433.84$$

Hence, the profit margin is

$$\frac{NPV}{\text{EPV Prems}} = \frac{123.42}{10,433.84} = 1.18\%$$

(c) Now there is no uncertainty with respect to survival.
The NPV is

$$\begin{aligned} \text{PV initial expenses} &= -200 \\ \text{PV profit in first year} &\ ((3,000 - 50)(1.06) - 2,799.73) = 327.27 \\ \text{PV profit in second year} &\ ((2,799.73 + 3,000 - 50)(1.06) - 0.9(5,833.95)) = 844.16 \\ \Rightarrow \quad NPV &= -200 + 327.27v_{8\%} + 844.16v_{8\%}^2 = 826.76 \end{aligned}$$

DHW3e Chapter 15, Part 1 – Equity Linked Insurance – Deterministic Methods

- Equity-linked insurance – an example of insurance contract in which the main purpose of the contract is investment.
- DHW3e p.571: Includes a death benefit, predominantly as a way of distinguishing from pure investment products, but designed to emphasize investment opportunity, with a view to competing with pure investment products sold by other financial institutions.
- Premium paid goes two places:
    - Allocated premium (AP) – goes into an investment account owned by the policyholder (managed by insurance company). Account invested in a fund that is selected by policyholder. Variations of the symbol F_t are used for the time-t value of this fund.
    - Unallocated premium (UAP) – the remainder of the premium, which goes to the insurer's funds.

DHW3e Example 15.1

- A **10-year** equity-linked contract issued to **(55)**.
- Annual **premium** $\pi = 5000$
- **Expenses** deducted by insurer: 5% of first premium and 1% of subsequent premiums.

  So allocated premium $AP_{t=0}$ = ______ and $AP_{t \geq 1}$ = ______

  and unallocated premium $UAP_{t=0}$ = _______ and $UAP_{t \geq 1}$ = _______

- At the **end** of each year, the policyholder is charged a **management charge** (MC) of **.**75% of policyholder's fund, which is transferred to the insurer's fund.

- **Death benefit** is equal to 110% of the policyholder's fund at end of year of death, less that year's end-of-year management charge.

- **Cash value** (amount the policyholder can keep if surrenders contract): Value of fund at the end of year of surrender *after* the management fee has been deducted.

- **Maturity value with Guaranteed Minimum Maturity Benefit "GMMB":**

  If contract held to maturity date, policyholder receives the greater value between value of fund and total of premiums paid.

a. Project the year-end fund values for the policyholder's fund *for a contract that remains in-force* for 10 years. (Notation: F_t , F_{t-}) Assumptions: Fund earns 9% interest per year.

t (1)	MC_t (5)	F_t (6)
1	38.83	5 138.67
2	82.47	10 914.17
3	129.69	17 162.26
4	180.77	23 921.60
5	236.03	31 234.01
6	295.80	39 144.77
7	360.47	47 702.83
8	430.44	56 961.14
9	506.12	66 977.02
10	588.00	77 812.45

b. Compute the profit vector, profit signature, and NPV at 15% using the following profit test assumptions. (**Deterministic** profit testing.)

- Interest for policyholder fund: 9% per year.
- Survival model: $q_x^{(d)}$ = .005 for all x = 55, 56, … , 64
- Lapses:
  - 10% of (lives in-force at year end) surrender in the 1st year.
  - 5% of (lives in-force at year end) surrender in the 2nd year.
  - No lapses after the 2nd year.
  - All surrenders assumed to occur after MC is deducted.
- Initial expenses: 10% of first premium, plus 150. So, $650 total.
- Renewal expenses: .5% of second and subsequent premiums. So, $25 each.
- Interest for insurer fund: 6% per year.
- No reserves.

$\Pr_t$
−650.00
301.26
103.52
147.61
195.31
246.91
302.73
363.12
428.46
499.14
575.60

c. Suppose we project the policyholder's fund using a much lower earnings rate, such that F_{10} turns out to be $44,000. Write an expression for $\Pr_{10}$ under these modified assumptions.

<u>DHW3e Example 15.2</u> A **5-year** equity-linked insurance policy issued to **(60)**

- The **single premium** is \$10,000. The insurer deducts an **expense charge** of 3% and invests remainder in policyholder's fund.

  So AP = _______ and UAP = ________.

- A **management charge** (MC) of **.**06% is deducted from policyholder's fund at <u>start</u> of each <u>subsequent</u> month and transferred to the insurer.

- **Death benefit**: 101% of all money in the policyholder fund,
  or **guaranteed minimum death benefit (GMDB)** of 10,000$\times$ 1.05^{t-1} for *t*-th year;
  paid at end of year.

- **Cash Values / Surrender benefits**: 90% of fund value during 1st year; 95% during 2nd year; 100% after 2 years.

- **Maturity guarantee (GMMB):** If contract held to maturity date, policyholder receives money in the fund with guarantee that payout not less than 10,000.

Test profitability (which includes determining fund values F_t) at under the following **profit test assumptions:**

- **Survival model:** Constant $\mu^{(d)} \equiv .006$ so that ${}_{1/12}p_x = e^{.006/12} \approx .9995$
- **Lapse rates:** Surrender can only occur at end of month. *Given an in-force policy at month's end*, probability that policy is surrendered is .004, .002, .001 for the 1st, 2nd, 3rd years, respectively.
- **Interest:** 8% for policyholder fund, 5% for insurer's fund
- **Insurer expenses:**
  Initial: 1% of the single premium plus \$150 (so \$100 + 150 = \$250)
  Renewal: After 1st month: payable at <u>start</u> of month .008% of single premium plus .01% of policyholder's fund at end of previous month. (So \$0.80 + .01% of previous month's ending fund value)
- **Risk discount rate:** 12%

