

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 0.2–3

0 Introduction

0.2 Introduction to differential equations

Differential Equation. A *differential equation* is an equation with a derivative in it.

Example 1.

$$\frac{d^2x}{dt^2} + x \frac{dx}{dt} = 6t$$

- What is x ? *dependent variable*
- What is t ? *independent variable*

$$\begin{aligned}\frac{dy}{dx} &\text{ vs } y' \text{ vs } \dot{y} \\ \frac{d^2y}{dx^2} &\text{ vs } y'' \text{ vs } \ddot{y}\end{aligned}$$

$$y'' + xy' = 6x$$

- What's the difference between this differential equation and the one before it?

Higher order! Also, odd (calc 1)

Solution. A *solution* for a differential equation is a function that satisfies the equation (makes the equation true). Any single solution is called a *particular solution*. The set of all solutions is called the *general solution*.

Example 2. The differential equation

$$y' = 3x^2$$

is very boring. Why?

- A particular solution is *specified constants*
- The general solution is *unknown constants (of integration)
i.e. all solutions*

Why is the equation in Example 1 *much* harder to solve?

ODE: single independent variable

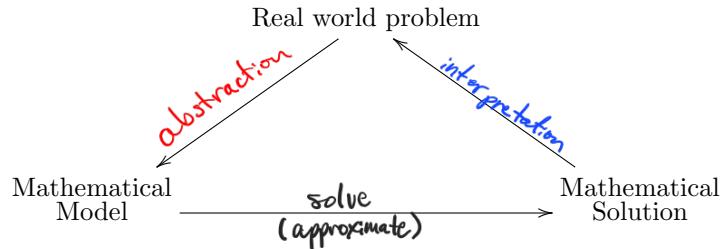
PDE: multiple independent variables

We will learn when and how differential equations can be solved analytically (almost never).

Barring that, we will learn how to approximate and use solutions.

¹We should probably come up with some more specific terminology.

Who cares about these things? Right.



Example 3. $P(t) = Ce^{kt}$ is the general solution for $\frac{dP}{dt} = kP$. Check this.

$$\frac{dP}{dt} = Ck e^{kt} = k(Ce^{kt}) = kP \quad \checkmark$$

- What does this have to do with the flow chart above?

Example 4. Show $y = \cosh t = \frac{1}{2}(e^t + e^{-t})$ is a particular solution for $\frac{d^2y}{dt^2} - y = 0$ on the interval $(-\infty, \infty)$.

$$\frac{dy}{dt^2} = \frac{1}{2}(e^t + (-(-e^{-t}))) = \frac{1}{2}(e^t + e^{-t}) = y \Rightarrow \frac{dy}{dt^2} = y \quad \checkmark$$

Example 5. For what values of r is $y = e^{rt}$ a solution for $y'' + y' - 6y = 0$?

$$\begin{aligned} r^2 e^{rt} + r e^{rt} - 6e^{rt} &= 0 \\ (r-2)(r+3) &= 0 \\ r = 2, -3 & \end{aligned}$$

0.3 Classification of differential equations

Here is a terrible wall of definitions. Enjoy!

Order. The *order* of a differential equation is the order of the highest derivative that appears in the equation. More specifically,^a a differential equation of order n is of the form

$$F\left(t, x(t), \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}\right) = 0,$$

where F is a function.

^aor is this more generally?

Autonomous. If F (as above) is independent of t , the differential equation is called *autonomous*. Otherwise, it is called *nonautonomous*.

the independent variable

Linear and homogeneous. A differential equation of order n is called *linear* if it is of the form

$$F\left(t, x(t), \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}\right) = a_n(t) \frac{d^n x}{dt^n} + \dots + a_1 \frac{dx}{dt} + a_0 x + b(t),$$

where the a_i 's and b are all functions of t . If $b(t) = 0$, then the differential equation is called *homogeneous*; otherwise, it is called *nonhomogeneous*.

“What is all this madness?” you may ask. Well, different classifications of differential equations require different techniques and strategies.

Example 6. Classify the following differential equations:

- $t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + 2y - \sin t = 0$
 - Non-autonomous : not indep of t
 - Linear : l.c. of $\frac{d^n x}{dt^n}$
 - 2nd order
 - non homogeneous : $b(t) \neq 0$
- $y'' + \underline{yy'} = 0$
 - 2nd order
 - Autonomous
 - non linear

Eg: $y' = yx$ Problem w/ integrating directly

$$\text{Eg } y' = xe^x \quad \text{or} \quad \frac{dy}{dx} = xe^x$$

$$y = \int xe^x dx = [xe^x - e^x + C] \rightarrow \begin{aligned} &+ x \quad e^x \\ &- 1 \quad e^x \\ &+ C \quad e^x \end{aligned} \quad \begin{aligned} &\text{General} \\ &\text{Solution} \end{aligned}$$

↳ initial condition
 $y_0 = y(x_0)$ to identify
 a particular solution

3

$$y(0) = 0 \rightarrow e^0(0-1) + C = 0 \rightarrow C = 1$$

An ODE w/
initial conditions
is an Initial Value
Problem (IVP)

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Sections 1.1–2

1 First order ODEs

In case no one mentioned it, and *ODE* is an ordinary differential equation, which is just a differential equation with no partial derivatives (those are called PDEs). The word “ordinary” is just used to let you know that since there are no partial derivatives, you won’t have to do anything too silly. While this course deals exclusively in ODEs, we maintain the right to do silly things.

1.1 Integrals as solutions

Which is easier to solve?

- $\frac{dy}{dx} = f(x, y)$
- $\frac{dy}{dx} = f(x)$

Why?

Example 1. Solve $y' = xe^x$. What do you need to identify a single particular solution?

Example 2. Solve $y' = xe^x$, $y(0) = 0$.

IVP. An *IVP*, or *initial value problem*, is an ODE with enough initial conditions to identify a single particular solution.

Can we solve $\frac{dy}{dx} = f(y)$? Why is this harder?

Here's a fun fact from Calculus 1 that will help:

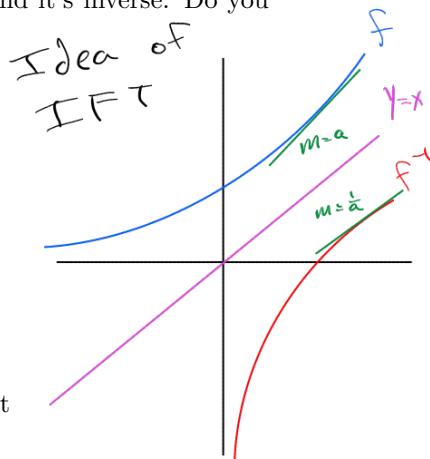
Inverse Function Theorem. If $y(x)$ is continuously differentiable and has a nonzero derivative at x_0 , then

$$(y^{-1})'(y(x_0)) = \frac{1}{y'(x_0)}.$$

That is, the derivative of the inverse at $y(x_0)$ is the reciprocal of the derivative at x_0 .

