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subject: Linear Algebra

date: 26 Feb 2024

keywords: Linear Functions, Kernel, Image

topic: 3.1 Functions on Vector Spaces

0 Quiz Review from Friday

Def: <u>Dimension</u>: # basis vectors

Know these definitions!

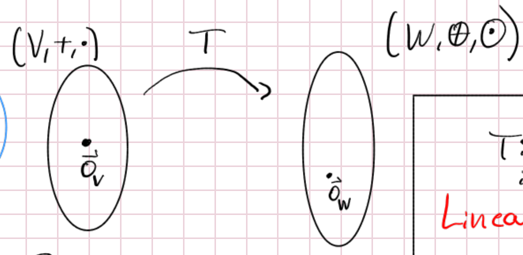
$$H = \text{span}\left\{ \begin{bmatrix} \vdots \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} \vdots \\ \vdots \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

$$\dim H = 2 \neq 4 = \dim \mathbb{R}^4$$

$$\Rightarrow \dim H^\perp = 4 - 2 = 2 \rightarrow \text{basis of } H^\perp \text{ has 2 vectors}$$

$$\vec{x} \in H^\perp \Leftrightarrow \vec{x} \text{ orthogonal to } H \Leftrightarrow \vec{x} \text{ orthogonal to basis vectors of } H$$

1 Functions



Oh, Great Potato!

Definition

$T: V \rightarrow W$ is a **Linear Transformation** if

$\forall \vec{u}, \vec{v} \in V,\ a \in \mathbb{R},$

- $T(\vec{u} + \vec{v}) = T(\vec{u}) \oplus T(\vec{v})$
- $T(a \cdot \vec{u}) = a \odot T(\vec{u})$

This has a lot of nice structure!

10 U.S. Axioms
Bases
Orthogonality
Norm
⋮

Examples

1) $T: \mathbb{R}^3 \rightarrow P_2$ via $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto (x+y) + zt + yt^2$

Is T linear?

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3,\ a \in \mathbb{R}$$

1) $$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = (x_1+x_2+y_1+y_2) + (z_1+z_2)t + (y_1+y_2)t^2$$

$$= (x_1+y_1) + z_1 t + y_1 t^2 + (x_2+y_2) + z_2 t + y_2 t^2$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \checkmark$$

2) $$T\left(a\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} ax_1 \\ ay_1 \\ az_1 \end{bmatrix}\right) = (ax_1 + ay_1) + az_1 t + ay_1 t^2$$

$$= a(x_1 + y_1 + z_1 t + y_1 t^2)$$

$$= a\,T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) \checkmark$$

$\Rightarrow T$ is a linear transformation!

subject	date	keywords
topic		

Fun Fact

If T is linear, then $T(\vec{0}) = \vec{0}$ (Say $T: V \to W$)

Proof: $T(\vec{0}_V) = T(\vec{v} - \vec{v}) = T(\vec{v} + (-\vec{v})) = T(\vec{v}) + T(-\vec{v})$
$= T(\vec{v}) - T(\vec{v}) = \vec{0}_W$ $\blacksquare$

Examples

$T: \mathbb{R}^3 \to \mathbb{R}^3$ via $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ z \\ 7 \end{bmatrix}$

Not linear! $T(\vec{0}_{\mathbb{R}^3}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

But also...

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ z_1+z_2 \\ 7 \end{bmatrix}$$

$$\neq \begin{bmatrix} x_1+x_2+y_1+y_2 \\ z_1+z_2 \\ 14 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ z_1 \\ 7 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ z_2 \\ 7 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \quad ✗$$

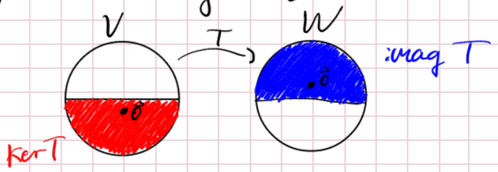
can also verify w/ counterexample or scalar axiom

Kernel and Image

Def: Kernel of T, denoted $\ker T = \{\vec{x} \in \operatorname{dom} T : T(\vec{x}) = \vec{0}\} \subseteq \operatorname{dom} T$

Def: Image of T, denoted $\operatorname{imag} T = \{T(\vec{x}) \in \operatorname{codom} T : \vec{x} \in \operatorname{dom} T\} \subseteq \operatorname{codom} T$

Pokéballs



Examples

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ via $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ 0 \end{bmatrix}$

Is T linear? If so...

- What are bases for $\ker T$ or $\operatorname{imag} T$?
- Is T injective and/or surjective?
- What are dimension of $\ker T$, $\operatorname{imag} T$, $(\ker T)^\perp$, $(\operatorname{imag} T)^\perp$

Is T injective? No! $T\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right)$

Is T surjective? No! $\nexists \begin{bmatrix} x \\ y \end{bmatrix}$ s.t. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

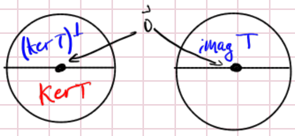
Is T linear? Yes! Check:

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1+x_2+y_2 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \checkmark$$

$$T\left(a\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} ax \\ ay \end{bmatrix}\right) = \begin{bmatrix} ax+ay \\ 0 \end{bmatrix} = a\begin{bmatrix} x+y \\ 0 \end{bmatrix} = aT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \checkmark$$

$$\ker T = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \vec{0}\right\} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x+y=0\right\} = \left\{\begin{bmatrix} x \\ -x \end{bmatrix} : x \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$\Rightarrow \dim \ker T = 1 \Rightarrow \dim(\ker T)^\perp = 1$



to be continued...