Table 15.4 *Deterministic projection of the policyholder's fund for Example 15.2.*

t	AP_t	F_{t-1}	MC_t	F_t	GMDB
$\frac{1}{12}$	9700	0.00	0.00	9762.41	10000.00
$\frac{2}{12}$	0	9762.41	5.86	9819.33	10000.00
$\frac{3}{12}$	0	9819.33	5.89	9876.57	10000.00
$\frac{4}{12}$	0	9876.57	5.93	9934.16	10000.00
$\frac{5}{12}$	0	9934.16	5.96	9992.07	10000.00
$\frac{6}{12}$	0	9992.07	6.00	10050.33	10000.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	10346.74	6.21	10407.07	10000.00
$1\frac{1}{12}$	0	10407.07	6.24	10467.74	10500.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	0	11094.29	6.66	11158.97	10500.00
$2\frac{1}{12}$	0	11158.97	6.70	11224.03	11025.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	0	11895.85	7.14	11965.20	11025.00
$3\frac{1}{12}$	0	11965.20	7.18	12034.96	11576.25
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	0	12755.32	7.65	12829.68	11576.25
$4\frac{1}{12}$	0	12829.68	7.70	12904.48	12155.06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	0	13676.89	8.21	13756.62	12155.06

Table 15.5 *Deterministic projection of the insurer's fund for Example 15.2.*

t	UAP_t	MC_t	E_t	I_t	EDB_t	ECV_t	$\Pr_t$
0	0	0.00	250.00	0.00	0.00	0.00	−250.00
$\frac{1}{12}$	300	0.00	0.80	1.22	0.12	−3.90	304.20
$\frac{2}{12}$	0	5.86	1.78	0.02	0.09	−3.93	7.93
$\frac{3}{12}$	0	5.89	1.78	0.02	0.06	−3.95	8.01
$\frac{4}{12}$	0	5.93	1.79	0.02	0.05	−3.97	8.08
$\frac{5}{12}$	0	5.96	1.79	0.02	0.05	−3.99	8.13
$\frac{6}{12}$	0	6.00	1.80	0.02	0.05	−4.02	8.18
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	6.21	1.83	0.02	0.05	−4.16	8.50
$1\frac{1}{12}$	0	6.24	1.84	0.02	0.05	−1.05	5.42
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2	0	6.66	1.91	0.02	0.06	−1.12	5.83
$2\frac{1}{12}$	0	6.70	1.92	0.02	0.06	0.00	4.74
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3	0	7.14	1.99	0.02	0.06	0.00	5.11
$3\frac{1}{12}$	0	7.18	2.00	0.02	0.06	0.00	5.14
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
4	0	7.65	2.08	0.02	0.06	0.00	5.54
$4\frac{1}{12}$	0	7.70	2.08	0.02	0.06	0.00	5.57
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5	0	8.21	2.17	0.02	0.00	0.00	5.99

Table 15.6 *Calculation of the profit signature for Example 15.2.*

		Probability	
t	Pr_t	in force @ $t-\frac{1}{12}$	Π_t
0	−250.00	$\cancel{1.0000}$	−250.00
$\frac{1}{12}$	304.20	1.0000	304.20
$\frac{2}{12}$	7.93	0.9955	7.90
$\frac{3}{12}$	8.01	0.9910	7.94
$\frac{4}{12}$	8.08	0.9866	7.97
$\frac{5}{12}$	8.13	0.9821	7.98
$\frac{6}{12}$	8.18	0.9777	8.00
⋮	⋮	⋮	⋮
1	8.50	0.9516	8.09
$1\frac{1}{12}$	5.42	0.9492	5.14
⋮	⋮	⋮	⋮
2	5.83	0.9235	5.38
$2\frac{1}{12}$	4.74	0.9221	4.37
⋮	⋮	⋮	⋮
3	5.11	0.9070	4.63
$3\frac{1}{12}$	5.14	0.9056	4.66
⋮	⋮	⋮	⋮
4	5.54	0.8908	4.93
$4\frac{1}{12}$	5.57	0.8895	4.96
⋮	⋮	⋮	⋮
5	5.99	0.8749	5.24

At risk discount rate 12%,

$$\text{NPV} = \sum_{k=0}^{60} \Pi_{\frac{k}{12}} (1+r)^{-\frac{k}{12}} = \$302.42.$$

Stochastic Techniques in Equity-Linked Insurance

<u>DHW3e Example 15.1</u>

- A **<u>10-year</u>** equity-linked contract issued to **<u>(55)</u>**.
- Annual **<u>premium</u>** $\pi = 5000$
- **<u>Expenses</u>** deducted by insurer: 5% of first premium and 1% of subsequent premiums.

  So allocated premium $AP_{t=0}$ = ______ and $AP_{t\geq 1}$ = ______

  and unallocated premium $UAP_{t=0}$ = _______ and $UAP_{t\geq 1}$ = _______

- At the **<u>end</u>** of each year, the policyholder is charged a **<u>management charge</u>** (MC) of **.**75% of policyholder's fund, which is transferred to the insurer's fund.

- **<u>Death benefit</u>** is equal to 110% of the policyholder's fund at end of year of death, less that year's end-of-year management charge.

- **<u>Cash value</u>** (amount the policyholder can keep if surrenders contract): Value of fund at the end of year of surrender *after* the management fee has been deducted.

- **<u>Maturity value</u> with Guaranteed Minimum Maturity Benefit "GMMB":**

  If contract held to maturity date, policyholder receives the greater value between value of fund and total of premiums paid.

a. Use Excel to project the year-end fund values for the policyholder's fund *for a contract that remains in-force* for 10 years. (Notation: F_t , F_{t-}) Assumptions: Fund earns 9% interest per year.

b. Compute the profit vector, profit signature, and NPV at 15% using the following profit test assumptions. (**<u>Deterministic</u>** profit testing.)

- Interest for policyholder fund: 9% per year.
- Survival model: $q_x^{(d)} = .005$ for all x = 55, 56, … , 64
- Lapses:
  - 10% of (lives in-force at year end) surrender in the 1st year.
  - 5% of (lives in-force at year end) surrender in the 2nd year.
  - No lapses after the 2nd year.
  - All surrenders assumed to occur after MC is deducted.
- Initial expenses: 10% of first premium, plus 150. So, \$650 total.
- Renewal expenses: .5% of second and subsequent premiums. So, \$25 each.
- Interest for insurer fund: 6% per year.
- No reserves.