This is a really neat theorem. Draw the graph of a nonlinear one-to-one function and its inverse. Do you see why this theorem is true?

Don't forget that $\frac{dy}{dx} = f(y)$. When $x(y) = y^{-1}$ is differentiable, we have



Then from the Inverse Function Theorem, we know that

(when y is continuously differentiable and has a nonzero derivative). Now we can just

Example 3 (Exercise 1.1.6). Solve $y' = (y-1)(y+1)$, $y(0) = 3$.

$$\frac{dy}{dx} = -\quad \text{Not integrable}$$

See next page
for sol'n

$$\frac{dx}{dy} = (y^{-1})' \quad \text{and} \quad \frac{1}{y'} = \frac{1}{f(y)} \quad (\text{not precise, but ignore it})$$

$$\begin{cases} \text{IFT} \\ \frac{dx}{dy} = \frac{1}{f(y)} \end{cases}$$

$$2x - 2c = \ln|y+1| - \ln|y-1|$$

$$2x - 2c = \ln\left|\frac{y+1}{y-1}\right|$$

$$e^{2x-2c} = \frac{y+1}{y-1}$$

$$ye^{2x-2c} - e^{2x-2c} = y+1$$

$$\begin{aligned} & \text{General Solution: } y = \frac{1 + e^{2x} e^{-2c} C_0}{e^{2x} e^{-2c} - 1} \\ & \text{Particular solution: } y = \frac{1 + 2e^{2x}}{2e^{2x} - 1} \\ & 1 = A(y+1) + B(y-1) \\ & B = \frac{1}{2}, A = \frac{1}{2} \\ & y(0) = 3 \rightarrow 3 = \frac{1 + C_0}{e^{-2c} - 1} \rightarrow 3e^{-2c} - 3 = 1 + C_0 \rightarrow C_0 = 2 \end{aligned}$$

$$\begin{aligned} & \frac{1}{y-1} dy - \frac{1}{2} \int \frac{1}{y+1} dy \\ & = \frac{1}{2} \ln|y-1| - \frac{1}{2} \ln|y+1| + C \\ & \text{Solve for } y \end{aligned}$$

$$\int \frac{dx}{dy} dy = \int \frac{1}{(y-1)(y+1)} dy$$

$$y' = (y-1)(y+1); y(0) = 3$$

Partial
Fractions

$$\frac{dy}{dx} \xrightarrow[\text{Theorem}]{\text{Inverse Function}} \frac{dx}{dy} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

$$\Rightarrow 1 = A(y+1) + B(y-1)$$

$$y=1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$y=-1 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$\Rightarrow x = \int \frac{dx}{dy} dy = \int \frac{1}{(y-1)(y+1)} dy = \frac{1}{2} \int \frac{1}{y-1} dy - \frac{1}{2} \int \frac{1}{y+1} dy$$

$$= \frac{1}{2} \ln|y-1| - \frac{1}{2} \ln|y+1| + C$$

$$\Rightarrow 2x - 2C = \ln \left| \frac{y-1}{y+1} \right|$$

$$\Rightarrow e^{2x} \underbrace{e^{-2C}}_{C_0} = \frac{y-1}{y+1} \Rightarrow C_0 e^{2x} y + C_0 e^{2x} = y-1$$

$$\Rightarrow (C_0 e^{2x} - 1)y = - (1 + C_0 e^{2x}) \Rightarrow y = \frac{1 + C_0 e^{2x}}{1 - C_0 e^{2x}}$$

$$\underline{y(0)=3} \Rightarrow 3 = \frac{1 + C_0 e^{2(0)}}{1 - C_0 e^{2(0)}}$$

$$\Rightarrow 3 - 3C_0 = 1 + C_0 \Rightarrow 2 = 4C_0 \Rightarrow C_0 = \frac{1}{2}$$

Check: $\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = \frac{\cancel{3}/2}{\cancel{1}/2} = 3 \checkmark$

$$\Rightarrow y = \frac{1 + \frac{1}{2} e^{2x}}{1 - \frac{1}{2} e^{2x}} \left(\frac{2}{2} \right) = \boxed{\frac{2 + e^{2x}}{2 - e^{2x}}}$$

1.2 Slope fields

Recall that, in general, first order equations are of the form

$$y' = f(x, y),$$

where f is any function you like, depending on *both* x and y . If f depends on just one of these variables, we saw in the last section that you can just integrate to solve.

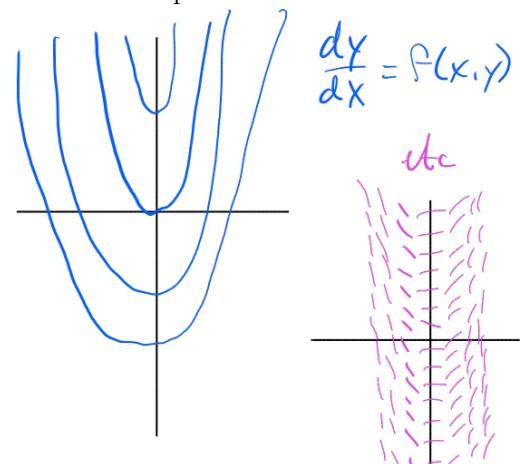
What does the equation $y' = f(x, y)$ mean? It takes x and y values and assigns (by f) a value to y' , often interpreted as *slope*. That is,

We can graph this!

Example 4. Let $y' = 2x$. Plot the slope field by hand and find the general solution. Compare them.

$$y = x^2 + C \text{ or general solution}$$

ODE relate slope to "indep. and dep
fxn values"



Google “bluffton slope field” and plot a slope field by way of internet.

Example 5. Plot a slope field (via computer) for $y' = x/y$. Beware computers.

What's wrong here?

Problems!
Infinite slope?
Multiple solns?

Example 6. Plot a slope field (via computer) for $y' = 2\sqrt{|y|}$. Beware intuition.

What's wrong here?

Given a problem, there are two basic questions:

- 1.
- 2.

$$\frac{dy}{dx} = f(x, y) \text{ is ODE}$$

Picard's Theorem.^a If $f(x, y)$ is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0) , then a solution to the IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

exists near x_0 and is unique.

^aAlso commonly referred to as the Fundamental Theorem of Existence and Uniqueness (FEU)

Example 7. $x' = x^{1/3}, x(0) = 0$ is a sufficiently simple-looking IVP, right? Show $x = 0$ is a solution, and for any nonnegative real α ,

$$x(t) = \begin{cases} (\frac{2}{3}t)^{3/2}, & |t| < \alpha \\ 0, & t \leq -\alpha, \alpha \leq t \end{cases}$$

is also a solution. There are an uncountable number of solutions to this IVP.

