Geometric Limiting Behavior of Sums of Iterates of Polynomial Functions

Introduction

Fractals are geometric shapes that contain details on infinitesimal scales. Often, fractals are self-similar and repeating, and they show up throughout nature. Such examples include the distribution of copper ore, leaves on succulents, broccoli, and blood veins in organisms. Contemporary developments in mathematics have viewed fractals through the lens of function theory, where a rich theory underpins these dynamics. How functions behave through repeated application of them to a set is called "limiting behavior."

Limiting behavior of functions has been studied by mathematicians for centuries. The exponential function, given by $y = e^x$, describes any quantity y that grows over time x proportionally to its' size (consider: disease infection through a population or growth of a financial account). One can express e^x as a limit of a sequence of polynomial functions, which converges for any x (real or complex). Other families of polynomials may only converge for specific inputs, while even other families may not converge at all. Arguably one of the most famous current open problems in mathematics, the Collatz conjecture (also know as the $3n + 1$ problem) is a question on the limiting behavior of a specific sequence of functions. Determining exactly when, and with what conditions, a sequence of functions converges is of great interest for mathematicians, and there are numerous applications of this work in theoretical computer science, engineering, physics, financial mathematics, and beyond.

In discrete dynamics, these sequences of functions are defined by *iterates* of the function; that is, $f^k = f \circ f \circ \cdots \circ f$, the composition of the function f with itself k times. When f is a complex-valued function defined by a polynomial (with complex coefficients and a complex variable, z), the *filled Julia set* of f, denoted $K(f)$, is the set of points that remain bounded under iteration (that is, the set of all points z for which the distance between zero and points in the sequence $f^k(z)$ is bounded from above). These sets are almost always fractal; an example of a filled Julia set, Douady's Rabbit, can be seen in Figure 1.

In my work with Dr. Scott Kaschner (Butler) and Dr. Alex Kapiamba (Brown), rather than studying the limiting behavior of a single function, f, we have been developing a theory of the changes in limiting behavior for a sequence of functions, f_n, by studying the limiting behavior of filled Julia sets. In particular, two families of polynomial functions we studied are $f_n(z) = (p(z))^n + q(z)$ and $g_n(z) = p^n(z) + q(z)$. Similar families of polynomial maps have been well-studied. In a 2012 paper, Boyd and Schulz [2] and later Kaschner, Romero, and Simmons [4] considered $h_n(z) = z^n + c$, where c is a complex parameter; they were able to exactly describe the limiting behavior of $K(h_n)$ for almost all c and described how this behavior depends on c. Work from 2020 by Brame and Kaschner generalized this to maps of $h_n(z) = z^n + q(z)$ [3], where q is a polynomial function, and my work was directly motivated by and generalizes the work of these authors.

For families of maps of one complex variable, $f_n(z) = (p(z))^n + q(z)$, I showed that the limiting behavior of $K(f_n)$ almost always exists and is explicitly described its dependence on both p and q. On the other hand, for maps of the form $g_n(z) = p^n(z) + q(z)$, where p^n is the composition of p with itself n times, the limiting behavior of $K(g_n)$ is periodic (that is, it falls into a repeated cycle of sets, rather than to a single set) unless strict conditions on p and q are met; as a result, subsequential limits may exist for many choices of p and q.

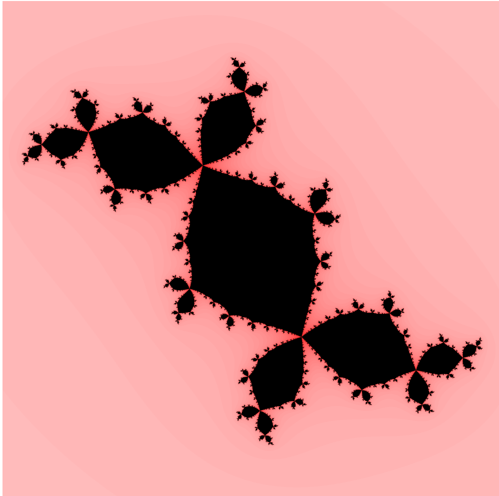

Figure 1: Douady's Rabbit, a filled Julia set, $K(f)$, for $f(z) = z^2 - 0.2 + 0.75i$; the black region is $K(f)$.