Stochastic methods

Instead of × 1.09 for the policyholder fund, we simulate a random accumulation factor $R_{[t-1,\,t]}$ for each interval.

For this example, we assume $R_{[t-1,\,t]}$ is <u>lognormally distributed</u> with parameters

$$\mu = \ln\left(1.09 - \tfrac{1}{2}(.15)^2\right)(1) \quad \text{and} \quad \sigma = .15.$$

Note: For a lognormal r.v. R, the symbols μ and σ are <u>not</u> the mean and stdev of R.
Rather, they are the mean and stdev of ln(R).
E[R] turns out to be $e^{\mu+\frac{1}{2}\sigma^2}$.

Simulate $NPV_{15\%}$ and Future Losses (def'n ↓) $L_{6\%}$

<u>Monte Carlo</u> simulation of n = 1000 NPVs ↓

Table 15.8 *Results from 1 000 simulations of the net present value.*

E[NPV]	380.91
SD[NPV]	600.61
95% CI for E[NPV]	(343.28, 417.74)
5th percentile	–859.82
Median of NPV	498.07
95th percentile	831.51
N^- #(negative NPV)	87
N^* #(GMMB not used)	897

<u>Some Uses of Monte Carlo Simulation Methods</u>

Stochastic Pricing

(DHW p.589) Modify design of contract such that, at 15%,

- NPV is + at bottom 5th percentile point
- E[NPV] is ≥ 65% of the $650 acquisition cost

Here are four (separate) suggested modifications to the original equity linked contract:

(1) Increasing the premium from $5 000 to $5 500, and hence increasing the GMMB to $55 000 and the acquisition costs to $700.
(2) Increasing the annual management charge from 0.75% to 1.25%.
(3) Increasing the expense deductions from the premiums from 5% to 6% in the first year and from 1% to 2% in subsequent years.
(4) Decreasing the GMMB from 100% to 90% of premiums paid.

In each of the four cases, the remaining features of the policy are as described in Example 15.1.

Table 15.9 *Results from changing the structure of the policy in Example 15.1.*

	Change			
	Increase *P* (1)	Increase *MC* (2)	Increase *UAP* (3)	Decrease *GMMB* (4)
E[*NPV*]	433.56	939.60	594.68	460.33
SD[*NPV*]	660.67	725.97	619.75	384.96
95% CI	(392.61, 474.51)	(894.60, 984.60)	(556.27, 633.09)	(436.47, 484.19)
5%-ile	−930.81	−617.22	−724.40	145.29
Median of *NPV*	562.87	1 065.66	721.74	500.00
95%-ile	929.66	1 625.44	1 051.78	831.51
N^-	86	78	80	46
N^*	897	882	894	939

Stochastic Reserving

(a) VaR reserving

- Value at Risk (VaR) – Example:
- Set up a 95% quantile reserve – Simulate e.g. N = 1000 values for future loss L, and find the loss amount for which it appears that Pr[L ≤ amt] = .95.

DHW simulated \$1259.56, so after acquisition costs are paid set aside $_0V$ = \$1259.56 per policy.

Re-evaluate at time 1, and could set $_1V$ equal to $_0V$. Considering expected future losses and whether current reserve was/seems to remain sufficient (or now seems excessive).

Calculations by the authors, with $N = 1000$ and $j = 0.06$, gave a value for $_0V$ of \$1259.56. Hence, if, after paying the acquisition costs, the insurer sets aside a reserve of \$1259.56 for each policy issued, it will be able to meet its future liabilities with probability 0.95 **provided** all the assumptions underlying this calculation are realized. These assumptions relate to

- expenses,
- lapse rates,
- the survival model, and, in particular, the diversification of the mortality risk,
- the interest rate earned on the insurer's fund,
- the interest rate earned on the reserve,
- the interest rate model for the policyholder's fund,
- the accuracy of our estimate of the upper 95th percentile point of the loss distribution.

(b) CTE reserving

- Conditional Tail Expectation (CTE), Tail Value at Risk (TVaR), Expected Shortfall – Example
- Set up a reserve at time 0 if the 50 worst (highest) simulated values for L ranged from 1260.76 to 7512.41 and had mean 3603.11.

Remarks: DHW32:

(1)
The CTE reserve was estimated using simulations based only on information available at the start of the policy.

(2)
In practice, would update the CTE reserve regularly, perhaps yearly, as more info becomes available, particularly about the realized value for $R_{t,t+1}$.

(3)
Holding a large CTE reserve, which earns interest at a lower rate than the risk discount rate, and which may not be needed when the policy matures, will have an adverse effect on the profitability of the policy.

Review topics for MA 398 final exam (Note: This is a 2-page document)

- DHW 3e 8.1-8.3: Probabilities and present values for insurance payments/continuous annuities in a continuous multi-state model. See esp. quiz and midterm problems. Includes multiple decrement models and permanent disability models (esp. constant forces) Maximum likelihood estimation of transition forces under constant forces assumption.

- DHW 3e Ch. 8: Present values of benefits and premium streams in a fully discrete multi-state model. See problems from notes, quizzes, midterm (recall the matrix-style presentation of transition probability data)

- DHW 3e Ch. 11 (2e Ch. 10) Using a salary scale. See, e.g., the homework problem from notes.

- DHW 3e Ch. 11 (2e Ch. 10) Projecting the value of the retirement fund in the context of a defined contribution plan, amount of annuity income, replacement ratio. Can include reversionary annuities as well as life annuities with a guarantee period (during which the annuity is an annuity certain and not life contingent).