What is happening here?

$$f(t, x) = x^{1/3} \quad \frac{\partial f}{\partial x} = \frac{1}{3} x^{-2/3}$$

← not defined at $x=0$, which is the initial condition

Example 8. Show $y' = 1 + y^2, y(0) = 0$ has a unique solution $y = \tan x$ on $(-\pi/2, \pi/2)$.

$$\frac{dx}{dx} = \sec^2 x = 1 + \tan^2 x \quad \checkmark \quad \left. \begin{array}{l} \text{sln to IVP} \\ \checkmark \end{array} \right\}$$

$$y(0) = \tan(0) = 0$$

Picard's Theorem

$$f(x, y) = 1 + y^2$$

$$\frac{\partial f}{\partial y} = 2y \quad \text{Continuous everywhere!}$$

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Sections 1.3

Recall in Section 0.2–3 we agreed $\frac{dy}{dx} = f(x, y)$ tends to be harder than $\frac{dy}{dx} = f(x)$. That doesn't mean they are impossible.

Separable Equation. A first order ODE is *separable* if it can be written as $y' = f(x)g(y)$, where f and g are functions

Separable equations can be solved with Integration!

1.3 Separable equations

How can we manipulate $\frac{dy}{dx} = f(x)g(y)$ to solve the ODE?

Do you want to just multiply dx by both sides? What does that even mean?

$$\begin{aligned} \frac{1}{g(y)} \frac{dy}{dx} &= f(x) \\ y = h(x) \quad \frac{dy}{dx} &= h'(x)dx \\ \frac{1}{g(h(x))} h'(x) &= f(x) \Rightarrow \int \frac{1}{g(h(x))} h'(x) dx = \int f(x) dx \end{aligned}$$

$$\Rightarrow \int \frac{1}{g(y)} dy = \int f(x) dx$$

Despite the wondrous power of separable equations, there is still one minor issue. What happens when we can integrate, but we can't solve for y in a reasonable way?

Implicit Solutions. A solution to an ODE not of the explicit form $y = h(x)$.

$$\begin{aligned} \text{Example 1. Solve } (1+x)dy - ydx &= 0. \Rightarrow (1+x)dy = ydx \Rightarrow \frac{1}{y} dy = \frac{1}{1+x} dx \\ \Rightarrow \ln|y| &= \ln|1+x| + C \end{aligned}$$

$$y = C_0(1+x)$$

We may not want to, but we can actually solve for y for this solution. Let's do that.

$$\cot y \, dy = \frac{x}{\sec x} dx \Rightarrow \int \tan y \, dy = \int x \cos x \, dx$$

$$\ln |\sec y| = x \sin x + \cos x + C$$

$$\sec y = e^{\ln |\sec y|} = e^{x \sin x + \cos x + C}$$

Example 2. Solve $\sec(x)dy = x \cot(y)dx$

Example 3. You've found a dead body! Its temperature is $88.6^\circ F$ at 2am and $78.6^\circ F$ at 3am. The ambient air temperature is $68.6^\circ F$ from midnight to 3am. Estimate the time of death.

$$\frac{dT}{dt} = K(T - T_{\text{air}})$$

T temp (${}^\circ F$) Find murder o' clock
t time (h)

Newton's Law of Cooling

$$T(0) = 98.6 \quad \text{cooling began}$$

$$\frac{dT}{dt} = K(T - 68.6) \rightarrow \frac{1}{T-68.6} dT = -K dt$$

$$\ln |T - 68.6| = -kt + C$$

$$\rightarrow T - 68.6 = C_0 e^{-kt}$$

$$T = C_0 e^{-kt} + 68.6$$

$$T(0) = C_0 + 68.6 \rightarrow C_0 = 30$$

$$T(t) = 30e^{-kt} + 68.6$$

See next page for
Solu' →

$$\frac{2}{3} = e^{-kt_0} \rightarrow \ln \frac{2}{3} = -kt_0$$

$$\rightarrow k = \frac{1}{t_0} \ln \frac{2}{3}$$

$$\frac{1}{3} = e^{-k(t_0+1)} = e^{\frac{1}{t_0} \ln \left(\frac{2}{3}\right)(t_0+1)}$$

$$= e^{\frac{1}{t_0} \left(t_0 \ln \frac{2}{3} + \ln \frac{2}{3}\right)}$$

$$= e^{\ln \frac{2}{3} + \frac{1}{t_0} \ln \frac{2}{3}}$$

$$= e^{\ln \frac{2}{3}}$$

$$= \frac{2}{3} e^{\frac{1}{t_0} \ln \frac{2}{3}}$$

$$\rightarrow \frac{1}{2} = e^{\frac{1}{t_0} \ln \frac{2}{3}}$$

$$\rightarrow \ln \frac{1}{2} = \frac{1}{t_0} \ln \frac{2}{3} \rightarrow t_0 = \frac{\ln \frac{2}{3}}{\ln \frac{1}{2}} \stackrel{?}{=} 0.5849 \text{ h} \approx 35 \text{ min}$$

MURDER O'CLOCK
1:25 AM

T = body temperature ($^{\circ}\text{F}$)

t = time (h)

Newton's Law of Cooling: $\frac{dT}{dt} = k(T - T_{\text{air}})$

$$\sim \frac{dT}{T-68.6} = kt \Rightarrow \ln |T-68.6| = kt + C$$

$$\Rightarrow T-68.6 = C_0 e^{kt}$$

$$\Rightarrow T = C_0 e^{kt} + 68.6$$

Let $t=0$ be the time the body started cooling (98.6°F)

$$\Rightarrow 98.6 = C_0 e^{k(0)} + 68.6 \Rightarrow C_0 = 30$$

We have a system of 2 unknowns and 2 variables

$$\begin{cases} 88.6 = 30 e^{kt} + 68.6 \rightarrow \frac{2}{3} = e^{kt} \\ 78.6 = 30 e^{k(t+1)} + 68.6 \rightarrow \frac{1}{3} = e^{k(t+1)} = e^{kt} e^k \end{cases}$$
$$\frac{1}{3} = \frac{2}{3} e^{-k} \rightarrow e^{-k} = \frac{1}{2} \rightarrow k = \ln\left(\frac{1}{2}\right)$$
$$\rightarrow \frac{2}{3} = e^{(-\ln\frac{1}{2})t} \rightarrow \frac{2}{3} = \left(\frac{1}{2}\right)^t \rightarrow t = \frac{\ln \frac{2}{3}}{\ln \frac{1}{2}} = 0.5849 \text{ h} \approx 35 \text{ min} *$$

* = time after start of cooling

$$\Rightarrow \text{Murder 0'Clock} = 2 \text{ am} - 35 \text{ min} =$$

1:25 am

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Sections 1.4

Recall in the last section we looked at some “easy” cases of $y' = f(x, y)$. Here’s a slightly less easy one.

First Order Linear Equation. An ODE of the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is called *first order linear*. Additionally, we call this standard form for the first order linear equation.