Methods

To begin this project, I studied maps of the form $f_n(z) = (p(z))^n + c$. In their 2012 paper, Boyd and Schulz [2] studied the family $h_n(z) = z^n + c$, and they showed that the limiting behavior of $K(h_n)$ only relied on whether c was inside or outside of the unit circle (points whose distance from zero is exactly 1). Heuristically, the z^n term, called the power map, dominates the dynamics of h_n outside the unit circle, but c controls the dynamics inside the unit circle. Building off this work, we developed a theory for the family $f_n(z) = (p(z))^n + c$. A similar intuitive understanding applies: after taking the image of z by p, the power map dominates outside the circle. Thus, a first corollary was developed, where the limiting behavior of $K(f_n)$ only relies on if c is inside or outside of the preimage of the unit circle by p. Turning to a more general family of maps $f_n(z) = (p(z))^n + q(z)$, the previous theory from [3] is applicable. The arguments presented therein were modified and extended to this family.

The dynamics for the family of maps $g_n(z) = p^n(z) + q(z)$ is much more interesting. Unlike the previously studied families, the dynamics of $K(g_n)$ no longer depend solely on one parameter or the two polynomials p and q independently; the roles of both p and q are inherently dependent on each other. It is because of this that the limit fails to exist in most cases. One necessary condition for $\lim_{n\to\infty} K(g_n)$ existing is p must have a singular attracting fixed point in the complex plane (as opposed to an attracting cycle, which occurs more often). However, we can modify the index by which we consider the limiting behavior. If we assume that p is hyperbolic, then all points in $\mathcal{A}$, the interior of $K(p)$ will be attracted to periodic attracting cycles, which we denote $z_1, \ldots, z_k$. Thus, p^k, will have fixed points. We define a "good" subsequence n_m as one for which p^{n_m} converges on the interior of the set $K(p)$. Thus, we can define $\hat{g}(z) = q(z) + \lim_{m\to\infty} p^{n_m}(z)$; this function can be used to approximate g_n, and the approximation gets better with larger n. Thus, when we consider a good subsequence, the limiting behavior of $K(g_n)$ is controlled (see Figure 2).

Theorem 1. *For any polynomials p and q and good sequence n_m for p,*

$$\left(\partial \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\mathcal{A}) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j\right) \subseteq \liminf_{m\to\infty} K(g_{n_m}) \subseteq \limsup_{m\to\infty} K(g_{n_m}) \subseteq \left(\bigcap_{j=0}^{\infty} \hat{g}^{-j}(\mathcal{A}) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j\right), \quad (1)$$

where $\mathcal{J}_j = \{\hat{g}^j(z) \in J(p)$ and $\hat{g}^\ell(z) \in \mathcal{A}$ for $\ell = 1, \ldots, j-1\}$.

Theorem 2. *Let p and q be polynomials such that n_m is good for p, $\lim_{k\to\infty} \hat{g}^k(\bigcap_{j=0}^{\infty} \hat{g}^{-j}(\mathcal{A}))$ exists, and $\hat{g}$ has no attracting periodic points on the boundary of $K(p)$. Then*

$$\lim_{m\to\infty} K(g_{n_m}) = \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\mathcal{A}) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j.$$