- DHW 3e Ch. 11 (2e Ch. 10) Computing actuarial liability and normal cost in a final-salary-style defined benefit pension plan. Be able to use either the projected unit method or traditional/current unit method. Similar to review problem & problem covered in lecture. (I'll ask about AL at an age that is well before retirements are possible/permitted)

- DHW 3e Sec. 11.12 Computing actuarial liability in a retiree health plan (consider inflation, age cost factor, discount factor)

- DHW 3e Ch. 13 (2e Ch. 12) – The basics of profit testing:
  - Calculating Pr_t for the t^{th} year, given premium/contribution, expenses, previous reserves, reserves to set up, mortality info, etc.
  - Using mortality info and a given profit vector to compute a profit signature.
  - Obtaining a NPV of a contract from a profit signature at a specified hurdle rate.

- DHW 3e Ch. 14 – Projecting account value for UL and profit testing type-A or type-B.

- DHW 3e Ch. 15 – Equity-linked insurance: Computing policyholder's projected fund value after management charge; computing Pr_t and Π_t for insurer in a profit test.

We've covered a lot! I'll try to keep things straightforward and unambiguous. The exam will probably be around 20-30% pre-midterm material and the rest drawn from more recent material.

I will <u>not</u> ask anything outside of what's listed above. If you see a formula on the review problems, I will supply that formula in the exam.

MA398 Final Exam Review

1. Consider an employee a defined contribution plan.

   Her salary during the year [27, 28] was \$55,000, and she began contributing to her retirement fund at the end of that year.

a. Project the value of the retirement fund at retirement using the following assumptions:
   - She retires at exact age 60.0.
   - She contributes 5% of each year's annual salary at the end of the year.
   - Her salary increases by 2% on her birthday each year.
   - Money invested in the retirement earns 9% effective per year.

b. Upon retirement, the employee plans to purchase a fully discrete whole life annuity due with level payments of X, and with payments guaranteed for 10-years. (Take the view that it's a 10-year annuity certain with a 10-year deferred life annuity.)

   The annuity is priced using $i = 6\%$ with $\ddot{a}_{70} = 9.5$ and ${}_{10}p_{60} = .98$. Use (a) to **project the value of X** if the retiree spends the full account value to purchase this annuity product.

c. **Find the replacement ratio** using the assumptions in (a) and the projected annual annuity income payment X in (b).

2. Consider the following excerpt from a pension service table.

(Key: s_x = salary scale; $r_{64}{}^{exact}$ and $r_{65}{}^{exact}$ are expected numbers of age-retirements at exact ages 64 & 65; r_{64} is expected number of age-retirements at exact age 64.5)

x	ℓ_x	r_x	$\ddot{a}_x^{(12)}$
45	1,000,000	0	
⋮			
64^{exact}		5,000	*given*
64		4,000	*given*
65^{exact}	40,000	40,000	*given*

<u>Data</u>: (i) A 45-year old employee has a final-salary-style defined benefit pension plan allowing age-retirement only at ages 64, 64.5, or 65.

(ii) She earns 60,000 during [44, 45].

(iii) Salaries are projected using $s_y = (1.04)^y$

(iv) She has accumulated 15 years of pensionable service by age 45.

(v) The plan has an accrual rate of 1.5%, that is, the annual pension benefit is defined to be 1.5% of (salary in final year of employment) per year of service. There is no penalty for retiring at age 64 or 64.5. Pension benefits are paid monthly in advance.

a. Give an expression for the actuarial liability (i.e. reserve for pension benefit liability) at age 45 using the **projected unit** approach. (Your expression will be in terms of $\ddot{a}_x^{(12)}$ and v.)

b. Show how to find the normal cost in (a).

c. Give an expression for the actuarial liability using the **traditional/current unit** approach.

d. Show how to find the normal cost in (c).

Note: *<u>In both cases</u> the accrual factor used to compute* AL_x *is multiplied by the <u>number of years worked by age x</u> (<u>not</u> the number of years that will have been completed by retirement). The only liability that has actually accrued by age xis the result of employee work actually completed by that time.) The difference between (a) and (b) is in the way that the "final salary" is projected.*

PUC: use projected future final salary. TUC: use current/most recent salary.

3. Consider the problem of reserving for retirement health benefits for a 50-year-old employee. Make the following assumptions:

- Today, when the employee is exactly 50 years old, the premium to purchase one-year of coverage for a 60-year-old is \$10,000.
- At any point in time, it costs 2.5% more to purchase a year's coverage for a 61-year-old than for a 60-year-old, 2.5% more to purchase coverage for a 62-year-old than for a 61-year-old, etc. (2.5% more to purchase coverage for an $(n + 1)$-year old than for an n-year-old for each integer $n \geq 60$.)
- For each fixed $n = 60, 61, \ldots$, the premium to insure an n-year-old increases by 5% every year due to inflation.
- The effective rate of interest is $i = 4\%$.
- Out of $\ell_{50} = 100$ fifty-year-olds, we expect $r_{60}^{exact} = 30$ retirements at age 60 and $r_{65}^{exact} = 45$ retirements at age 65. No other retirements are expected.

a. Write a "term-by-term" expression for the actuarial value of total health benefits for a single 50-year-old employee.

b. Identify coefficients and a decimal number j for which the expression in (a) can be written in the form

$$\underline{\quad (coefficient) \quad}(\ddot{a}_{60@j}) + \underline{\quad (coefficient) \quad}(\ddot{a}_{65@j})$$

You should accomplish this by rewriting (a) in terms of life annuities with level annual payment amounts, discounted at a different rate j. Identify the coefficients and the rate j.

4. Consider a fully discrete 5-year term life insurance policy on (60).

   Project the profit Pr_4 emerging at time 4 for a policy that is in-force at time 3. under the following assumptions:

   - ${}_3V = 1500$
   - $\pi = 3000$
   - The insurer experiences expenses at time 3 of 5% of the premium.
   - Interest on invested premium and reserves is 4.5%/year.
   - The death benefit is 500,000.
   - $q_{63} = .002$
   - ${}_4V = 1500$

5. A four-year term insurance policy on (60) has profit vector

   $(Pr_0\ , 100, 200, 300, 400).$

   Assume the survival model ${}_tp_{60} = 1 - .001t$ for $t = 0, 1, 2, 3, 4$.