1.4 First Order Linear Equations

How can we solve $\frac{dy}{dx} + P(x)y = f(x)$?

$$q(x) \frac{dy}{dx} + p(x)y + g(x) = 0$$

i) Divide by $q(x)$

$$\rightarrow \frac{dy}{dx} + \frac{p(x)}{q(x)}y = -\frac{g(x)}{q(x)}$$

$$\rightarrow \frac{dy}{dx} + P(x)y = f(x)$$

standard form

Looks vaguely like a product rule

1. Write in Standard Form
2. Find integrating factor

$$\mu(x) = M(x) = e^{\int P(x) dx}$$

3. Multiply both sides of standard form by μ

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

4. Undo product rule

$$\frac{d}{dx} [\mu(x)y] = \mu(x)f(x)$$

$$\begin{aligned} \frac{d}{dx} [\mu(x)y] &= \mu'(x)y + \mu(x)y' \\ &= P(x)\mu(x)y + \mu(x)\frac{dy}{dx} \end{aligned}$$

5. Integrate!

$$\int \frac{d}{dx} [\mu(x)y] dx = \int \mu(x)f(x) dx$$

|| // ↗
 $\mu(x)y$ 1 $y = \frac{\int \mu(x)f(x) dx}{\mu(x)}$

Let's look at an example!

Example 1. Find a general solution and find an interval on which the solution is defined.

$$\frac{dy}{dx} = y + e^x \quad \Rightarrow \quad y' - y = e^x$$

$$\mu(x) = e^{\int p(x) dx} = e^{\int -dx} = e^{-x}$$

$$\rightarrow e^{-x} y' - e^{-x} y = e^{-x} e^x \quad |$$

$$\frac{d}{dx}[e^{-x} y] = 1 \rightarrow e^{-x} y = x + C \rightarrow \boxed{y = x e^x + C e^x}$$

Example 2. Solve $x dy = (x \sin(x) - y) dx$

$$x \frac{dy}{dx} = x \sin x - y \rightarrow x y' + y = x \sin x \rightarrow y' + \frac{y}{x} = \sin x \quad (x \neq 0)$$

$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x \quad \leadsto \quad x y' + y = x \sin x \rightarrow \frac{d}{dx}[xy] = x \sin x$$

$$\rightarrow xy = \int x \sin x dx \quad \begin{matrix} t & \sin x \\ -1 & -\cos x \\ 0 & -\sin x \end{matrix} \quad \rightarrow xy = -x \cos x + \sin x + C \quad \rightarrow \boxed{y = -\cos x + \frac{\sin x}{x} + C \cdot \frac{1}{x}}$$

Example 3. Solve $y' = 2y + x(e^{3x} - e^{2x})$ given the initial condition of $y(0) = 2$.

$$y' - 2y = x e^{3x} - x e^{2x} \quad \mu(x) = e^{-2x} \quad 2 = -1 + C \rightarrow C = 3$$

$$\frac{d}{dx}[e^{-2x} y] = x e^{3x} - x \quad \begin{matrix} t & e^{3x} \\ -1 & e^{2x} \\ 0 & e^x \end{matrix} \quad \rightarrow e^{-2x} y = x e^{3x} - e^{2x} - \frac{1}{2} x^2 + C \quad \text{IVP} \quad \downarrow \quad \text{plug back in}$$

$$\rightarrow \boxed{y = -\frac{1}{2} x^2 e^{2x} + x e^{3x} - e^{3x} + 3 e^{2x}}$$

Example 4. Initially, 50 pounds is dissolved in a large tank holding **300 gallons of water**. A brine solution is pumped into the tank at a rate of **3 gallons per minute**, and the well-stirred solution is then pumped out at the same rate. If the concentration of the solution entering is **2 pounds per gallon**, determine the amount of salt in the tank at time t .

How much salt is present after 50 minutes? After a long time?

$$A(t) \text{ amount of salt at time } t \text{ (lbs)}$$

$$t \text{ time (min)} \quad \frac{dA}{dt} = \left(\frac{\text{rate in}}{\text{min}} - \frac{\text{rate out}}{\text{min}} \right) = \left(\frac{3 \text{ gal}}{\text{min}} \cdot \frac{2 \text{ lbs}}{\text{gal}} - \frac{3 \text{ gal}}{\text{min}} \cdot \frac{A(t) \text{ lbs}}{300 \text{ gal}} \right) = 6 - \frac{A(t)}{100}$$

$$A' + \frac{1}{100} A = 6$$

$$\mu = e^{0.01t} \quad \leadsto \quad A = 600 + \frac{C e^{-t}}{100} \quad \rightarrow \quad C = -550$$

Begin 12 Sept

Review: - Separable $\frac{dy}{dx} = f(x)g(y)$

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

- Linear $\frac{dy}{dx} + P(x)y = f(x)$

$$\text{Let } \mu(x) = e^{\int P(x) dx}$$

$$\text{then } y = \frac{1}{\mu(x)} \int \mu(x) f(x) dx$$

"The Day of Weird Subs"

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Sections 1.5

We have learned some really neat tricks to leverage separability and linearity and solve ODEs. When all of those things fail, here's the next thing you try:

Homogeneous ODE. A first order ODE is called homogeneous if it can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

You may notice that this word has been used before. We give a different definition here because it made sense to someone at some point. Use context to determine which version of "homogeneous" you're dealing with.

1.5 Substitution

How can we solve $xy' + y + x = 0$ with initial condition $y(1) = 1$?

Subtract x , divide by it!

$$\rightarrow y' + \frac{1}{x}y = -1$$

We could use linear techniques ... or substitute

$$v = \frac{y}{x} \Rightarrow y = xv \quad \text{so } y' = v + xv'$$

$v + xv' + v = -1$
 $v' + \frac{2}{x}v = -\frac{1}{x}$
this eqn is also linear... so we got
Nowhere. Do Integrating factor method on either \star
 $\Rightarrow y = \frac{3-x^2}{2x}$ (use IVP to solve for C)

y is our dependent variable, so
get rid of them all w/ v

Substitution problems are a lot like ice cream. They come in many flavors, and if you have too many, your brain freezes.

Example 1. Solve the IVP $2yy' + 1 = y^2 + x$, $y(0) = 1$.

$v = y^2 \rightarrow v' = 2yy'$

Not linear!

Some times substitutions help

Linear!

$$\begin{aligned} v' + 1 &= v + x \\ \Rightarrow v' - v &= x - 1 \\ \Rightarrow \int (v' - v) dx &= \int (x - 1) dx \\ \Rightarrow v &= \frac{1}{e^{-x}} \int (xe^{-x} - e^{-x}) dx \\ &= \frac{1}{e^{-x}} (-xe^{-x} - e^{-x} + e^{-x} + c) \\ &= Ce^x - x \end{aligned}$$

$M(x) = e^{\int -dx} = e^{-x}$

+x	e^{-x}
-1	$-e^{-x}$
+0	e^{-x}

You may find it helpful to know the contents of this chart:

If you see...	Try this substitution!
xy'	$v = \frac{y}{x}$
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Example 2. Bernoulli's Equation!