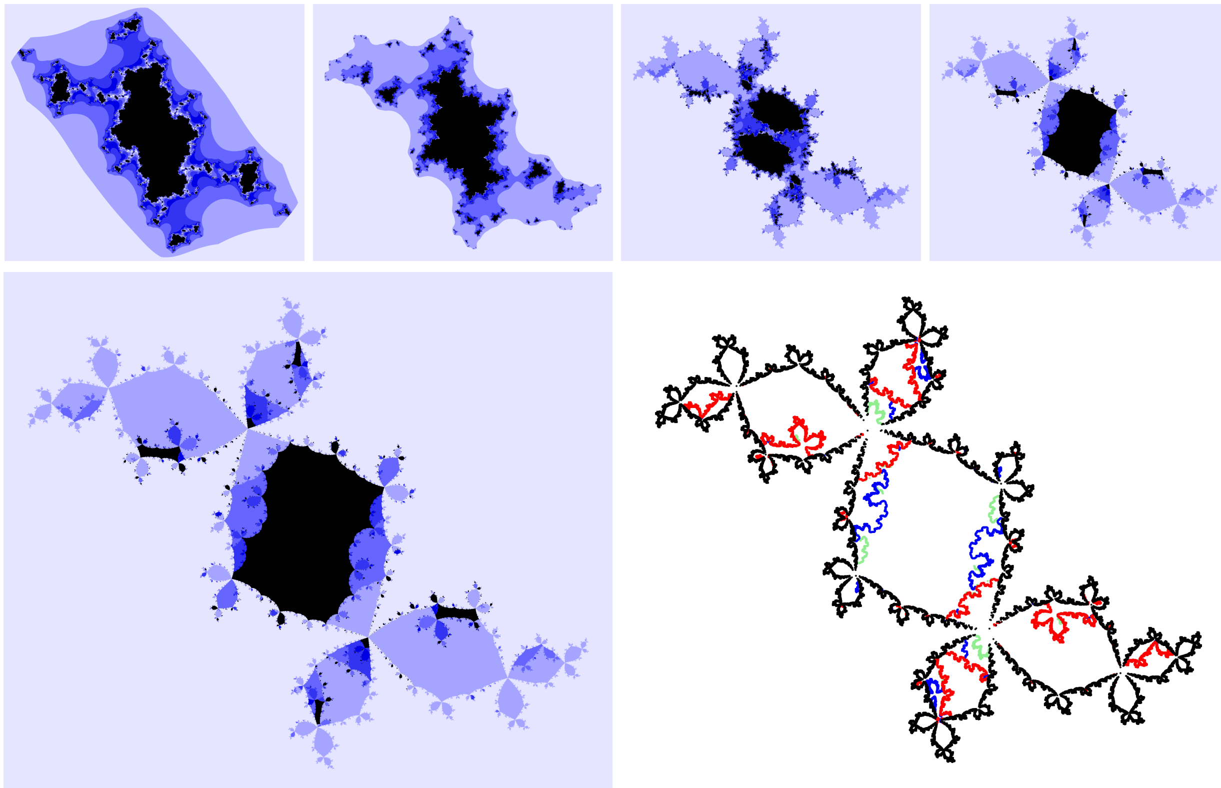

Figure 2: Top: $K(g_n)$ for $p(z) = z^2 - 0.123 + 0.745i$, $q(z) + z^2 + 0.12 - 0.3i$, $n_m = 3, 6, 15, 150$ (i.e. along the good subsequence $0 \mod 3$). Bottom left: $K(g_{1500})$ for the same p and q. Bottom right: boundary of $K(p)$ (black) and $\mathcal{J}_j$ for $j = 1, 2, 3$ (red, blue, green).

Conclusion

The theory developed in this work is consistent with and has generalized the recent developments over the past decade into one comprehensive family. Additionally, a new class of families, considering iterates of polynomial maps, was introduced. There remain open questions regarding the need for technical hypotheses on either p or q, such as hyperbolicity; even in the quadratic case it is unknown whether or not almost every polynomial is hyperbolic. Additionally, various proofs for the work presented herein rely on a potential theoretic argument; simpler proofs are desired. Finally, extensions to the family of iterates of rational maps are welcome.

References

[1] A. M. Alves. Geometric limits of julia sets and connectedness locus of the family of polynomials $p_c(z) = z^n + cz^k$. *Dynamical Systems*, (just-accepted):1–17, 2019.

[2] S. H. Boyd and M. J. Schulz. Geometric limits of Mandelbrot and Julia sets under degree growth. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 22(12):1250301, 21, 2012.

[3] M. Brame and S. Kaschner. Geometric limits of Julia sets for sums of power maps and polynomials. *Complex Analysis and its Synergies*, 6:1–8, 2020.

[4] S. R. Kaschner, R. Romero, and D. Simmons. Geometric limits of Julia sets of maps $z^n + \exp(2\pi i\theta)$ as $n \to \infty$. *International Journal of Bifurcation and Chaos*, 25(08):1530021, 2015.