   The acquisition cost per policy is \$1000.

   Find the profit signature and compute the NPV at hurdle rate 10%.

6. Consider a fully discrete 10-year equity-linked contract issued to (40).

a. Determine the time-5 management charge and the fund balance after the management charge is deducted for a contract that remains in-force at time 5. Assume the following:

- The insured party's investment fund balance at time 4, immediately before premium payment, is \$24,100.
- At time 4, the insured party pays a premium of \$5000. 95% of the premium is allocated to the policyholder's investment fund, and the unallocated portion of the premium is transferred to the insurer.
- The fund earns 12% interest per year.
- At the end of each year, a management charge equal to .1% of the fund value is transferred to the insurer.

b. Project the profit Pr_5 emerging for the insurer for a contract that is assumed to be in-force at time 4. Assume:

- The insurer has expenses of \$10 due at the time the premium is paid.
- The insurer's funds earn 6% per year.
- The death benefit for each year is equal to 125% of the fund balance after the management fee has been deducted.
- $q_{44} = .0005$.
- The cash value (for surrendered policies) is equal to the year-end account value immediately after the management charge has deducted.
- Reserves are 0.

c. Compute the element π_5 of the profit signature under the following additional assumptions:

- The mortality decrement is modeled by $q_{40+t} = .0005$ for $t = 0, 1, 2, 3, 4, 5$.
- During the first year, 10% of those who survive to the year's end will choose to surrender their policies. No other surrenders occur.

(Do not compute the entire profit signature.)

7. Consider a continuous multistate model with the following states:

0 – healthy 1 – permanently disabled 2 – dead

Suppose that μ^{01}, μ^{02}, and μ^{12} are constant forces of transition, and that these are the only transitions possible.

a. Find the probability that a healthy person dies within 5 years. (Just set up the integral(s)—you do not need to simplify at all.)

b. Find the expected present value of a 5-year term insurance of 100 issued to a healthy life. Assume that the force of interest is a constant δ. Again, do not simplify the integral(s).

c. Find an expression for the probability that a person who is healthy at time 0 is in the disabled state at time 5.

d. Find an expression for ${}_5p_x^{\overline{00}}$.

8. For k = 0, 1, 2, p_{x+k}^{ij} is given by the *i, j* entry of

$$\begin{bmatrix} .8-.05k & .1 & .1+.05k \\ .2 & .6 & .2 \\ \_\_ & \_\_ & \_\_ \end{bmatrix}.$$ (0 – healthy; 1 – sick; 2 – dead)

Find EPV of death benefit of B for a 3-year term insurance policy on (x), assuming a start in State 0. *(Make a tree!)*

9. Consider a temporary disability model with states

   0 = healthy, 1 = temporary disability, 2 = dead.

   Assume constant forces of transition between integer ages, and consider the following individuals:

   - A disabled 40-year-old who became healthy at age 40.5 but then died at age 40.6.

     $\overset{40}{\boxed{1}} \longrightarrow \overset{40.5}{\boxed{0}} \longrightarrow \overset{40.6}{\boxed{2}}$

   - A healthy 40-year-old who remained healthy the whole year.

     $\overset{40}{\boxed{0}} \longrightarrow \overset{41}{\boxed{0}}$

   - A healthy 40.8-year-old who remained healthy for the rest of the year. This person was not under observation prior to age 40.8.

     $\overset{40.8}{\boxed{0}} \longrightarrow \overset{41}{\boxed{0}}$

   - A healthy 40-year-old who became disabled at age 40.1 and then died at age 40.5.

     $\overset{40}{\boxed{0}} \longrightarrow \overset{40.1}{\boxed{1}} \longrightarrow \overset{40.5}{\boxed{2}}$

   Estimate the transition probabilities μ_{40}^{01} and μ_{40}^{12}. (The MLE estimate for μ_{40}^{ij}, $i \neq j$, is given by $\hat{\mu}_{40}^{ij} = \frac{\#\text{ transitions } i \to j}{\text{Total of all time spent in } (i)}$.)

$\hat{\mu}_{40}^{01} = \frac{1}{.1+.2+1+.1}$ $\qquad$ $\hat{\mu}_{40}^{12} = \frac{1}{.4+.5}$

10 Universal life insurance. Assume that reserves are set equal to account values.

Assume that withdrawals occur at the end of any year with independent rate 4%.

(a) For a Type A product with 100,000 level death benefit:

${}_t\mathrm{AV} =$

${}_t\mathrm{Pr} =$

(b) For a Type B product with face amount 100,000 (so d.b. = 100,000 + ${}_t\mathrm{AV}$)

***cw:** If needed, just say how to modify (a).*

${}_t\mathrm{AV} =$

${}_t\mathrm{Pr} =$

MA398 Final Exam Review

1. Consider an employee a defined contribution plan at exact age 28.
Her salary during the year [27, 28] was $55,000.

a. Project the value of the retirement fund at retirement using the following assumptions:
- She retires at exact age 60.0.
- She contributes 5% of her annual salary at the end of each year.
- Her salary increases by 2% on her birthday each year. → $S_y = (1.02)^y$
- Money invested in the retirement earns 9% effective per year.

b. Upon retirement, the employee plans to purchase a fully discrete whole life annuity due with level payments of X, and with payments guaranteed for 10-years. (Take the view that it's a 10-year annuity certain with a 10-year deferred life annuity.)

The annuity is priced using $i = 6\%$ with $\ddot{a}_{70} = 9.5$ and ${}_{10}p_{60} = .98$. Use (a) to project the value of X if the retiree spends the full account value to purchase this annuity product.

c. Find the replacement ratio using the assumptions in (a) and the projected annual annuity income payment X in (b).