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \quad n \in \mathbb{R}$$

sub: $v = y^{1-n}$ *NOT $n-1$, easy mistake*

$$v' = (1-n)y^{-n}y'$$

divide by y^n

$$\begin{cases} y' + P(x)y = f(x)y^n \\ y^{-n}y' + P(x)y^{1-n} = f(x) \end{cases}$$

$$\frac{1}{1-n}v' + P(x)v = f(x)$$

$$v' + (1-n)P(x)v = f(x)(1-n) \quad \text{Linear!}$$

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 1.6

Idea: we can qualitatively study autonomous eqns w/o solving them!

Recall,

Autonomous Equations. First order autonomous ODEs are of the form

$$\frac{dx}{dt} = f(x) \quad \leftarrow \begin{matrix} \text{No indep variables} \\ \text{as function input!} \end{matrix}$$

Also recall,

Newton's Law of Cooling.

$$\frac{dx}{dt} = -k(x - A) \quad \text{Autonomous}$$

Note that $x = A$ is a constant solution to any Newton's Law of Cooling problem.

1.6 Autonomous Equations

Constant solutions for an ODE are called equilibrium solutions (or equilibria solutions if you have more than one).

Any point x_0 on the x-axis where $\frac{dx}{dt} = f(x_0) = 0$ is called a critical point. Why?

derivative is zero! see Calc. I

Stability of Equilibria.

An equilibrium is *stable* (or attracting) if nearby solutions approach it as $t \rightarrow \infty$.
unstable (or repelling) if nearby solutions move away from it as $t \rightarrow \infty$.

Equilibria that are not stable or unstable are called *shunt* (or indifferent).

Goal: understand behavior of autonomous eqns through the study of
critical/equilibrium points

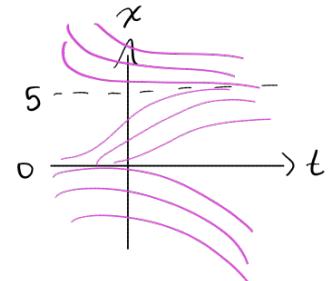
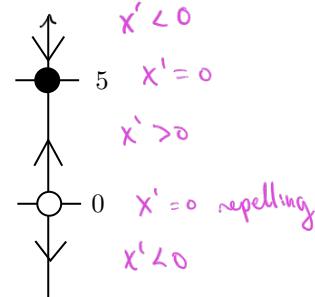
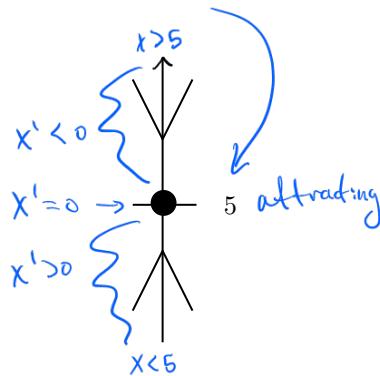
Compare the phase diagrams or phase portraits of the following ODEs equilibria.

$$x' = -0.3(x - 5) \quad \text{and} \quad x' = 0.1x(5 - x)$$

$$x' = 0 \text{ at } x = 5$$

$$x' = 0 \text{ at } 0 \text{ and } 5$$

*Think first
derivative test*



How do we construct these phase diagrams?

- 1.
- 2.
- 3.
- 4.

Example 1. Logistic growth with harvesting:

$$\frac{dx}{dt} = kx(M - x) - h \quad \text{where } k = 1 \text{ and } M = 2$$

general logistic growth

k = growth constant

M = carrying capacity

h = harvesting parameter

$$x' = x(2-x) - h$$

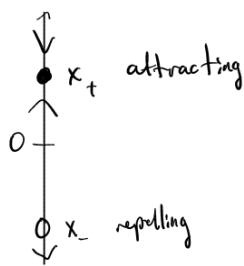
$$= -x^2 + 2x - h$$

$$\frac{-2 \pm \sqrt{4-4h}}{-2}$$

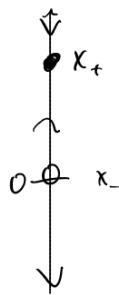
$$\rightarrow x_{\pm} = 1 \pm \sqrt{1-h}$$

Bifurcation theory!

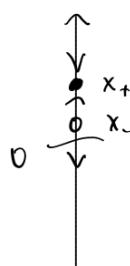
$$h < 0$$



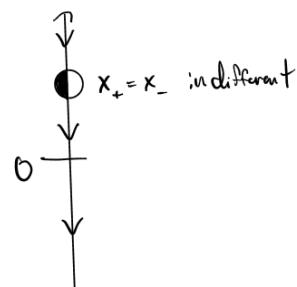
$$h = 0$$



$$0 < h < 1$$



$$h = 1$$



$$h > 1 \text{ no crit. points!}$$

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Section 1.7

Sometimes, we can't find a solution. If I just pick an ODE out of a bag, it is not going to be solved through any of the techniques we've looked at so far. So what can we do?

1.7 Euler's Method

Euler's Method is a way to approximate $x(t_1), x(t_2), x(t_3), \dots$ where $t_0 < t_1 < t_2 < \dots$

We accomplish this through the definite integral of both sides of

$$x' = f(t, x) \\ x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(t, x(t)) dt$$

Integrate, FTC says...

This implies that

$$x(t_1) = x_0 + \boxed{\int_{t_0}^{t_1} f(t, x(t)) dt}$$

Almost certainly impossible
to directly compute

We can use your favorite Riemann Sum evaluation technique. We'll use the Left Hand Rule.

What is $x(t_1)$? It's our first approximation; let's call it x_1 .

$$x_1 = x_0 + \underbrace{(t_1 - t_0)}_{\text{step } s} f(t_0, x(t_0))$$

We tend to make our t_i 's evenly spaced apart to create consistent step size s .

How can we approximate $x(t_2)$ (which we call x_w)?

$$x_2 = x_1 + s f(t_1, x_1)$$

How can we approximate $x(t_n)$ (that is, x_n)?

$$\sim x_{n+1} = x_n + s f(t_n, x_n)$$

What do we need to consider when determine how many steps to take in our Euler Method approximation?

Let's look at an example!

Example 1. $x' = x$, $x(0) = 1$. Given a step size of 0.2 and $t_0 < t < 1$.

↳ general soln: $x(t) = e^t$
 $x(0) = 1$
 $x(t) = e^t$

$$\begin{array}{ll} t=0 \quad x_0 = 1 & t=0.2 \quad x_1 = 1 + 0.2(1) = 1.2 \\ t=0.4 \quad x_2 = 1.2 + 0.2(1.2) = 1.44 & t=0.6 \quad x_3 = 1.728 + 0.2(1.728) = 2.0736 \\ t=0.8 \quad x_4 = 2.0736 + 0.2(2.0736) = 2.48832 & \\ t=1 \quad x_5 = 1.44 + 0.2(1.44) = 1.728 & \end{array}$$

For a more in depth analysis of step size, see page 24 of Lebl.

Example 2. Computer Time

Excel!



Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Section 1.8

1.8 Exact Equations

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, so we could graph f in \mathbb{R}^3 by $z = f(x, y)$. We could also take the *total differential* of f as follows:

$$\begin{aligned} z &= f(x, y) \\ dz &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \end{aligned}$$

For example, if $f(x, y) = x^2 + y^2$, then

$$dz = 2x dx + 2y dy$$

Thus, $2x dx + 2y dy = 0$ has

$$x^2 + y^2 = C$$

as the general solution.

Exact Equations. The differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is an *exact differential equation* if the left hand side of the equation is an exact differential.

! In other words, $M(x, y) dx + N(x, y) dy = 0$ is an exact differential equation if there is some function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, often called a *potential function*, such that

$$df = M(x, y) dx + N(x, y) dy.$$

Criterion for Exactness. Let $M(x, y)$ and $N(x, y)$ be continuous with continuous partial derivatives in some rectangular region R in \mathbb{R}^2 . Then $M(x, y) dx + N(x, y) dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Why? Note that $M(x, y) dx + N(x, y) dy = 0$ is exact if and only if there is a function f such that

Then by *Clairaut's Theorem*, this is true if and only if

↳ Symmetry of partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 1. Is $\frac{dy}{dx} = \frac{-2x-y}{x-1}$ exact?

$$(x-1)dy = -(2x+y)dx$$

$$(2x+y)dx + (x-1)dy = 0$$

$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1 \quad \checkmark \text{ Exact!}$$

Example 2. Solve $2xy dx + (x^2 - 1) dy = 0$

$$\text{Exact!} \quad \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Guess of potential function:

$$M = \frac{\partial F}{\partial x} \rightarrow F = \int M dx = \int 2xy dx = x^2y + C(y)$$

$$F = \int N dy = \int (x^2 - 1) dy = x^2y - y + C(x)$$

but also

$$x^2 - 1 = N = \frac{\partial F}{\partial y} = x^2 + C'(y)$$

$$\rightarrow F = x^2y - y + C_0$$

$$\rightarrow C'(y) = -1 \rightarrow C(y) = -y + C_0$$

$$\boxed{\text{answer: } F(x,y) = x^2y - y = C}$$

Example 3. Solve $\frac{(\sin(y) - y \sin(x))}{N} dx + \frac{(\cos(x) + x \cos(y) - y)}{N} dy = 0$

$$\frac{\partial M}{\partial y} = \cos y - \sin x \quad \frac{\partial N}{\partial x} = -\sin x + \cos y \quad \text{Exact!}$$

$$F = \int \frac{\partial F}{\partial x} dx = \quad \text{Complete later}$$

Example 4. Solve $(3x^2y + e^y) dx + (x^3 + xe^y - 2y) dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + e^y \quad \frac{\partial N}{\partial x} = 3x^2 + e^y$$

$$F = \int \frac{\partial F}{\partial x} dx = \int M dx = \int (3x^2y + e^y) dx = x^3y + xe^y + C(y)$$

$$\boxed{f(x,y) = x^3y + xe^y - y^2 = C}$$

$$N = \frac{\partial \int \frac{\partial F}{\partial x} dx}{\partial y} = x^3 + xe^y + C'(y) \rightarrow C(y) = -y^2$$

Example 5. Solve $(3x \cos(3x) + \sin(3x) - 3) dx + (2y + 5) dy = 0$

It's a trap! Separable

Example 6. Solve $(x + y) dx + (x \ln(x)) dy = 0$

$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = \ln x + 1 \quad \text{Not exact}$$

Linear in y

$$(x+y) + x \ln x \frac{dy}{dx} = 0$$

$$x \ln x \frac{dy}{dx} = -(x+y)$$

$$\frac{dy}{dx} = -\frac{(x+y)}{x \ln x} \Rightarrow \frac{dy}{dx} + \frac{1}{x \ln x} y = -\frac{1}{\ln x}$$

Example 7. Solve $y(x+y+1) dx + (x+2y) dy = 0$

$$\frac{\partial M}{\partial y} = x+y+1+y \quad \frac{\partial N}{\partial x} = 1 \quad \text{Not exact}$$

$$(x+2y) \frac{dy}{dx} = -yx - y^2 - y \quad \text{Not linear!}$$

$$V = xy \rightarrow y = \frac{V}{x} \rightarrow \frac{dy}{dx} = \frac{1}{V} - \frac{x}{V^2} \frac{dV}{dx}$$

Example 8. Solve $(-xy \sin(x) + 2y \cos(x)) dx + (2x \cos(x)) dy = 0$

Not Exact

$$\frac{\partial M}{\partial y} = -x \sin x + 2 \cos x \quad \frac{\partial N}{\partial x} = -2x \sin x + 2 \cos x$$

$$F = \int \frac{\partial F}{\partial y} dy = \int N dx = \int 2x \cos x dy = 2xy \cos(x) + C(x)$$

$$\frac{\partial}{\partial x} \left(\int \frac{\partial F}{\partial y} dy \right) = 2y \cos x - 2xy \sin x + C'(x) = 0$$

$$\Rightarrow f(x, y) = 2xy \cos(x) = c$$

Example 9. Solve $2xe^x - y + 6x^2 = \frac{dy}{dx}$

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Section 2.1

2.1 Second Order ODEs

Second Order Linear ODEs A second order linear ODE is of the form

$$A(x)y'' + B(x)y' + C(x)y = D(x)$$

However, we can always make our lives easier and divide by $A(x)$ to achieve

$$y'' + p(x)y' + q(x)y = f(x) \quad \text{General form}$$

Superposition Theorem. If y_1 and y_2 are solutions to the second order linear homogenous equation $y'' + p(x)y' + q(x)y = 0$, then for any constants C_1, C_2 ,

$$y = C_1y_1 + C_2y_2$$

is also a solution.

Let's take another look at the Fundamental Theorem for Existence and Uniqueness!

Fundamental Theorem for Existence and Uniqueness (revisited). Suppose p, q , and f are continuous on some interval I and a, b_0, b_1 are constants such that $a \in I$. The ODE

must contain initial conditions! $y'' + p(x)y' + q(x)y = f(x)$

has exactly one solution y on I satisfying $y(a) = b_0$ and $y'(a) = b_1$.

Example 1. Verify $y = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx)$ is a unique solution to $y'' + k^2y = 0, y(0) = b_0, y'(0) = b_1$.

$p(x) = 0, q(x) = k^2, f(x) = 0$
Continuous everywhere on any interval! Bahahaha!

$$\begin{aligned} \frac{dy}{dx} &= -kb_0 \sin(kx) + b_1 \cos(kx) \\ \frac{d^2y}{dx^2} &= -k^2 b_0 \cos(kx) - kb_1 \sin(kx) \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} + k^2y &= \left(-k^2 b_0 \cos(kx) - kb_1 \sin(kx)\right) + k^2 \left(b_0 \cos(kx) + \frac{b_1}{k} \sin(kx)\right) \stackrel{\checkmark}{=} 0 \\ y(0) &= b_0 \cos(0) + \frac{b_1}{k} \sin(0) = b_0 \checkmark & y'(0) &= -kb_0 \sin(0) + b_1 \cos(0) = b_1 \checkmark \end{aligned}$$

What does it mean for a set of functions to be linearly dependent?