(a) $.05(55000)(1.09)^{32} + .05(55000 \times 1.02)(1.09)^{31} + .05(55000 \times (1.02)^2)(1.09)^{30} + \cdots + (33^{rd}\ term)$

one term for each year in [28, 60)

$= .05(55000)(1.09)^{32} \times \dfrac{1 - (1.02/1.09)^{33}}{1 - (1.02/1.09)}$

$= 599,492.02$

(b) $599,492.02 = X \cdot (\underbrace{\ddot{a}_{\overline{10|}}}_{1+v+\cdots+v^9 = \frac{1-v_{6\%}^{10}}{1-v_{6\%}}} + \underbrace{{}_{10}E_{60}}_{\underbrace{{}_{10}p_{60}}_{.98} \times v_{6\%}^{10} \times 9.5} \cdot \ddot{a}_{70})$

$599,492.02 = X \cdot (7.80169 + .98 \times v_{6\%}^{10} \times 9.5)$

$46,113.54 = X$

$S_y = (1.02)^y$

(c) $\dfrac{Sal[59,60]}{Sal[27,28]} = \dfrac{1.02^{59}}{1.02^{27}} \rightarrow Sal[59,60] = 55000 \times 1.02^{32} = 103649.73 \rightarrow$

Replacement Ratio $= \dfrac{46113}{103649} \approx .445$ 44.5%

2. Consider the following excerpt from a pension service table.

(Key: s_x = salary scale; r_{64}^{exact} and r_{65}^{exact} are expected numbers of age-retirements at exact ages 64 & 65; r_{64} is expected number of age-retirements at exact age 64.5)

x	ℓ_x	r_x	$\ddot{a}_x^{(12)}$
45	1,000,000	0	
⋮			
64^{exact}		5,000	*given*
64		4,000	*not given*
65^{exact}	40,000	40,000	*given*

<u>Data</u>: (i) A 45-year old employee has a final-salary-style defined benefit pension plan allowing age-retirement only at ages 64, 64.5, or 65.

(ii) She earns 60,000 during [44, 45].

(iii) Salaries are projected using $s_y = (1.04)^y$

(iv) She has accumulated 15 years of pensionable service by age 45.

(v) The plan has an accrual rate of 1.5%, that is, the annual pension benefit is defined to be 1.5% of (salary in final year of employement) per year of service. There is no penalty for retiring at age 64 or 64.5. Pension benefits are paid monthly in advance.

a. Give an expression for the actuarial liability (i.e. reserve for pension benefit liability) at age 45 using the **projected unit** approach.

b. Show how to find the normal cost in (a).

c. Give an expression for the actuarial liability using the **traditional/current unit** approach.

d. Show how to find the normal cost in (c).

(a) $AL_{45} = \frac{5000}{1,000,000} \times v^{19} \times .015\,(60,000 \times 1.04^{19}) \times \underset{\text{yrs}}{15}\, \ddot{a}_{64}^{(12)}$

$+ \frac{4000}{1,000,000} \times v^{19.5} \times .015\,(60000 \times 1.04^{19.5}) \times 15\, \ddot{a}_{64.5}^{(12)}$

$+ \frac{40,000}{1,000,000} \times v^{20} \times .015\,(60,000 \times 1.04^{20}) \times 15\, \ddot{a}_{65}$

(b) For TUC before mid-yr ret's start happening,
$NC_{45} = \frac{1}{15} \times AL_{45}$

(c) Same as (a), but delete the salary increase factors $1.04^{19}, \ldots, 1.04^{20}$.

(d) For TUC before mid-yr ret's:
$AL_{45} + NC_{45} = (\text{no mid-yr}) + \frac{16}{15} \times 1.04 \times AL_{45}$

rescale yrs accrued

upgrade salary

Solve

$NC_{45} = \left(\frac{16}{15} \times 1.04 - 1\right) \cdot AL_{45}$

3. Consider the problem of reserving for retirement health benefits for a 50-year-old employee. Make the following assumptions:

- Today, when the employee is exactly 50 years old, the premium to purchase one-year of coverage for a 60-year-old is $10,000.
- At any point in time, it costs 2.5% more to purchase a year's coverage for a 61-year-old than for a 60-year-old, 2.5% more to purchase coverage for a 62-year-old than for a 61-year-old, etc. (2.5% more to purchase coverage for an $(n+1)$-year old than for an n-year-old for each integer $n \geq 60$.)
- For each fixed $n = 60, 61, \ldots$, the premium to insure an n-year-old increases by 5% every year due to inflation.
- The effective rate of interest is $i = 4\%$.
- Out of $\ell_{50} = 100$ fifty-year-olds, we expect $r_{60}^{exact} = 30$ retirements at age 60 and $r_{65}^{exact} = 45$ retirements at age 65. No other retirements are expected.

a. Write a "term-by-term" expression for the actuarial value of total health benefits for a single 50-year-old employee.

b. Identify coefficients and a decimal number j for which the expression in (a) can be written in the form

$j = -2.8945\%$

$\underset{3301.27}{\underline{(coefficient)}}\ (\ddot{a}_{60@j}) + \underset{5735.25}{\underline{(coefficient)}}\ (\ddot{a}_{65@j})$

You should accomplish this by rewriting (a) in terms of life annuities with level annual payment amounts, discounted at a different rate j. Identify the coefficients and the rate j.

$$\underbrace{\frac{30}{100}}_{\Pr[\text{ret } @ 60]} \times \underbrace{(1.04)^{-10}}_{\text{discount}} \times \Big[10{,}000 \underbrace{(1.05)^{10}}_{\text{inflation}} + p_{60} \times 10{,}000 \underbrace{(1.05)^{11}}_{\text{infl.}} \underbrace{(1.02)}_{\text{age cost}} \underbrace{(1.04)^{-1}}_{\text{discount}} + {}_2p_{60} \times 10{,}000 (1.05)^{12} (1.02)^2 (1.04)^{-2} + \cdots \Big]$$

$$\text{Let } v_j = \frac{1.05 \times 1.02}{1.04}$$

$$\text{So } 1+j = \frac{1.04}{1.05 \times 1.02} \approx .971055$$

$$j = -.028945$$

$$+ \frac{45}{100} \times (1.04)^{-15} \Big[10{,}000 (1.02)^5 (1.05)^{15} + p_{65} \times 10000 (1.02)^6 (1.05)^{16} (1.04)^{-1} + {}_2p_{65} \times 10000 (1.02)^7 (1.05)^{17} (1.04)^{-2} + \cdots \Big]$$

$$= \underbrace{\frac{30}{100} \times (1.04)^{-10} \times 10000 (1.05)^{10}}_{3301.27} \ddot{a}_{60@\underbrace{(-2.8945\%)}_{j\%}} + \underbrace{\frac{45}{100} (1.04)^{-15} (10000) (1.02)^5 (1.05)^{15}}_{5735.25} \ddot{a}_{65@j\%}$$

same -2.8945%

4. Consider a fully discrete 5-year term life insurance policy on (60).

Project the profit Pr_4 emerging at time 4 for a policy that is in-force at time 3. under the following assumptions:

- ${}_3V = 1500$
- $\pi = 3000$
- The insurer experiences expenses at time 3 of 5% of the premium.
- The death benefit is 500,000.
- $q_{63} = .002$ $\quad i = 4.5\%$
- ${}_4V = 1500$

$$Pr_4 = (\underbrace{{}_3V}_{1500} + \underbrace{\pi}_{3000} - \underbrace{.05\pi}_{5\%(3000)}) \times 1.045 - \underbrace{.002}_{q_{63}} \times 500000 - \underbrace{.998}_{p_{63}} \times \underbrace{{}_4V}_{1500}$$

$$= 2048.75$$

5. A four-year term insurance policy on (60) has profit vector

$(Pr_0, 100, 200, 300, 400)$.

Assume the survival model $\cancel{p_{60+t}}\ {}_tp_{60} = 1 - .001t$ for $t = 0, 1, 2, 3, 4$.

The acquisition cost per policy is \$1000.

Find the profit signature and compute the NPV at hurdle rate 10%.

$\Pi_0 = Pr_0 = -1000$

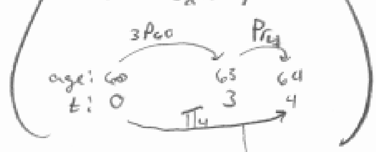
$\Pi_1 = Pr_1 = 100$

$\Pi_2 = p_{60} \times Pr_2 = .999 \times 200 = 199.8$

$\Pi_3 = {}_2p_{60} \times Pr_3 = .998 \times 300 = 299.4$

$\Pi_4 = {}_3p_{60} \times Pr_4 = .997 \times 400 = 398.8$

For example



$$NPV_{10\%} = -1000 + 100\, v_{10\%} + 199.8\, v_{10\%}^2 + 299.4\, v_{10\%}^3 + 398.8\, v_{10\%}^4 = -246.64$$

6. Consider a fully discrete 10-year equity-linked contract issued to (40).

a. Determine the time-5 management charge and the fund balance after the management charge is deducted for a contract that remains in-force at time 5. Assume the following:

- The insured party's investment fund balance at time 4, immediately before premium payment, is \$24,100.
- At time 4, the insured party pays a premium of \$5000. 95% of the premium is allocated to the policyholder's investment fund, and the unallocated portion of the premium is transferred to the insurer.
- The fund earns 12% interest per year.
- At the end of each year, a management charge equal to .1% of the fund value is transferred to the insurer.

$\to 95\% (5000) = 4750 \leftarrow AP$

So $UAP = 250$

$$F_5 = \underbrace{(24{,}100}_{F_4} + \underbrace{4750}_{AP})(1.12) - \underbrace{MC}_{32.31} = \overline{F_5} = 32279.69$$

$$C = .001[(24100 + 4750)(1.12)]$$

b. Project the profit Pr_5 emerging for the insurer for a contract that is assumed to be in-force at time 4. Assume:

- The insurer has expenses of \$10 due at the time the premium is paid.
- The insurer's funds earn 6% per year.
- The death benefit for each year is equal to 125% of the fund balance after the management fee has been deducted. ↳ part of which is the fund itself.
- $q_{44} = .0005$.
- The cash value (for surrendered policies) is equal to the year-end account value immediately after the management charge has deducted.
- Reserves are 0.

c. Compute the element π_5 of the profit signature under the following additional assumptions:

- The mortality decrement is modeled by $q_{40+t} = .0005$ for $t = 0, 1, 2, 3, 4, 5$.
- During the first year, 10% of those who survive to the year's end will choose to surrender their policies. No other surrenders occur.

(Do not compute the entire profit signature.)

(b)

$$Pr_5 = (\underbrace{0}_{4V} + \underbrace{250}_{UAP} - \underbrace{10}_{exp.})(1.06) - \underbrace{q_{44}}_{.0005}^{die} \times .25(\overline{F_5}\ 32279.69) - q^{withdraw} \times 0 - p^{(\tau)} \times {}_5V\ 0 = \boxed{250.365}$$

(c)

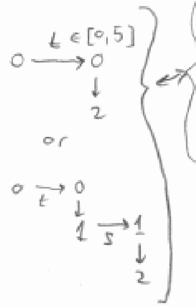
$${}_4p_{40}^{(\tau)} = \underbrace{(.9995)}_{\text{alive @ } t=1}\underbrace{(.9)}_{\text{Don't surrender}}\ \underbrace{(.9995)^3}_{\text{survive each of next 3 yrs}} \to S_0$$

$$\boxed{\pi_5 = .9 \times (.9995)^4 \times Pr_5 = 224.88}$$

7. Consider a continuous multistate model with the following states:

0 – healthy 1 – permanently disabled 2 – dead

Suppose that μ^{01}, μ^{02}, and μ^{12} are constant forces of transition, and that these are the only transitions possible.



a. Find the probability that a healthy person dies within 5 years. (Just set up the integral(s)—you do not need to simplify at all.)

$$\int_{t=0}^{5} e^{-(\mu^{01}+\mu^{02})t}\cdot\mu^{02}\,dt + \int_{t=0}^{5}\int_{s=0}^{5-t} e^{-(\mu^{01}+\mu^{02})t}\cdot\mu^{01}\cdot e^{-\mu^{12}s}\cdot\mu^{12}\,ds\,dt$$

b. Find the expected present value of a 5-year term insurance of 100 issued to a healthy life. Assume that the force of interest is a constant δ. Again, do not simplify the integral(s).