$y_1, \dots, y_n : I \rightarrow \mathbb{R}$ are linearly dependent if $\exists c_1, \dots, c_n \in \mathbb{R}$ (not all zero)

when $\sum_{k=1}^n c_k y_k = 0 \quad \forall x \in I$

Example 2. Show $\sinh(x)$ and $\cosh(x)$ are linearly independent. (Recall, $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

$$c_1 \sinh(x) + c_2 \cosh(x) = \frac{c_1}{2}(e^x - e^{-x}) + \frac{c_2}{2}(e^x + e^{-x}) = \frac{(c_1+c_2)}{2}e^x + \frac{(c_2-c_1)}{2}e^{-x}$$

Sps Bwoc $\{\sinh x, \cosh x\}$ dependent. Then $c_1 \sinh(x) + c_2 \cosh(x) = 0 \forall x \in \mathbb{R}$. Note $e^x \neq 0 \forall x \in \mathbb{R}$, so $c_1 \sinh(x) + c_2 \cosh(x) = 0$ only if $\frac{c_1+c_2}{2} = 0$ and $\frac{c_2-c_1}{2} = 0$. $\hookrightarrow c_1 = c_2 = 0 \Rightarrow$ linearly independent!

Theorem. Let p, q be continuous functions and y_1, y_2 solutions to the ODE

$$y'' + p(x)y' + q(x)y = 0. \quad \text{linearly independent solns}$$

Then $y = c_1 y_1 + c_2 y_2$ is the general solution to the ODE. for $c_1, c_2 \in \mathbb{R}$

Example 3. Find the general solution to $y'' + y = 0$

$$\begin{cases} y_1 = \sin x \\ y_2 = \cos x \end{cases} \text{ Both solns to } y'' + y = 0$$

$$\text{Assume Bwoc } c_1 \sin x + c_2 \cos x = 0$$

$$\Rightarrow \frac{-c_1}{c_2} \frac{\sin x}{\cos x} = 1 \Rightarrow \frac{-c_1}{c_2} \tan x = 1$$

↑
Not true for $x=0$
 $\frac{-c_1}{c_2} \tan x = 0$

Lemma: $\sin x$ and $\cos x$ are linearly independent.

What do we do when we already have one solution?

y_1 is a soln to $y'' + p(x)y' + q(x)y = 0$

$y_2 = v(x)y_1(x)$ for some $v(x)$

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

$$v''y_1 + 2v'y_1' + vy_1'' + p(x)(v'y_1 + vy_1') + q(x)v(y_1) = 0$$

$$v(y_1'' + p(x)y_1' + q(x)y_1) + v''y_1 + 2v'y_1' + p(x)v'y_1 = 0$$

$$\begin{aligned} & \underbrace{v(y_1'' + p(x)y_1' + q(x)y_1)}_{W=v'} + v''y_1 + (2v'y_1' + p(x)y_1)v' = 0 \\ & \Rightarrow y_1 w' + (2y_1' + p(x)y_1)w = 0 \end{aligned}$$

$$w' + \left(\frac{2y_1'}{y_1} + p(x) \right)w = 0 \quad \text{First order linear}$$

$$M = e^{\int \left(\frac{2y_1'}{y_1} + p(x) \right) dx} = e^{2 \ln |y_1| + \int p(x) dx} = y_1^2 e^{\int p(x) dx}$$

$$\Rightarrow \frac{d}{dx}[Mw] = 0$$

$$Mw = C$$

$$y_1^2 e^{\int p(x) dx} w = C$$

$$V = W = \frac{C}{y_1^2 e^{\int p(x) dx}} = C e^{-\int p(x) dx} y_1^{-2}$$

let $C=1$ (only need one soln)

$$V = \int e^{-\int p(x) dx} y_1^{-2} dx$$

$$\Rightarrow y_2 = y_1 \int e^{-\int p(x) dx} \frac{dx}{y_1^2}$$

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 2.2–3

2.2 Constant Coefficient Second Order Linear ODEss

Constant Coefficient Second Order Linear ODEs A second order linear ODE is of the form

$$ay'' + by' + cy = f(x)$$

However, for right now we are going to focus on the much easier to solve:

$$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$$

Let's guess a solution of $y = e^{rx}$. What does this achieve?

$$e^{rx} (ar^2 + br + c) = 0$$

Auxiliary equs

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall from prior courses,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What can our roots look like?

C

We have a strategy to find solutions based on the form our roots take.

- 2 Real Roots:

$$r_1 \neq r_2$$
$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x} \quad \leftarrow \text{linearly indep.!}$$

$$Y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

• 1 Real Root:

1 rep. RR root \Rightarrow discriminant = 0
 $\Rightarrow r = \frac{-b}{2a}$

$$Y_1 = e^{rx} \Rightarrow Y_2 = Y_1 \int \frac{e^{-\int p(t)dt}}{Y_1^2} dx = e^{rx} \int \frac{e^{-\int \frac{b}{a}dx}}{e^{2rx}} dx = e^{rx} \int \frac{e^{-\frac{b}{a}x}}{e^{2rx}} dx = e^{rx} \int \frac{e^{-\frac{b}{a}x}}{e^{-2rx}} dx = e^{rx} \int 1 dx = xe^{rx}$$

$$\sim \boxed{y = C_1 e^{rx} + C_2 x e^{rx}}$$

• 2 Complex Roots: \mathbb{C} my beloved

$$r_1 = \alpha + i\beta$$

$$r_2 = \alpha - i\beta$$

$$\text{Euler: } e^{i\theta} = \cos \theta + i \sin \theta$$

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

$$Y_2 = \underline{\quad} = e^{\alpha x} (\cos \beta x - i \sin \beta x) \quad \leftarrow \quad \mathbb{C} \text{ solns!}$$

$$Y_3 = \frac{1}{2}(Y_1 + Y_2); \quad Y_4 = \frac{1}{2}(Y_1 - Y_2)$$

$$= e^{\alpha x} \cos \beta x; \quad = e^{\alpha x} \sin \beta x$$

$$\boxed{y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x}$$

$\leftarrow \mathbb{R}$ solns!

Example 1. Find the general solution for

$$y^{(4)} + y^{(3)} - 3y'' - 2y' = 0. \quad \text{Aux eqn}$$

Guess $y = e^{rt}$

$$e^{rt} (r^4 + r^3 - 3r^2 - 2r) = 0$$

$\hookrightarrow r=0$ trivial soln It's golden!

$$e^{rt} (r)(r+2)(r-\varphi)(r-\bar{\varphi})$$

$$y = C_1 + C_2 e^{-2t} + C_3 e^{\varphi t} + C_4 e^{\bar{\varphi} t}$$

Idea: Higher order constant coeff. linear diff eqs can be solved using $y = e^{rt}$ and factoring the aux eqn into products of quadratic roots!

Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 2.5

2.5 Nonhomogeneous Equations

Constant Coefficient Linear Nonhomogeneous ODEs. A linear nonhomogeneous ODE with constant coefficients is of the form

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x),$$

where $f(x) \neq 0$.