$$100\left[\int_{t=0}^{5} e^{-(\mu^{01}+\mu^{02})t}\cdot\mu^{02}\cdot e^{-\delta t}\,dt + \int_{t=0}^{5}\int_{s=0}^{5-t} e^{-(\mu^{01}+\mu^{02})t}\cdot\mu^{01}\cdot e^{-\mu^{12}s}\cdot\mu^{12}e^{-(s+t)\delta}\,ds\,dt\right.$$

c. Find an expression for the probability that a person who is healthy at time 0 is in the disabled state at time 5.

$$\int_{t=0}^{5} e^{-(\mu^{01}+\mu^{02})t}\cdot\mu^{01}\cdot e^{-\mu^{12}(5-t)}\,dt$$

and stay there! to t=5

d. Find an expression for ${}_5p_x^{\overline{00}}$.

$$e^{-(\mu^{01}+\mu^{02})\times 5}$$

Matrix Top row

$k=0$

p_x^{0j}: .8 .1 .1

$k=1$

p_{x+1}^{0j}: .75 .1 .15

$k=2$

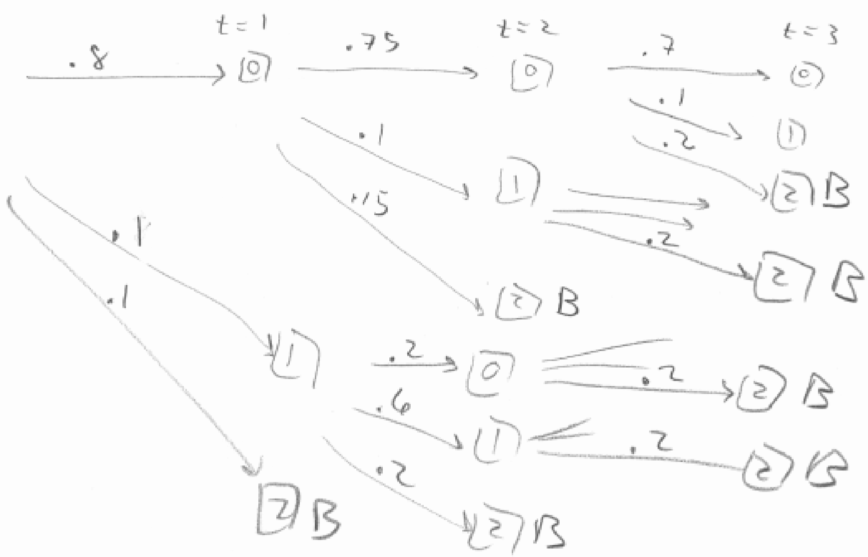
p_{x+2}^{0j}: .7 .1 .2

8. For $k = 0, 1, 2$, p_{x+k}^{ij} is given by the i, j entry of

$$\begin{bmatrix} .8-.05k & .1 & .1+.05k \\ .2 & .6 & .2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find EPV of death benefit of B for a 3-year term insurance policy on (x), assuming a start in State 0. *(Make a tree!)*

$$EPV = B\times\left[\begin{array}{l}(.8)(.75)(.2)v^3 + (.8)(.1)(.2)v^3 \\ \quad + (.8)(.15)v^2 \\ + (.1)(.2)(.2)v^3 + (.1)(.6)(.2)v^3 \\ \quad + (.1)(.2)v^2 \quad + .1v\end{array}\right]$$



9.

Consider a temporary disability model with states
0 = healthy, 1 = temporary disability, 2 = dead.

Assume constant forces of transition between integer ages, and consider the following individuals:

- A disabled 40-year-old who became healthy at age 40.5 but then died at age 40.6.

$$\underset{40}{\boxed{1}} \xrightarrow[\text{in } \boxed{1}]{.5\text{ yr}} \underset{40.5}{\boxed{0}} \xrightarrow[\text{in } \boxed{0}]{.1\text{ yr}} \underset{40.6}{\boxed{2}}$$

- A healthy 40-year-old who remained healthy the whole year.

$$\underset{40}{\boxed{0}} \xrightarrow[1\text{ yr in } \boxed{0}]{} \underset{41}{\boxed{0}}$$

- A healthy 40.8-year-old who remained healthy for the rest of the year. This person was not under observation prior to age 40.8.

$$\underset{40.8}{\boxed{0}} \xrightarrow[\text{in } \boxed{0}]{.2\text{ yr}} \underset{41}{\boxed{0}}$$

- A healthy 40-year-old who became disabled at age 40.1 and then died at age 40.5.

$$\underset{40}{\boxed{0}} \xrightarrow[\text{in } \boxed{0}]{.1\text{ yr}} \underset{40.1}{\boxed{1}} \xrightarrow[\text{in } \boxed{1}]{.4\text{ yr}} \underset{40.5}{\boxed{2}}$$

Estimate the transition probabilities μ_{40}^{01} and μ_{40}^{12}. (The MLE estimate for μ_{40}^{ij}, $i \neq j$, is given by $\hat{\mu}_{40}^{ij} = \frac{\#\text{ transitions } i \to j}{\text{Total of all time spent in } (i)}$.)

Time in $\boxed{0}$: $.1 + 1 + .2 + .1 = 1.4$

(To get variance, square only the denominator)

$$\hat{\mu}_{40}^{01} = \frac{1}{1.4} \qquad \left(\text{and } \operatorname{var}\left[\hat{\mu}_{40}^{01}\right] = \frac{1}{(1.4)^2}\right)$$

$$\hat{\mu}^{12} = \frac{1}{.9} \qquad \left(\text{and } \operatorname{var}\left[\hat{\mu}_{40}^{12}\right] = \frac{1}{(.9)^2}\right)$$

Time in $\boxed{1}$: $.5 + .4 = .9$