Lebl calls the LHS of this equation Ly , where L is a linear transformation. That is,

$$Ly = a_n y^{(n)} + \cdots + a_1 y' + a_0 y,$$

$$L: C \rightarrow C \quad \text{via} \quad L(y) = \sum_{k=0}^n a_k y^{(k)}$$

Set of continuous fns

where L is the function that turns a function y into this very specific linear combination of y and its derivatives. Can you show that L is a linear transformation?

To solve a nonhomogeneous equation, first solve the *associated homogeneous equation*,

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0, \iff Ly = 0 \quad \text{or} \quad L[y] = 0$$

and call the general solution y_c (c for “complementary”). That’s right. Just pretend that $f(x)$ was never there.

Next, find a particular solution for the original nonhomogeneous equation (drat! $f(x)$ has returned!)

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x), \iff Ly = f(x) \quad \text{or} \quad L[y] = f(x)$$

and call it y_p (p for “particular”).

Sometimes called the forcing fn

Theorem. The general solution to the nonhomogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$$

is

$$y = y_c + y_p,$$

where y_c is the general solution to the associated homogeneous equation, and y_p is *any* particular solution to the original nonhomogeneous equation.

Proof of this theorem follows from the linearity of L .

One question remains: How do we get that one particular solution we need? Yep. That is the hard part. We’ll study two methods:

1. The Method of Undetermined Coefficients¹ *Algebra intensive*
2. Variation of Parameters² *Calculus intensive*

¹This is glorified guess and check.

²This is often called “Var of Parm,” which definitely sounds more delicious.

2.5.1 The Method of Undetermined Coefficients

Example 1. Find the general solution for

$$y'' - 4y' - 12y = \sin 2t.$$

$$e^{rt}(r^2 - 4r - 12) = \sin 2t$$

$$y_h + y_p = C_1 e^{6t} + C_2 e^{-2t}$$

$$\sin 2t = \frac{(-4A + 8B - 12t) \sin 2t + (-4B - 8A - 12B) \cos 2t}{-16A + 8B} = \frac{-8A - 16B}{-16A + 8B} \cos 2t$$

Guess $y_p = A \sin 2t + B \cos 2t$

the undetermined
coefficient(s)

$$y_p' = 2t \cos 2t - 2B \sin 2t$$

$$y_p'' = -4A \sin 2t - 4B \cos 2t$$

$$\Rightarrow A = \frac{1}{20}, B = \frac{1}{40}$$

$$y = y_h + y_p = C_1 e^{6t} + C_2 e^{-2t} + \frac{1}{20} \sin 2t + \frac{1}{40} \cos 2t$$

Example 2. Find the general solution for

$$y'' - 4y' - 12y = 2t^3 - t + 3.$$

$$y_h = C_1 e^{6t} + C_2 e^{-2t} \text{ again!}$$

$$y_p = At^3 + Bt^2 + Ct + D$$

$$y_p' = 3At^2 + 2Bt + C$$

$$y_p'' = 6At + 2B$$

$$A = -\frac{1}{6}, B = \frac{1}{6}, C = -\frac{1}{9}, D = -\frac{5}{27}$$

$$y = C_1 e^{6t} + C_2 e^{-2t} - \frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

Example 3. Find the general solution for

$$y'' - 4y' - 12y = te^{4t}.$$

Superposition Revisited.^a Let y_1 be a solution for

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f_1(x),$$

and y_2 be a solution for

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f_2(x),$$

Then for any constants k_1 and k_2 , $k_1 y_1 + k_2 y_2$ is a solution for

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = k_1 f_1(x) + k_2 f_2(x).$$

^aNow with more super-ness!

2.5.2 Variation of Parameters

Here's a fun thing:

The Wronskian. Let y_1 and y_2 be continuous on some interval I . Then the *Wronskian* of y_1 and y_2 , denoted by $W(y_1, y_2)$, is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}.$$

Theorem: This is not the Wronskian you're looking for.

Let y_1 and y_2 be continuous on some interval I . Then $W(y_1, y_2) = 0$ for all $x \in I$ if and only if y_1 and y_2 are *linearly dependent* on I .

Example 4. Show that $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are linearly independent if and only if $r_1 \neq r_2$.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t} = (r_2 - r_1) e^{(r_1+r_2)t} \\ &\neq 0 \quad \forall t \in \mathbb{R} \\ &\text{iff } r_1 \neq r_2 \end{aligned}$$

Var of parm is great if you have a second order nonautonomous, nonhomogeneous equation and you really like integrals. Suppose first that you have

$$y_c = c_1 y_1 + c_2 y_2,$$

the complementary solution for the second order nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Note that we've normalized our equation so that there is no coefficient on y'' . The big advantage of Var of Parm is that you *don't have to have constant coefficients*. Indeed, p and q can be any gross function of x you want.

Since we have y_c , all we need is y_p , so let's guess $y_p = u_1(x)y_1 + u_2(x)y_2$

where u_1 and u_2 are nonconstant functions of x . This looks gross, so we'll suppress all the (x) 's and have

$$y_p'$$

To get started, we need derivatives of y_p . Well,

which is, again, gross. Now we're gonna make an assumption that may seem like a total scam. This is fine. I promise it will be fine... eventually. For now, though, let's just assume

With this assumption, we now have

Plugging this in to our original nonhomogeneous equation, we have

After some algebra, we have

which is pretty great. Now we can combine this with the assumption we made earlier. It turns out that making this assumption *does* eliminate some of the possible solutions. Do we care? Not really. We only need one y_p ! Now we have two equations:

Solving the first equation for u_1 , we have

Substituting this into the second equation, we have

or, after algebra,

We could do some similar algebra to solve for u_1 . Ultimately, we end up with

This gives us the following fun theorem:

Var of Parm. For the ODE

$$y'' + p(x)y' + q(x)y = f(x)$$

with complementary solution $y_c = c_1y_1 + c_2y_2$, a particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

Example 5. Find the general solution for TYPO:

$$\begin{aligned} r^2 - 2r + 1 \\ (r-1)^2 \end{aligned} \quad W(y_1, y_2) = e^{2t} \quad y'' - 2y' = \frac{e^x}{x^2 + 1}.$$

$$Y_h = C_1 e^t + C_2 t e^t$$

$$Y_p = -e^t \int \frac{te^t \cdot \frac{e^t}{t^2+1}}{e^{2t}} dt + te^t \int \frac{e^t \cdot \frac{e^t}{t^2+1}}{e^{2t}} dt$$

$$= -e^t \int \frac{t}{t^2+1} dt + te^t \int \frac{1}{t^2+1} dt = -\frac{1}{2} e^t \ln(1+t^2) + te^t \arctan t$$

$$Y = C_1 e^t + C_2 t e^t + \cancel{\dots}$$

Example 6. Find the general solution for

$$2y'' + 18y = 6 \tan 3x.$$

Example 7. Find the general solution for

$$xy'' - (x+1)y' + y = x^2.$$